Monte Carlo Simulations of an *Ising* Model on a *Sierpiński* Carpet

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Abstract

We study by Monte Carlo simulations the equilibrium and dynamical critical properties of fractal lattices with non-integer Hausdorff dimension. These lattices are known as good candidates to fill the gap between integer dimensions. Focusing on the Sierpiński carpet with Ising spins we are able to obtain the critical exponents which are comparable to the predictions of the Renormalization Group. Moreover we show that the use of Finite Size Scaling is forbidden for these systems and therefore explain the different exponents observed previously.

Introduction

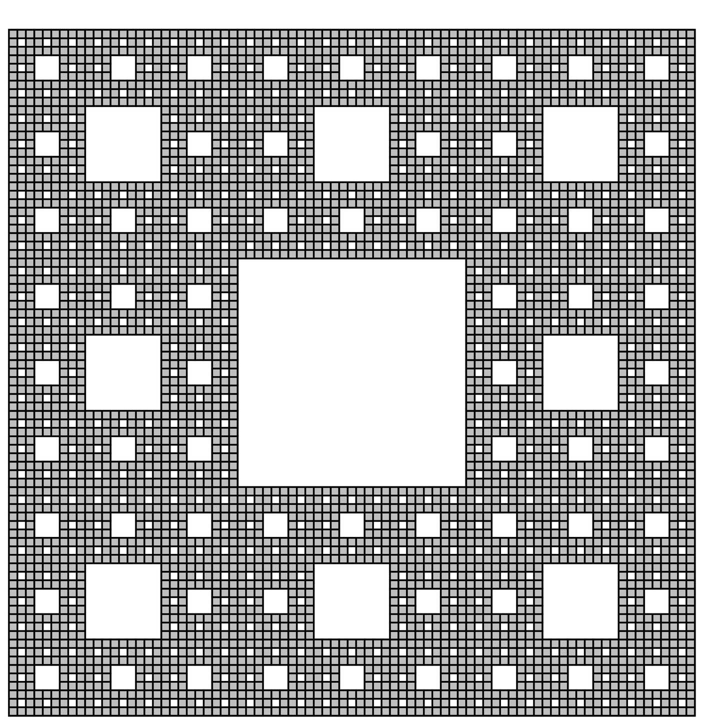
Renormalization Group makes predictions regarding non integer dimensions:

- $4-\epsilon$ and $2+\epsilon$ treatment
- change of order of phase transitions in *Potts*–models [1] and frustrated systems [2]

Is it possible to study critical behavior of systems with non integer dimension **numerically**?
Are the results compatible with those from Renormalization Group?

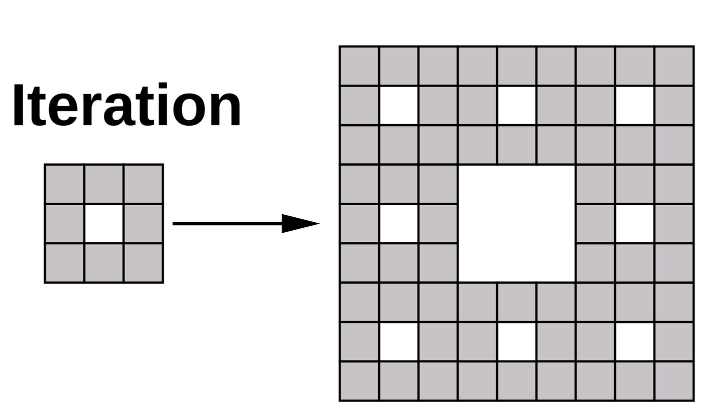
Proposal: Investigate the critical behavior of fractal lattices with infinite ramification order.
Problem: There is no known fractal system, which is translation invariant and has a non integer spatial dimension.

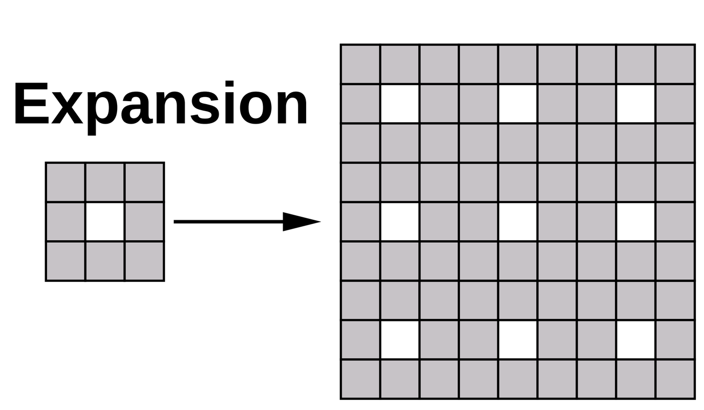
The Model



The *Sierpiński* carpet 3-1 as shown above has *Hausdorff* dimension $d_{\mathcal{H}} = 1.8927\cdots$. We place *Ising* spins with next neighbor interaction on the gray squares [3, 4] and apply periodic boundaries.

There are two different ways of enlarging such a carpet:





Repeating the **iteration** one gets a fractal in mathematical sense. Repeating the **expansion** the average number of next neighbors

$$\bar{n}_{ki} = 4 - 4q\,\frac{1 - l^k/(l^2-q^2)^k}{(l^2-q^2)-l} \qquad (1)$$

is conserved, where l denotes the length of the carpet's generator, which is $l=3$ in our case, and q denotes the size of the centered hole, which means $q=1$ here. In the following we will use the term $SC(k,\,i)$ for a *Sierpiński* carpet 3-1 produced by k iterations followed by i expansions.

Calculated Quantities

By means of the *Wolff*–Algorithm [5] in LTP (Low Temperature Phase) and of the *Swendsen–Wang*–Algorithm [6] in HTP (High Temperature Phase) the system is simulated in the range $1.4 \le T \le 1.82$ and the following quantities are calculated:

$$E = -J\sum_{(ij)} \mathbf{S_i}\cdot\mathbf{S_j} \quad , \qquad (2)$$

$$M = \sum_{i=1}^{N} \mathbf{S}_i \quad , \qquad (3)$$

$$\chi = \frac{\langle M^2\rangle - \langle M\rangle^2}{NT} \quad , \qquad (4)$$

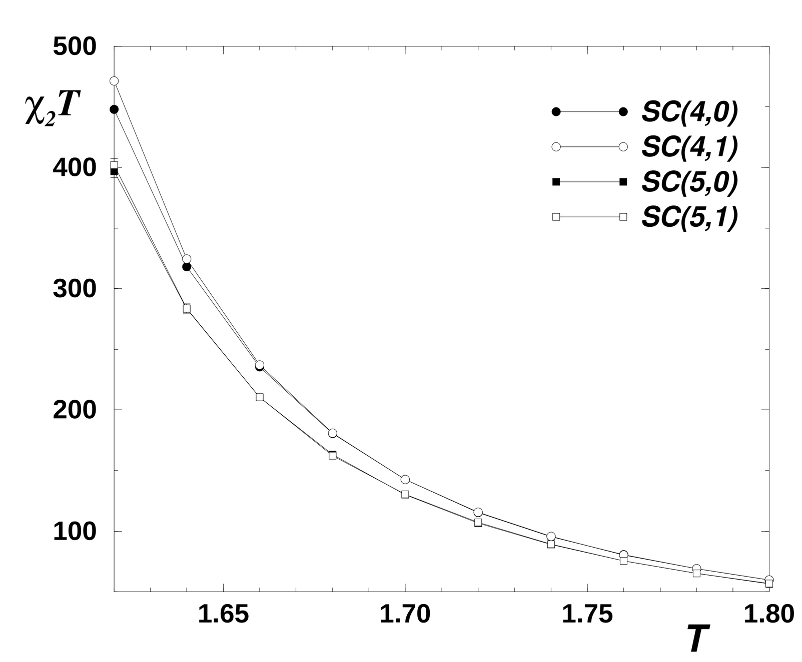
$$\partial_\beta \chi = \langle M\rangle\langle E\rangle - \langle ME\rangle \quad . \qquad (5)$$

Critical Behavior

Problems of FSS

FSS (Finite Size Scaling) relies on a smooth dependence of finite size corrections on the system's size. But there is no unambiguous way of scaling a fractal. Two possible ways are expansion and iteration, but only iteration leads to a system with non integer dimension in mathematical sense. For FSS it must be proved that the iteration process does not add any corrections.

Since an expanded carpet $SC(k,\,i)$ consists of l^i carpets of type $SC(k,\,0)$ without removing or adding bonds or sites, their thermodynamic behavior *must* be the same, as long as the correlation length ξ is sufficiently smaller than the smaller system size, $\xi \ll l(k,\,0)$. If FSS is possible, $SC(k+1,\,0)$ must behave as $SC(k,\,0)$ in this region.



In the figure above χT is depicted for different lattices. While the larger iterated one $SC(5,0)$ does not behave as the original $SC(4,0)$, the expanded $SC(4,1)$ of $SC(4,0)$ and the expanded $SC(5,1)$ of $SC(5,0)$ do, as long as the correlation length is small enough.

FSS is **not** applicable to *Sierpiński*–carpets.

Static critical behavior

Critical exponents must now be obtained outside the FSS region. In a **direct approach** one could fit the quantities, for example the magnetization (3) directly to

$$\langle M\rangle = M_0\,|T-T_c|^\beta\,(1 + \text{correct.}) \quad . \qquad (6)$$

For χ it turns out to be more efficient [7] to fit by means of the **slope-method**:

$$\frac{\chi T}{(\chi T)'} = -\frac{(T-T_c)}{\gamma}\Big(1 + \text{correct.}\Big) \quad . \qquad (7)$$

Since we already know that the carpets must be treated as individual systems for different iteration, we find different exponents for different systems:

Lattice	T_c (slope-method)	γ by slope (slope-method)	γ directly (HTP only)	β directly (LTP only)
$SC(4,0)$	1.5266(11)	1.911(15)	1.901(36)	0.1043(9)
$SC(5,0)$	1.5081(12)	2.030(35)	2.069(8)	0.1221(6)
$SC(6,0)$	1.4992(11)	2.055(54)		

Due to limited resources the results for $SC(6,0)$ are less reliable.

Dynamic critical behavior

For comparison we obtain critical exponents additionally by means of **dynamic critical behavior**. The method consists of annealing the spin system from $T=0$ to T_c, which has been determined previously. After a short time t_0 the dynamic of the systems shows universal, critical behavior [8, 9, 10, 11] without finite size corrections for very long times compared to the times we are using here. Here the algorithm must be local as heat bath [12], which is used here.
One assumes that the free energy per site f has the usual scaling form

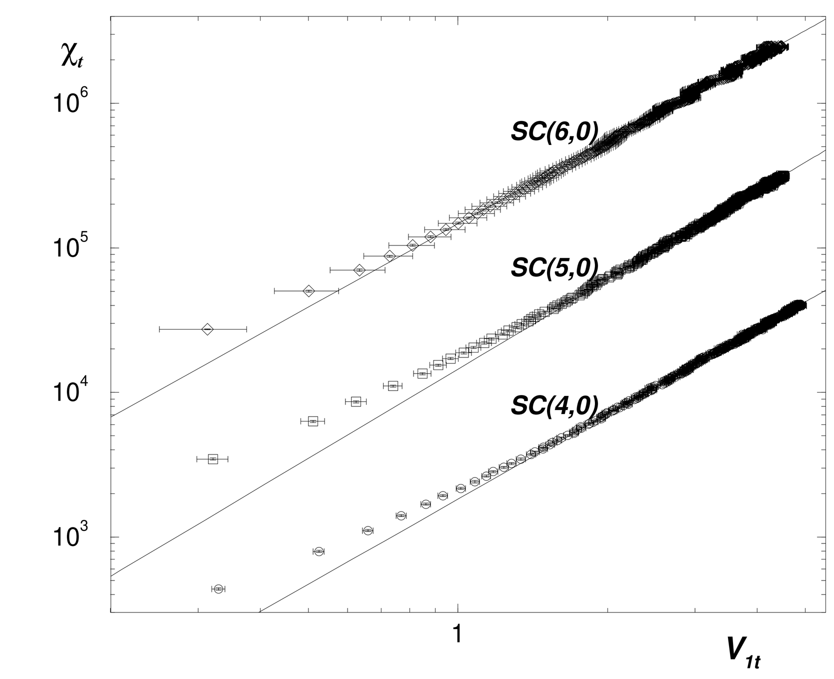
$$f\big((T-T_c),h,t\big) = b^{-d}\,f\big((T-T_c)\,b^{y_t},h\,b^{y_h},t/b^z\big) \qquad (8)$$

with the time t measured in MC steps, the external magnetic field h, and the dynamical exponent z included. Using $y_t = 1/\nu$ and $y_h = d-\beta/\nu$ one obtains at $T=T_c$ and putting $b = t^{1/z}$

$$M_t = \frac{\partial f}{\partial h}\bigg|_{h=0} \propto t^{-\beta/(z\nu)} \qquad (9)$$

$$\chi_t = \frac{\partial^2 f}{\partial h^2}\bigg|_{h=0} \propto t^{\gamma/(z\nu)} \qquad (10)$$

$$V_{1t} = \frac{\partial \log(M)}{\partial T}\bigg|_{h=0} \propto t^{1/(z\nu)} \qquad (11)$$



As exemplary shown in the figure above in a double–logarithmic plot M_t against V_{1t} or χ_t against V_{1t} one obtains as slope $-\beta$ and γ respectively.

Lattice	$V_1(t) \longrightarrow \nu z$	$\chi(V_1) \longrightarrow \gamma$	$M(V_1) \longrightarrow \beta$
$SC(4,0)$	3.06(11)	1.959(32)	0.1154(29)
$SC(5,0)$	3.21(15)	2.048(49)	0.1200(55)

Due to insufficient statistics is was not possible to obtain an reliable value and error of $SC(6,0)$.

Conclusion

An analysis of the static critical behavior of an Ising–spin–system on differently enlarged *Sierpiński*-carpets gives evidence that FSS is impossible for these systems. Therefore our static studies are based on the behavior outside the FSS regime and the results are substantiated by an investigation of dynamic critical behavior.

The results from both methods the static and the dynamic are well compatible, while significantly different from 2D–*Ising* ($\gamma = 1.75$ and $\beta = 0.125$) and earlier studies based on FSS:

	γ	β
Bonnier et al. [4]	2.2	
Monceau et al. [3]	2.754(25)	0.127(6)
this work	≈ 2.05	≈ 0.12

In conclusion we compare our results with the Renormalization Group calculations [13]. With the *Hausdorff* dimensions of our *Sierpiński* carpet $d_{\mathcal{H}} \approx 1.893$, the $4-\epsilon$ Renormalization Group gives $\gamma \approx 1.85$ and $\beta \approx 0.10$. The results which we get would be more compatible with $d_{\mathcal{H}} \approx 1.70$. Although we cannot exclude other corrections which could cause these deviations (as lacunarity [14, 15, 16]) our results suggest that the Hausdorff dimension of the lattice is not exactly the non-integer space dimension for the phase transition even if it should give a rough estimate.

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