Field theory for Wiener Sausages and Self-Organised Criticality

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Outline

1. What is SOC?
   - SOC Models
   - The Manna Model
2. Field theory
   - Simplifications
   - Diagrams
   - Tree level
   - The SOC mechanism
3. The Wiener Sausage Problem
   - Spattering random walk
   - Statistical field theory
   - Renormalisation
   - Results

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Summary: SOC

- A brief reminder of **Self-Organised Criticality (SOC)**.
- An exact representation of the **Manna model** as a **field theory**.
- Results at **tree level**,
  i.e. the mean field theory of the Manna model (valid above the upper critical dimension)
- The field-theoretic **mechanism of SOC**.

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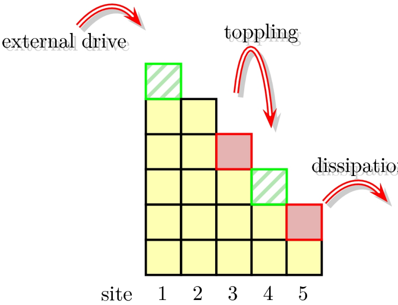
What is Self-Organised Criticality (SOC)?



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

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What is Self-Organised Criticality (SOC)?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.
- **Scale invariance in space and time: Emergence! Universality!**

Universal (?) exponents τ and D

$$\mathcal{P}(s;L) = a s^{-\tau} \mathcal{G}\left(\frac{s}{bL^{D}}\right)$$

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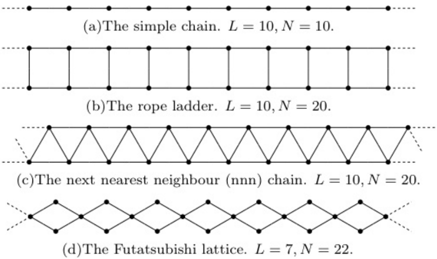
SOC Models

BUT: SOC Models notorious for **not** displaying systematic, robust, clean scaling behaviour. “Key ingredients” may not suffice.

Controversies: **Conservation**, **Stochasticity**, **Separation of time scales**, **Abelianness**.

Oslo Model and **Manna Model** both display **systematic**, **robust**, **clean** scaling behaviour:



(a)The simple chain. $L = 10, N = 10$.

(b)The rope ladder. $L = 10, N = 20$.

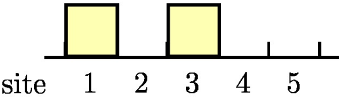
(c)The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

(d)The Futatsubishi lattice. $L = 7, N = 22$.

Same scaling exponents independent from lattice topology in $d = 1, 2, 3$ (From N Huynh, GP and Chew, 2011).

The Manna Model

Manna 1991, Dhar 1999



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Robust, solid, universal, reproducible.
- Defines a universality class.

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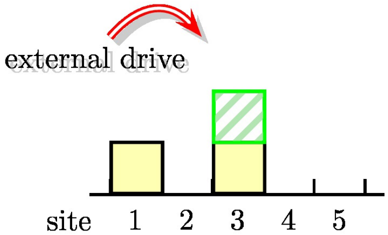

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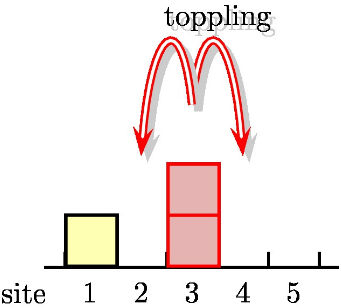

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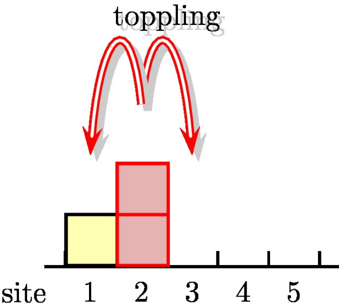

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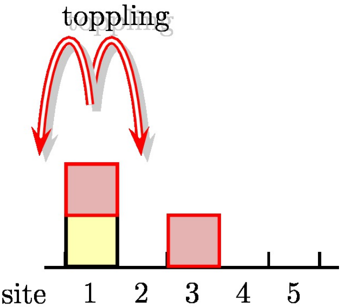

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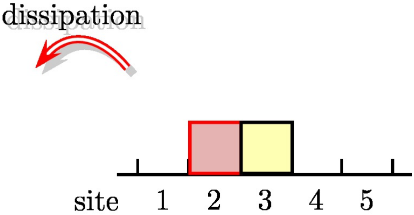

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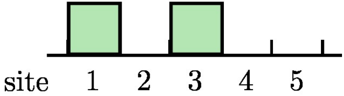
The Manna Model

Revised version



Problem: Manna Model appears to be **excluded volume ("fermionic")** — don't smooth out!
At most one particle per site.
Solution: Introduce **carrying capacity n** and make toppling probabilistic (occupation over **n**).

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The SOC mechanism

So how does it work then?

Symmetry of vertices and stationarity.

- Mass is attenuation of activity.
- Conservation links attenuation to (additional) substrate deposition…
- or equivalently, symmetry of vertices equates mass terms of activity and substrate deposition terms.
- Additional substrate deposition vanishes *as we choose to consider stationarity.*
- Thus mass vanishes *in the particular ensemble*.
- **The activity propagator is not renormalised at any order.**

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What are the key findings?

- **Field theory for the Manna Model derived from microscopic rules**.
- Now we know **why** and **how** the propagator is massless.
- **Symmetry of vertices**, reflecting conservation (**conservation not necessary!**),
- …ensures that the renormalisation of the propagator vanishes at **stationarity**.
- Criticality is a matter of the (stationary) **ensemble**.
- Correlations in the bulk are non-trivial and shift the **local branching ratio**.
- Other mechanisms challenged: Absorbing states, sweeping across the critical point, Goldstone bosons, no criticality …

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Volume of a Wiener by field theory

Results

- In **one** dimensions: Length covered proportional to square root of time, $\sqrt{t}$.
- In **two** dimensions: Area covered linear in time, t.
- Finite size scaling?
- In three dimensions and higher: Volume linear in time, t.
- ...random walker may never return.
- Well known results (Leontovich and Kolmogorov, Berezhkovskii, Makhnovskii and Suris)...
- ...but, hey, what a nice playground for field theory (fermionicity, renormalisation, calculating moments easily ...sort of).

Thank you!

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