Multiscaling in the areas enclosed by coalescing random walkers

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Abstract

We consider the correlation functions of areas between coalescing random walkers as a candidate for a simple, analytically tractable process to display multiscaling. We are able to calculate, to our knowledge for the first time, the second moment and the two point correlation function. In addition, we investigate higher order correlations numerically and can thereby confirm the presence of multiscaling.

Introduction

Motivated by recent research in cluster aggregation [2], we study the scaling of correlation functions of the area size between the trajectories of coalescing random walkers, as illustrated in Figure 1.

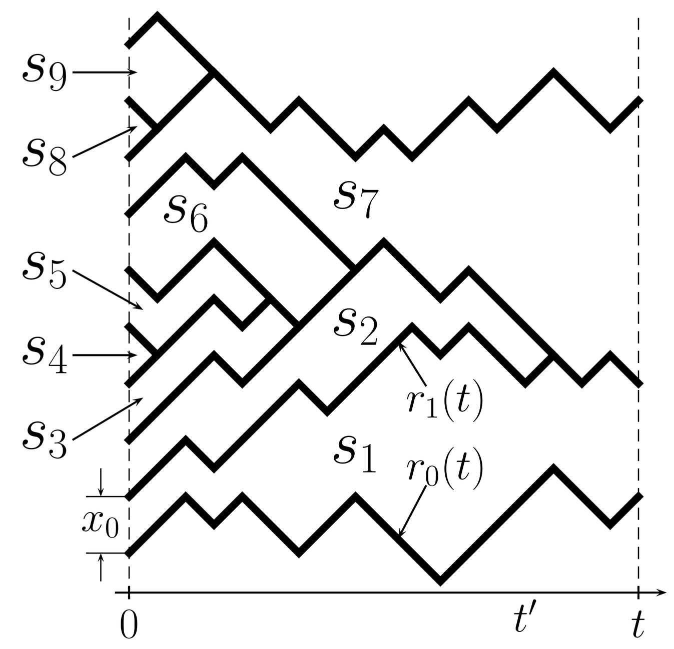

Figure 1. A sequence of trajectories $r_0(t'), r_1(t'), \dots$ starting at $t' = 0$ with initial spacing x_0 and demarcating areas $s_1, s_2, \dots$ between them. The dashed lines indicate the "walls" at $t' = 0$ and the terminating time $t' = t$. Whenever random walkers meet, they coalesce and perform the rest of the walk together.

Gap and multiscaling refers to the scaling γ_n of the nth moment of some observable s, for example

$$\langle s^n \rangle (t) \to t^{\gamma_n} \quad \text{for large } t.$$

Then an example of gap scaling is

$$\gamma_n = \frac{3}{2}\left(1 + n - \frac{4}{3}\right) \quad \text{for } n = 1, 2, \dots.$$

For a system displaying multiscaling one has, for example,

$$\gamma_1 = 1 \quad \text{and} \quad \gamma_n = \frac{3}{2} \text{ for } n = 2, 3, \dots.$$

with logarithmic corrections.

To show the presence of multiscaling, the scaling of at least three correlation functions has to be determined. The first moment of the average area size between coalescing random walkers is trivial and known to be $\langle s \rangle = x_0 t$ [1]. We go a step further and calculate the second moment exactly. The behaviour of the third moment is characterised numerically.

Model

Figure 1 shows a lattice realisation of a sequence of random walkers, which start with initial spacing x_0. Whenever two random walkers meet, they coalesce and the resulting, single walker continues its walk for the remaining time. The expectation value of the moments of the area between two random walkers is denoted $\langle s^n \rangle$.

The main aim of this work is to calculate the correlation functions $\langle s_1 s_2 \rangle$, $\langle s_1 s_2 s_3 \rangle$, and $\langle s_1 s_2 s_3 s_4 \rangle$ and determine their asymptotes, in order to test for multiscaling.

Analytical Results

By using the scheme for finding $\langle s^n \rangle$ presented in [1], (XXX Peter: Say a few words about this scheme!) we were able to find the second moment $\langle s^2 \rangle$ exactly:

$$\left\langle s^2 \right\rangle (t; x_0) = \frac{1}{180}\left(\frac{x_0^3}{D}\right)^2 \left[(1 + 28T + 132T^2)\frac{2}{\sqrt{\pi}}\sqrt{T}e^{-\frac{1}{4T}} - (1+30T) + (1 + 30T + 180T^2 + 120T^3)\mathcal{E}\left(\frac{1}{\sqrt{4T}}\right)\right]$$

where $T(t; x_0) = t/(x_0^2/D)$ is a dimensionless quantity and $\mathcal{E}$ is the error function $\mathcal{E}(u) = (2/\sqrt{\pi})\int_0^u \mathrm{d}x\, e^{-x^2}$.

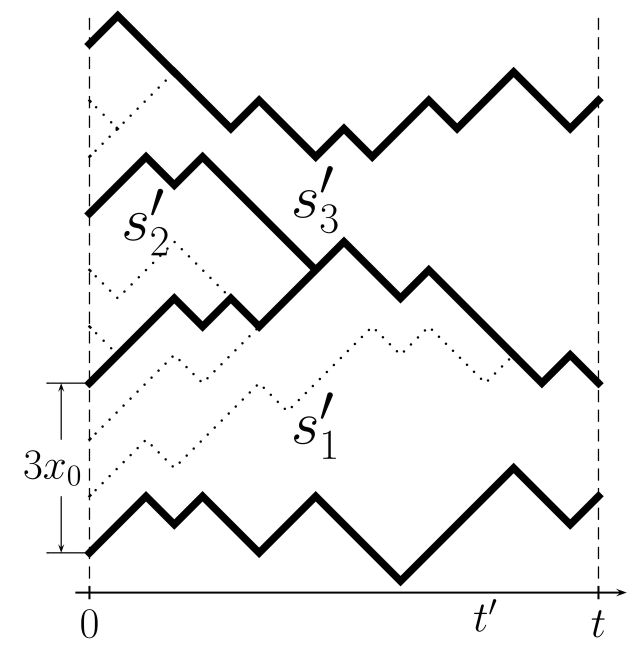

Figure 2. An example of merged areas with initial spacing $3x_0$, so that $s_1' = s_1 + s_2 + s_3$, $s_2' = s_4 + s_5 + s_6, \dots$.

Correlation functions of the form $\langle s_1 s_2 \rangle$ can be obtained by considering moments of the form $\langle s^2 \rangle$, but with an integer multiple of the initial spacing (we thank Alan Bray for sharing this insight with us), see Figure 2. One can therefore derive the two point correlation function by noting that

$$\left\langle (s_1 + s_2)^2 \right\rangle - \left\langle s_1^2 \right\rangle - \left\langle s_2^2 \right\rangle = 2 \langle s_1 s_2 \rangle,$$

and recognising that $\left\langle (s_1 + s_2)^2 \right\rangle (x_0; t) = \left\langle s^2 \right\rangle (2x_0; t)$. For a more general result, the network of trajectories is pruned by joining n consecutive areas $s_1, \dots, s_n$ to one large area. This has the same effect as releasing random walkers with initial spacing nx_0. The exact two point correlation function is therefore,

$$\langle s_1 s_{n+1} \rangle (t; x_0) =$$

$$\frac{1}{2}\left(\left\langle s^2 \right\rangle (t; |n-1|x_0) - 2\left\langle s^2 \right\rangle (t; |n|x_0) + \left\langle s^2 \right\rangle (t; |n+1|x_0) \right)$$

with the exact expression of $\left\langle s^2 \right\rangle (t; x_0)$ given above. Figure 3 shows $\langle s_1 s_{1+n} \rangle$ together with its numerical estimate for $x_0 = 2, t = 4096$ and $D = 1$.

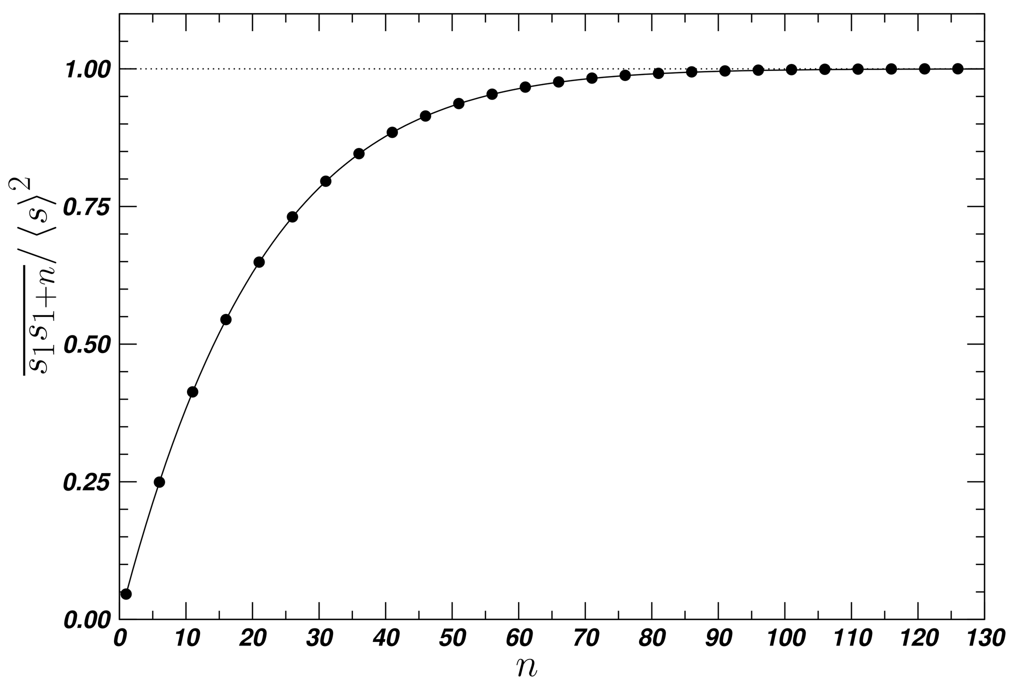

Figure 3. Numerical estimate for the correlation function $\langle s_1 s_{1+n} \rangle$ for $x_0 = 2, t = 4096$ and $D = 1$.

Numerics

The higher correlation functions were found numerically. The result of the numerical simulations for $\langle s_1 s_2 s_3 \rangle$ is shown in Figure 4 in the form $\overline{s_1 s_2 s_3}/[(x_0^3/D)^3 T^{3/2}]$, where we divide by $(x_0^3/D)^3$ to render the result dimensionless and by $T^{3/2}$ to remove the suspected leading order.

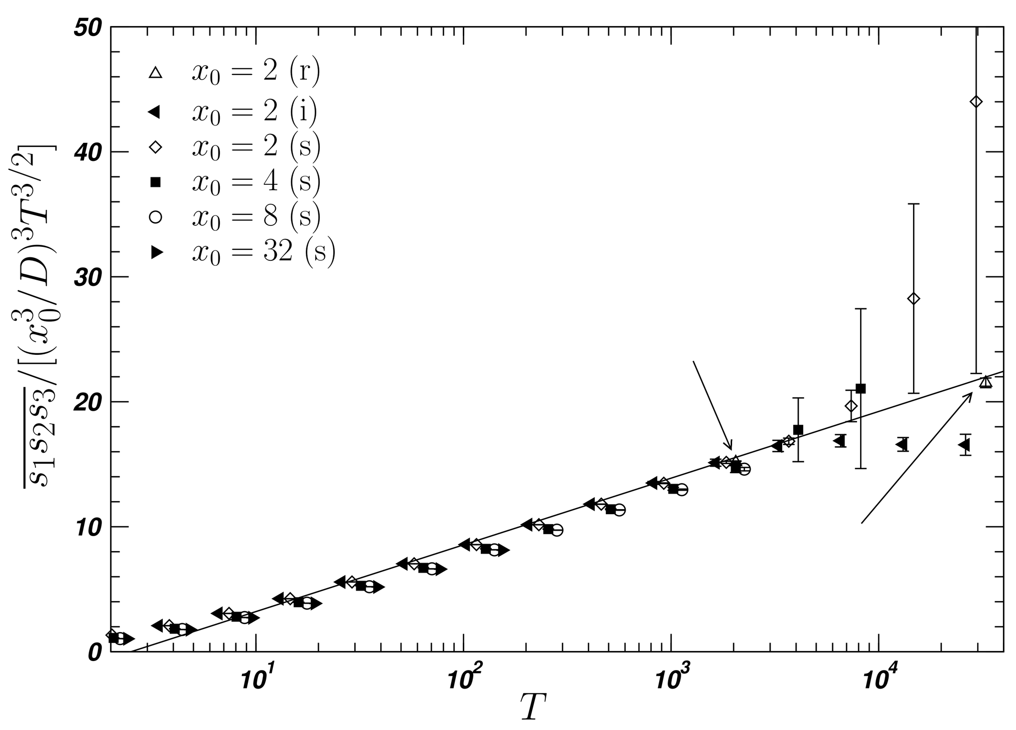

Figure 4. The numerical estimate of $\overline{s_1 s_2 s_3}$. Data marked (i) is from independent samples, (s) from sequential samples, (r) from importance sampling with repulsive potential (see arrows). The straight line is a fit to a logarithm $a \ln(T/b)$ to guide the eye.

From the result in Figure 4 we can quite confidently **rule out gap scaling**, which would require $\langle s_1 s_2 s_3 \rangle / t^2$ to converge in the limit of large t to a non-vanishing value.
(XXX Peter: Put all that g_3 stuff in a grey box?) A direct measure for the validity of gap scaling is a moment ratio of the form

$$g_3 = \frac{\langle s_1 s_2 s_3 \rangle \langle s \rangle}{\langle s_1 s_2 \rangle^2}$$

which converges to a non-vanishing value if the observables obey gap scaling, irrespective of the value of the exponents. The numerical estimate of g_3 is shown in Figure 5 (using the analytical results for $\langle s \rangle$ and $\langle s_1 s_2 \rangle$). Its decay to 0 indicates once more that $\langle s_1 s_2 s_3 \rangle$ does not diverge fast enough.

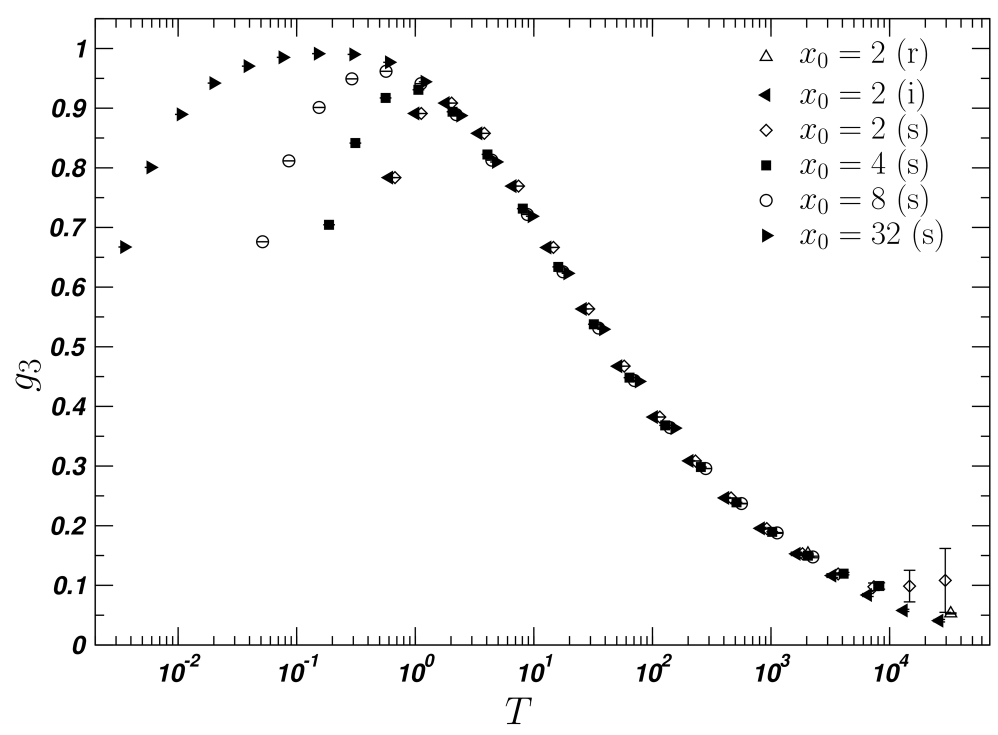

Figure 5. The numerical estimate of the moment ratio g_3.

Four point correlation function

Similar to Figure 4, Figure 6 shows $\overline{s_1 s_2 s_3 s_4}/[(x_0^3/D)^4 T^{3/2}]$ together with a fit to a parabola in $\ln(T)$. While the fit is not perfect and the data is, apart from the result from importance sampling, slightly inconsistent for large T, the suggested parabola seems plausible. The data is certainly not consistent with gap scaling.

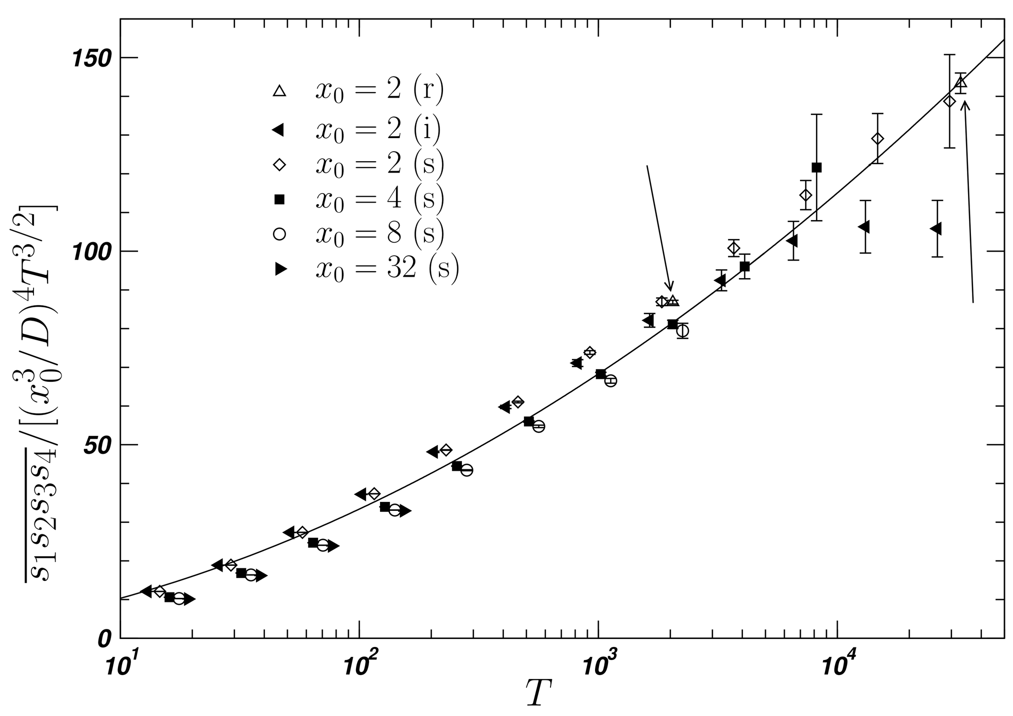

Figure 6. The numerical estimate of $\overline{s_1 s_2 s_3 s_4}$. The solid line is a fit to a parabola in the logarithm of T, $a \ln(T/b) + c \ln(T/b)^2$

Summary

Motivated by the question whether the sequence of the areas between trajectories of coalescing random walkers exhibits multiscaling, we have calculated the second moment and subsequently the two point correlation function exactly. Furthermore, we provide numerical results for a three point correlation function to expose the absence of gap scaling. While the analytical proof for multiscaling in the present system is still lacking, the numerical results indicate it convincingly.

References

[1] Pruessner G 2004 *J. Phys. A: Math. Gen.* **37** 7455–7471 (*Preprint* `cond-mat/0402564`)

[2] Connaughton C, Rajesh R and Zaboronski O V 2005 *Phys. Rev. Lett.* **94** 194503