Lack of Scaling in Complex Systems: The Forest Fire Model Revisited

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Abstract

The Forest Fire Model (FFM) [1] is one of the best known complex systems, which becomes scale free in a certain limit. Looking closely enough it becomes unclear what "scale free" really means for this model.

Introduction

The **Forest Fire Model (FFM)** [1] is a stochastic cellular automaton, which serves as a toy model for fire spreading in forests. It has an involved history (for an overview see [2]), but usually it refers to the following system: On a d-dimensional square lattice of lateral size L ($d = 2$ in the following) each node can be either occupied (tree) or empty (ash). It evolves as follows:

- *Growing:* Pick $1/\theta$ sites at random, one after the other, and occupy it if empty.
- *Burning:* Pick a site at random. If it is occupied remove the entire cluster connected to it by nearest neighbor interaction. Usually periodic boundaries are applied.
- Go back to the first step.

The figure below shows a configuration of the model after several million steps:

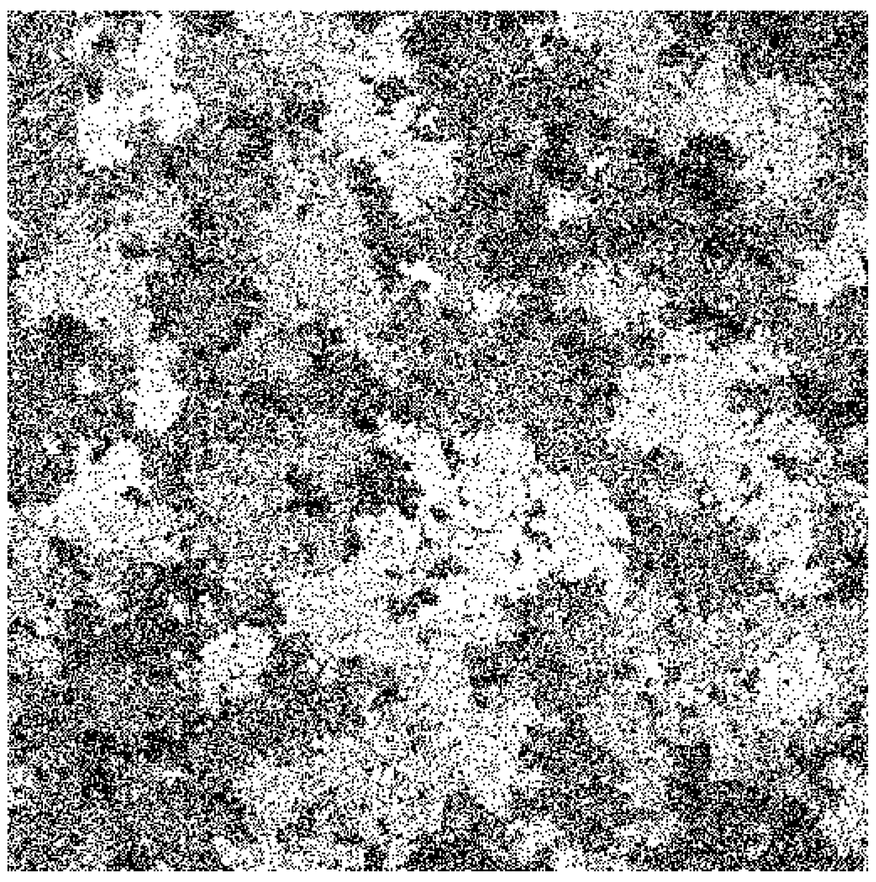

Fig. 1. A configuration of the model of size 512×512.

If $n(s)$ is the distribution of cluster sizes, the distribution of clusters destroyed during the burning procedure is given by $sn(s)$.
From the definition of the model it is clear that $1/\theta$ is restricted by the system size. This leads to a scale s_0 in the distribution $n(s)$; the simplest ansatz for it is known as **simple scaling** [3]:

$$n(s) \propto s^{-\tau} \mathcal{G}(s/s_0) \tag{1}$$

Here $\mathcal{G}(s/s_0)$ is the (universal) **cutoff** function, i.e.

$$\mathcal{G}(x) = \begin{cases} 1 & \text{for } x \ll 1 \\ 0 & \text{for } x \gg 1 \end{cases} \tag{2}$$

It is well known that simple scaling in the FFM breaks down, when finite size effects are present [3], so that s_0 is mainly determined by the system size. Outside the finite size scaling regime, i.e. with the correlation length $\xi \ll L$, Eqn. 1 is the *definition* of τ.
The exponent usually found in $d = 2$ is $\tau = 2.14$.

Numerics

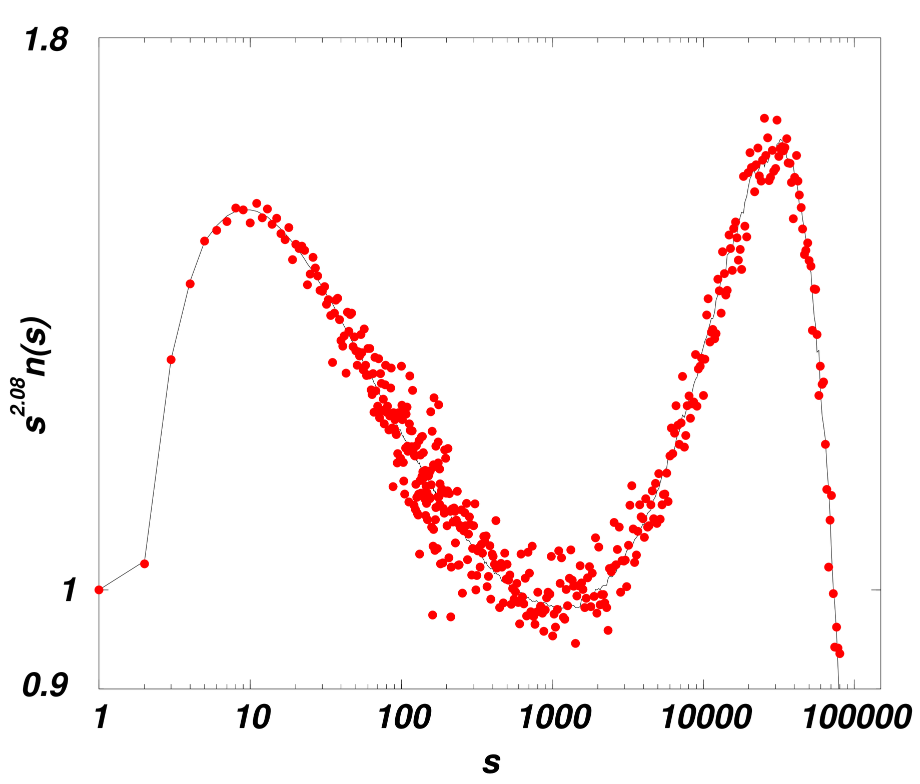

Fig. 2. Comparison between $n(s)$ as obtained from counting *all* clusters (line) and counting *burned* clusters (circles) only.

Usually $n(s)$ is *not* measured by counting *all* clusters, but only by the histogram of the clusters actually burned down. However, using a variant of the Hoshen-Kopelman algorithm [4], it is possible to count all clusters without increasing the computing time significantly. This algorithm gives extremely good statistics, especially for large system sizes, and the high quality of the results allows new insights in the nature of the distribution $n(s)$.
Fig. 2. shows $n(s)$ as obtained from counting *burned* clusters only (circles) together with $n(s)$ as obtained from counting *all* clusters (line). To make the difference clearer, $n(s)s^{2.09}$ is plotted against s (s. below).
For a sufficiently small correlation length, the system does not show finite size effects; by comparing $n(s)$ for different system sizes a (sufficiently small) $1/\theta$ can be found, so that finite size effects are not visible [5]. By comparing $n(s;L)$ and $n(s;L')$ for two different system sizes L and L' one can therefore ensure $\xi \ll L$ without actually measuring it. The following graph shows the ratio $n(s;L=4000)/n(s;L=8000)$ for $1/\theta = 6000$ as obtained from a short run (500.000 steps only).

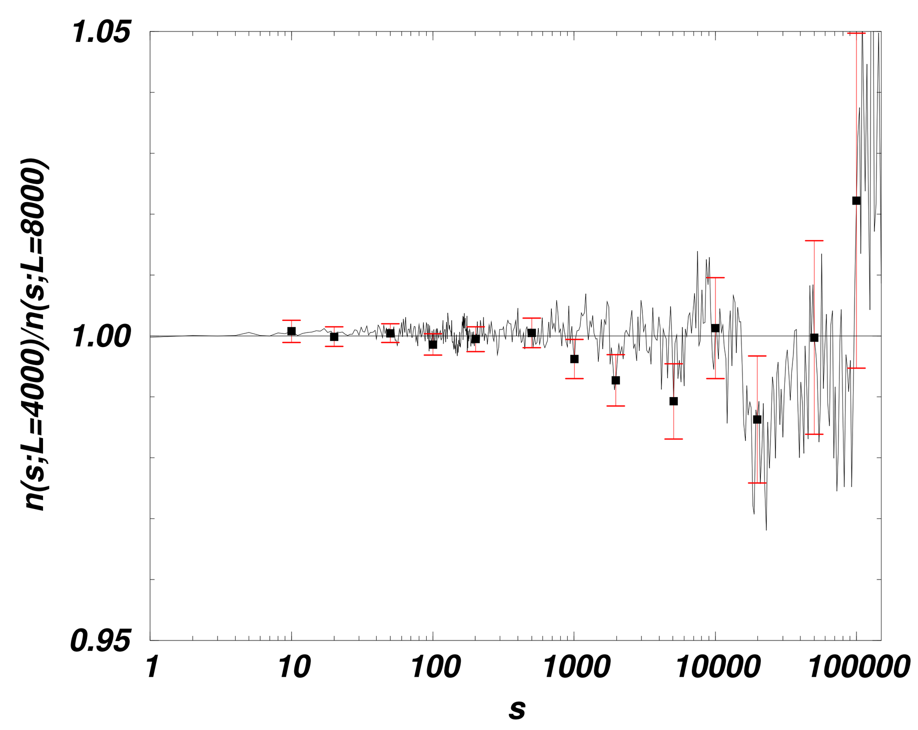

Fig. 3. Two histograms obtained from two different system sizes would fall on top of each other; their ratio is very close to 1 and deviations are well covered by the error bars (data obtained in a very short run).

Results

The most obvious way to obtain τ is to draw $n(s)$ for different $1/\theta$ in a double logarithmic plot and to move them vertically (multiplying $n(s)$) and horizontally (varying s_0) in order to collapse the data. In the plot below $sn(s)$ is shown for various values of $1/\theta$:

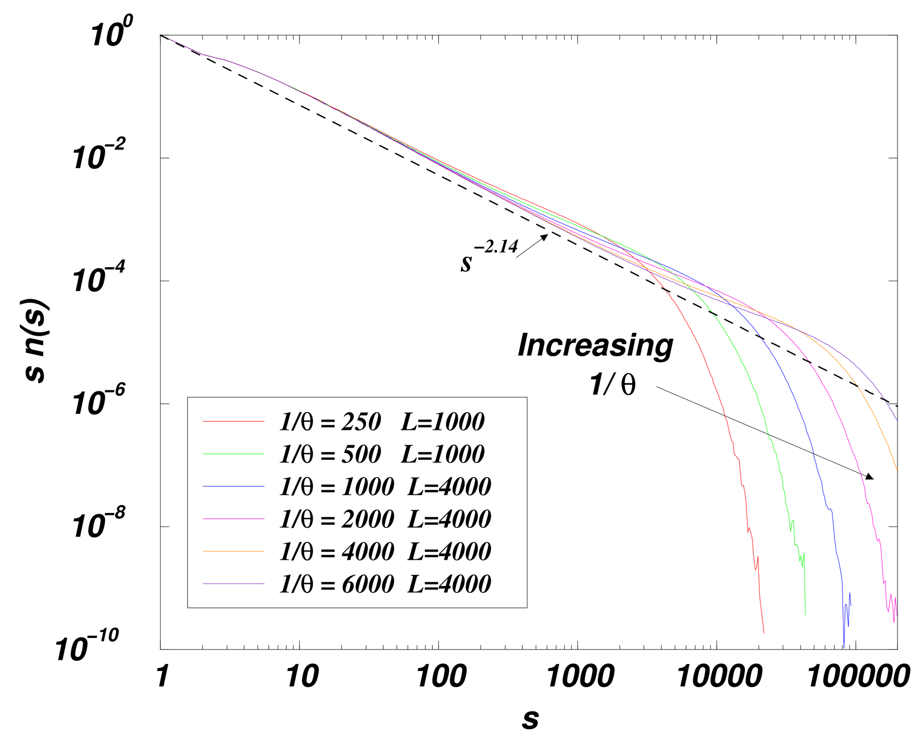

Fig. 4. Estimating τ: The obvious way.

It is much more instructive to look directly for $\mathcal{G}(x)$, which is supposed to be a straight line and should become horizontal for a properly chosen τ. However, plotting $n(s)s^\tau$, with $\tau = 2.09$ in the plot below, the following graphs are obtained.

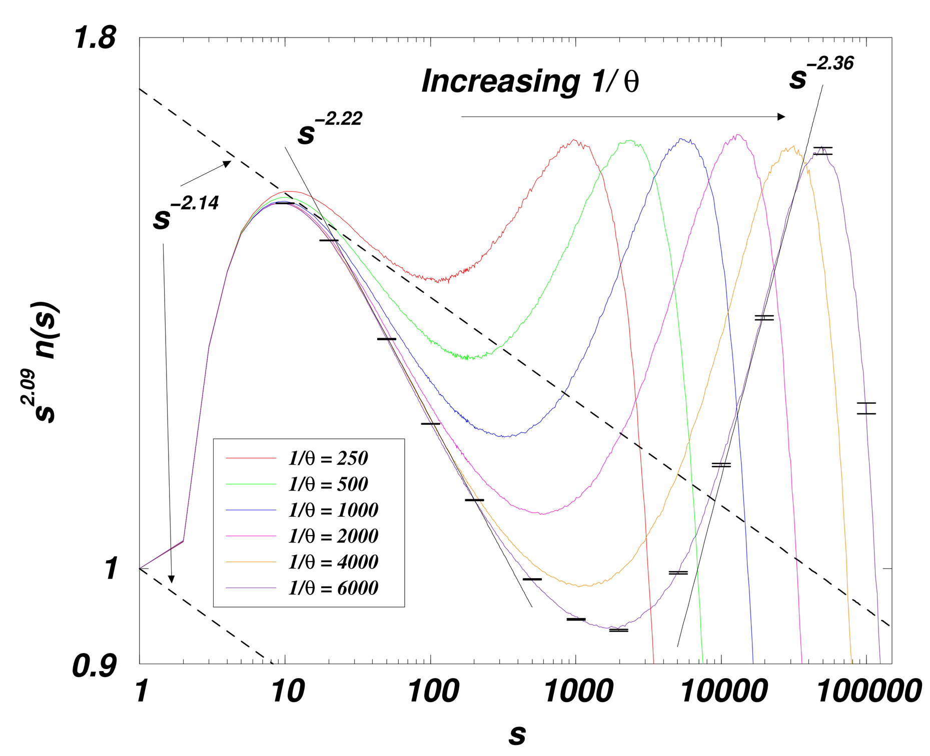

Fig. 5. Rescaled histograms $n(s)$ (the error bars are estimated from the short run presented in Fig. 3).

From this plot it is clear that

$$\boxed{n(s) \text{ cannot scale as } s^{-\tau}\mathcal{G}(s/s_0)}$$

- there is no way to collapse the data.
These results therefore substantiate the (largely ignored) suggestion by P. Grassberger [6] that simple scaling is violated in the FFM. Chosing other values for τ cannot cure the mismatch.

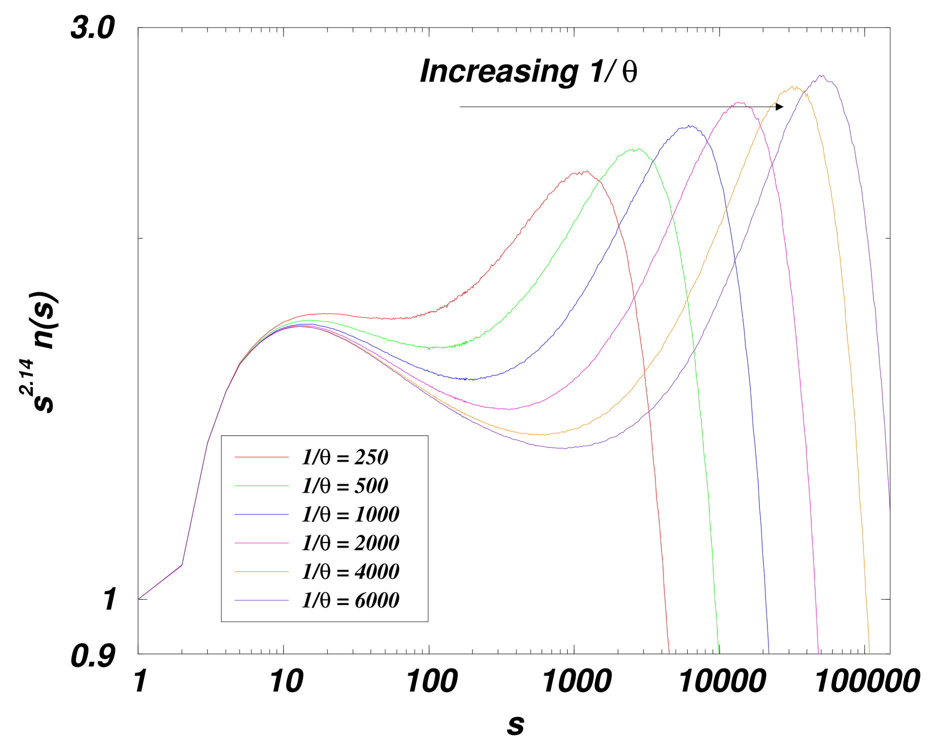

Fig. 6. Rescaled histograms $n(s)$ with $\tau_{\text{est.}} = 2.14$.

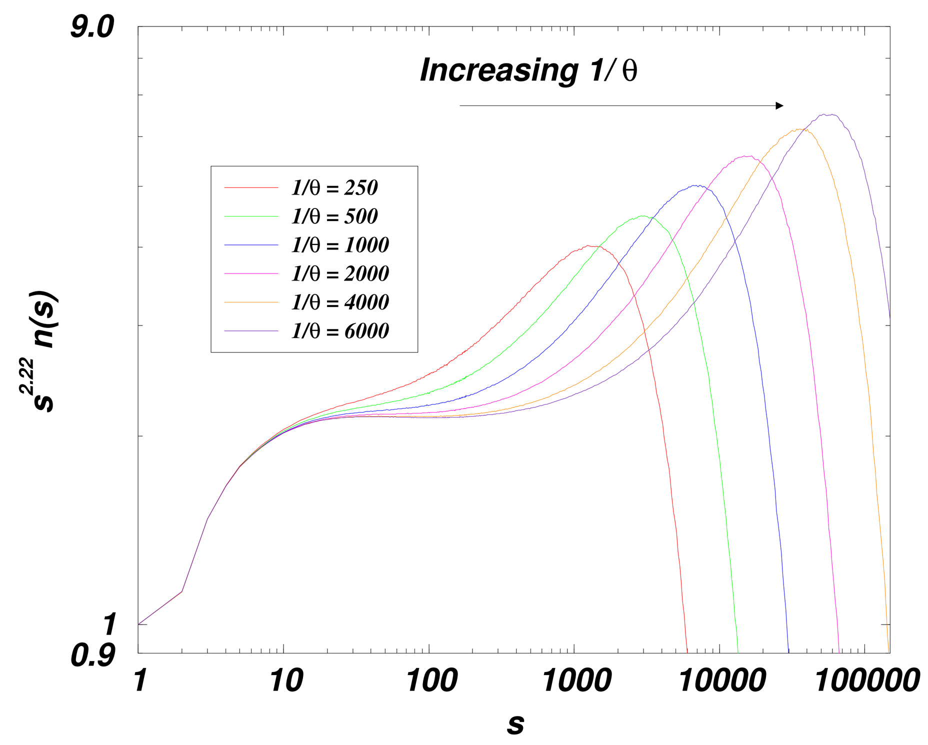

Fig. 7. Rescaled histograms $n(s)$ with $\tau_{\text{est.}} = 2.22$.

Why?!

We cannot offer an explanation for this behavior yet. The central question is:

$$\boxed{\text{Is there an asymptotic shape for } n(s)s^\tau\text{?}}$$

The following points might be considered in a discussion:

- $n(s)$ is a PDF with divergent first moment for divergent $1/\theta$; therefore $n(s)$ has no cutoff in that limit.
- The tree-tree correlation length is known to diverge in that limit ($\nu_T \approx 0.54$ [7]).
- The length of correlations inside a given cluster diverges also, but with a slightly different exponent ($\nu \approx 0.57$ [7])
- One length scale, roughly defined by the first maximum, seems to be fixed.
- However, the shape of $n(s)s^\tau$ depends directly on τ, so that the maxima and the minimum in the plot above might move or even disappear.
- Is there any connection to multifractality or multifractal scaling?
- Is it possible to find an expression for the size independent (low s) part of the curve above?

References

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