The zero dimensional contact process

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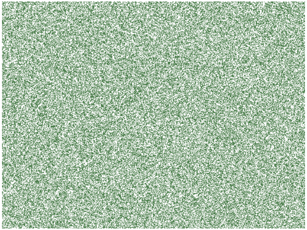
Physical motivation



- Contact process around for about 50 years.
- *Enormous* universality class (Janssen, Grassberger, de la Torre).
- Field theory not fully understood.
- Applications: Dynamics in ecology.

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$$\partial_t \phi(\mathbf{x}, t) = \lambda\phi(1-\phi) - \phi + \nabla^2\phi + \sqrt{\phi(\mathbf{x}, t)}\eta(\mathbf{x}, t)$$

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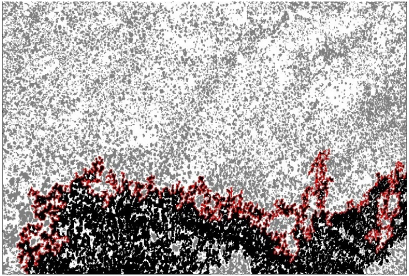

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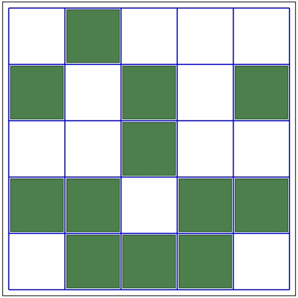
Outline

Contact Process

Definition of the model



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- Extinction with rate e
- Colonisation with rate c
- Driving parameter: $\lambda = c/e$

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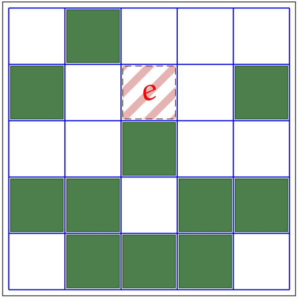

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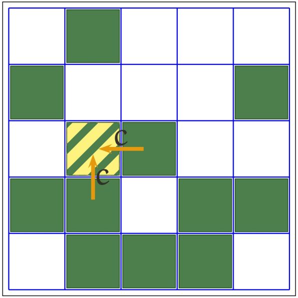

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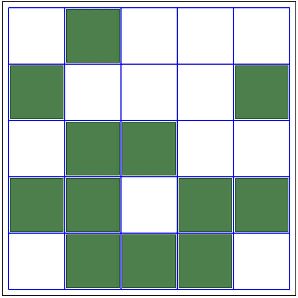

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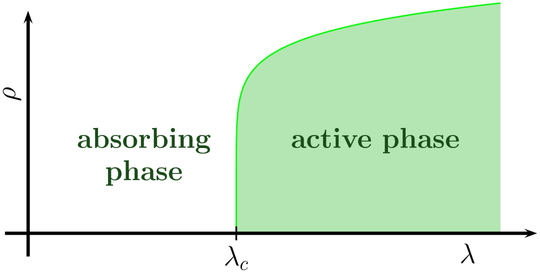

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- Order parameter: Density of occupied sites ρ
- Continuous phase transition (absorbing state)
  Directed percolation universality class
- **Universality!**
- Field theory understood, but generally hindered by non-linear noise.

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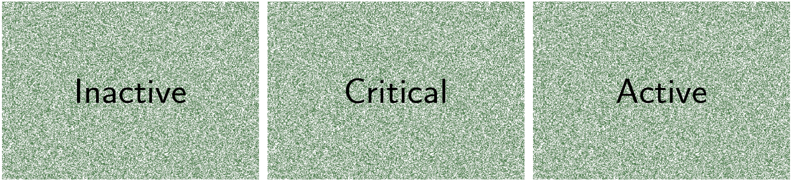

$\lambda < \lambda_c$ $\quad$ $\lambda = \lambda_c$ $\quad$ $\lambda > \lambda_c$

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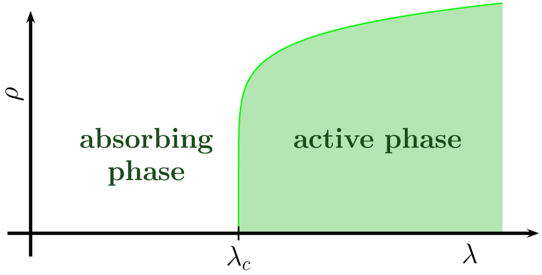

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Directed Percolation (DP) should be everywhere!

(But isn't)

The DP conjecture by Janssen [1981] and Grassberger [1982]:

Under very general circumstances, every system that has a unique absorbing state belongs to the *directed percolation* (DP) universality class (same exponents, same scaling functions, same amplitude ratios etc.).

However...

To date, the DP universality has not/hardly been observed in nature (maybe due to: anisotropy, quenched noise...)

Recent experiments: Takeuchi *et al.*

Langevin equation of the contact process

Basic ingredients

$\rho(\mathbf{x}, t)$ is the density of particle at time t and position $\mathbf{x}$.
Equation of motion:

$$\partial_t \rho = c\rho(\rho_0 - \rho) - e\rho$$

- Colonisation provided carrying capacity ρ_0 (uniform, constant) is not exceeded.
- Extinction with constant rate.
- Rescale time and particle density.
- Solve Mean Field Theory — stationarity state: $\rho = 1 - 1/\lambda$
- Phase transition at $\lambda = \lambda_c = 1$.

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Space and noise

$\rho(\mathbf{x}, t)$ is the density of particle at time t and position $\mathbf{x}$.
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$$\partial_t \rho = \lambda \rho (1 - \rho) - \rho + D\nabla^2 \rho$$

- Add effective diffusion.
- Add noise.
- Noise is Gaussian and white, $\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2\Gamma^2 \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$.
- Amplitude accounts for "number of attempts" (Ito convention).

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$$\partial_t \rho = \lambda\rho(1-\rho) - \rho + D\nabla^2\rho + \sqrt{\rho}\eta(\mathbf{x}, t)$$

Build a field theory from it:

$$\int \mathcal{D}\rho\mathcal{D}\tilde{\rho}\,\mathrm{exp}\left(\int \mathrm{d}^d x \mathrm{d}t\ \tilde{\rho}(\partial_t - D\nabla^2 - (1-\lambda))\rho + \Gamma^2\tilde{\rho}(\lambda\rho - \tilde{\rho})\rho\right)$$

Noise is *not* bilinear: $\Gamma^2\tilde{\rho}\tilde{\rho}\rho$.

Langevin equation of the contact process

$$\partial_t \rho = \lambda\rho(1-\rho) - \rho + D\nabla^2\rho + \sqrt{\rho}\eta(\mathbf{x}, t)$$

Which term can be removed without making it trivial?

- Interaction? No!
- Noise? No!
- Diffusion? . . . YES! . . .

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Langevin equation of the contact process

In 0 spatial dimensions

$$\partial_t \rho = \lambda\rho(1-\rho) - \rho + \sqrt{\rho}\eta$$

- All space dependence gone.
- Rescale time: $s(t) = \int_0^t \mathrm{d}t' \rho(t')$
- Redefine density: $\rho(t) = \hat{\rho}(s(t))$
- It follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(s(t)) = \rho\hat{\rho}' = \lambda\rho(1-\rho) - \rho + \sqrt{\rho}\eta$$

Langevin equation of the contact process

In 0 spatial dimensions

$$\hat{\rho}'(s) = \lambda(1 - \hat{\rho}(s)) - 1 + (\hat{\rho}(s))^{-1/2}\eta(t)$$

- Now note: $\langle \eta(t)\eta(t') \rangle = 2\Gamma^2\delta(t - t') = 2\Gamma^2\rho(s)\delta(s - s')$.
- For bijection to exist, need $\hat{\rho} > 0$.

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Langevin equation of the contact process

Ornstein-Uhlenbeck process

$$\hat{\rho}'(s) = -\lambda\hat{\rho}(s) + (\lambda - 1) + \eta(s)$$

Stationary state? Potential

$$U(\hat{\rho}) = \frac{1}{2}\lambda(\rho - \frac{\lambda - 1}{\lambda})$$

and

$$\hat{\rho}'(s) = -\frac{\mathrm{d}U}{\mathrm{d}\hat{\rho}} + \eta(s)$$

so that at stationarity

$$\mathcal{P}_0(\hat{\rho}) \propto \exp\left(-U(\hat{\rho})\right)$$

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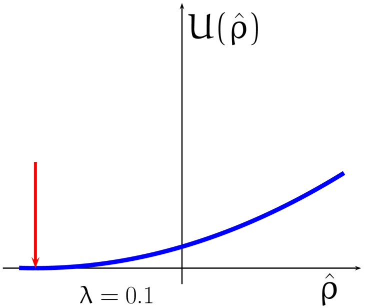

Ornstein-Uhlenbeck process with absorbing wall at $\hat{\rho} = 0$.
Reparameterise:

$$\dot{\rho} = -u\rho(t) + \eta(t)$$

with $u = \lambda$ and absorbing wall at $a = -(\lambda - 1)/\lambda$.
Is there a phase transition at finite a?

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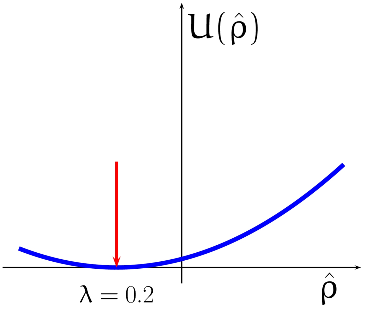

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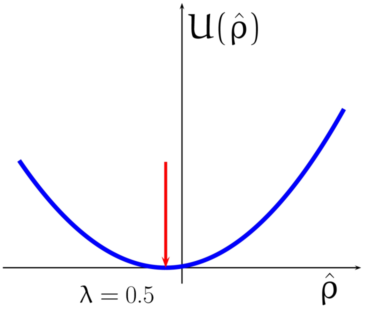

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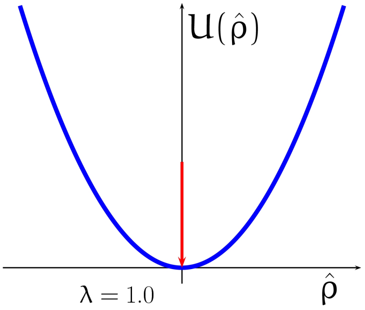

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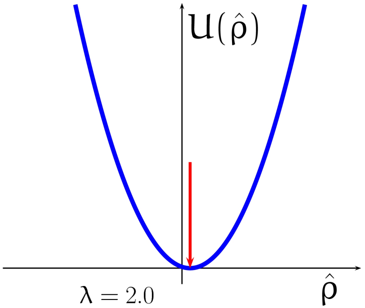

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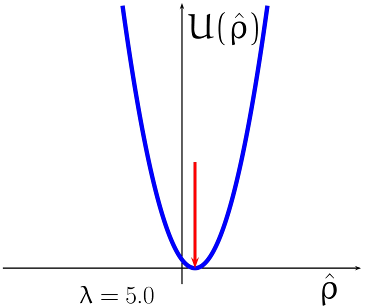

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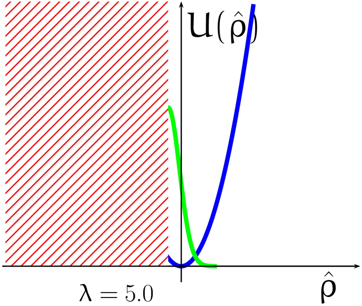

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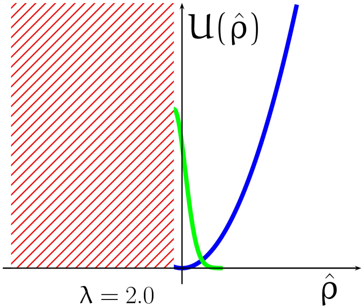

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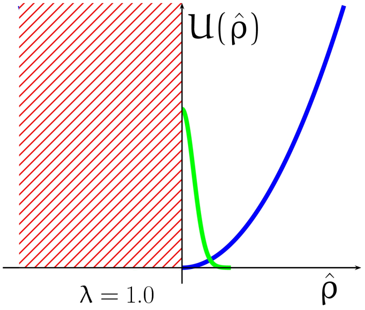

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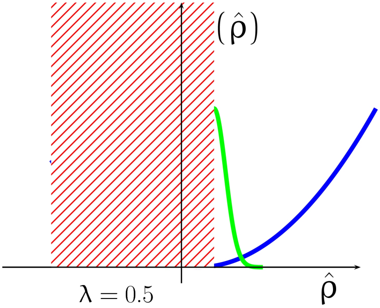

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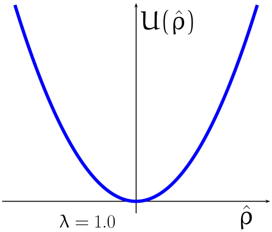
Fokker-Planck equation



Turn Langevin equation into Fokker-Planck equation:

$$\partial_t P(\rho, t) = u\partial_\rho \Big(\rho P(\rho, t)\Big) + \Gamma^2 \partial_\rho^2 P(\rho, t)$$

with Dirichlet boundary condition, $P(a, t) = 0$.
At $a = 0$, this is standard OU, except for the BC. Apply mirror charge trick. $\longrightarrow$ Hermite polynomials.
Can't do it easily for other a, because potential has a kink.

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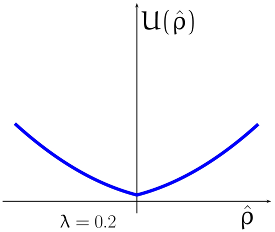

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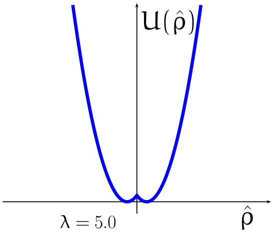

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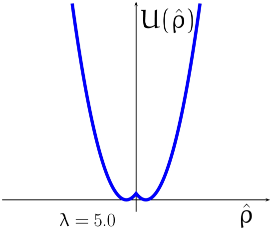


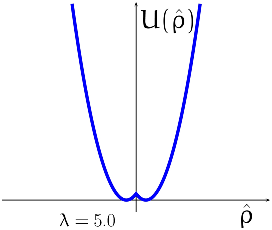
- Turn into eigenvalue problem, say

$$\lambda_n f_n(\rho) = u \frac{\mathrm{d}}{\mathrm{d}\rho}(\rho f_n(\rho)) + \Gamma^2 \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} f_n(\rho)$$

- Too naive! Polynomial solution divergent in time, even at $a = 0$.
- Consider $f_n(\rho) = \exp(-\frac{u}{2\Gamma^2}\rho^2) g_n(\rho)$.
- In the light of large t, better use

$$-\lambda_n g_n(\rho) = \Gamma^2 g_n''(\rho) - u\rho g_n'(\rho)$$

Fokker-Planck equation



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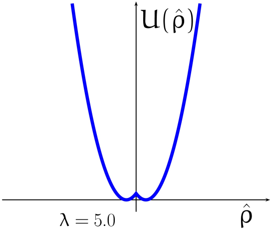

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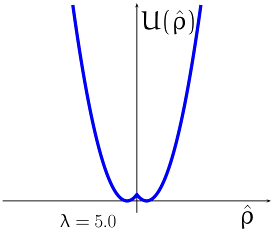

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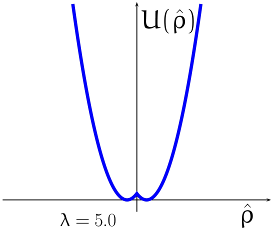

- Problem 1: Recurrence relation does not allow for polynomial solution.
- Problem 2: Is this a Sturm-Liouville problem? Infinite domain?!
- Problem 3: Kummer functions! Not analytic at $\rho = 0$, issue at $a < 0$.

Power series solution

Try

$$g_n(\rho) = \sum_{m=0}^{\infty} b_m(\rho - a)^m$$

and obtain with $c_n = n!b_n$:

$$c_m = ac_{m-1} + (m - 2 - \lambda_n)c_{m-2}$$

for $m \geqslant 2$. Boundary condition means $c_0 = 0$. Since $c_1 \neq 0$ for non-trivial solution, set $c_1 = 1$. Then λ_n is to be found from integrability.

Power series solution

Integrability I

For normalisation of probability density we need

$$\left|\int \mathrm{d}\rho\, e^{-\frac{u}{2\Gamma^2}\rho^2} g_n(\rho)\right| < \infty$$

For orthogonality we need more. Because the operator $(\Gamma^2\partial_\rho^2 + u\rho\partial_\rho)$ can be shown to Hermitian under

$$\langle \bullet \rangle = \int_a^\infty \mathrm{d}\rho \bullet e^{-\frac{1}{2}\frac{u}{\Gamma^2}\rho^2}$$

we impose

$$\int_a^\infty \mathrm{d}\rho\, g_n^2(\rho) e^{-\frac{1}{2}\frac{u}{\Gamma^2}\rho^2} = h_n \ .$$

This condition should determine λ_n. It does so for $a = 0$ where the g_n become Hermite polynomials.

Power series solution

Integrability II

Schemes to determine λ_n:

- Perturbation theory $\lambda_n(a)$ about $a = 0$ difficult to handle.
- For $a > 0$ one can show that c_n have to have alternating signs for g_n to have finite norm. $\longrightarrow$ numerical scheme.
- Resulting λ_n correspond to those found using special functions.
- For arbitrary a, one can minimise $|g_n(z)|$ at some suitably large z, which will in fact produce the correct eigenvalues λ_n.

Sturm-Liouville theory

The original ODE can be written as a Sturm-Liouville problem

$$\frac{\lambda_n - 1}{\Gamma^2} u e^{-\frac{1}{2}\frac{u}{\Gamma^2}\rho^2} g_n(\rho) = -\frac{u}{\Gamma^2}\rho e^{-\frac{1}{2}\frac{u}{\Gamma^2}} g_n'(\rho) - e^{-\frac{1}{2}\frac{u}{\Gamma^2}} g_n''(\rho)$$

Problem: $\rho \in (-\infty, \infty)$, infinite domain.
Physics: No problem to make domain finite.
Mathematics: Boundary not F natural, therefore spectrum not necessarily discrete, because Elliott's theorem does not apply (Horsthemke and Lefever).
If we can find a suitable set of special functions, the normalisation will be taken care of. Impose boundary condition at $\rho = a$ to determine λ_n.

Sturm-Liouville theory

Special functions

- Kummer U has a jump at $\rho = 0$.
- Just (QMUL) + Majumdar (LPTMS): Parabolic cylinder functions; Solution to SL problem, orthogonal, normalisable, boundary condition generates λ_n:

$$D_{\lambda_n}\left(\frac{u}{\Gamma^2}a\right) = 0$$

- Done:

$$P(\rho, t; \rho_0) = \sum_{n=0}^{\infty} h_n^{-1} e^{-\lambda_n u t} e^{-\frac{u(\rho-\rho_0)^2}{4\Gamma^2}} D_{\lambda_n}\left(\sqrt{\frac{u}{\Gamma^2}}\rho\right) D_{\lambda_n}\left(\sqrt{\frac{u}{\Gamma^2}}\rho_0\right)$$

$D_{\lambda_0}(\sqrt{u}\rho_0/\Gamma)$ has no nodes on $[a, \infty) \Rightarrow \lambda_0$ dominates the long time behaviour (survival, expected position, etc.)

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Sturm-Liouville theory

Special functions

…Solution:

$$P(\rho, t; \rho_0) = \sum_{n=0}^{\infty} h_n^{-1} e^{-\lambda_n u t} e^{-\frac{u(\rho-\rho_0)^2}{4\Gamma^2}} D_{\lambda_n}\left(\sqrt{\frac{u}{\Gamma^2}}\rho\right) D_{\lambda_n}\left(\sqrt{\frac{u}{\Gamma^2}}\rho_0\right)$$

$D_{\lambda_0}(\sqrt{u}\rho_0/\Gamma)$ has no nodes on $[a, \infty) \Rightarrow \lambda_0$ dominates the long time behaviour (survival, expected position, etc.)

P also solves the Kolmogorov backward equation.

Note: Once the problem is boiled down to a standard SL problem, nothing exciting can happen in the spectrum $\longrightarrow$ *no phase transition!*

Summary

- Start with the full contact proceess.
- Take away space.
- Transform noise.
- Solve OU process with absorbing boundary.
- ...parabolic cylinder functions do the trick.
- No phase transition (visible in the spectrum).