Critical phenomena, the volume of a Wiener and the feeling of confusion

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Imperial College London

Imperial College London, Lunchtime seminar, 1 Nov 2012

Outline

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Phase transitions

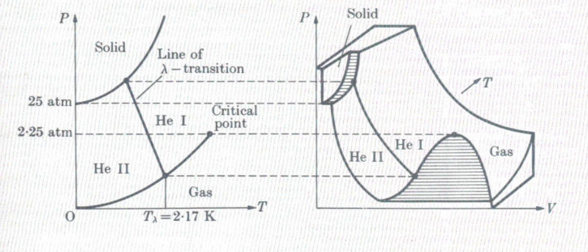

FIG. 1.14. Sketch (not to scale) of the equilibrium PVT surface of ^{4}He in the low-temperature region, and the projection of this surface on to the PT plane. Contrast this phase diagram with that of a normal fluid shown in Fig. 1.1(a). From Huang (1963).

From Stanley, 1971

Phase transition: Sharp change of the properties of a material.
Second order (continuous) transition: Weird and wonderful!

Universality at the gas/liquid critical point

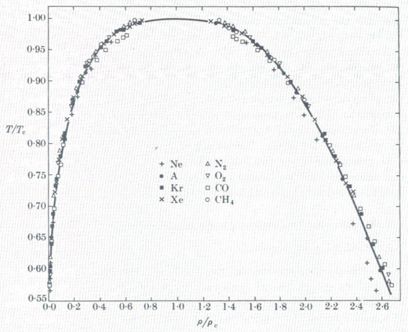

FIG. 1.8. Measurements on eight fluids of the coexistence curve (a reflection of the $P\rho T$ surface in the ρT plane analogous to Fig. 1.3). The solid curve corresponds to a fit to a cubic equation, i.e. to the choice $\beta = \frac{1}{3}$, where $\rho - \rho_c \sim (-\epsilon)^\beta$. From Guggenheim (1945).

From Stanley, 1971

Critical point: T_c, p_c. There and in the approach, very different materials display the same features:

Universality!

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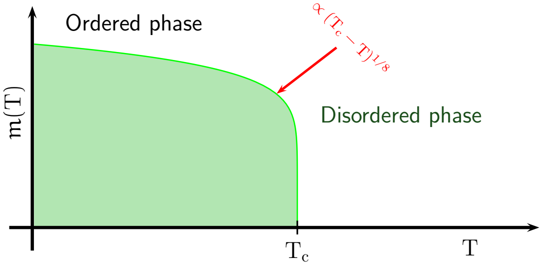
Order parameter



The order parameter indicates the change (positive in the ordered phase, vanishing in the disordered phase).

Universal exponents characterise continuous phase transitions. For example:

$$m(T) = \begin{cases} 0 & \text{for } T > T_c \\ a(T_c - T)^{1/8} + \dots & \text{for } T \leqslant T_c \end{cases}$$

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Correlation function

$m(\mathbf{r})$ is the local order parameter (a measure of local order, say, the local amount of condensate).

Two-point correlation function $C(\mathbf{r})$ measures how local fluctuations are related:

$$C(\mathbf{r}) = \left\langle \left(m(\mathbf{r}+\mathbf{r}') - \left\langle m(\mathbf{r}+\mathbf{r}')\right\rangle\right) \left(m(\mathbf{r}') - \left\langle m(\mathbf{r}')\right\rangle\right)\right\rangle$$

Translational invariance: $C(\mathbf{r})$ independent from $\mathbf{r}'$.
At critical point:

$$C(\mathbf{r}) = |\mathbf{r}|^{-1/4}$$

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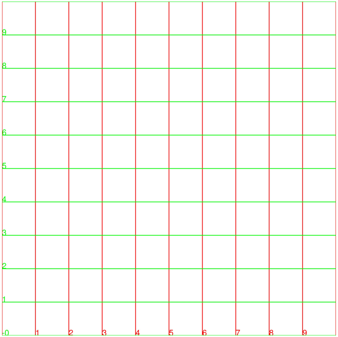
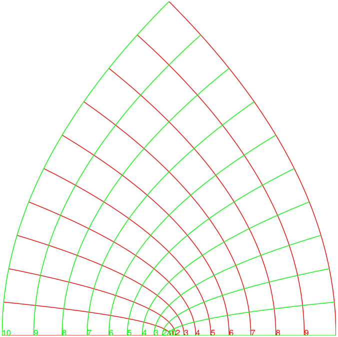
Scale invariance

Power laws are the signature of scale invariance: “Same” features on every scale.

Conformal invariance: Statistics invariant under conformal maps



Original lattice



z^2 map

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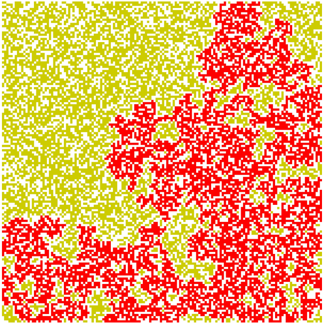
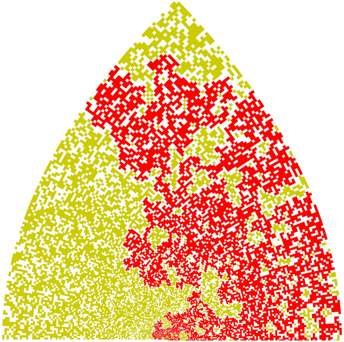


Original cluster



Mapped cluster

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Power law correlation function

"Why is a power law any different from any other functional dependence? What is the physical significance of scaling?"

Full scaling[1] — pure power law: No scale from within.
Example:

- Exponential correlations, $C(r) = \exp(-x/\xi)$. Correlation length[2] = distance over which correlations decay by e^{-1}.

$$C(r+\xi) = C(r)/e$$

- Power law, $C(r) = ar^{-2}$: Correlations decay by the same factor at every multiple:

$$C(r\sqrt{e}) = C(r)/e$$

[1] As opposed to finite size scaling with intermediate power law scaling.
[2] In general, this holds only asymptotically.

Phase transitions

Key features

- Appear at some critical point (temperature, probability . . .)
- Universal
- Observables governed by power laws
- Emergence (more is different!)
- Scale invariance
- 2D: Conformal invariance
- (SLE: Oded Schramm, Stanislav Smirnov, Wendelin Werner, Terence Tao)

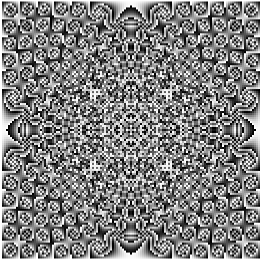
Outline

1. Critical phenomena

2. Self-organised criticality
   - Prelude: The physics of fractals — the BTW Model
   - Why SOC?
   - Experiments
   - More models? Better models!

3. Field theory

Prelude: The physics of fractals



Question: Where does scale invariant behaviour in nature come from?

Answer: Due to a phase transition, self-organised to the critical point.

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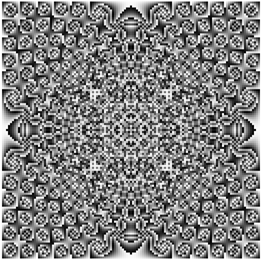
Prelude: The physics of fractals



- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- $1/f$ noise "everywhere" (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where's the Physics?*
- Bak, Tang and Wiesenfeld, 1987: *Self-Organized Criticality: An Explanation of $1/f$ Noise*

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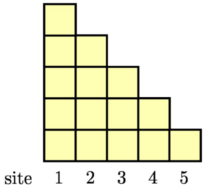
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

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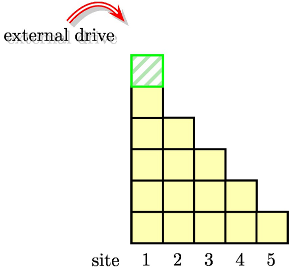

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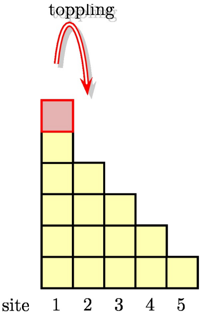

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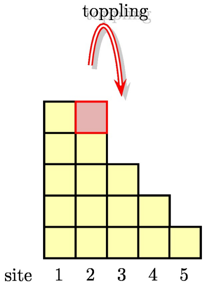

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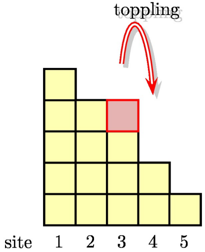

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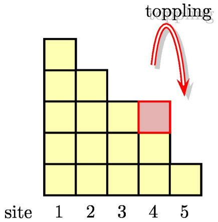

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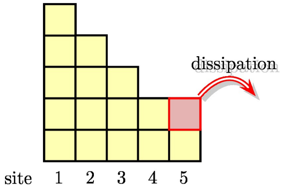

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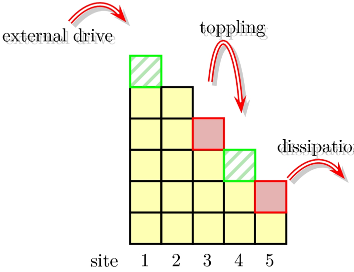

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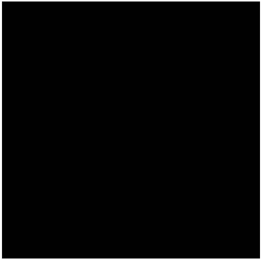

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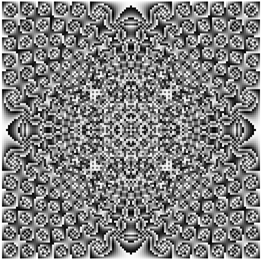


Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.

Why is SOC important?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Emergence!

- Explanation of emergent,
- …cooperative,
- …long time and length scale
- …phenomena,
- …as signalled by power laws.

Why is SOC important?

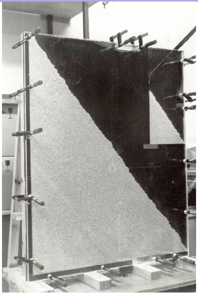
SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Universality!

- Understanding and classifying natural phenomena
- …using *Micky Mouse Models*
- …on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?

Experiments:

Granular media, superconductors, rain...



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al.* (Physica C, 1991).
- Ricepiles experiments by Frette *et al.* (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

More models

- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models…

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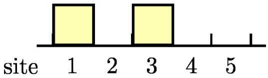
More models

The failure of SOC?

- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, $\alpha \approx 0.05$ (localisation), $\alpha = 0.22$ (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]
- Directed Models: Exactly solvable (lack of correlations)

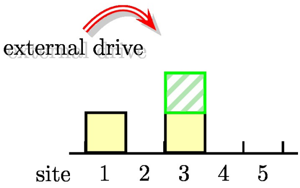
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.
- **Working field theory exists!**

Manna Model



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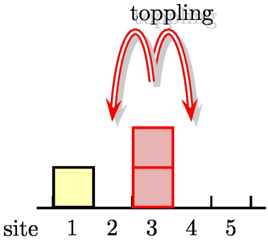

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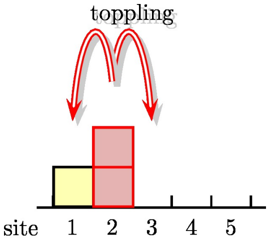

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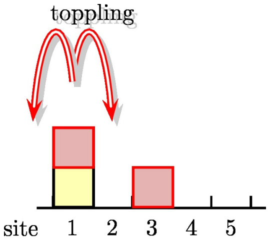

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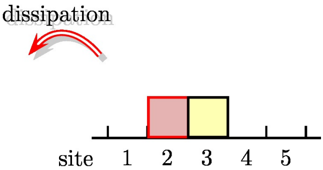

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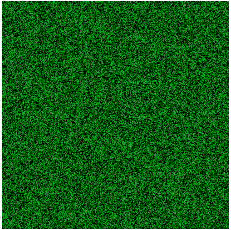

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Outline

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Programme

Take the stochastic process of interest and rewrite it in terms of creation and annihilation operators:

$$\begin{aligned} a^\dagger(\mathbf{x})\,|n_\mathbf{x}\rangle &= |n_\mathbf{x}+1\rangle \\ a(\mathbf{x})\,|n_\mathbf{x}\rangle &= n_\mathbf{x}\,|n_\mathbf{x}-1\rangle \end{aligned}$$

$|n_\mathbf{x}\rangle$ is a configuration with $n_\mathbf{x}$ particles at site $\mathbf{x}$. These “coherent states” are eigenstates of the particle number operator

$$a^\dagger(\mathbf{x})a(\mathbf{x})\,|n_\mathbf{x}\rangle = n_\mathbf{x}\,|n_\mathbf{x}\rangle$$

$|0\rangle$ is the empty system.

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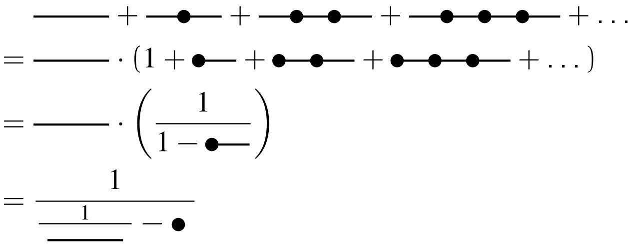
Perturbative Field Theory

Integrals are written in diagrams.
Loops and multiple interactions can be (re)summed into effective couplings:



on confusion

Youtube clip

Field Theory for SOC

- Rewrite Manna Model in terms of creation and annihilation operators
- Calculate diagrams
- Sum over diagrams (renormalisation)
- Find critical properties
- Identify mechanism of SOC

Which Wiener?

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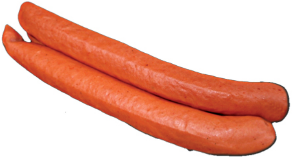

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Wiener process

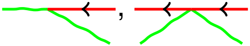
(named after Norbert Wiener)

Consider a random walker on a 2D lattice:

What is the (blue) area of the trace (volume of a “Wiener sausage”)?

Volume of a Wiener

Originally calculated by Kolmogorov and Leontovich (1933).
Field theory:

- Walker walks:  $= \frac{1}{-\imath\omega + D\mathbf{k}^2}$
- ...and leaves behind a trace in the form of branched-off particles 
- ...which are stuck on the lattice $= \frac{1}{-\imath\omega + \epsilon}$
- No deposition if there is a particle already 

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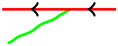
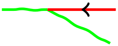
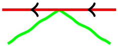
Details of the diagrams

Deposition is suppressed in the presence of deposits. *Without* that, deposits could be found all along the walker's trajectory (multiple deposits at revisited sites):

 and 

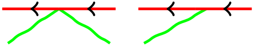
These two diagrams probe the lattice for deposits:

 and 

Renormalisation

Calculate features (such as the volume of the Wiener) using **renormalisation**.
Example: Deposit along the trajectory

. . . is reduced by suppressed deposition



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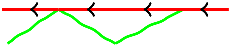
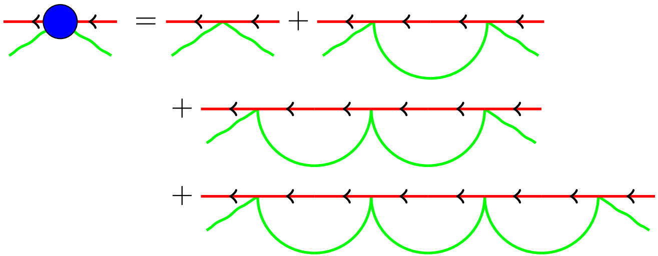


Renormalisation

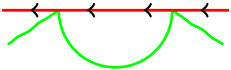
At the heart of the theory is only one diagram that needs renormalisation:



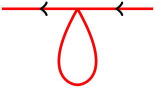
Renormalisation

What are the loops?

What physical process do the loops



correspond to? Trajectory intersecting itself:



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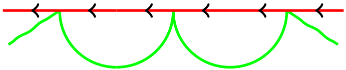
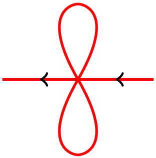


correspond to? Trajectory intersecting itself:



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Results

- Field theory reproduces earlier results.
- Different boundary conditions easily accessible.
- Perfect playground to understand renormalisation, the nature of "fermionicity" and boundary conditions.
- Huge interest in the Wiener sausage phenomenon from chemistry and medical science.

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Summary

- Phase transitions: Singularities, universality, emergence.
- Self-organised Criticality: All of the above by self-tuning to critical point.
- Field theory: Diagrammatic representation of complicated physics (Manna Model, Wiener sausage)

Thanks!

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