Perimeter scaling in ordinary percolation

$\frac{7}{4}$, $\frac{4}{3}$ and all that

Gunnar Pruessner

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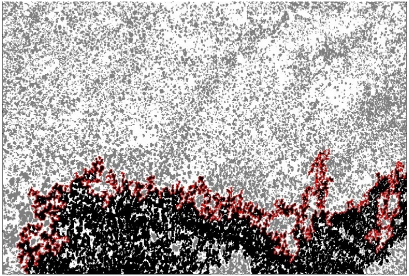
Biological motivation

- Characterise natural borderlines.
- Input to conservation and biomonitoring.
- Identification of the underlying process (universality class).
- Accepted ecological model: Contact process.
- Directed percolation universality class in nature.
- Relation to (gradient) percolation.

Biological motivation



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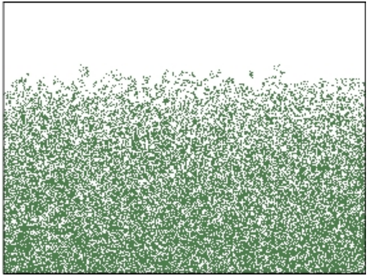

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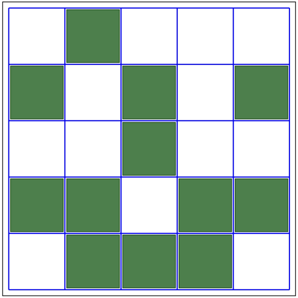
Outline

Contact Process

Definition of the model



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- Extinction with rate e
- Colonisation with rate c
- Driving parameter: $\lambda = c/e$

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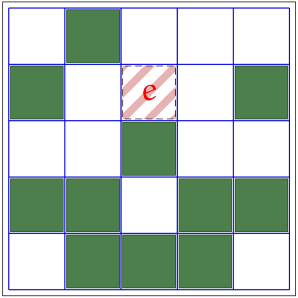

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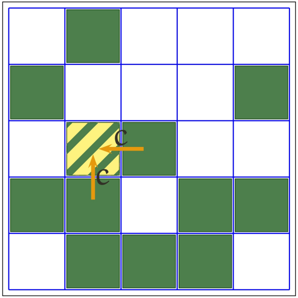

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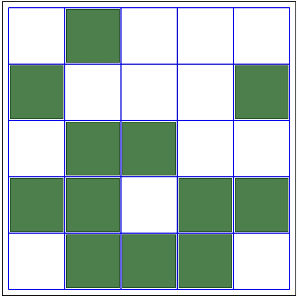

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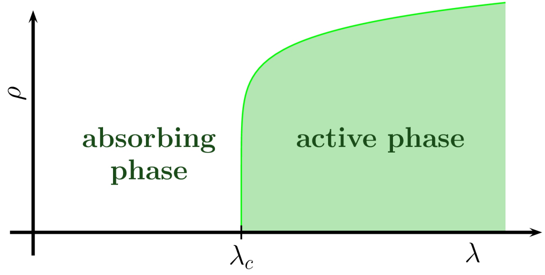

- Driving parameter: $\lambda = c/e$
- Order parameter: Density of occupied sites ρ
- Continuous phase transition (absorbing state)
  Directed percolation universality class
- Gradient contact process: $\lambda(x) = c(x)/e = \lambda' x$
- “Tuning parameter”: Space
- Borrow definition of borderlines from gradient percolation.

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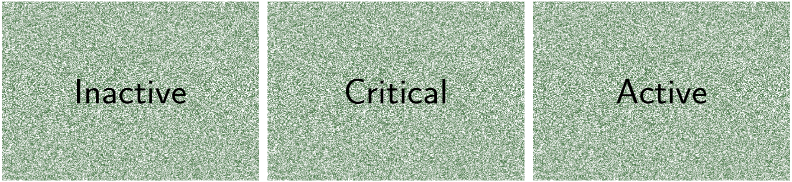

$\lambda < \lambda_c$ $\lambda = \lambda_c$ $\lambda > \lambda_c$

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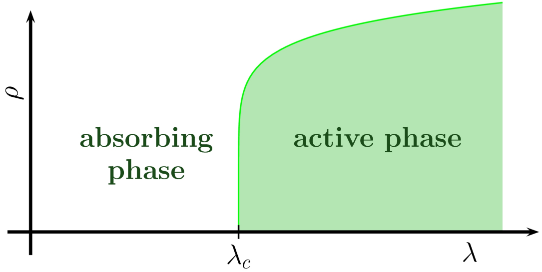

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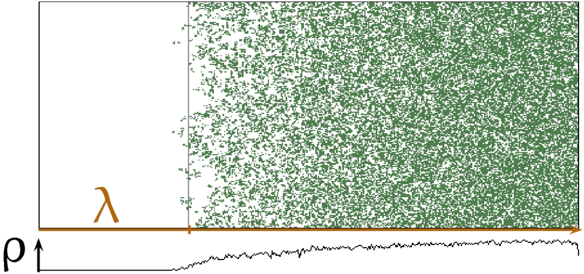

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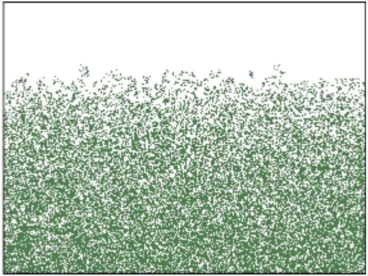

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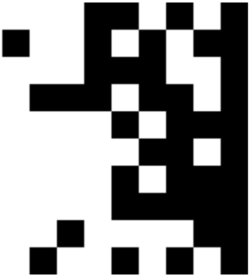
Outline

Motivation to study borderlines

- Biological relevance:
  - Species borders of great ecological importance
  - Changes in borderlines “just fluctuations”?
  - Identify different regions across the borderline (connectivity to large clusters)
- Questions and tasks
  - Define borderlines
  - Scaling of borderlines
  - How to control borderlines
  - Dynamics of borderlines
  - Field data readily available

Gradient percolation

Null model of the gradient contact process



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- …on a *big* lattice.
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- Find largest (spanning) cluster.
- Characterise that cluster.

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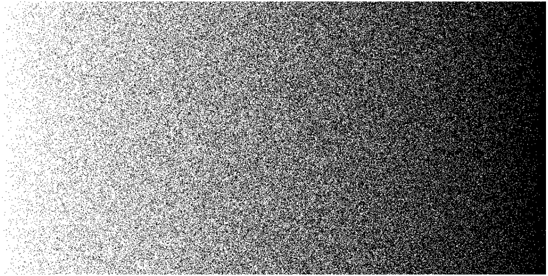

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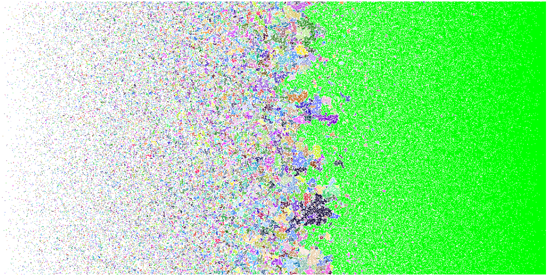

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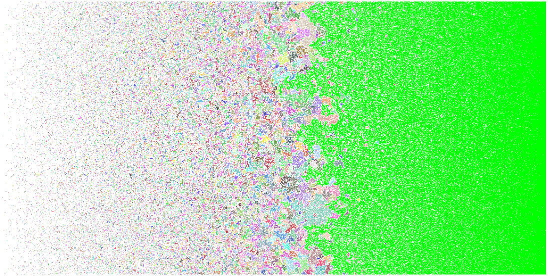

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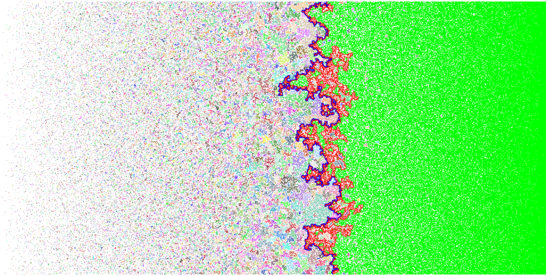

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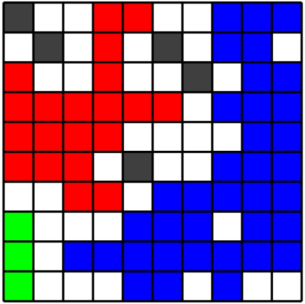
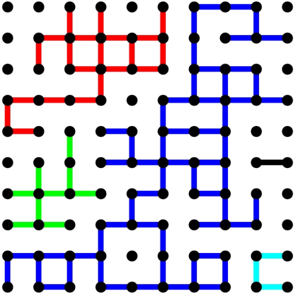
Percolation

Illustration of the model



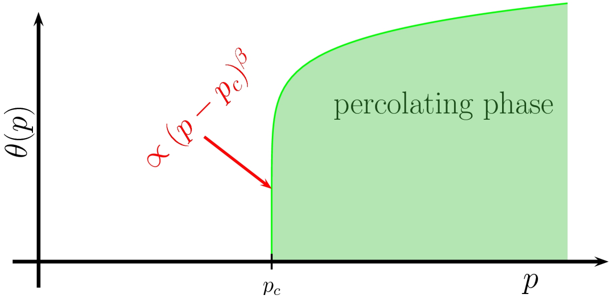
Sites occupied with probability p



Bonds active with probability p

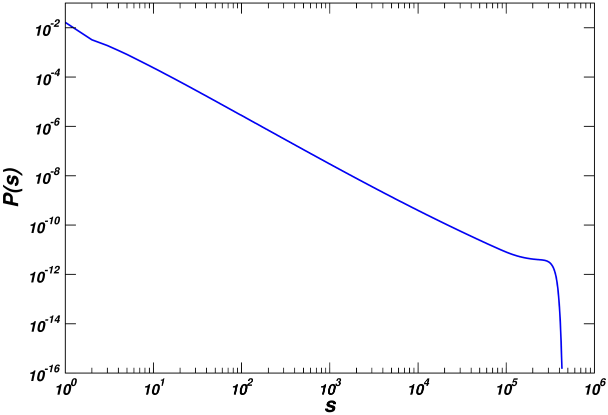
Cluster: Sites connected through occupied sites and active bonds.
Imposed interaction.

Percolation: Key features I



Order parameter *P*: fraction in the "infinite" cluster
In 2D: $\beta = 5/36$

Percolation: Key features II

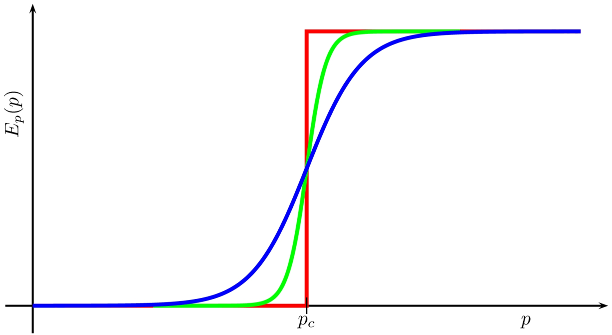


Cluster size distribution (density of *s*-clusters per site):

$$P(s) = as^{-\tau}\mathcal{G}(b(p-p_c)s^{\sigma})$$

In 2D: $\tau = 187/91$ and $\sigma = 36/91$

Percolation: Key features III



Crossing probability $E_p(p)$ for different system sizes.
At p_c crossing probability $0 < E_p(p) < 1$ even in the thermodynamic limit.

Percolation

A brief history I

- Three dimensional polymers: Flory 1941
- Mathematics: Hammersley and Broadbent 1954
- $p_c = 1/2$ in 2D bond percolation conjectured in 1955
- $\theta(1/2) = 0$ by Harris, 1960
- $p_c = 1/2$ tackled by Sykes and Essam, 1963
- “Dormant state”

Details: Grimmet, *Percolation*, 1997

Percolation

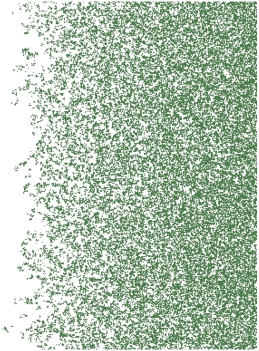
A brief history II

- Back on stage: Russo, and Seymour and Welsh, 1978
- Kesten: $p_c = 1/2$, 1980
- Uniqueness of infinite cluster: Newman and Schulman, 1981
- Renaissance because of Conformal Field Theory for crossing probabilities: Langlands *et al.* 1992, Cardy 1992
- Multiple spanning clusters: Hu and Lin 1996, Aizenman 1997, Cardy 1998
- Percolation is SLE with $\kappa = 6$, Smirnov 2001

Details: Grimmet, *Percolation*, 1997

Gradient percolation

Features of the borderline



- Borderline characterised by (gradient) percolation.
  Two definitions of borderline:
  hull (length l_h) and perimeter (length l_p)
- Borderline away from critical region.

Gradient percolation

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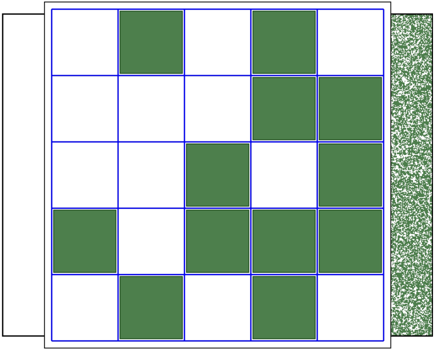

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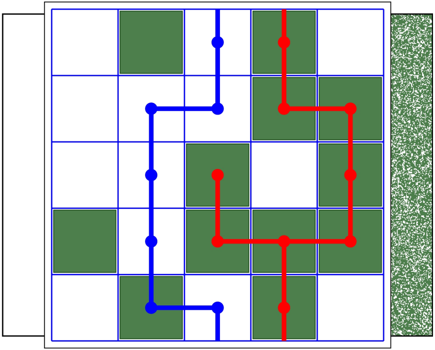

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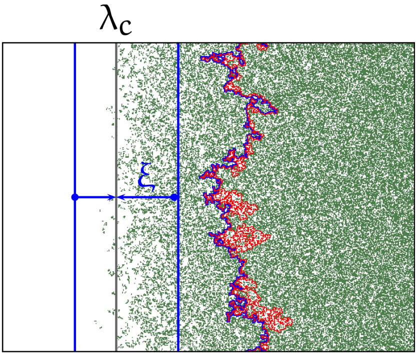

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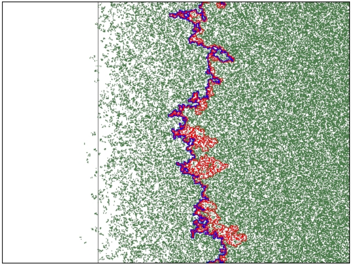

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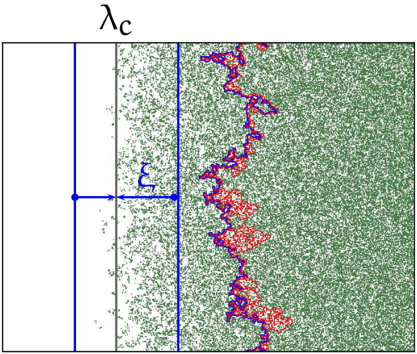
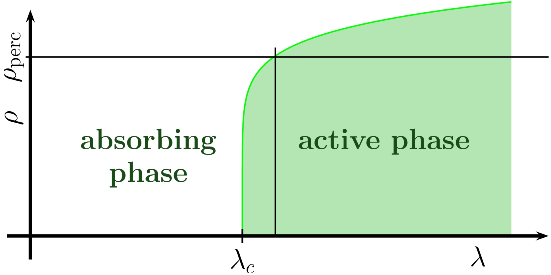


- Sapoval, Rosso and Gouyet (1985), Saleur and Duplantier (1987), Smirnov and Werner (2001): Fractal dimension of hull $D_h = 7/4$ (fragile!), fractal dimension of perimeter $D_p = 4/3$ (SAW)
- Surprise(?): Despite correlations, hull scales as in gradient percolation, i.e. both exponents are found numerically

Borderlines are a percolation effect



Key observation: Borderline exponents are due to percolation, independent of the underlying process.
In fact, percolation is “imposed” and exponents an artefact.

If correlation length is small compared to percolation feature studied, the underlying process has no effect:

$7/4$ and $4/3$ everywhere

— the blessing and curse of universality.

Gradient percolation

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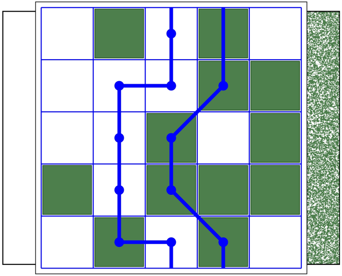
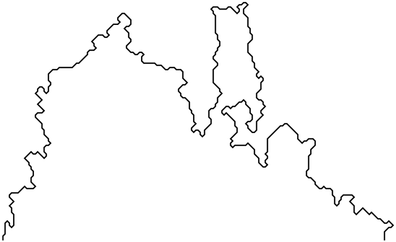


- Generalised definition of hull:
- l is the steplength to define the cluster.
- $s \geqslant l$ is the steplength to define an s-hull.
- $s = l$ produces the standard hull $(7/4)$,
  $s > l$ produces a perimeter $(4/3)$.

Measuring the fractal dimension



Box counting

Number k of discs of radius r needed to cover a d_f dimensional object:

$$k \propto r^{-d_f}$$

Plot k versus r

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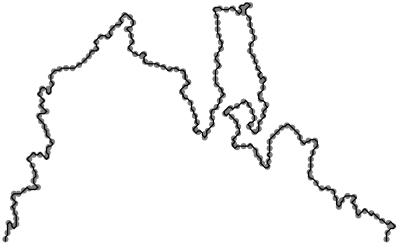

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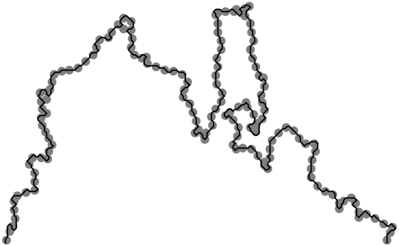

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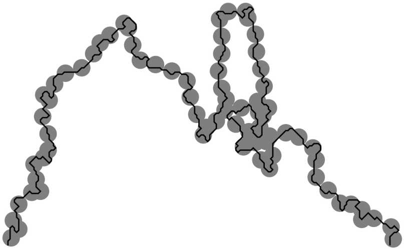

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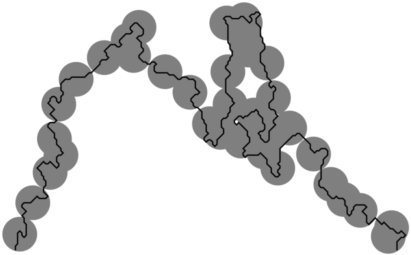

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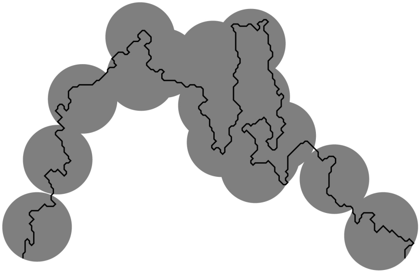

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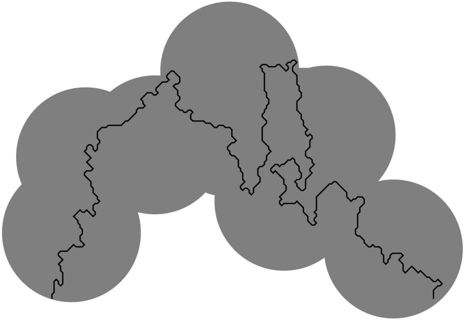

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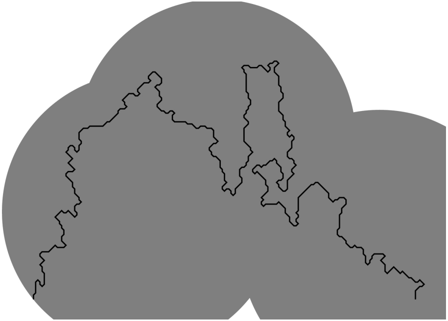

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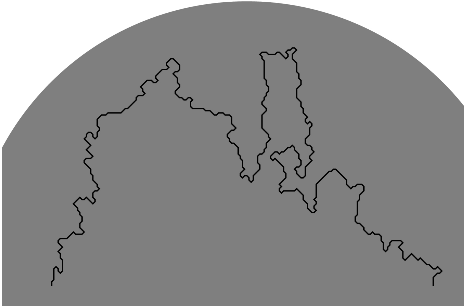

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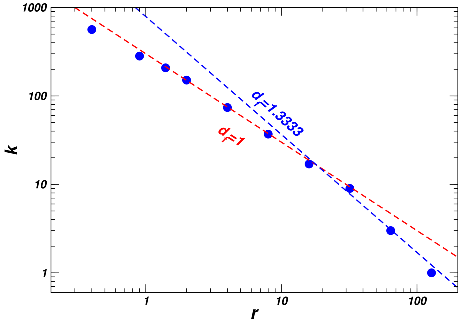

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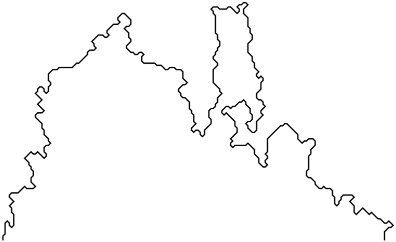

Equipaced polygon method

Average distance $\bar{r}$ between points on curve skipping l intermediate steps:

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Using $k \propto l^{-1} \propto \bar{r}^{-d_f}$.

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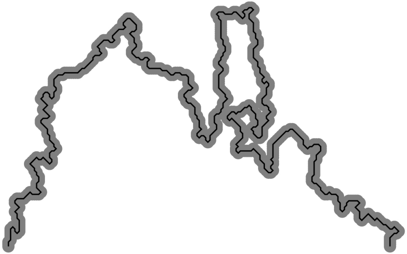

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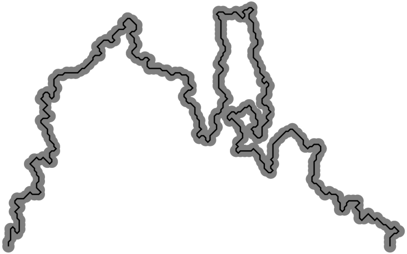

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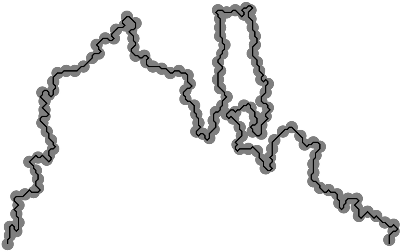

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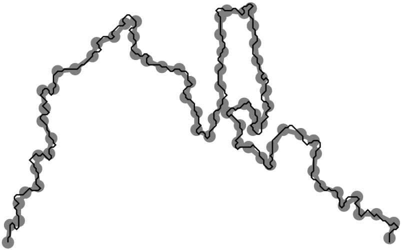

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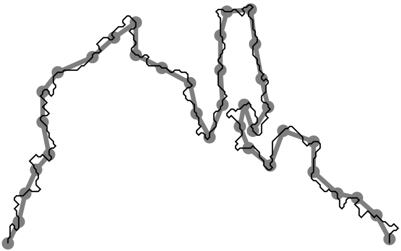

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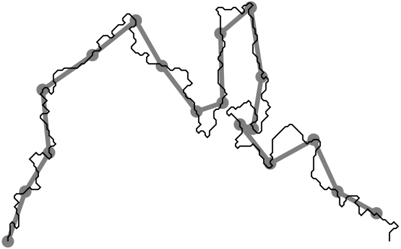
Average distance $\bar{r}$ between points on curve skipping l intermediate steps:

$$\bar{r} \propto l^{1/d_f}$$

Using $k \propto l^{-1} \propto \bar{r}^{-d_f}$.

Measuring the fractal dimension



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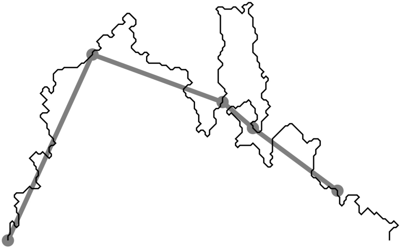

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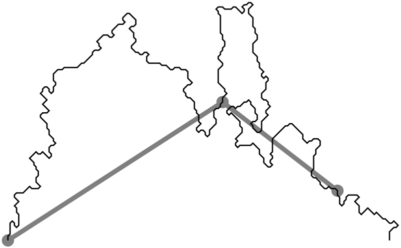

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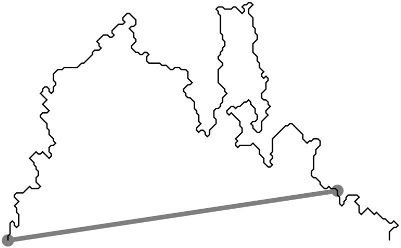

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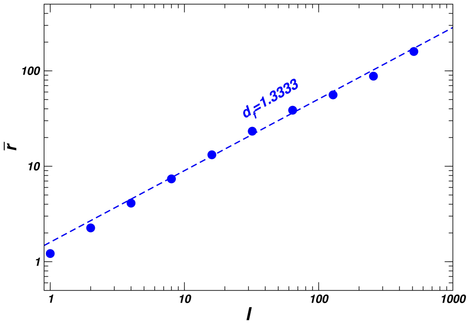

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- Original: Nice and colourful.
- Introduce grey levels and threshold
- Determine largest cluster and characterise
- $7/4$ ($s = l$, hull) and $4/3$ ($s > l$, perimeter) well reproduced.

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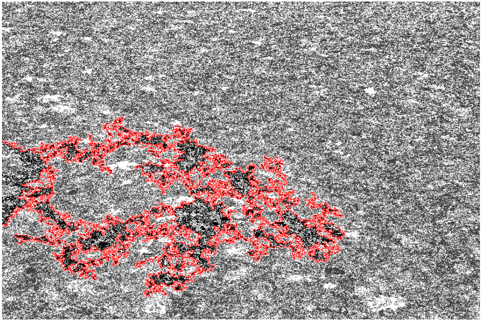

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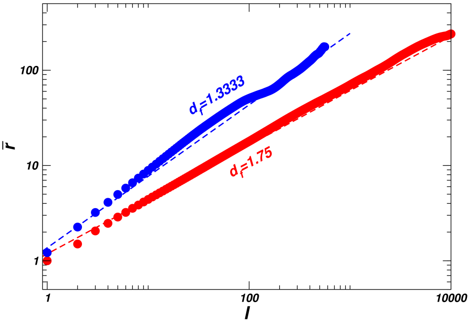

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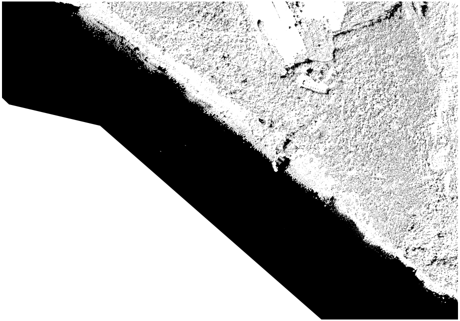
7/4 versus 4/3 in environmental gradients

7/4 slipping away. . .



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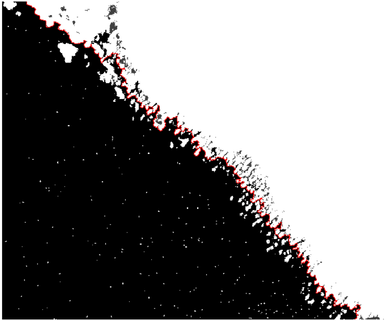

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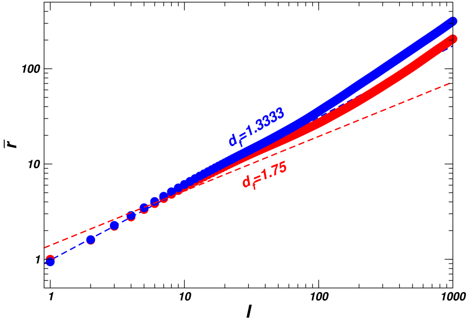

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7/4 versus 4/3 in environmental gradients

7/4 slipping away…

- Theory: 7/4 on the short scale, 4/3 on the large scale.
- Simulations ([gradient] percolation and [gradient] contact process): 7/4 for ($s = l$, hull) and 4/3 for ($s > l$, perimeter), as expected.
- Empirical data (mostly) produce 4/3 for all s.
- Some data compatible with 7/4 on the *large* scale.

Why?

7/4 versus 4/3

Why almost always 4/3?

Correlations (underlying process + mapping on lattice)?
Scales involved:

- Lattice spacing (disc size, reed size) r_0.
- System size L.
- Definition of cluster l.
- Determination of hull s.
- Rule (Kolb): Crossover from 7/4 to 4/3 on scales bigger than $(s/l-1)^{-\alpha}$.
- Do finite correlations modify Kolb's rule even at large s, l and L?

Summary
7/4 versus 4/3

- Percolation properties independent from underlying process.
- Expected hull-exponent: 7/4
- Expected perimeter-exponent: 4/3
- Both exponents are man-made and bear little physical relevance.
- Confirmed numerically.
- Empirical evidence: perimeter-exponent 4/3 often prevails even for the hull.
- Why? Additional length scale?

Thank you!