Chapter 6

Field theory for reaction-diffusion processes

Word of caution: These are draft notes, which I prepared for the Barcelona summer school 2015, on the basis of some of the notes mentioned below, my own notes and discussions with friends, colleagues and/or students, in particular SAOIRSE AMARTEIFIO, SIMONE CENCI, TIM EVANS, CHIU FAN LEE, AMAN PUJARA, and NANXIN WEI. A lot of the emphasis draws on what I found difficult to comprehend or difficult to explain. As with so many things, the deeper one drills and the more answers one finds, the more complicated the problem becomes.

The canonical approach to non-equilibrium statistical mechanics suggests that there are three different equivalent views: Equation of motions of the degree of freedom (the observable), *i.e.* LANGEVIN equations, equations of motion of the probability density, *i.e.* FOKKER-PLANCK equations, and finally a path integral or, more specifically, field theoretic approach which, in a sense, proceeds through both. This view parallels that of quantum mechanics, which has the HEISENBERG picture, describing observables as the time dependent quantities, SCHRÖDINGER equations, which describes the time-evolution of the probability density wave, and finally field theory, namely FEYNMAN's path integral approach. There are wonderful text books available that are concerned with the field theoretic approach to non-equilibrium[1], such as Täuber's (2014) recent textbook and the one by Vasil'ev (2004).

However, there is a fourth view, which is also field theoretic, but allows for an (almost?) independent route to a description of a reaction-diffusion process. It does not start with LANGEVIN or FOKKER-PLANCK, but with a master equation, which is found on the basis of "updating rules" of a stochastic process. In that sense, this field-theoretic approach, known as the DOI-PELITI approach (Doi, 1976; Peliti, 1985) is independent. This approach also provides fairly easy access to alternative description, *i.e.* once the "action"

[1] Non-equilibrium can be further divided into out-of-equilibrium (relaxation to equilibrium) and far-from-equilibrium.

has been found, LANGEVIN and FOKKER-PLANCK equations can be determined. Much of this field theory is developed along the same lines as the theory of second quantisation. It remains, however, nothing more than a classical theory with a few dirty tricks.

There are two key-advantages of the DOI-PELITI approach over other techniques: Firstly there is no need to find an effective theory. One starts from a set of rules, which are cast into a master equation and derives an action from that. It is, in principle, an exact method, which can be applied at and away from a critical point, but provides the full power of field theory if one is interested in exploring asymptotia (and even when not). Secondly, if non-linearities are dealt with by means of a loop expansion, this provides a systematic approximation scheme even in mesoscopic systems and even in the presence of lattice effects. By construction, there is no need to enforce the "physical boundary condition" that particle densities cannot be negative and that particles are quantised, *i.e.* using DOI-PELITI, particles are not smeared out, even when their *expected* density is.

In the following, the DOI-PELITI approach is described in great detail, mostly with some very basic examples in mind, to which I will add bells and whistles as I go along. I will not follow the dry route of first introducing all the language and tools and use them in some general form — I set the problem and take the tools out of the toolbox as I go along. Alternative descriptions exist, in particular the wonderful and wonderfully accessible tutorials by (Cardy, 1999, 2006, 2008), which are also very succinct (much more than I could ever be, and certainly much more than I am trying), but also by Täuber (2014), who presents the material in the greater context of field theoretic renormalisation.

6.1 The Process

To avoid drowning in notation and details, the following simple process is considered: Particles hop on a d-dimensional hypercubic lattice $\mathbb{Z}^d$, independently with POISSONian rate H to nearest neighbouring sites, of which there are q (coordination number).[2] As a reminder, I suggest the following exercise:

[2]Reminder: Poissonian means that an event takes place with the same probability in each infinitesimal snippet of time, Δt. The snippets need to be small, so that we don't run into trouble with having more than one event in a snippet. If the rate is H, then the probability that an event takes place in Δt is $H\Delta t$ and, therefore the probability that no event takes place over time t is $\lim_{\Delta t\to 0}(1-H\Delta t)^{t/\Delta t} = \exp(-Ht)$.

Exercise 8:

- *What is the probability for n events to take place over the time period t in a Poisson process with rate H? What is the expected number of events that take place during t? What is the variance of that number?*
- *What is the probability density of an event taking place after waiting time t?*
- *What is the average waiting time for an event?*
- *What is the average waiting time between any two events?*
- *What is the average waiting time for two events?*

In other words, each particle picks with rate H one of its nearest neighbouring sites at random and uniformly, and "relocates" there. We will find that in the continuum limit this is diffusion. We may want to consider this very same process on a *finite* lattice (or a finite interval, slab *etc.*), in which case we need to introduce boundary conditions.

In addition to the hopping, we will consider spontaneous extinction and spontaneous creation. The former means that particles may disappear with Poissonian rate ϵ, the latter that particles come into existence somewhere with a certain rate and density β. Later (Section 6.3.3), we will also touch on branching, *i.e.* particles producing offspring, which will take place with rate σ and further down the line (Section **??**) an excluded volume constraint, whereby particles cannot simply hop anywhere, unless there is enough space for them. To keep things simple, we will limit this *carrying capacity* to *one particle per site*.

6.1.1 The Master equation

At the very beginning we write down a master equation (or, to follow VAN KAMPEN, an M-equation) for the hopping only and we will do that on the lattice.[3] The processes of extinction and spontaneous creation will be added afterwards. We will aim to write a field theory that is an *exact* representation of what is happening on the lattice and only much later take a continuum limit, to simplify calculations.

In the following a lattice site is denoted $\mathbf{x}$ and a unit vector $\mathbf{e}$, which displaces by a single lattice spacing.[4] On a regular, d-dimensional lattice, $q = 2d$ such unit vectors exist. They come in pairs of vectors pointing in opposite directions. The probability of finding the system in the state where $n_{\mathbf{x}}$ particles are to be found at site $\mathbf{x}$ is $P(\{n_{\mathbf{x}}\}; t)$, where $\{n_{\mathbf{x}}\}$ is the set of all particles numbers of all the sites on the finite lattice and t is the time (this argument will very often be dropped). In the following, the notation is slightly abused — $P(n_{\mathbf{x}} + 1)$ is the probability P for the configuration $\{n_{\mathbf{x}}\}$ with only one specific $\mathbf{x}$ having

[3] Taking a different approach, one may want to consider particles hoping between only two sites.

[4] This is the set of *all* displacement vectors. On a centrosymmetric lattice one may think of them as the primitive basis vectors of the lattice including every inversion, $-\mathbf{e}$.

in fact $n_{\mathbf{x}} + 1$ particles. I will highlight the problems with the notation occasionally.

Let's first consider only diffusion. The probability $P(\{n_{\mathbf{x}}\}; t)$ of finding the system in state $\{n_{\mathbf{x}}\}$ evolves in time t as the particular state $\{n_{\mathbf{x}}\}$ is reached from other states or as $\{n_{\mathbf{x}}\}$ decays into other ("neighbouring") states. As time is continuous, there are no simultaneous events, so if $\{n_{\mathbf{x}}\}$ is generated, then because it is reached by a single particle making a single hop, *i.e.* a jump by $\mathbf{e}$ which happens with frequency H/q for each particle. To systematically tally all processes that lead to $\{n_{\mathbf{x}}\}$, we look at the *target site* $\mathbf{x}$ of a particle jumping. If that is to create state $\{n_{\mathbf{x}}\}$, then because $\mathbf{x}$ carried one particle too few and one of the surrounding sites one particle too many, so the total influx to the probability $P(\{n_{\mathbf{x}}\}; t)$ is

$$\frac{H}{q} \sum_{\mathbf{e}} (n_{\mathbf{x}+\mathbf{e}} + 1) P(\{n_{\mathbf{x}} - 1, n_{\mathbf{x}+\mathbf{e}} + 1\}; t)$$

where the sum $\sum_{\mathbf{e}}$ runs over all neighbours (of $\mathbf{x}$), $\{n_{\mathbf{x}} - 1, n_{\mathbf{x}+\mathbf{e}} + 1\}$ denotes the state that differs from $\{n_{\mathbf{x}}\}$ by site $\mathbf{x}$ having one particle fewer and site $\mathbf{x}+\mathbf{e}$ having one particle more. The probability on the right is multiplied by $n_{\mathbf{x}+\mathbf{e}} + 1$ because *each* of the particles at $\mathbf{x} + \mathbf{e}$ may make a jump independently, so the total rate of these concurrent Poisson processes grows proportionally with the particle number on the neighbouring site.

At the same time state $\{n_{\mathbf{x}}\}$ is depleted by any particle at site $\mathbf{x}$ jumping to a neighbouring site, so that there is an outflow from $P(\{n_{\mathbf{x}}\}; t)$ of $H n_{\mathbf{x}} P(\{n_{\mathbf{x}}\}; t)$. This can happen at every site on the lattice, so the total rate of change of $P(\{n_{\mathbf{x}}\}; t)$ is

$$\begin{aligned} \frac{d}{dt} P(\{n_{\mathbf{x}}\}; t) &= -H \sum_{\mathbf{x}} n_{\mathbf{x}} P(\{n_{\mathbf{x}}\}; t) + \frac{H}{q} \sum_{\mathbf{x}} \sum_{\mathbf{e}} (n_{\mathbf{x}+\mathbf{e}} + 1) P(\{n_{\mathbf{x}} - 1, n_{\mathbf{x}+\mathbf{e}} + 1\}; t) \\ &= \frac{H}{q} \sum_{\mathbf{x}} \sum_{\mathbf{e}} \left((n_{\mathbf{x}+\mathbf{e}} + 1) P(\{n_{\mathbf{x}} - 1, n_{\mathbf{x}+\mathbf{e}} + 1\}; t) - n_{\mathbf{x}} P(\{n_{\mathbf{x}}\}; t) \right) . \end{aligned} \tag{6.1}$$

It is worth considering boundary conditions at this stage: For DIRICHLET (open) boundary conditions, we will *define* $P(\{n_{\mathbf{x}}\}; t) = 0$ for a state where any of the particles would have to reside outside the lattice. For example, if $\mathbf{x}$ is a boundary site and $\mathbf{x} + \mathbf{e}$ is outside the system,[5] we will require $P(\{n_{\mathbf{x}+\mathbf{e}}\}; t) = 0$ for $n_{\mathbf{x}+\mathbf{e}} > 0$ if $\mathbf{x} + \mathbf{e}$ is outside. That way, "particles cannot hop back in" and yet boundary sites still lose particles with rate $n_{\mathbf{x}} P(\{n_{\mathbf{x}}\}; t)$. It is interesting to note the lack of a causal link between events in Eq. (6.1), as the depletion of $\{n_{\mathbf{x}}\}$ due to particles leaving $\mathbf{x}$ and its repopulation by particles arriving from other sites are separate terms.

NEUMANN (reflecting) boundary conditions may be implemented a bit more elegantly by identifying $\mathbf{x} + \mathbf{e}$ with $\mathbf{x} - \mathbf{e}$ if $\mathbf{x} + \mathbf{e}$ were outside the system.

We will also impose $P(\{n_{\mathbf{x}}\}; t) = 0$ for any $n_{\mathbf{x}} < 0$, so that we can evaluate Eq. (6.1) for any state $\{n_{\mathbf{x}}\}$. Demanding that an allowed state must have non-negative particle num-

[5] Of course, this isn't particularly neat, having introduced $\{n_{\mathbf{x}}\}$ as the particle numbers on the hypercube, not including any notion of outside.

bers is not enough, because of terms like $(n_{x+e}+1)P(\{n_x-1, n_{x+e}+1\};t)$ which need to evaluate to something, even when $n_x = 0$. Setting $P(\{n_x\};t) = 0$ for any $n_x < 0$ then simply suppresses events whereby a site becomes empty by a particle hopping onto it.[6] We are entering dangerous territory, as this trick of defining $P(\{n_x\};t) = 0$ in certain cases suggests that boundary conditions can be met simply by *imposing* them somewhere. Yet, we must make sure that whatever we impose is compatible with all of the equations that we are using.[7] In the present case, according to Eq. (6.1), a state with negative occupation is never populated, *i.e.* $\frac{d}{dt}P(\{n_x\};t) = 0$ if any of the n_x are negative, so imposing $P(\{n_x\};t) = 0$ for $n_x < 0$ is compatible with Eq. (6.1).

Next, we will consider extinction with rate ϵ, and spontaneous creation with rate β and ignore diffusion for a moment. The extinction is proportional to the number of particles present, whereas the spontaneous creation is not,

$$\frac{dP(\{n_x\};t)}{dt} = \sum_y \epsilon\Big((n_y+1)P(\{n_y+1\};t) - n_y P(\{n_x\};t)\Big) + \sum_y \beta\Big(P(\{n_y-1\};t) - P(\{n_y\};t)\Big)\,, \tag{6.2}$$

where **y** now stresses our focus on "that one site **y** in question". I suppose we could have done this above in Eq. (6.1) as well...

As the POISSON processes on the right of Eq. (6.1) and (6.2) are independent, so they can be superimposed. In fact, the rates with which $P(\{n_x\};t)$ changes can simply be added, resulting in a total rate of change of $P(\{n_x\};t)$ subject to all processes concurrently.[8] The resulting expression

$$\begin{aligned}\frac{dP(\{n_x\};t)}{dt} = {} & \frac{H}{q}\sum_y\sum_e \big((n_{y+e}+1)P(\{n_y-1, n_{y+e}+1\};t) - n_y P(\{n_x\};t)\big) \\ & + \epsilon\sum_y \big((n_y+1)P(\{n_y+1\};t) - n_y P(\{n_x\};t)\big) \\ & + \beta\sum_y \big(P(\{n_y-1\};t) - P(\{n_x\};t)\big)\end{aligned} \tag{6.3}$$

is the process we will consider for the time being: Diffusing particles, which are also subject to spontaneous extinction and creation.

[6] An intermediate event with negative particle numbers, à la "if three people are in a room and four leave, then one has to come back in for it to be empty", is *not* allowed.

[7] For example, if we want to solve $f_t(x,t) = Df_{xx}$ with $f(x=0,t) = 0$, then imposing the latter makes sense only if we never consider $f_t(x,t) = Df_{xx}$ at $x = 0$, because it is likely to be incompatible with $f(x = 0,t) = 0$.

[8] Obviously, these rates are *not* the Poisson rates themselves, which enter only as a pre-factor. The rates of change I am talking about here are terms like $-\epsilon n_y P(\{n_x\};t)$ in Eq. (6.2).

6.2 Operators and states

To transform the master equation Eq. (6.3) into a field theory, we introduce pure, normalised state vectors, $|\{n_{\mathbf{x}}\}\rangle$, describing a system with configuration $\{n_{\mathbf{x}}\}$ (better known as the occupation number representation.[9] These states fulfil the orthogonality condition

$$\langle\{n_{\mathbf{x}}\}|\{m_{\mathbf{x}}\}\rangle = \prod_{\mathbf{x}} \delta_{n_{\mathbf{x}},m_{\mathbf{x}}} \,. \tag{6.4}$$

The empty state, $\forall_{\mathbf{x}} n_{\mathbf{x}} = 0$ is known as the **vacuum** — this is a special state only in as far as it is the empty state, otherwise it is a state like every other state, so in a vector representation this is a (unit) vector like all other vectors. The left and right vacuum will be denoted as $\langle 0|$ and $|0\rangle$ in the following. Technically, the occupation number vectors $\{\langle n_{\mathbf{x}}|\}$ and $\{|n_{\mathbf{x}}\rangle\}$ form a bra and ket-basis of the FOCK space we are interested in.[10] Occasionally, we may write a particular state in terms of its occupation numbers, say $|\{3,17,0,42,0,1\}\rangle$ and so $\langle\{3,17,0,42,0,1\}|\{3,17,0,42,0,1\}\rangle = 1$ and $\langle\{3,17,0,42,0,1\}|\{3,17,0,41,0,1\}\rangle = 0$. We note in passing that of course the vacuum projects to itself:

$$\langle\{0,0,0,\ldots\}|\{0,0,0,\ldots\}\rangle = 1 \,. \tag{6.5}$$

In the following, we will focus on the mixed state

$$|\psi(t)\rangle = \sum_{\{n_{\mathbf{x}}\}} P(\{n_{\mathbf{x}}\};t)\,|\{n_{\mathbf{x}}\}\rangle \tag{6.6}$$

where the sum runs over all possible "pure" states, *i.e.* all occupation numbers. Each term is weighted by the probability of this state. The vector $|\psi(t)\rangle$ will serve as a moment

[9] To call it a "pure" state may be misleading, because pure states may be eigenstates of some relevant operator, which we have not yet introduced. Strictly, the states we are using here are occupation number representations, which are product states of single-particle states (and actually sums of permutations of such product states, if we insist on indexing particles). These single particles states are, in the present case, particle positions. So a "pure state" in the following is a state which is fully described by stating how many particles are at what position (this is beautifully in line with standard thermodynamics, where an argument like this must be invoked in support of the mystical, magical GIBBS-factor). Mixed states would be states where each such states has a probability less than unity. The pure states may be seen as unit vectors in the (FOCK) space of states described by particles numbers. If we were dealing with a system containing one particle that could be at site X or at site Y, the "pure" states would be $(1,0)$ and $(0,1)$ respectively, so a *two*-dimensional vector space. If we have two particles, then we have a *three*-dimensional vector space, $(1,0,0)$ corresponding to 2 particles at X, $(0,1,0)$ to one at X and one at Y and $(0,0,1)$ to two particles at Y. One may feel generous (but not for long) and use an s^n-dimensional vector space for a system of s sites or states and n particles. Some states would not be occupied —

Exercise 9: *Which?*

For example, the empty state might correspond to $(1,0,0,0,\ldots)$. We will, however, consider arbitrary n and arbitrary $s\ldots$

[10] If we think of the $\{|n_{\mathbf{x}}\rangle\}$ as representing pure occupation numbers, then we may as well consider the $\{\langle n_{\mathbf{x}}|\}$ as their transpose.

generating function and a probability generating function. The mechanics of the latter can be seen instantly by projecting out any state of interest,

$$\langle\{n_{\mathbf{x}}\}|\psi(t)\rangle = P(\{n_{\mathbf{x}}\};t)$$

where $\langle\{n_{\mathbf{x}}\}|$ denotes one particular state of interest.

To introduce creation and annihilation operators, we again soften the notation — $|n_{\mathbf{x}}\rangle$ means a state with certain particle numbers throughout, *in particular* $n_{\mathbf{x}}$ particles at $\mathbf{x}$. The following is then the *definition* of the creation operator $a^\dagger(\mathbf{x})$ thought to create a particle at $\mathbf{x}$ and the annihilation operator $a(\mathbf{x})$:

$$a^\dagger(\mathbf{x})\,|n_{\mathbf{x}}\rangle = |n_{\mathbf{x}}+1\rangle \tag{6.7a}$$
$$a(\mathbf{x})\,|n_{\mathbf{x}}\rangle = n_{\mathbf{x}}\,|n_{\mathbf{x}}-1\rangle\ . \tag{6.7b}$$

This definition differs from the usual definition in quantum mechanics in the factor that is being pulled out, which is $\sqrt{n_{\mathbf{x}}+1}$ and $\sqrt{n_{\mathbf{x}}}$ respectively there. In the following we may take Eq. (6.7) and Eq. (6.4) as all we need to know about the algebra and the space we are considering. The occupation number vectors are eigenvectors of the **particle number operator** $a^\dagger(\mathbf{x})a(\mathbf{x})$, as

$$a^\dagger(\mathbf{x})a(\mathbf{x})\,|n_{\mathbf{x}}\rangle = n_{\mathbf{x}}\,|n_{\mathbf{x}}\rangle \tag{6.8}$$

which follows instantly from Eq. (6.7).

A number of interesting features follow from Eq. (6.7b), which are most easily derived using the basis $|\{n_{\mathbf{x}}\}\rangle$ of the FOCK space (Negele and Orland, 1988). Firstly, the commutator

$$\left[a(\mathbf{x}),a^\dagger(\mathbf{x})\right]_- = a(\mathbf{x})a^\dagger(\mathbf{x}) - a^\dagger(\mathbf{x})a(\mathbf{x}) = 1 \tag{6.9}$$

which can be seen by applying $\left[a,a^\dagger\right]_-$ to any particular occupation number representation vector $|\{n_{\mathbf{x}}\}\rangle$. Trivially

$$[a(\mathbf{x}),a(\mathbf{x})]_- = 0 \tag{6.10a}$$
$$\left[a^\dagger(\mathbf{x}),a^\dagger(\mathbf{x})\right]_- = 0\,, \tag{6.10b}$$

i.e. operators acting at different points in space commute (note that operators do not depend on time) — position is a "quantum number" here, so in effect, operators operating at different points create particles with different quantum numbers. Therefore

$$\left[a(\mathbf{x}),a^\dagger(\mathbf{x}')\right]_- = 0 \tag{6.11}$$

for any $\mathbf{x}\neq\mathbf{x}'$.

What about the action of the operators to bras? Dropping the position $\mathbf{x}$ for a moment,

$$\langle m|\, a^\dagger\, |n\rangle = \langle m|n+1\rangle = \delta_{m,n+1} = \delta_{m-1,n} = \langle m-1|n\rangle \tag{6.12}$$

from Eq. (6.7) and similarly

$$\langle m|\, a\, |n\rangle = n\, \langle m|n-1\rangle = n\delta_{m,n-1} = (m+1)\delta_{m+1,n} = (m+1)\, \langle m+1|n\rangle \tag{6.13}$$

and so

$$\langle n_{\mathbf{x}}|\, a^\dagger(\mathbf{x}) \;=\; \langle n_{\mathbf{x}}-1| \tag{6.14a}$$

$$\langle n_{\mathbf{x}}|\, a(\mathbf{x}) \;=\; (n_{\mathbf{x}}+1)\, \langle n_{\mathbf{x}}+1|\ , \tag{6.14b}$$

with the one catch that $\langle 0|\, a^\dagger(\mathbf{x}) = 0$, as can be seen in Eq. (6.12), chosing $m = 0$ and any $n \geqslant 0$. This is the price we have to pay for not having $\sqrt{n_{\mathbf{x}}+1}$ on the right hand side of Eq. (6.7a).

We are now in the situation to calculate expectation values on the basis of $|\psi(t)\rangle$, as defined Eq. (6.6) above. For example, the expected particle number at some position $\mathbf{y}$ is

$$\langle n\rangle\,(\mathbf{y};t) = \sum_{\{n_{\mathbf{x}}\}} P(\{n_{\mathbf{x}}\};t) n_{\mathbf{y}} \tag{6.15}$$

where the sum runs over all states. Based on $|\psi(t)\rangle$ and the particle number operator $a^\dagger(\mathbf{y})a(\mathbf{y})$, Eq. (6.8), we can *almost* write this instantly as

$$a^\dagger(\mathbf{y})a(\mathbf{y})\,|\psi(t)\rangle = \sum_{\{n_{\mathbf{x}}\}} P(\{n_{\mathbf{x}}\};t) a^\dagger(\mathbf{y})a(\mathbf{y})\,|\{n_{\mathbf{x}}\}\rangle = \sum_{\{n_{\mathbf{x}}\}} P(\{n_{\mathbf{x}}\};t) n_{\mathbf{y}}\,|\{n_{\mathbf{x}}\}\rangle\ , \tag{6.16}$$

where we have, without much thinking, taken the operator into the sum.[11] To reduce Eq. (6.16) to Eq. (6.15) we need a vector that projects to unity with *every* other vector. This vector is

$$\langle \text{☼}| = \sum_{\{n'_{\mathbf{x}}\}} \left\langle \{n'_{\mathbf{x}}\}\right| \tag{6.17}$$

which I may call the **abyss** in the following, because all vectors project, indiscriminately to the same with it, $\langle \text{☼}|\{n'_{\mathbf{x}}\}\rangle = 1$, it is the "nemesis" of any structure. Applying it from the left to Eq. (6.16) gives

$$\langle \text{☼}|\, a^\dagger(\mathbf{y})a(\mathbf{y})\,|\psi(t)\rangle = \sum_{\{n_{\mathbf{x}}\}} P(\{n_{\mathbf{x}}\};t) n_{\mathbf{y}}\, \langle \text{☼}|\{n_{\mathbf{x}}\}\rangle = \sum_{\{n_{\mathbf{x}}\}} P(\{n_{\mathbf{x}}\};t) n_{\mathbf{y}} = \langle n\rangle\,(\mathbf{y};t) \tag{6.18}$$

precisely what we were after. We may be looking for a more elegant representation of the

[11] Simply on the basis of its linearity, but not considering any possible convergence problems. Apeirophobia kicking in, `http://www.karinkihlberg-reubenhenry.org/moving_image/apeirophobic-framework/`.

abyss $\langle ☼|$. Dropping the position index $\mathbf{x}$ for a moment, we note that based on Eq. (6.14b), $n!\,\langle n| = \langle 0|\,a^n$, *i.e.* the left vectors that make up $\langle ☼|$ can be written in terms of powers of the annihilation operators,

$$\langle ☼| = \sum_{n=0}^{\infty} \frac{1}{n!}\,\langle 0|\,a^n = \langle 0|\,e^a\,. \tag{6.19}$$

The abyss has the wonderful property that it is invariant under $a^\dagger$ (from the right),

$$\langle ☼|\,a^\dagger = \sum_{n=0}^{\infty} \langle n|\,a^\dagger = \sum_{n=1}^{\infty} \langle n-1| + 0 = \langle ☼| \tag{6.20}$$

using Eq. (6.14a) and $\langle 0|\,a^\dagger = 0$. In other words, $\langle ☼|$ is an eigenstate of the creation operator, known as a **coherent state**.[12]

Using coherent states for the definition of the abyss Eq. (6.17), which is defined in all space means that we need to apply $\exp(a)$ everywhere,

$$\langle ☼| = \langle 0|\,e^{\sum_x a(x)}\,. \tag{6.21}$$

It is the ease with which we can write the abyss (the sum of all state vectors) and its resulting relation to the creation operator that makes the definitions Eq. (6.7) so useful (in contrast to the symmetric, quantum-mechanical definition involving $\sqrt{n}$).

Looking back at how we introduced the abyss, it is not surprising that it is an eigenvector of the creation operator. After all, we demanded that it projects everything to unity. So, promoting anything by applying the creation operator first is not going to make any difference. That (correctly) suggests that we may drop $a^\dagger(\mathbf{y})$ in the expectation Eq. (6.18),

$$\langle n\rangle\,(\mathbf{y};t) = \langle ☼|\,a(\mathbf{y})\,|\psi(t)\rangle\,, \tag{6.22}$$

as we do not need to recreate what has been annihilated by $a(\mathbf{y})$ if it is all going to be projected into the abyss anyway. The virtue of keeping the $a^\dagger(\mathbf{y})$ becomes clearer if higher moments of the density are needed, because

$$\left(a^\dagger a\right)^m |n\rangle = n^m\,|n\rangle \tag{6.23}$$

whereas

$$(a)^m\,|n\rangle = \frac{n!}{(n-m)!}\,|n-m\rangle\,, \tag{6.24}$$

provided $n \geqslant m$.

In the following we will assume normal ordering, which means that the creation op-

[12] One may be tempted to generalise, using $\langle 0|\exp(a\alpha) = \sum_n \alpha^n\,\langle n|$ with $\alpha \in \mathbb{C}$. Applying $a^\dagger$ from the right then gives $\langle 0|\exp(a\alpha)\,a^\dagger = \sum_{n=1} \alpha^n\,\langle n-1| = \alpha\,\langle 0|\exp(a\alpha)$, forming an eigenvector of $a^\dagger$ with eigenvalue α. As $\alpha \in \mathbb{C}$ one may suspect coherent states to form an overcomplete basis.

erators stand on the left and the annihilation operators on the right. Using $aa^\dagger = 1 + a^\dagger a$, Eq. (6.9), that ordering can always be established, even among operators that do not commute. It is obviously particularly useful if the operators are eventually projected to the abyss.

6.2.1 Creation and annihilation for diffusion with creation and extinction

In the following we will construct the creation and annihilation operators for the master equation Eq. (6.3) we developed in Section 6.1.1, p. 51. As a reminder, that consistent of three ingredients: Diffusion with a hopping rate H/q, spontaneous extinction with rate ϵ and spontaneous creation with rate β. We will start with spontaneous extinction, as the easiest and maybe most important example.

Using the definition of $|\psi(t)\rangle$, Eq. (6.6), we note that its derivative with respect to time for the present case can apparently be written in three terms, namely the three terms of $\dot{P}(\{n_{\mathbf{x}}\}; t)$, Eq. (6.3),

$$\frac{d}{dt}|\psi(t)\rangle = (\widehat{\mathcal{A}}_H + \widehat{\mathcal{A}}_\epsilon + \widehat{\mathcal{A}}_\beta)\,|\psi(t)\rangle \ . \tag{6.25}$$

Each of these operators $\widehat{\mathcal{A}}_H$, $\widehat{\mathcal{A}}_\epsilon$ and $\widehat{\mathcal{A}}_\beta$ are **time evolution operators** in their own right. Below, we will obviously use their sum

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_H + \widehat{\mathcal{A}}_\epsilon + \widehat{\mathcal{A}}_\beta \ , \tag{6.26}$$

as the **LOUVILLian**.[13]

There are two equivalent perspectives on $d/dt\,|\psi\rangle$: It is either the time derivative of $|\psi(t)\rangle$ with $\dot{P}(\{n_{\mathbf{x}}\}, t)$ replaced by Eq. (6.3), or the master equation multiplied from the right by $|\{n_{\mathbf{x}}\}\rangle$ and summed over all configurations $\{n_{\mathbf{x}}\}$. That latter view means that $d/dt\,|\psi\rangle$ is a *transform* of the master equation.

We will "translate" the master equation term by term into an operator. For the time being, we focus on $\widehat{\mathcal{A}}_\epsilon$,

$$\widehat{\mathcal{A}}_\epsilon\,|\psi(t)\rangle = \epsilon \sum_{\mathbf{y}} \sum_{\{n_{\mathbf{x}}\}} \left((n_{\mathbf{y}}+1)P(\{n_{\mathbf{y}}+1\}; t) - n_{\mathbf{y}} P(\{n_{\mathbf{x}}\}; t)\right) |\{n_{\mathbf{x}}\}\rangle \ , \tag{6.27}$$

where we have introduced $\mathbf{y}$ to avoid a clash in notation and swapped summation over states $\{n_{\mathbf{x}}\}$ and sites $\mathbf{y}$. Using Eq. (6.6), we can straight-forwardly determine

$$a^\dagger(\mathbf{y})a(\mathbf{y})\,|\psi(t)\rangle = \sum_{\{n_{\mathbf{x}}\}} n_{\mathbf{y}} P(\{n_{\mathbf{x}}\}; t)\,|\{n_{\mathbf{x}}\}\rangle \ . \tag{6.28}$$

[13] Nomenclature is somewhat confused here. In quantum systems, the operator usually called the time evolution operator, in statistical mechanism, the evolution of the density is given by the LOUVILLian. What evolves here, however, is not the probability density, but its transform, and some authors prefer the use of HAMILTONian or action.

This is essentially the second term of Eq. (6.27). To avoid drowning in notation, we write the first part as

$$\sum_{n=0}(n+1)P(n+1)\,|n\rangle = \sum_{n=0}P(n+1)a\,|n+1\rangle = a\sum_{n=1}P(n)\,|n\rangle = a\sum_{n=0}P(n)\,|n\rangle \quad (6.29)$$

where the last identity holds only because $a\,|0\rangle = 0$. We can therefore write

$$\widehat{\mathcal{A}}_\epsilon = \epsilon\sum_{\mathbf{y}}\Big(a(\mathbf{y}) - a^\dagger(\mathbf{y})a(\mathbf{y})\Big)\ . \quad (6.30)$$

We proceed similarly with the other terms. Spontaneous creation has one term of the form (see Eq. (6.2))

$$\sum_{n=0}P(n-1)\,|n\rangle = \sum_{n=0}P(n-1)a^\dagger\,|n-1\rangle\ , \quad (6.31)$$

whose right hand side looks a bit strange as $|n-1\rangle$ does not exist for $n=0$, yet $P(-1)=0$ avoids that term. As mentioned above, p. 50, this is dangerous territory. In the present case, we can safely perform the summation from $n=1$ and so Eq. (6.31) is in fact $a^\dagger\sum_0 P(n)\,|n\rangle$. The second term in the spontaneous creation part of Eq. (6.2) gives just $-\beta\,|\psi\rangle$, so

$$\widehat{\mathcal{A}}_\beta = \beta\sum_{\mathbf{y}}\Big(a^\dagger(\mathbf{y}) - 1\Big)\ . \quad (6.32)$$

We see that the key to writing the master equation in terms of operators is to match-up the occupation numbers featuring in the ket-vector and in the probability, so that any operator can be moved outside of the summation and what remains is $|\psi\rangle$.

Finally, we need to consider the diffusion term. Again, we ought to simplify notation to avoid overlooking the crucial steps. On *a single site* $\mathbf{y}$, the terms are

$$\frac{H}{q}\sum_{\mathbf{e}}\left((n_\mathbf{e}+1)P(\{n-1,n_\mathbf{e}+1\}) - nP(n)\right) \quad (6.33)$$

where $n_\mathbf{e}$ denotes the occupation at a site displaced from $\mathbf{y}$ by the unit vector $\mathbf{e}$. The first term describes hopping of a particle from $\mathbf{y}+\mathbf{e}$ to $\mathbf{y}$, the second term the hopping from $\mathbf{y}$ to any of the neighbours. There are q of them, so the second term will in total just be $-Ha^\dagger(\mathbf{y})a(\mathbf{y})$, just like the term we encountered in the spontaneous extinction, Eq. (6.30). This is not a coincidence: Any Poisson process that can "befall" every particle individually and independently will take place *in toto* across the entire system with a rate proportional to the total particle number. The first term, $(n_\mathbf{e}+1)P(\{n-1,n_\mathbf{e}+1\})$ is the one "that makes the music". It is the only term that carries any signature of the lattice structure. This term features in our transform as

$$\sum_{\{n\}}(n_\mathbf{e}+1)P(\{n-1,n_\mathbf{e}+1\})\,|\{n\}\rangle\ . \quad (6.34)$$

If $|n-1, n_{\mathbf{e}}+1\rangle$ is a configuration $\{n_{\mathbf{x}}\}$ except for the two occupation numbers $n_{\mathbf{y}}$ and $n_{\mathbf{y}+\mathbf{e}}$, which we write as n and $n_{\mathbf{e}}$ respectively, then

$$\begin{aligned}(n_{\mathbf{e}}+1)P(\{n-1, n_{\mathbf{e}}+1\})\,|n, n_{\mathbf{e}}\rangle &= a^\dagger(\mathbf{y})(n_{\mathbf{e}}+1)P(\{n-1, n_{\mathbf{e}}+1\})\,|n-1, n_{\mathbf{e}}\rangle \\ &= a^\dagger(\mathbf{y})a(\mathbf{y}+\mathbf{e})P(\{n-1, n_{\mathbf{e}}+1\})\,|n-1, n_{\mathbf{e}}+1\rangle\ .\end{aligned} \tag{6.35}$$

Now that the occupation numbers in the ket-vector match that in the probability, the summation over all states $\{n\}$ simply generates $|\psi\rangle$. We thus have

$$\widehat{\mathcal{A}}_H = \frac{H}{q}\sum_{\mathbf{y}}\sum_{\mathbf{e}}\left(a^\dagger(\mathbf{y})a(\mathbf{y}+\mathbf{e}) - a^\dagger(\mathbf{y})a(\mathbf{y})\right) = \frac{H}{q}\sum_{\mathbf{y}}\sum_{\mathbf{e}} a^\dagger(\mathbf{y})\left(a(\mathbf{y}+\mathbf{e}) - a(\mathbf{y})\right)\ , \tag{6.36}$$

where we have split the extinction $Ha^\dagger(\mathbf{y})a(\mathbf{y})$ among the q nearest neighbours again. There is an elegant identity that we may wish to use, namely

$$\begin{aligned}&\left(a^\dagger(\mathbf{y}) - a^\dagger(\mathbf{y}+\mathbf{e})\right)\left(a(\mathbf{y}+\mathbf{e}) - a(\mathbf{y})\right) \\ &\qquad = a^\dagger(\mathbf{y})\left(a(\mathbf{y}+\mathbf{e}) - a(\mathbf{y})\right) + a^\dagger(\mathbf{y}+\mathbf{e})\left(a(\mathbf{y}) - a(\mathbf{y}+\mathbf{e})\right)\ .\end{aligned} \tag{6.37}$$

Summing over all $\mathbf{y}$ and all $\mathbf{e}$ and ignoring any problems at boundaries for a moment,[14] the second term reproduces the first one. If the first one describes hopping onto and away from site $\mathbf{y}$, the second one describes the same two processes on $\mathbf{y}+\mathbf{e}$. Below we will make use of this doubling by considering only half of the $\mathbf{e}$. For the time being we write

$$\widehat{\mathcal{A}}_H = -\frac{H}{2q}\sum_{\mathbf{y}}\sum_{\mathbf{e}}\left(a^\dagger(\mathbf{y}+\mathbf{e}) - a^\dagger(\mathbf{y})\right)\left(a(\mathbf{y}+\mathbf{e}) - a(\mathbf{y})\right)\ . \tag{6.38}$$

Together with Eq. (6.30) and (6.32) we have therefore completed the programme of writing all three processes in terms of operators.

We have cast the master equation in the language of creation and annihilation operators. This is actually a pretty straight-forward procedure. There are certain terms, that *cannot* be "turned into" operators. For example, $(n+1)P(n+1)\,|n\rangle$ can be written as $aP(n+1)\,|n+1\rangle$ and then summed over, but that also means that $P(n+1)\,|n\rangle$ *cannot* be written in that form. This is not a coincidence. It indicates that a certain process (here, the transition from $n+1$ to n particles) is not due to *concurrent Poisson processes* "befalling" $n+1$ individual particles. Instead, it is more akin to a cooperative phenomenon at the microscopic scale. In some cases, this type of transition can be written in terms of operators, once a different degree of freedom is considered, *i.e.* when we ended up with $P(n+1)\,|n\rangle$, we may have chosen the wrong entity as a particle. In other cases introducing a second particle species may result in a more suitable process. We have to keep in mind that we

[14] In fact, a very long moment. Considering different boundary conditions is not part of the present chapter.

are aiming for a perturbation theory over a bilinear form, the GAUSSian theory. So, we must avoid writing an operator, which may be very elegant and compact, describing the process fully, yet leads to completely undesired processes once expanded perturbatively.

In general, the processes that the present formalism is capable to describe are

- indvidual to each particle,
- local in time (Poissonian),
- local in space.

6.3 Derivation of the field theory

We are now in the position to make the transition to the field theory. Technically, this is probably the most demanding part of the procedure, but it will turn out that it can be expressed as a set of rules as simple as "an $a^\dagger$ becomes a $\phi^\dagger$ and an a becomes a ϕ". As a matter of pragmatism, I could just state those rules, but that would leave the reader ill-equipped to resolve ambiguities and other problems that, sadly, arise.

If the result of the above procedure is an operator $\widehat{\mathcal{A}}$ such that

$$\frac{d}{dt}\left|\psi(t)\right\rangle = \widehat{\mathcal{A}}\left|\psi(t)\right\rangle \tag{6.39}$$

with $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_H + \widehat{\mathcal{A}}_\epsilon + \widehat{\mathcal{A}}_\beta$ in our example above, see Eq. (6.25), then the formal solution is simply

$$\left|\psi(t)\right\rangle = \exp\left(\widehat{\mathcal{A}}t\right)\left|\psi(0)\right\rangle \ . \tag{6.40}$$

This is a formal solution not least because we do not know whether $\exp\left(\widehat{\mathcal{A}}t\right)$ is well defined.[15] Proceeding nevertheless, we can rewrite, Eq. (6.18), the expected particle number at time t, as

$$\langle n\rangle(\mathbf{y};t) = \left\langle \text{☼}\middle|a^\dagger(\mathbf{y})a(\mathbf{y})\middle|\psi(t)\right\rangle = \left\langle \text{☼}\middle|a^\dagger(\mathbf{y})a(\mathbf{y})\exp\left(\widehat{\mathcal{A}}t\right)\middle|\psi(0)\right\rangle \ . \tag{6.41}$$

In fact, the expectation of any (operator) observable $\widehat{\mathcal{O}}$ may be written as

$$\left\langle\widehat{\mathcal{O}}\right\rangle(t) = \left\langle \text{☼}\middle|\widehat{\mathcal{O}}\exp\left(\widehat{\mathcal{A}}t\right)\middle|\psi(0)\right\rangle = \left\langle \text{☼}\middle|\widehat{\mathcal{O}}\exp\left(\widehat{\mathcal{A}}t\right)\mathcal{I}\middle|0\right\rangle \ , \tag{6.42}$$

where we have introduced the initialisation operator $\mathcal{I}$ that generates $\left|\psi(0)\right\rangle$ from the vacuum, $\mathcal{I}\left|0\right\rangle = \left|\psi(0)\right\rangle$.

The expectation of unity, $\langle 1\rangle$ is obviously of particular interest. If the master equation

[15] In the following we will be very generous with issues of convergence and freely interpret any exponential $\exp\left(\widehat{\mathcal{A}}t\right)$ as a power series, $\sum_{n=0}^{\infty}(\widehat{\mathcal{A}}t)^n/n!$.

fulfils

$$\sum_{\{n_x\}} \dot{P}(\{n_x\};t) = 0\,, \tag{6.43}$$

then this conservation of probability ought to be reflected by

$$\langle 1\rangle\,(t) = \langle \doteq|\psi(t)\rangle = \left\langle \doteq\middle|\exp\left(\widehat{\mathcal{A}}t\right)\middle|\psi(0)\right\rangle = 1\,. \tag{6.44}$$

Expanding in small (but in fact arbitrary) t, we first note that

$$\langle \doteq|\psi(0)\rangle = \sum_{\{n_x\}} P(\{n_x\};0)\,\langle \doteq||\{n_x\}\rangle\rangle = \sum_{\{n_x\}} P(\{n_x\};0) = 1 \tag{6.45}$$

and therefore we require $\left\langle \doteq\middle|(\exp\left(\widehat{\mathcal{A}}t\right)-1)\middle|\psi(0)\right\rangle = 0$ and thus,

$$\left\langle \doteq\middle|(\widehat{\mathcal{A}}t) + (\widehat{\mathcal{A}}t)^2/2 + (\widehat{\mathcal{A}}t)^3/3\ldots\middle|\psi(0)\right\rangle = 0\,, \tag{6.46}$$

which can hold for arbitrary t only if in fact $\left\langle \doteq\middle|\widehat{\mathcal{A}}^n\middle|\psi(0)\right\rangle = 0$ for all $n \geqslant 1$, which is the case as soon as $\left\langle \doteq\middle|\widehat{\mathcal{A}}\middle|\psi(0)\right\rangle = 0$. Because $|\psi(0)\rangle$ is arbitrary, we need to demonstrate that

$$\left\langle \doteq\middle|\widehat{\mathcal{A}}\middle|\{n_x\}\right\rangle = 0 \tag{6.47}$$

for every state $|\{n_x\}\rangle$. In the following, we will demonstrate the consequences of probability conservation by focusing on a single occupation number n. As $\widehat{\mathcal{A}}$ is normal ordered, all creation operators are standing on the left, and because $\langle \doteq|$ is the left eigenvector of the creation operator with eigenvalue unity, Eq. (6.20), we have $\langle \doteq|\,\widehat{\mathcal{A}} = \langle \doteq|\,\widehat{\mathcal{A}}|_{a^\dagger=1}$, where

$$\widehat{\mathcal{A}}|_{a^\dagger=1} = \sum_{m=0} \frac{\alpha_m}{m!} a^m \tag{6.48}$$

denotes the operator created by setting all creation operators equal to unity. Anticipating some of the below, the coefficients $\alpha_m/m!$ are chosen so that

$$\left\langle \doteq\middle|\frac{\alpha_m}{m!}a^m\middle|n\right\rangle = \binom{n}{m}\alpha_m \tag{6.49}$$

with the tacit convention that $\binom{n}{m} = 0$ for $m > n$ (as will be used below). In summary,

$$\left\langle \doteq\middle|\widehat{\mathcal{A}}\middle|n\right\rangle = \sum_{m=0}^{n}\binom{n}{m}\alpha_m\,, \tag{6.50}$$

which has to vanish for all $n \geqslant 0$. Choosing $n = 0$ thus gives $\alpha_0 = 0$, choosing subsequently $n = 1$ then implies $\alpha_1 = 0$. In fact, if $\alpha_i = 0$ for all $0 \leqslant i < n$, then $\alpha_n = 0$

follows and by induction we have not only $\left\langle \text{☼}\middle|\widehat{\mathcal{A}}\middle|n\right\rangle = 0$ for all n and thus

$$\langle \text{☼}|\,\widehat{\mathcal{A}} = \langle \text{☼}|\,\widehat{\mathcal{A}}|_{a^\dagger=1} = 0 \tag{6.51}$$

and

$$\langle \text{☼}|\exp\left(\widehat{\mathcal{A}}t\right) = \langle \text{☼}| \tag{6.52}$$

we have, even stronger, $\alpha_n = 0$ for all n. In other words,

$$\widehat{\mathcal{A}}|_{a^\dagger=1} = 0\,. \tag{6.53}$$

The fundamental theorem of algebra therefore implies that $\widehat{\mathcal{A}}$ can always be written as $\widehat{\mathcal{A}} = (a^\dagger - 1)\widehat{\mathcal{A}}'$. Further down, we will introduce the DOI-shifted operator $a^\dagger = \tilde{a} + 1$ and so the LIOUVILLian is always a (right) multiple of the DOI-shifted creator, $\tilde{a}$. I should stress that this means *any* DOI-shifted operator (for any point in space, any species *etc.*), *i.e.*

$$\widehat{\mathcal{A}} = \sum_{\mathbf{x}} \tilde{a}(\mathbf{x})(\ldots \text{operators, maybe "associated" with } \mathbf{x}\ldots)\,. \tag{6.54}$$

As nice as Eq. (6.40) may look, in its current form, $\exp\left(\widehat{\mathcal{A}}t\right)$ is not of much help, because it is very difficult to make use of this very complicated operator (this is an opportunity to consider coherent states again). To make progress, we ought to get rid of all operators and retain only "their effect" (whatever that means). As a first step, we write

$$\exp\left(\widehat{\mathcal{A}}t\right) = \underbrace{(1+\Delta t\widehat{\mathcal{A}})(1+\Delta t\widehat{\mathcal{A}})\ldots(1+\Delta t\widehat{\mathcal{A}})}_{(t/\Delta t)\text{terms}} \tag{6.55}$$

for some finite Δt, our finite time slice. In other words, as time goes by, time evolution of the system is realised by "multiplying" its state $|\psi\rangle$ by $(1+\Delta t\widehat{\mathcal{A}})$, taking $\Delta t \to 0$ at some stage. Between any those brackets we will slot in some weird and wonderful identity, $\mathbf{1}$, so

$$\exp\left(\widehat{\mathcal{A}}t\right) = \mathbf{1}(1+\Delta t\widehat{\mathcal{A}})\mathbf{1}(1+\Delta t\widehat{\mathcal{A}})\mathbf{1}(1+\Delta t\widehat{\mathcal{A}})\ldots\mathbf{1}(1+\Delta t\widehat{\mathcal{A}})\mathbf{1}\,. \tag{6.56}$$

To obtain an expression for this identity, we first note the magical complex integral

$$I = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-i z^* z} z^{*n} z^m \tag{6.57}$$

where $z = x + iy$. Writing $z = r\exp(i\theta)$ in polar coordinates, the product $z^{*n}z^m$ becomes simply $r^{n+m}\exp(i\theta(m-n))$ and $\exp(-z^*z) = \exp\left(-r^2\right)$, which renders the integral essentially a GAUSSian. Carrying the coordinate transform through (with suitable Jacobian), the integral vanishes for $n-m \neq 0$ because $\int_0^{2\pi} d\theta\, \exp(in\theta) = 0$ for all non-zero

$n \in \mathbb{Z}$ and gives 2π otherwise, so that

$$I = \int_0^\infty dr \int_0^{2\pi} d\theta\, r^{1+n+m} e^{-r^2} e^{i\theta(m-n)} = \delta_{n,m}\pi \int_0^\infty du\, u^n e^{-u} = \delta_{n,m}\pi n! \tag{6.58}$$

using the substitution $u = r^2$. Although the pre-factors won't actually matter, the integral may be written as

$$\int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} z^{*n} z^m = \delta_{n,m} n! \tag{6.59}$$

where $v \wedge w$ denotes the **anticommutative multiplication (wedge or exterior product)**, $v \wedge w = -w \wedge v$ and therefore $v \wedge v = 0$, so

$$dz^* \wedge dz = dx \wedge dx + dy \wedge dy - i dy \wedge dx + dx \wedge i dy = 2 dx \wedge i dy\,.$$

This notation alludes to the fact that we are dealing with GRASSMANN variables. Below, the orthogonality relation (6.59) will turn into the path integral over the field $\phi(\mathbf{x}, t)$. It is worth contrasting that with the path integral normally encountered in JANSSEN-DE DOMINICIS (or MARTIN-SIGGIA-ROSE) field theories based on a LANGEVIN equation or a HAMILTONian (relaxation to equilibrium), where *two* independent fields are introduced and integrated over: The physical field $\phi(\mathbf{x}, t)$ and the auxiliary field $\tilde{\phi}(\mathbf{x}, t)$, a remnant of the explicit noise term. As we are drawing on a master equation, we no longer have a noise term. The differential equations to solve (the master equation) are *deterministic*!

As a brief intermezzo, I would like to point out that the orthogonality relation Eq. (6.59) has a rather surprising property: There is only *one* dummy variable, z, so by substitution it follows immediately that

$$\int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\zeta)} (z+\zeta)^{*n} (z+\zeta)^m = \delta_{n,m} n!\,, \tag{6.60}$$

for arbitrary $\zeta \in \mathbb{C}$. It turns out however, that z and z^* can be treated *as if* they were independent. In fact,

$$I_{n,m}(\zeta, \xi) = \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)} (z+\zeta)^{*n} (z+\xi)^m = \delta_{n,m} n!\,, \tag{6.61}$$

even if $\zeta \neq \xi$. To see this, we substitute to arrive at

$$\begin{aligned} I_{n,m}(\zeta, \xi) &= \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)} (z+\zeta)^{*n} (z+\xi)^m \\ &= \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^*(z+\xi-\zeta)} z^{*n} (z+\xi-\zeta)^m \\ &= \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} \sum_{i=0}^{\infty} \frac{(-z)^{*i}}{i!} u^i z^{*n} \sum_{j=0}^{m} z^j u^{m-j} \binom{m}{j} \end{aligned} \tag{6.62}$$

with $u = \xi - \zeta$. Orthogonality according to Eq. (6.59) means that only terms with $i + n =$

$j \leqslant m$ contribute. Because $i, j, n, m \geqslant 0$, we instantly have $I_{n,m}(\zeta, \xi) = 0$ for $n > m$. Of course, the shift $z' = z + \zeta$ could have been implemented as $z' = z + \xi$ which would result in the statement $I_{n,m}(\zeta, \xi) = 0$ for $n < m$. We thus have recovered orthogonality. It is instructive, however, to calculate $I_{n,m}(\zeta, \xi)$ explicitly for $n \leqslant m$; because $i \geqslant 0$ only $i = j - n$ contribute for $j \geqslant n$:

$$I_{n,m}(\zeta, \xi) = \sum_{j=n}^{m} \frac{(-)^{j-n}}{(j-n)!} u^{j-n} u^{m-j} \binom{m}{j} j! \tag{6.63}$$

where the trailing $j!$ is due to Eq. (6.59). Shifting the summation to $k = j - n = 0, \ldots, m-n$ leaves us with

$$\begin{aligned} I_{n,m}(\zeta, \xi) &= \sum_{k=0}^{m-n} (-)^k u^k u^{m-n-k} \frac{(k+n)!}{k!} \binom{m}{k+n} \\ &= \frac{m!}{(m-n)!} \sum_{k=0}^{m-n} (-)^k u^k u^{m-n-k} \binom{m-n}{k} \\ &= \frac{m!}{(m-n)!} (u-u)^{m-n} \,. \end{aligned} \tag{6.64}$$

For $m - n > 0$ this is clearly 0 and we therefore have recovered orthogonality. For $m = n$ the outcome is thus crucially dependent on our interpretation of $(u-u)^0$. The answer can be found in the very expression above, where we wrote $(-u)^0 u^0 \binom{0}{0} = (u-u)^0$, so $(u-u)^0 = 1$. This concludes the proof of Eq. (6.61).

If we write that relation (6.61) as

$$\begin{aligned} \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} z^{*n} z^m &= \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)} (z+\zeta)^{*n} (z+\xi)^m \\ &= \delta_{n,m} n! \,, \end{aligned} \tag{6.65}$$

then for every polynomial $f(a, b)$ in a and b, we also have

$$\int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} f(z^*, z) = \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)} f(z^* + \zeta^*, z + \xi) \,. \tag{6.66}$$

If we wish to write that same relation for an arbitrary function $f(a, b)$, then we need that it is expandable in a and b about $a = 0$ and $b = 0$ and that the sum of the resulting power series can be swapped with the integration,[16] as Eq. (6.61) needs to be applied term by term to the power series $f(a, b)$. We may wish to use Eq. (6.66) either in an existing *perturbative* field theory to justify a shift of field ϕ and ϕ^* independently by some (in principle time and space dependent) amount. This would need to be implemented under the assumption that the field theory before the shift is convergent.[17] However, Eq. (6.66)

[16] Apeirophobia, again.

[17] We would take the existing field theory, claim that we expand things perturbatively (*i.e.* that the theory

may also be used to write a different identity $\mathbf{1}$, see Eq. (6.67) below, which results in a different representation of the field theory (namely one with shifted fields), starting from a given action $\widehat{\mathcal{A}}$. This avoids the strong convergence assumption. The same can be done, after some painful calculation, at operator level.

In the following, to ease notation, we will deal with a field theory in a point, *i.e.* without space, so that the occupation numbers to consider are not $\{n_x\}$, but just $\{n\} = \mathbb{N}$. We use the integral (6.59) above to determine

$$\begin{aligned}\int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp(-\phi^*\phi) \exp\left(\phi a^\dagger\right) |0\rangle \langle 0| \exp(\phi^* a) \\ = \int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp(-\phi^*\phi) \sum_{n,m=0}^{\infty} \frac{\phi^n \phi^{*m}}{n!} |n\rangle \langle m| = \sum_n |n\rangle \langle n| = \mathbf{1}\end{aligned} \quad (6.67)$$

using Eq. (6.7a) and Eq. (6.14b), which, in particular, gives $\langle 0| a^n = n! \langle n|$. This identity can be slotted in between the brackets as indicated in Eq. (6.56). The fields ϕ in different brackets are independent and can be labelled according to the "time slot" they represent, say in the form

$$\int \frac{d\phi^*(t) \wedge d\phi(t)}{2\pi i} \exp(-\phi^*(t)\phi(t)) \exp\left(\phi(t) a^\dagger\right) |0\rangle \langle 0| \exp(\phi^*(t) a) = \mathbf{1}_t \,. \quad (6.68)$$

Using that notation, the terms we will consider in the following have time labels from 0 to t in steps of Δt, say

$$\mathbf{1}_{t+\Delta t}(1 + \Delta t \widehat{\mathcal{A}}) \mathbf{1}_t \quad (6.69)$$

which means that each bracket is contained in a sandwich of the form

$$\begin{aligned}\Xi(t + \Delta t, t) &= \langle 0| \exp(\phi^*(t + \Delta t) a) \, (1 + \Delta t \widehat{\mathcal{A}}) \exp(-\phi^*(t)\phi(t)) \exp\left(\phi(t) a^\dagger\right) |0\rangle \\ &= \exp(-\phi^*(t)\phi(t)) \sum_{n,m=0}^{\infty} \frac{\phi^*(t + \Delta t)^n \phi(t)^m}{m!} \langle n| \, (1 + \Delta t \widehat{\mathcal{A}}) \, |m\rangle\end{aligned} \quad (6.70)$$

where all scalar pre-factors have been taken out of the sandwich. We can thus write Eq. (6.55) as

$$\begin{aligned}\exp\left(\widehat{\mathcal{A}} t\right) = \int \frac{d\phi^*(t + \Delta t) \wedge d\phi(t + \Delta t)}{2\pi i} \frac{d\phi^*(t) \wedge d\phi(t)}{2\pi i} \cdots \frac{d\phi^*(0) \wedge d\phi(0)}{2\pi i} \\ \cdot \exp(-\phi^*(t + \Delta t)\phi(t + \Delta t)) \exp\left(\phi(t + \Delta t) a^\dagger\right) |0\rangle \\ \cdot \Xi(t + \Delta t, t) \Xi(t, t - \Delta t) \ldots \Xi(\Delta t, 0) \cdot \\ \langle 0| \exp(\phi^*(0) a) \,.\end{aligned} \quad (6.71)$$

is convergent), so that we only ever deal with integrals like $\int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp(-\phi^*\phi A) \, f(\phi, \phi^*)$, with coupling A and polynomial f, and then use Eq. (6.66) to shift the fields.

To make further progress, we need to determine $\Xi(t+\Delta t,t)$ as defined in Eq. (6.70). Considering the two terms in the bracket $(1+\Delta t\widehat{\mathcal{A}})$ separately, the unity trivially gives rise to a term $\langle n|m\rangle = \delta_{n,m}$. What deserves our full attention is the term $\left\langle n\middle|\widehat{\mathcal{A}}\middle|m\right\rangle$. We can safely assume that $\widehat{\mathcal{A}}$ is generally a polynomial in $a^\dagger$ and a. If we assume that $\widehat{\mathcal{A}} = a^{\dagger\gamma}a^\sigma$ with integers $\gamma,\sigma \geqslant 0$, then this will easyly be generalised to linear combinations thereof:

$$\left\langle n\middle|a^{\dagger\gamma}a^\sigma\middle|m\right\rangle = \frac{m!}{(m-\sigma)!}\delta_{n-\gamma,m-\sigma} \tag{6.72}$$

for $n \geqslant \gamma$ and $m \geqslant \sigma$ and 0 otherwise. Using that in Eq. (6.70) we have

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} \frac{\phi^*(t+\Delta t)^n\phi(t)^m}{m!}\langle n|\, a^{\dagger\gamma}a^\sigma\,|m\rangle \\
&= \sum_{n=\gamma,m=\sigma}^{\infty} \frac{\phi^*(t+\Delta t)^n\phi(t)^m}{m!}\frac{m!}{(m-\sigma)!}\delta_{n-\gamma,m-\sigma} \\
&= \phi(t)^\sigma\phi^*(t+\Delta t)^\gamma \sum_{n=0,m=0}^{\infty} \frac{\phi^*(t+\Delta t)^m\phi(t)^m}{m!} \\
&= \phi(t)^\sigma\phi^*(t+\Delta t)^\gamma \exp(\phi^*(t+\Delta t)\phi(t))
\end{aligned} \tag{6.73}$$

We thus have

$$\Xi(t+\Delta t,t) = \exp\left\{(\phi^*(t+\Delta t)-\phi^*(t))\phi(t)\right\}\left\{1+\Delta t\phi^*(t+\Delta t)^\gamma\phi(t)^\sigma\right\} \tag{6.74}$$

which now can be dealt with algebraically like any other old term — we do not need to worry about operators and their order any more. In fact it seems that we can essentially rewrite an action $\widehat{\mathcal{A}} = a^{\dagger\gamma}a^\sigma$ by "translating" any $a^\dagger$ into a ϕ^* (more elegantly a $\phi^\dagger$) and every a into a ϕ. Below we will see that this indeed the case and in fact carries through to the observable.[18]

So far, we have managed to rewrite Eq. (6.55) as Eq. (6.71), now with a simplified $\Xi(t+\Delta t,t)$. Next, we turn our attention to the "caps" (comprising the left ket and the right bra) as they show in Eq. (6.71). According to Eq. (6.42), $\left\langle\widehat{\mathcal{O}}\right\rangle(t) = \left\langle\text{☼}\middle|\widehat{\mathcal{O}}\exp\left(\widehat{\mathcal{A}}t\right)\mathcal{I}\middle|0\right\rangle$, we need to determine the right cap

$$C_r = \langle 0|\exp(\phi^*(0)a)\,\mathcal{I}\,|0\rangle \tag{6.75}$$

[18]So, one might shrug for a moment: If we can shuffle terms (fields, *i.e.* scalars, no longer operators) around freely now, but not before (when "they were" operators) then surely we must arrive at different results by starting off with differently ordered operators first. But the explanation is trivial: We have used normal ordering, most recently when assuming that the action could be written as $a^{\dagger\gamma}a^\sigma$, which is in normal ordered form. Only once we have normal ordering does the translation above work as simply and smoothly as outlined.

and the left cap

$$C_\ell = \langle \stardiamond | \, \widehat{\mathcal{O}} \exp(-\phi^*(t+\Delta t)\phi(t+\Delta t)) \exp\Big(\phi(t+\Delta t) a^\dagger\Big) \, |0\rangle \ . \tag{6.76}$$

Starting with the right cap, C_r, we can safely assume that $\mathcal{I}$ is a power of creators — what's the point initialising from the vacuum and using annihilators. We are starting with a blank canvas! In any case, if the operator is normal ordered, then its annihilators are on the right, and $a\,|0\rangle = 0$. Let's say then $\mathcal{I} = a^{\dagger\, r}$ and so

$$C_r = \sum_{n=0}^{\infty} \langle n | \, \phi^{*n}(0) \, |r\rangle = \phi^{*r}(0) \tag{6.77}$$

using the result for the abyss, Eq. (6.17) and Eq. (6.19), above. The left cap involves the observable operator $\widehat{\mathcal{O}}$, which we may choose to be a^ℓ, omitting any creators on the left (normal ordering!) which would be swallowed by the abyss anyway:

$$\begin{aligned} C_\ell &= \exp(-\phi^*(t+\Delta t)\phi(t+\Delta t)) \sum_{n=0}^{\infty} \langle \stardiamond | \, a^\ell \frac{\phi^n(t+\Delta t)}{n!} \, |n\rangle \\ &= \exp(-\phi^*(t+\Delta t)\phi(t+\Delta t)) \sum_{n=\ell}^{\infty} \frac{\phi^n(t+\Delta t)}{n!} \frac{n!}{(n-\ell)!} \langle \stardiamond | n-\ell \rangle \\ &= \exp(-\phi^*(t+\Delta t)\phi(t+\Delta t)) \, \phi^\ell(t+\Delta t) \exp(\phi(t+\Delta t)) \end{aligned} \tag{6.78}$$

Putting it all together, we have (see in particular Eq. (6.71))

$$\begin{aligned} \Big\langle \widehat{\mathcal{O}} \Big\rangle (t) &= \Big\langle \stardiamond \Big| \widehat{\mathcal{O}} \exp\Big(\widehat{\mathcal{A}} t\Big) \, \mathcal{I} \Big| 0 \Big\rangle \\ &= \int \left(\prod_{t'=0}^{t+\Delta t} \frac{d\phi^*(t') \wedge d\phi(t+\Delta t)}{2\pi i} \right) C_\ell \left(\prod_{t'=0}^{t} \Xi(t'+\Delta t, t') \right) C_r \\ &= \int \mathcal{D}\phi \, \phi^\ell(t+\Delta t) \exp\Bigg\{ (1-\phi^*(t+\Delta t))\phi(t+\Delta t) + \\ &\qquad \sum_{t'=0}^{t} [\phi^*(t'+\Delta t) - \phi^*(t')]\phi(t') \Bigg\} \left(\prod_{t'=0}^{t} (1 + \Delta t \phi^{*\gamma}(t'+\Delta t)\phi^\sigma(t')) \right) \phi^{*r}(0) \end{aligned} \tag{6.79}$$

where the sums and products over t' are understood to proceed in steps of Δt. In the expression above we have used the shorthand

$$\mathcal{D}\phi \ = \prod_{t'=0}^{t+\Delta t} \frac{d\phi^*(t') \wedge d\phi(t+\Delta t)}{2\pi i} \tag{6.80}$$

suggesting that we are on track to turn our master equation into a path integral. In the

limit of small Δt, the product over $(1 + \Delta t \phi^{*\gamma} \phi^{\sigma})$ can be written as an exponential again

$$\prod_{t'=0}^{t} \left\{ 1 + \Delta t \phi^{*\gamma}(t' + \Delta t) \phi^{\sigma}(t') \right\} = \exp\left(\sum_{t'=0}^{t} \Delta t \phi^{*}(t' + \Delta t)^{\gamma} \phi(t')^{\sigma} \right) \tag{6.81}$$

so that

$$\left\langle \widehat{\mathcal{O}} \right\rangle (t) = \int \mathcal{D}\phi \, \phi^{\ell}(t+\Delta t) \exp\left(\big(1 - \phi^{*}(t + \Delta t)\big) \phi(t + \Delta t) + \sum_{t'=0}^{t} \mathcal{A}_{\Delta t}(t') \right) \phi^{*r}(0) \tag{6.82}$$

with the action

$$\mathcal{A}_{\Delta t}(t') = \big(\phi^{*}(t' + \Delta t) - \phi^{*}(t') \big) \phi(t') + \Delta t \phi^{*\gamma}(t' + \Delta t) \phi^{\sigma}(t') \tag{6.83}$$

no longer in terms of operators but in terms of fields. We will undertake a few more algebraic manipulations but in principle Eq. (6.82) and Eq. (6.83) are what we were aiming for.

Firstly, we want to take the limit $\Delta t \to 0$, in which case the sum over the action turns into a RIEMANN sum,

$$\mathcal{A}_{\Delta t}(t') = \Delta t \left(\mathring{\phi}^{*}(t') \phi(t') + \phi^{*\gamma}(t') \phi^{\sigma}(t') \right) + \Delta t^{2} \ldots \tag{6.84}$$

and thus

$$\mathcal{A}' = \lim_{\Delta t \to 0} \sum_{t'=0}^{t} \mathcal{A}_{\Delta t}(t') = \int_{0}^{t} dt' \, \mathring{\phi}^{*}(t') \phi(t') + \phi^{*}(t')^{\gamma} \phi^{(}t')^{\sigma} \, . \tag{6.85}$$

To ease notation and more importantly, as we will see later, to remove the term $(1 - \phi^{*}(t + \Delta t)) \phi(t + \Delta t)$ in Eq. (6.82), we introduce the **DOI-shifted** field

$$\tilde{\phi}(t) = \phi^{*}(t) - 1 \tag{6.86}$$

which at this stage looks like an innocent substitution that produces a JACOBian of unity in the path integral. Strictly, however, this is not a substitution. Rather, we have used Eq. (6.61) and in fact Eq. (6.66) which assumes that all other functions can be expanded in small $\phi^{*}(t)$.[19]

[19] The DOI-shift can be performed elegantly at operator level, by using the orthogonality relation

Exercise 10:

$$\langle 0 | \exp(a) \, a^{m} \tilde{a}^{n} | 0 \rangle = n! \delta_{n,m} \tag{6.87}$$

with $\tilde{a} = a^{\dagger} - 1$. *The expression to consider,* $(1 - 1)^{n-m}$, *is similar to the one in Eq. (6.64).*

In terms of the shifted field, the action becomes

$$\begin{aligned}\mathcal{A}' &= \int_0^t dt' \, \mathring{\tilde{\phi}}(t')\phi(t') + \phi^\sigma(t')(\tilde{\phi}(t')+1)^\gamma \\ &= \tilde{\phi}(t)\phi(t) - \tilde{\phi}(0)\phi(0) + \int_0^t dt' \, \left\{-\tilde{\phi}(t')\mathring{\phi}(t') + \phi^\sigma(t')(\tilde{\phi}(t')+1)^\gamma\right\} ,\end{aligned} \tag{6.88}$$

where the first (surface) term, $\tilde{\phi}(t)\phi(t)$ cancels with the $\big(1-\phi^*(t+\Delta t)\big)\phi(t+\Delta t)$ of Eq. (6.82) in the limit of small Δt. The last integral (without the surface terms) defines our new action $\mathcal{A}$, which means that our path integral becomes, in the limit of small Δt

$$\left\langle \widehat{\mathcal{O}} \right\rangle (t) = \int \mathcal{D}\phi \, \phi^\ell(t) \exp\big(-\tilde{\phi}(0)\phi(0) + \mathcal{A}\big) \, (\tilde{\phi}(0)+1)^r \, . \tag{6.89}$$

The term $\exp\big(-\tilde{\phi}(0)\phi(0)\big)$ requires a bit more attention than some authors suggest and yet, it will turn out, that it can essentially be dropped. Going all the way back to the orthogonality relation Eq. (6.59), we see that a term like that would enforce $r = 0$,

$$\int \frac{d\tilde{\phi}^*(0) \wedge d\phi(0)}{2\pi\mathring{\imath}} \exp\big(-\tilde{\phi}(0)\phi(0)\big) \, (\tilde{\phi}(0)+1)^r = \delta_{r,0} \, . \tag{6.90}$$

Does that mean that we cannot, by construction, have any initialisation other than the vacuum, $r = 0$? This was certainly not the case when we started out writing our observable, so why does the *field-theory* enforce this when it wasn't there to start with?

The answer is that we have been a bit nonchalant with our surface terms and the continuum limit $\Delta t \to 0$, which we *have* to take because of Eq. (6.55). We note that (see Eq. (6.79))

$$\begin{aligned}&\sum_{t'=0}^{t} (\phi^*(t'+\Delta t) - \phi^*(t'))\phi(t') \\ &\qquad = -\sum_{t'=0}^{t} \Big\{\phi^*(t')\big(\phi(t') - \phi(t'-\Delta t)\big)\Big\} - \phi^*(0)\phi(-\Delta t) + \phi^*(t+\Delta t)\phi(t)\end{aligned} \tag{6.91}$$

produces a slightly different surface term, which will force $\phi^*(-\Delta t)^r = (\tilde{\phi}(-\Delta t)+1)^r$ to have $r = 0$, *i.e.* having a vacuum just before we created the first particle using $\mathcal{I}$. That *is*, in fact, what we had started out with in Eq. (6.42) and later more specifically Eq. (6.75), namely an operator $\mathcal{I}$ creating particles from the vacuum.[20] If the vacuum is an absorbing state, *i.e.* once the system is empty it is bound to stay empty, then it makes no difference

[20] This result looks coincidental, after all, we never explicitly made use of the fact that we create particles over the vacuum. But it isn't coincidental, considering that the path integral is constructed to "probe what's there and to propagate it subject to infinitesimal evolution". There is just nothing to be propagated before the creation of particles over the vacuum and so we end up with only $r = 0$. At times $t' \geqslant 0$ there are annihilator terms (with amplitudes), such as $\phi(t')^q$ times some pre-factor, which means that $r = q$ also contributes.

to push back the enforcement of the vacuum from initial time[21] $t_0 = -\Delta t$ to $t_0 \to -\infty$. The structure of our approach Eq. (6.42) would need to be adapted slightly to

$$\left\langle \widehat{\mathcal{O}} \right\rangle (t) = \left\langle \text{☼} \middle| \widehat{\mathcal{O}} \exp\left(\widehat{\mathcal{A}} t\right) \mathcal{I} \exp\left(\widehat{\mathcal{A}}(-t_0)\right) \middle| 0 \right\rangle \tag{6.92}$$

where $\exp\left(\widehat{\mathcal{A}}(-t_0)\right)$ evolves the system from time t_0 to 0 and $\exp\left(\widehat{\mathcal{A}} t\right)$ from 0 to t. In fact, this provides a route to probing the system at different times and to let it evolve before and after, say

$$\begin{aligned}
&\left\langle \widehat{\mathcal{O}} \right\rangle (t_0, t_1, t_2, t_3, t) \\
&\quad = \left\langle \text{☼} \middle| \widehat{\mathcal{O}}_3 \exp\left(\widehat{\mathcal{A}}(t_3 - t_2)\right) \widehat{\mathcal{O}}_2 \exp\left(\widehat{\mathcal{A}}(t_2 - t_1)\right) \widehat{\mathcal{O}}_1 \exp\left(\widehat{\mathcal{A}} t_1\right) \mathcal{I} \exp\left(\widehat{\mathcal{A}} t_0\right) \middle| 0 \right\rangle \\
&= \left\langle \text{☼} \middle| \exp\left(\widehat{\mathcal{A}}(t - t_3)\right) \widehat{\mathcal{O}}_3 \exp\left(\widehat{\mathcal{A}}(t_3 - t_2)\right) \widehat{\mathcal{O}}_2 \exp\left(\widehat{\mathcal{A}}(t_2 - t_1)\right) \widehat{\mathcal{O}}_1 \exp\left(\widehat{\mathcal{A}} t_1\right) \mathcal{I} \exp\left(\widehat{\mathcal{A}} t_0\right) \middle| 0 \right\rangle ,
\end{aligned} \tag{6.93}$$

where we have used $|\text{☼}\rangle \exp\left(\widehat{\mathcal{A}}(t - t_3)\right) = |\text{☼}\rangle$, see Eq. (6.52), *i.e.* the observable is independent of t. Following the derivation above, the sandwiched operators $\widehat{\mathcal{O}}_2$ and $\widehat{\mathcal{O}}_1$, say density operators, for example, will no longer have the abyss to their left. Rather they appear in a term of the form

$$S = \langle 0 | \exp(\phi^*(t + \Delta t) a) \, (1 + \Delta t \widehat{\mathcal{A}}) \widehat{\mathcal{O}}_i \exp(-\phi^*(t)\phi(t)) \exp\left(\phi(t) a^\dagger\right) |0\rangle \tag{6.94}$$

for $i = 1, 2$, namely a mixture of Ξ, Eq. (6.70), and the left cap, Eq. (6.76). In principle, we need to consider the commutation of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{O}}_i$, but omitting the former produces an error only of order Δt, so we will drop $\widehat{\mathcal{A}}$ (obviously, that is justfied only if it is done for a fixed number of "time slots" and $\Delta t \to 0$ taken afterwards. If $\widehat{\mathcal{O}}_i = a^{\dagger k} a^\ell$, now allowing for (re-) creation as the left bra is no longer the abyss and particles have to be "reinstated" once being killed to probe their number, we find using the corresponding expression for the action, Eq. (6.3),

$$S = \exp\left\{ \left(\phi^*(t + \Delta t) - \phi^*(t)\right)\phi(t) \right\} \phi(t)^\ell \phi^{*k}(t + \Delta t) , \tag{6.95}$$

again, coming down to translating each $a^\dagger$ to a ϕ^* and each a to a ϕ. Apart from these extra fields popping up precisely "where" (at which point in time) we want to measure them, everything else remains unchanged.[22]

[21] A corresponding argument can be made for the left cap and the abyss: If the system "always" ends up in the vacuum state after some time, it makes no difference to cap on the left with the abyss or with the vacuum. One can show this explicitly for example for a massive random walker.

[22] Very importantly, creation fields ϕ^* are ahead in time of annihilation fields ϕ and because we will find that $\langle \phi(t)\tilde{\phi}(t') \rangle \propto \theta(t - t')$, the Heaviside theta function, that means that contractions of terms like those in Eq. (6.95), $\phi^*(t')\phi(t)$ with $t' = t + \Delta t > t$, do not contribute to observables: The densities of fields recreated

At first the above looks like a technical marginal aimed at the construction of more complicated observables, such as higher point correlation functions. Yet, it applies much more generally. If we wish to apply the operator $\mathcal{I}$ after the system has evolved from the vacuum all the way from $t' = t_0$ to $t' = 0$, then $\mathcal{I}$ does not enter into the right cap as in Eq. (6.75), because the right vector is now $\exp\left(\widehat{\mathcal{A}} t_0\right)|0\rangle$, not just $|0\rangle$. But inspecting Eq. (6.93), it turns out that the operators of $\mathcal{I}$ will get translated into fields in exactly the same way as the operators of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{O}}_i$.

More importantly, the recipe Eq. (6.93) provides a path to push the initial time t_0 back to $-\infty$ and the final time t forward to $+\infty$, so that we can apply FOURIER transforms (from $\mathbf{x}, t$ to $\mathbf{k}, \omega$) to make the bilinear part of the action local (for the time being we will stay with time only and leave space for later).

If we move the beginning of time back to $-\infty$, we can write the action as

$$\mathcal{A} = \int_{-\infty}^{\infty} dt' \left(-\tilde{\phi}(t')\mathring{\phi}(t') + \phi^{\sigma}(t')(\tilde{\phi}(t') + 1)^{\gamma} \right), \tag{6.96}$$

to extract observables at time t originating from perturbations at t_0 using

$$\left\langle \widehat{\mathcal{O}} \right\rangle (t) = \int \mathcal{D}\phi \, \phi^{\ell}(t) \exp(\mathcal{A}) \, (\tilde{\phi}(t_0) + 1)^{r} \, . \tag{6.97}$$

The discussion above still sounds like a marginal. Isn't it all very simple? We have to translate $a^{\dagger}$ to $\phi^*(t)$ and a to $\phi(t)$ and then we express things in terms of FOURIER modes $\phi^*(\omega)$ and $\phi(\omega)$. There is one very important aspect of moving t_0 to $-\infty$ (where the field theory will enforce vaccum): If the vacuum is *not* absorbing, for example because there is spontaneous creation, then the "initialisation" $\mathcal{I}$ in $\mathcal{I}\exp\left(\widehat{\mathcal{A}} t_0\right)|0\rangle$ provides *additional* particles. In that case, we need to decide how we wish to "initialise" our system. In Eq. (6.93) the initialisation operator $\mathcal{I}$ is applied after the system has evolved from $t' = t_0$ to $t' = 0$. So, we may as well replace $\mathcal{I}$ by some more complicated operator (but see below) that terminates all particles ("resetting" the system) and (re-)creates only those that we actually wanted to allow for. If we don't do that, we may end up studying a system very different from the intended one (namely one in a stationary state, not starting from the vacuum).[23]

> **Exercise 11:** *Try to construct an operator that removes all particles and creates precisely* n*. Doesn't work? How about slotting in* $|n\rangle\langle\text{☼}|$*?*

We are now in the situation to FOURIER transform without too much ado (or at least one might think so), thereby changing the field $\phi(t)$ (we still have not allowed the space dependence to return, this will happen later) to $\phi(\omega)$ — we do not introduce any change

after killing them to probe their density do not themselves contribute to that density. This finding has a deeper mathematical meaning, in that it establishes in the present field theory effectively the ITŌ convention, $\theta(0) = 0$.

[23] We cannot move t_0 back to 0, because that invalidates the FOURIER transform below.

in notation to indicate that the parameterisation has changed. The sign convention we will follow is (this time, sorry, *with* space dependence):

$$\phi(\mathbf{k},\omega) = \int \mathrm{d}t\mathrm{d}^{\mathrm{d}}x\, \phi(\mathbf{x},t)\exp(\mathrm{i}\omega t - \mathrm{i}\mathbf{k}\mathbf{x}) \tag{6.98}$$

and correspondingly

$$\phi(\mathbf{x},t) = \int \text{đ}\omega \text{đ}^{\mathrm{d}}k\, \phi(\mathbf{k},\omega)\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\mathbf{x}) \tag{6.99}$$

where $\text{đ}\omega = \mathrm{d}\omega/(2\pi)$ *etc.*.

The reason why not everything is perfectly straightforward is the fact that $\phi^*(t)$ is the complex conjugate of $\phi(t)$ and the FOURIER transform of the complex conjugate is not just the complex conjugate of the FOURIER transform. This is, to large extent, a matter of notation — do we mean the former or the latter when we write $\phi^*(\omega)$. What makes matters worse (but actually easier as far as notation is concerned) is the DOI-shift. The FOURIER-transform of $\tilde{\phi}(t)$ is in fact (see Eq. (6.86))

$$\tilde{\phi}(\omega) = \int \mathrm{d}t\, \tilde{\phi}(t)\exp(\mathrm{i}\omega t) = \int \mathrm{d}t\, (\phi^*(t) - 1)\exp(\mathrm{i}\omega t) = \phi^*(-\omega) - \delta(\omega) \tag{6.100}$$

where $\delta(\omega) = 2\pi\delta(\omega)$ and $\phi^*(-\omega)$ is the complex conjugate of the FOURIER transform of ϕ, evaluated at $-\omega$.

In the perturbation theory to come, we will need to perform the path integral $\int \mathcal{D}\phi\, \exp(\tilde{\mathcal{A}})$ over the bilinear part of Eq. (6.96) which we write as

$$\tilde{\mathcal{A}}_0 = -\int_{-\infty}^{\infty} \mathrm{d}t'\, \tilde{\phi}(t')\mathring{\phi}(t') + \epsilon\tilde{\phi}(t')\phi(t')\,, \tag{6.101}$$

where we have anticipated the extinction (or mass) term $\epsilon\tilde{\phi}(t')\phi(t')$. To make each term local, we need to perform a FOURIER transform as suggested above,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t'\, \tilde{\phi}(t')\mathring{\phi}(t') + \epsilon\tilde{\phi}(t')\phi(t') = \int_{-\infty}^{\infty} \mathrm{d}t' \int \text{đ}\omega' \int \text{đ}\omega'' \Big\{ \tilde{\phi}(\omega'')(-\mathrm{i}\omega')\phi(\omega') \\ + \epsilon\tilde{\phi}(\omega'')\phi(\omega') \Big\} \exp\big(-\mathrm{i}(\omega''+\omega')t'\big) = \int \text{đ}\omega'\, \tilde{\phi}(-\omega')(-\mathrm{i}\omega' + \epsilon)\phi(\omega')\,,\end{aligned} \tag{6.102}$$

where

$$\tilde{\phi}(-\omega')(-\mathrm{i}\omega' + \epsilon)\phi(\omega') = \phi^*(\omega')(-\mathrm{i}\omega' + \epsilon)\phi(\omega') \tag{6.103}$$

for $\omega' \neq 0$. This is the local form we are after, because the bilinear part of the action is now local in ω,

$$\tilde{\mathcal{A}}_0 = -\int \text{đ}\omega'\, \big(\phi^*(\omega')(-\mathrm{i}\omega' + \epsilon)\phi(\omega')\big)\,, \tag{6.104}$$

except for the $\delta(\omega)$.

The trouble with the extra $\delta(\omega)$ is that, well, it doesn't just go away (not until we

think about this, anyway). The fundamental problem is that with it, we no longer have $\tilde{\phi}(-\omega) = \phi^*(\omega)$, but $\tilde{\phi}(-\omega) = \phi^*(\omega) - \delta(\omega)$. But in order to use the usual, GAUSSian result

$$\frac{1}{\pi}\int dx dy\, \exp(-z^* A z) = \frac{1}{A} \tag{6.105}$$

with arbitrary complex A in the exponent (arbitrary, except for the need of a positive realpart), we need the exponential's argument in the form $z^* A z$, not $(z+\zeta)^* A z$. At the moment we have

$$\int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp\bigl(-\tilde{\phi}(-\omega) A \phi(\omega)\bigr) \tag{6.106}$$

So, what is that? We cannot just replace ϕ^* by $\tilde{\phi}^*$ in the integration measure and pretend that we are dealing with Eq. (6.105) — we are integrating over only one complex field, namely ϕ and ϕ^* is its complex conjugate. What we need to do is, use Eq. (6.61) and then Eq. (6.105):

$$\begin{aligned}\int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp\bigl(-\tilde{\phi}(-\omega) A \phi(\omega)\bigr) &= \int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp(-(\phi^*(\omega) - \delta(\omega)) A \phi(\omega)) \\ &= \int \frac{d\phi^* \wedge d\phi}{2\pi i} \exp(-\phi^*(\omega) A \phi(\omega)) \ .\end{aligned} \tag{6.107}$$

Using that identity for every ω, we arrive at the GAUSSian path integral

$$\int \mathcal{D}\phi\, \exp\left(-\int d\omega\, \left(\phi^*(\omega)(-i\omega' + \epsilon)\phi(\omega')\right)\right) \tag{6.108}$$

and with that, at the usual field-theoretic results.

As far as normalisation is concerned, we may take the view that we have lost track of it as we keep ignoring the JACOBian and the difficulties that arise for it in the continuum. Given the form of Eq. (6.108), in order to produce

$$\langle 1 \rangle = \int \mathcal{D}\phi\, \exp\left(-\int d\omega\, \left(\phi^*(\omega)(-i\omega' + \epsilon)\phi(\omega')\right)\right) 1 = 1 \tag{6.109}$$

the JACOBian hidden in $\mathcal{D}\phi$ would need to be a complicated "mess" of $-i\omega + \epsilon$ (and, later, $\mathbf{k}$ as well), given that the GAUSSian has "variance" $1/(-i\omega + \epsilon)$. To avoid these complication, we should normalise explicitly, demanding that $\mathcal{D}\phi$ contains the right normalisation such that

$$\langle 1 \rangle = \int \mathcal{D}\phi\, \exp\bigl(\tilde{A}\bigr)\, 1 = 1 \ .$$

The standard field theoretical procedures then carry through, in particular cancellation of vacuum fluctuations. In order to perform the perturbation theory, all we need to know explicitly is the properly normalised bare propagator. So, we keep in mind that things will need to be normalised and deal with it when the problem arises. It turns out to be smaller than expected.

So, fair enough, life is complicated. But at least we can now FOURIER-transform the

entire action. When we wrote $\tilde{\mathcal{A}}_0$, Eq. (6.104), we have already dealt with the spontaneous extinction with rate ϵ and the time propagation ∂_t. There is still the space-dependence to incorporate, however. Originally, we were summing over a lattice, so the natural extension of $\tilde{\mathcal{A}}_0$ to the lattice is

$$\mathcal{A}_0 = -\sum_{\mathbf{y}} \int đ\omega' \, \left(\phi^*(\mathbf{y},\omega')(-\imath\omega' + \epsilon)\phi(\mathbf{y},\omega') \right) \,, \tag{6.110}$$

however it is much more convenient to take the continuum limit in space, interpreting the sum above as a RIEMANN-sum with mesh a^d on a lattice with lattice spacing a and absorbing this volume element into the fields, whose product therefore turns into a density:

$$\sum_{\mathbf{y}} a^d \frac{\phi^*(\mathbf{y},\omega')\phi(\mathbf{y},\omega')}{a^d} \,. \tag{6.111}$$

As far as the path integral is concerned, the fields are dummy variables and the rescaling becomes part of the normalisation (or, equivalently, the JACOBian). Again, to ease notation, we do not introduce new variable names to indicate the re-definition of the fields. The sum $\sum_{\mathbf{y}} a^d$, turns into an integral $\int d^d y$ and upon substituting in the FOURIER-transformed fields Eq. (6.99), we have

$$\mathcal{A}_0 = -\int đ^d k' đ\omega' \, \tilde{\phi}(-\mathbf{k}',-\omega')(-\imath\omega' + \epsilon)\phi(\mathbf{k}',\omega') \,, \tag{6.112}$$

writing it without making the relation $\tilde{\phi}(-\mathbf{k}',-\omega') = \phi^*(\mathbf{k},\omega)$ (up to a δ, as discussed above), explicit.[24]

There are two more terms, to deal with, the diffusion $\widehat{\mathcal{A}}_H$, Eq. (6.38), which is bilinear and the spontaneous creation with rate β, Eq. (6.32), which is linear. Further down, Section 6.3.3, we will consider non-linear terms, such as branching.

The translation of the operators in the particle hopping term $\widehat{\mathcal{A}}_H$ works just as above, so that

$$\mathcal{A}_H = -\frac{H}{2q} \int dt \, \sum_{\mathbf{y}} \sum_{\mathbf{e}} \left(\tilde{\phi}(\mathbf{y}+\mathbf{e},t) - \tilde{\phi}(\mathbf{y},t) \right) \left(\phi(\mathbf{y}+\mathbf{e},t) - \phi(\mathbf{y},t) \right) \tag{6.113}$$

where the unities of the DOI-shift in the first term cancel. Next we want to take a **con-**

[24] After FOURIER-transforming the only difference between $\tilde{\phi}$ and ϕ^* is a $\delta(\mathbf{k})\delta(\omega)$, which is irrelevant (shown above) as far as the GAUSSian integral is concerned, but might be important otherwise. In particular in real space and real time, we will keep $\phi^* = \tilde{\phi} + 1$, never ignoring the shift by unity.

tinuum approximation in space, approximating

$$\begin{pmatrix} \phi(\mathbf{y}+\mathbf{e}_x,t)-\phi(\mathbf{y}) \\ \phi(\mathbf{y}+\mathbf{e}_y,t)-\phi(\mathbf{y}) \\ \cdots \\ \phi(\mathbf{y}+\mathbf{e}_z,t)-\phi(\mathbf{y}) \end{pmatrix} = a\nabla\phi(\mathbf{y}) + \mathcal{O}(a^2) \tag{6.114}$$

where $a = |\mathbf{e}_x| = |\mathbf{e}_y| = \ldots = |\mathbf{e}_z|$ is the lattice spacing and the $\mathbf{e}_i$ are basis vectors of the lattice of that length. Dealing with a hypercubic lattice, they are obviously orthogonal. The sum in (6.113), however, runs over all displacements, *i.e.* $\{\mathbf{e}_x, \mathbf{e}_y, \ldots, \mathbf{e}_z, -\mathbf{e}_x, -\mathbf{e}_y, \ldots, -\mathbf{e}_z\}$, so that

$$\mathcal{A}_H = -\frac{H}{2q}\int dt \sum_{\mathbf{y}} a^d \left(2a^2 \frac{\nabla\tilde{\phi}(\mathbf{y},t)\cdot\nabla\phi(\mathbf{y},t)}{a^d} + \mathcal{O}(a^3)\right) . \tag{6.115}$$

To take the continuum limit, $a \to 0$, without losing the whole term, we need to keep $Ha^2/q = D$ constant. This is an interesting subtlety, which in some settings has much more severe implications (imagine hopping *by* branching): The hopping rate must diverge like $H \propto a^{-2}$ as $a \to 0$. This is not an artefact: If the mean square displacement of our walkers has to be maintained even when $a \to 0$, *i.e.* they have to hop more frequently, so that after FOURIER-transforming we have

$$\mathcal{A}_H = -D\int đ^d k' đ\omega'\, \mathbf{k}'^2 \tilde{\phi}(-\mathbf{k}', -\omega')\phi(\mathbf{k}', \omega') , \tag{6.116}$$

noticing the sign.[25]

Together with the spontaneous creation of particles, which we easily read off from Eq. (6.32), we finally arrive at "our" action:

$$\begin{aligned} \mathcal{A} = -\int đ^d k' đ\omega'\, \tilde{\phi}(-\mathbf{k}', -\omega')(-\mathring{\imath}\omega + D\mathbf{k}'^2 + \epsilon)\phi(\mathbf{k}', \omega') \\ + \int đ^d k' đ\omega'\, \beta\tilde{\phi}(\mathbf{k}', \omega')\delta(\mathbf{k}')\delta(\omega') \end{aligned} \tag{6.117}$$

where, in a sense, $\delta(\mathbf{k}')\delta(\omega')$ in the last integrand is the FOURIER-transform of unity. The field theory that we will be dealing with in the following is maybe best characterised by restating the observable $\left\langle \widehat{\mathcal{O}} \right\rangle(t)$ as of Eq. (6.82):

[25] The sign comes from $(\mathring{\imath}\mathbf{k}_1)\cdot(\mathring{\imath}\mathbf{k}_2)\delta(\mathbf{k}_1+\mathbf{k}_2) = \mathbf{k}_1\cdot\mathbf{k}_1\delta(\mathbf{k}_1+\mathbf{k}_2)$.

$$\left\langle \widehat{\mathcal{O}} \right\rangle (\mathbf{x}, t; \mathbf{x}_0, t_0) = \left\langle \phi^\ell(\mathbf{x}, t)(1 + \tilde{\phi}(\mathbf{x}_0, t_0))^r \right\rangle = \int \mathcal{D}\phi\, \phi^\ell(\mathbf{x}, t) \exp(\mathcal{A})\, (1 + \tilde{\phi}(\mathbf{x}_0, t_0))^r \,, \tag{6.118}$$

which we will take as the definition of $\langle \bullet \rangle$.

All in all, we are now in the situation, where we can invoke standard field theoretical techniques to analyse reaction-diffusion processes:

- Write down the master equation.
- Rewrite the master equation in terms of creation and annihilation operators.
- Extract the action $\widehat{\mathcal{A}}$, an operator such that $\widehat{\mathcal{A}} |\psi(t)\rangle = \frac{d}{dt} |\psi(t)\rangle$, see Eq. (6.39).
- Apply the Doi-shift.
- Map each operator to a "conjugate field", $\tilde{a} \to \tilde{\phi}$ and $a \to \phi$.
- Fourier transform the resulting action in terms of fields.
- To come: Extract the bilinear part and determine the bare propagator(s).
- Hard work: Start the perturbation theory, invoke renormalisation, go bonkers, avoid the looney bin.

6.3.1 The bilinear part

I am somewhat reluctant to discuss the actual integration of the path integral Eq. (6.118) in great detail. This is because apart from the spontaneous creation, our action (6.117) is bilinear (homogeneous of order 1 in both variables, $\tilde{\phi}$ and ϕ). Determining the path integral Eq. (6.118) at $\beta = 0$ is effectively part of the standard procedures of dealing with pretty much every field theory. What may or may not produce some headaches is the lack of a factor $1/2$ in our action and the fact that $\tilde{\phi}$ and ϕ are essentially complex conjugates of each other (these two properties are related).

Setting $\beta = 0$ for a moment, the standard result (Täuber, 2014), known as the **bare propagator** is

$$\begin{aligned}
&\left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}_0, \omega_0) \right\rangle \\
&= \int \mathcal{D}\phi\, \exp\left(- \int đ^d k' đ\omega'\, \tilde{\phi}(-\mathbf{k}', -\omega')(-i\omega + D\mathbf{k}'^2 + \epsilon)\phi(\mathbf{k}', \omega') \right) \phi(\mathbf{k}, \omega)\tilde{\phi}(\mathbf{k}_0, \omega_0) \\
&= \frac{\delta(\mathbf{k} + \mathbf{k}_0)\delta(\omega + \omega_0)}{-i\omega + D\mathbf{k}^2 + \epsilon} \,.
\end{aligned} \tag{6.119}$$

Another standard result is: This is all you need to know about the path integral. The rest is either perturbation theory or if not, trivially determined by the integral above. The qualitative reason for this statement is that a GAUSSian is fully determined by its variance and that is what we have extracted in (6.119). Quantitatively, the statement goes back to WICK's theorem, which states that any higher moment of GAUSSian random variables is determined by the sum over the products of all their possible pairings. For example

$$\begin{aligned}\left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_2,\omega_2)\phi(\mathbf{k}_1,\omega_1)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle &= \left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_2,\omega_2)\right\rangle \left\langle \phi(\mathbf{k}_1,\omega_1)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle \\ &+ \left\langle \phi(\mathbf{k}_3,\omega_3)\phi(\mathbf{k}_1,\omega_1)\right\rangle \left\langle \tilde{\phi}(\mathbf{k}_2,\omega_2)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle \\ &+ \left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle \left\langle \tilde{\phi}(\mathbf{k}_2,\omega_2)\phi(\mathbf{k}_1,\omega_1)\right\rangle \ , \end{aligned} \tag{6.120}$$

where, importantly, the second term vanishes, $\langle \phi(\mathbf{k}_3,\omega_3)\phi(\mathbf{k}_1,\omega_1)\rangle \left\langle \tilde{\phi}(\mathbf{k}_2,\omega_2)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle = 0$, on the basis what has been said above about WICK, or, equivalently, by consulting Eq. (6.59) and noticing that $\langle \phi(\mathbf{k}_3,\omega_3)\phi(\mathbf{k}_1,\omega_1)\rangle$ corresponds to $m = 2$ and $n = 0$ for any choice of $\mathbf{k}_3,\omega_3,\mathbf{k}_1,\omega_1$ (and equivalently for $\left\langle \tilde{\phi}(\mathbf{k}_2,\omega_2)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle$).

It will prove useful to have a diagrammatic language at our disposal if we proceed to take a perturbative approach to non-linearities. The symbol for $\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle$, Eq. (6.119), is simply a straight line, ⟵. The arrow points in the direction of time or, as we have FOURIER-transformed, in the direction of causality: The left end of the line should be thought of being an annihilator-field, $\phi(\mathbf{k},\omega)$, the right a creator-field, $\tilde{\phi}(\mathbf{k}_0,\omega_0)$. The propagator $\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle$ is often referred to as the **response propagator**, as it tells us how the systems responds (in terms of particle densities) to the creation of a particle somewhere.

There is a crucial difference between the response propagator $\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle$ and the correlation "propagator" $\langle \phi(\mathbf{k},\omega)\phi(\mathbf{k}_0,\omega_0)\rangle$, the latter best studied in connected form, *i.e.* by subtracting $\langle \phi(\mathbf{k},\omega)\rangle \langle \phi(\mathbf{k}_0,\omega_0)\rangle$ if that does not vanish. The former measures, well, the response, the latter, well, correlations. To draw an instructive parallel: The former measures the ripples on the pond somewhere after we have thrown in a pebble. The latter measures how ripples at different points in space and time are correlated (McAteer *et al.*, 2015). Maybe there aren't any, because no wind is going (there is no noise vertex), so to make a bit more sense of $\langle \phi(\mathbf{k}_2,\omega_2)\phi(\mathbf{k}_1,\omega_1)\rangle$, we have to give it a gentle kick in the back, $\left\langle \phi(\mathbf{k}_2,\omega_2)\phi(\mathbf{k}_1,\omega_1)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle$. While this object is a correlation function, it can no longer be called the correlation propagator, which has a very specific meaning in field theory, namely (the connected part of) $\langle \phi(\mathbf{k},\omega)\rangle \langle \phi(\mathbf{k}_0,\omega_0)\rangle$, non-zero only in the presence of a noise vertex.

The null-model of the response propagator, and normally, in fact, the bare propagator is a GAUSSian, as shown below. The null-model of the correlation propagator is $1/(\omega^2 + (D\mathbf{k}^2 + \epsilon)^2)$. Both are crucially different to the null-model of the equilibrium propagator (no "response", as there is no time), which is the non-equilibrium response propagator at $\omega = 0$, *i.e.* the LORENTZian $1/(D\mathbf{k}^2 + \epsilon)$ and produces the time-integral of

a GAUSSian, the classic ORNSTEIN-ZERNICKE correlation functions, generally of the form $|\mathbf{r}|^{-(d-2+\eta)} \exp(-|\mathbf{r}|/\xi)$ with correlation length ξ.

The literature is somewhat sloppy as far as the δ-functions in the propagator (and other objects) are concerned. It is common to write

$$\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle = \delta(\mathbf{k}+\mathbf{k}_0)\delta(\omega+\omega_0)G_0(\mathbf{k},\omega) \tag{6.121}$$

with $G(\mathbf{k},\omega) = (-\mathring{\imath}\omega + D\mathbf{k}^2 + \epsilon)^{-1}$ and refer to $G_0(\mathbf{k},\omega)$ as the (bare) response propagator. Or write $\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle = G_0(\mathbf{k},\omega)$, implying the δ-functions without stating them. This is possible as long boundary conditions do not spoil translational invariance in time and space (which is the symmetry that produces the δ-functions). As far as spatial translational invariance is concerned, a different route has to be taken as soon as particular boundary conditions are imposed. Time-translational invariance (or time-homogeneity) is often broken explicitly, *i.e.* one normally considers as part of the problem any breakdown of time-homogeneity. Solutions to the problem are constructed explicitly from the solution of the corresponding time-translational invariant problem.[26]

It is very instructive to FOURIER-transform the response propagator (6.119) back to real time and real space. It will turn out that the integral over space does not bear any surprises, so we will focus on the integral in time. We may want to "spend" the δ-function first,

$$\begin{aligned}\int đ\omega đ\omega_0 \, \exp(-\mathring{\imath}\omega t)\exp(-\mathring{\imath}\omega_0 t_0)\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle \\ = \int đ\omega đ\omega_0 \, \exp(-\mathring{\imath}\omega t)\exp(-\mathring{\imath}\omega_0 t_0)\,\delta(\mathbf{k}+\mathbf{k}_0)\delta(\omega+\omega_0)G_0(\mathbf{k},\omega) \\ = \delta(\mathbf{k}+\mathbf{k}_0)\int đ\omega \, \exp(-\mathring{\imath}\omega(t-t_0))\, G_0(\mathbf{k},\omega),\end{aligned} \tag{6.122}$$

and therefore focusing on the inverse FOURIER-transform of G_0,

$$\begin{aligned}G_0(\mathbf{k},t) &= \int đ\omega \, G(\mathbf{k},\omega)\exp(-\mathring{\imath}\omega t) \\ &= \int đ\omega \, \frac{1}{-\mathring{\imath}\omega + D\mathbf{k}^2+\epsilon}\exp(-\mathring{\imath}\omega t) = \theta(t)\exp\bigl(-t(D\mathbf{k}^2+\epsilon)\bigr)\end{aligned} \tag{6.123}$$

noticing that the pole is at $\omega = -\mathring{\imath}(D\mathbf{k}^2+\epsilon)$, we have

$$\int đ\omega đ\omega_0 \, \exp(-\mathring{\imath}\omega t)\exp(-\mathring{\imath}\omega_0 t_0)\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle = \theta(t-t_0)\exp\bigl(-(t-t_0)(D\mathbf{k}^2+\epsilon)\bigr) \tag{6.124}$$

The function $\theta(t-t_0)$ is the HEAVISIDE θ-function, which establishes causality: There is

[26] The observable Eq. (6.120) is an example: The creation $\tilde{\phi}(\mathbf{k}_2,\omega_2)$ and subsequent annihilation $\phi(\mathbf{k}_3,\omega_3)$ may be seen as occurring in a complicated system subject to creation by $\tilde{\phi}(\mathbf{k}_0,\omega_0)$ and annihilation by $\phi(\mathbf{k}_1,\omega_1)$, amounting to the calculation of $\left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_2,\omega_2)\right\rangle_{\text{complicated}}$ or simply as $\left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_2,\omega_2)\phi(\mathbf{k}_1,\omega_1)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle$ in an otherwise very simple, time-translational invariant system.

no particle density to be measured at $t < t_0$. In fact, we will implicitly use $\theta(0) = 0$, as suggested above.

Eq. (6.124) implies that all modes $\mathbf{k} \neq \mathbf{0}$ die off eventually and that $\mathbf{k} = \mathbf{0}$ survives indefinitely only if $\epsilon = 0$, *i.e.* when there is no dissipation. In other words, fine structures disappear exponentially fast and the zero-mode survives only if there is no dissipation.

FOURIER-transforming in space is an

Exercise 12: *... in* GAUSS*ian integrals, as*

$$\int \mathrm{d}^d k \, \exp\left(-tD\mathbf{k}^2\right) \exp(\mathrm{i}\mathbf{k}\mathbf{x}) = \frac{1}{(4\pi Dt)^{d/2}} \exp\left(-\mathbf{x}^2/(4Dt)\right) \ . \quad (6.125)$$

In summary, the propagator describes diffusion starting at $\mathbf{x}_0, t_0$, with evaporation ("attenuation") with rate ϵ:

$$\left\langle \phi(\mathbf{x}, t)\tilde{\phi}(\mathbf{x}_0, t_0)\right\rangle = G_0(\mathbf{x} - \mathbf{x}_0, t - t_0) = \theta(t - t_0) \frac{\exp(-(t - t_0)\epsilon)}{(4\pi D(t - t_0))^{d/2}} \exp\left(-\frac{(\mathbf{x} - \mathbf{x}_0)^2}{4D(t - t_0)}\right) \quad (6.126)$$

Staring at this for a moment, it looks as if "dilute" particles are everywhere.[27] Luckily, this is a misinterpretation. There is a finite *expectation* of finding a particle somewhere, but particles are still points that are not smeared out. This can be seen by probing the density at two different points in space at the same time, which vanishes strictly for the present process.[28]

Also, apeirophobia is taking hold of us. Eq. (6.126) diverges very dramatically for $\mathbf{x} = \mathbf{x}_0$ in the limit $t \to t_0$. Even when $\theta(t - t_0)$ prevents the worst, the integral over time at the origin $\mathbf{x} - \mathbf{x}_0 = \mathbf{0}$ is badly divergent in $d > 2$. This looks more than strange as Brownian particles do not return in dimensions $d > 2$ (random walkers have a positive probability to return in all dimensions, but that's a "lattice artefact"). This divergence occurs, however, for small $t - t_0$, *i.e.* it is an ultraviolet divergence. It comes down to the fact that the particle is *initially* so sharply focused.

[27] As far as the "everywhere" is concerned, this is not just another unfortunate result of having taken the continuum limit and thus having implemented an infinite hopping rate. The very POISSON process is to blame, which allows (with very small probability) any number of events to take place in a finite time interval. The diffusion equation does not respect special relativity. After any small time $t - t_0 > 0$ the particle density is strictly positive *everywhere*.

[28] Probing for the squared particle density somewhere is more subtle. Writing this observables in terms of operators involves the normal ordering effect $(a^\dagger a)^2 = a^{\dagger 2} a^2 + a^\dagger a$ and keeps us firmly in the realm of particle numbers, *i.e.* only the second term will contribute in the present case. This does not seem to square (pun intended) with the fact that this operator should be a squared density once we consider particle density fields. There are three ways of addressing this: 1) One may replace $aa^\dagger = 1 + a^\dagger a$ by $a(\mathbf{x}')a^\dagger(\mathbf{x}) = \delta(\mathbf{x}' - \mathbf{x}) + a^\dagger(\mathbf{x}')a(\mathbf{x})$ in the continuum. 2) Closely related, one may avoid such observables altogether, re-writing them as limits $\mathbf{x}' \to \mathbf{x}$ and $t' \to t$. 3) One may consider $\langle \phi\tilde{\phi}\rangle$ not as a particle density, but the probability density of finding a particle somewhere (or, more precisely, the weighted sum of finding n). Higher moments of that are not powers of particle densities.

6.3.2 The linear part

The linear part of the correlation function is the source term $\beta\tilde{\phi}$, see Eq. (6.117). We could have accounted for that by shifting $\phi(\mathbf{k}', \omega')$ by $\delta(\mathbf{k}')\delta(\omega')\beta/\epsilon$ in the bilinear part that was integrated out in Eq. (6.119). The result is a theory with shifted fields, which has consequences for all observables: Firstly, the observed $\phi(\mathbf{k}', \omega')$ has to be shifted by β/ϵ as well. Secondly, the shift implies a change of initialisation (see Eq. (6.75) and the discussion around Eq. (6.90)), which in the present case has no major repercussions. It has, though, consequences in systems where the vacuum is not stable against perturbation, *i.e.* when the system slips into some active state once a seed of activity has been put down. The problem is not that shifts are generally illegal, the problem is that implementing the shift mathematically correctly either results in extra terms to be considered at the initialisation or, ignoring them, in a different effective initialisation.

Therefore and because it is instructive, we will deal with the source term perturbatively. We start with a very general statement: If the action can be written in the form $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, then we may define

$$\left\langle \widehat{\mathcal{O}} \right\rangle_1 = \mathcal{N}_1^{-1} \int \mathcal{D}\phi\, \widehat{\mathcal{O}} \exp(\mathcal{A}_1) \tag{6.127}$$

as the expectation on the basis of $\mathcal{A}_1$ only. The normalisation $\mathcal{N}_1$ is chosen such that $\langle 1 \rangle_1 = 1$. What we are really interested in, however, is

$$\left\langle \widehat{\mathcal{O}} \right\rangle = \mathcal{N}^{-1} \int \mathcal{D}\phi\, \widehat{\mathcal{O}} \exp(\mathcal{A}) \tag{6.128}$$

which can be expressed in terms of the first expectation as

$$\left\langle \widehat{\mathcal{O}} \right\rangle = \frac{\mathcal{N}_1}{\mathcal{N}} \left\langle \widehat{\mathcal{O}} \exp(\mathcal{A}_2) \right\rangle_1 . \tag{6.129}$$

This is the starting point for a perturbation theory: If we are able to determine any expectation $\langle \bullet \rangle_1$ with polynomial $\bullet$, then we can determine $\left\langle \widehat{\mathcal{O}} \right\rangle$ as an expansion of $\widehat{\mathcal{O}} \exp(\mathcal{A}_2)$.

It gets better: If the only term in $\exp(\mathcal{A}_2)$ that contributes to $\langle 1 \rangle = \langle \exp(\mathcal{A}_2) \rangle_1$ is its zeroth order, *i.e.* 1, then we have $\langle 1 \rangle = \frac{\mathcal{N}_1}{\mathcal{N}}$ and imposing $\langle 1 \rangle = 1$ implies $\mathcal{N}_1 = \mathcal{N}$. In other words, we don't need to worry about the normalisation in $\left\langle \widehat{\mathcal{O}} \right\rangle$, as it's covered by $\mathcal{N}_1$. In fact, we will choose $\mathcal{A}_1$ to be the bilinear part of the action and so $\mathcal{N}_1$ is well known.[29] Generally, however, higher terms of $\exp(\mathcal{A}_2)$ *will* contribute, giving rise to so called **vacuum fluctuations** which (kindly!!) cancel.

[29] It isn't normally determined explicitly. Rather $\left\langle \widehat{\mathcal{O}} \right\rangle_1 = \langle \phi\tilde{\phi} \rangle_1$ is determined, using source fields, via a cumulant generating function (with vanishing means of ϕ and $\tilde{\phi}$), which is automatically normalised.

So, let's split our action Eq. (6.117) in two parts, $\mathcal{A} = \mathcal{A}_{\text{bil.}} + \mathcal{A}_{\text{src}}$, so that

$$\mathcal{A}_{\text{bil.}} = -\int đ^d k' đ\omega' \, \tilde{\phi}(-\mathbf{k}', -\omega')(-i\omega + D\mathbf{k}'^2 - \epsilon)\phi(\mathbf{k}', \omega') \tag{6.130}$$

and

$$\mathcal{A}_{\text{src}} = \int đ^d k' đ\omega' \, \beta\tilde{\phi}(\mathbf{k}', \omega')\delta(\mathbf{k}')\delta(\omega') \, . \tag{6.131}$$

The one and only expectation that we need to know of $\langle \bullet \rangle_{\text{bil.}}$ is the bare propagator (6.119)

$$\left\langle \phi(\mathbf{k}, \omega)\tilde{\phi}(\mathbf{k}_0, \omega_0) \right\rangle_{\text{bil.}} = \frac{\delta(\mathbf{k} + \mathbf{k}_0)\delta(\omega + \omega_0)}{-i\omega + D\mathbf{k}^2 + \epsilon} \, , \tag{6.132}$$

which we can find in the literature on field theory mentioned above, say (Täuber, 2014). To state WICK's theorem more explicitly, we have

$$\left\langle \phi(\mathbf{k}_1, \omega_1) \dots \phi(\mathbf{k}_n, \omega_n)\tilde{\phi}(\mathbf{k}'_1, \omega'_1) \dots \tilde{\phi}(\mathbf{k}'_m, \omega'_m) \right\rangle_{\text{bil.}} = 0 \text{ for all } n \neq m \text{ with } n, m \in \mathbb{N}_0 \, . \tag{6.133}$$

For $n = m = 0$ we have $\langle 1 \rangle_{\text{bil.}} = 1$ and for $n = m > 0$ we have,

$$\left\langle \phi_1 \dots \phi_n \tilde{\phi}_1 \dots \tilde{\phi}_n \right\rangle_{\text{bil.}} = \left\langle \phi_1 \tilde{\phi}_1 \right\rangle_{\text{bil.}} \dots \left\langle \phi_n \tilde{\phi}_n \right\rangle_{\text{bil.}} + \left\langle \phi_1 \tilde{\phi}_2 \right\rangle_{\text{bil.}} \dots + \dots \tag{6.134}$$

where we have written $\phi_i = \phi(\mathbf{k}_i, \omega_i)$ and $\tilde{\phi}_i = \tilde{\phi}(\mathbf{k}'_i, \omega'_i)$ to ease notation. The dots on the right hand side suggest *all possible pairings*, of which there are only[30] $n!$.

> **Exercise 13:** *How many non-vanishing terms does* $\left\langle \phi_1 \dots \phi_n \exp\left(-\gamma\tilde{\phi}^2\right) \right\rangle_{bil.}$ *generate.*

We proceed to calculate $\langle \bullet \rangle$, expanding

$$\exp(\mathcal{A}_{\text{src}}) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \tilde{\phi}^n(\mathbf{0}, 0) \tag{6.135}$$

as outlined above. The fields within $\mathcal{A}_{\text{src}}$ (or $\mathcal{A}_2$ above) that will appear in the expansion and thus feature in propagators, will be integrated over by the integrals contained in $\mathcal{A}_{\text{src}}$ (or $\mathcal{A}_2$). They known as **internal fields** as opposed to the **external fields** provided by $\widehat{\mathcal{O}}$.

[30] The situation is different compared to other field theories, where other propagators may exist, such as $\langle \phi\phi \rangle$. In general, there are $(2m-1)!!$ possible pairings of m fields, counting all of them, $m = 2n$ here.

Diagrammatically, much more as a matter of choice than a matter of convention, some authors use the symbol —● for the source. So, if the little bubble implies the strength of the source, β, we may write

$$\exp(\mathcal{A}_{\text{src}}) \hat{=} \sum_{n=0}^{\infty} \frac{(\text{—●})^n}{n!} \,. \tag{6.136}$$

One might think of the bubble as a "dead end", "not supplying any $\mathbf{k}$ or ω", thereby enforcing $\mathbf{k} = \mathbf{0}$ and $\omega = 0$ for any matching annihilator field.

Because the only non-zero contribution is due to the propagator Eq. (6.119) we have

$$\begin{aligned}\langle \phi(\mathbf{k}_1, \omega_1) \dots \phi(\mathbf{k}_n, \omega_n) \rangle &= \frac{\mathcal{N}_1}{\mathcal{N}} \langle \phi(\mathbf{k}_1, \omega_1) \dots \phi(\mathbf{k}_n, \omega_n) \exp(\mathcal{A}_{\text{src}}) \rangle_{\text{bil.}} \\ &= \frac{\mathcal{N}_1}{\mathcal{N}} \prod_{i=1}^{n} \beta^i \frac{\delta(\mathbf{k}_i)\delta(\omega_i)}{-\mathring{\imath}\omega_i + D\mathbf{k}_i^2 + \epsilon} = \frac{\mathcal{N}_1}{\mathcal{N}} \left(\frac{\beta}{\epsilon}\right)^n \prod_{i=1}^{n} \delta(\mathbf{k}_i)\delta(\omega_i)\end{aligned} \tag{6.137}$$

where the $1/n!$ of (6.135) is cancelled by the $n!$ arising from WICK's theorem, dictating that the n annihilator fields must be connected to the n sources in all possible ways. For $n = 0$ we have $1 = \langle 1 \rangle = \mathcal{N}_1/\mathcal{N}$, so this ratio can be omitted in the following.

For $n = 1$ we have the global particle density,

$$\langle \phi(\mathbf{k}, \omega) \rangle = \langle \phi(\mathbf{k}, \omega) \exp(\mathcal{A}_{\text{src}}) \rangle_{\text{bil.}} = \beta \left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{0}, 0) \right\rangle_{\text{bil.}} = \frac{\beta}{\epsilon} \delta(\mathbf{k})\delta(\omega) \tag{6.138}$$

and its FOURIER-transform gives

$$\langle \phi(\mathbf{x}, t) \rangle = \frac{\beta}{\epsilon} \tag{6.139}$$

independent of time and space. This might look surprising, if one assumes that there is a "beginning of time" — why is there no gradual, maybe exponential approach of the steady state? Because the "beginning of time" is infinitely far away. The very fact that we implemented a FOURIER transform means that we had to push back that beginning of time to $t \to -\infty$, so there is no approach to a steady state left any more. All that's left is uniformity.

Similarly, using the expansion of the source term in

$$\left\langle \phi(\mathbf{k}, t) \tilde{\phi}(\mathbf{k}_0, t_0) \right\rangle = \left\langle \phi(\mathbf{k}, t) \tilde{\phi}(\mathbf{k}_0, t_0) \exp(\mathcal{A}_{\text{src}}) \right\rangle_{\text{bil.}} \,, \tag{6.140}$$

again, only the zeroth order contributes and we have

$$\left\langle \phi(\mathbf{k}, t) \tilde{\phi}(\mathbf{k}_0, t_0) \right\rangle = \left\langle \phi(\mathbf{k}, t) \tilde{\phi}(\mathbf{k}_0, t_0) \right\rangle_{\text{bil.}} \tag{6.141}$$

as determined in Eq. (6.119).

We have managed to determine the spreading of an initial seed (subject to diffusion

and evaporation) in the form of the propagator (6.119), and using that, also moments of the density, Eq. (6.137). Looking at the propagator, in particular its FOURIER-transformed version (6.126), it looks as if all initial seeding eventually vanishes. That doesn't seem to square with the calculation that we have just completed, which indicates that there is always some density around. What has gone wrong?

The answer is simple: There is a confusion of $\tilde{\phi}$ and ϕ^*. Putting down an initial seed corresponds to a creator, $a^\dagger$, which is, in fact, $\tilde{\phi}+1$. If we ask for the density after placing a seed somewhere, the relevant observable is

$$\left\langle \phi(\mathbf{x},t)(1+\tilde{\phi}(\mathbf{x}_0,t_0))\right\rangle = \frac{\beta}{\epsilon} + \theta(t-t_0)\frac{\exp(-(t-t_0)\epsilon)}{(4\pi D(t-t_0))^{d/2}}\exp\left(-\frac{(\mathbf{x}-\mathbf{x}_0)^2}{4Dt}\right) \tag{6.142}$$

using Eq. (6.139) for $\langle\phi(\mathbf{x},t)\rangle$ and Eq. (6.126) for $\left\langle\phi(\mathbf{x},t)\tilde{\phi}(\mathbf{x}_0,t_0)\right\rangle$.

6.3.3 Non-linearity: Branching

Finally, I would like to say a few words about non-linear terms. As an example, I will use branchging. Just like for the source term, branching can be dealt with perturbatively, without producing infinitely many terms in the expansion.

The contribution to the master equation, see Eq. (6.3),, is

$$\dot{P}(\{n_\mathbf{x}\};t) = \ldots + \sigma\sum_\mathbf{y}(n_\mathbf{y}-1)P(\{n_\mathbf{y}-1\};t) - n_\mathbf{y}P(\{n_\mathbf{y}\};t) \tag{6.143}$$

where σ is the branching rate and $\ldots$ denotes the terms in (6.3) that have been omitted here. To find the corresponding operator, we consider, again, the problem at just one point. We need to express

$$\sum_n (n-1)P(n-1;t)\,|n\rangle - \sum_n nP(n;t)\,|n\rangle \tag{6.144}$$

in terms of an operator acting on $|\psi(t)\rangle = \sum_n P(n,t)\,|n\rangle$, Eq. (6.6). For the first term in (6.144), we note that

$$(n-1)P(n-1)\,|n\rangle = a^\dagger(n-1)P(n-1)\,|n-1\rangle = a^{\dagger 2}aP(n-1)\,|n-1\rangle\ .$$

The second term is far simpler, as it has the same structure as any other term "chipping away from the *status quo*", see Eq. (6.3), so $nP(n)\,|n\rangle = a^\dagger aP(n)\,|n\rangle$. The operator we are looking for is thus

$$\widehat{\mathcal{A}}_\sigma = \sigma\left(a^{\dagger 2}a - a^\dagger a\right) = \sigma\tilde{a}a^\dagger a = \sigma\tilde{a}a + \sigma\tilde{a}^2 a \tag{6.145}$$

Comparing to $\widehat{\mathcal{A}}_\epsilon$ in Eq. (6.30), which may be rewritten as $-\epsilon\tilde{a}a$, reveals that branching gives rise to **negative mass**, as the first term, $\sigma\tilde{a}a$ is plainly bilinear. This term is in fact the one and only term of the branching that accounts for the particle increase. The other

term, which is normally drawn as a branching vertex (see below), corresponds to the generation of correlation due to branching. This is a result of the DOI-shift: plain density changes are contained in the propagator, all other terms correspond to the generation of correlation (or, in case of a coagulation vertex, the probing thereof).

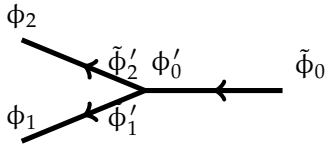
Without further ado, we translate the operators to fields. The new action reads

$$\mathcal{A} = -\int đ^d k' đ\omega' \, \tilde{\phi}(-\mathbf{k}', -\omega')(-\mathring{\imath}\omega + D\mathbf{k}'^2 + \epsilon - \sigma)\phi(\mathbf{k}', \omega') + \int đ^d k' đ\omega' \, \beta\tilde{\phi}(\mathbf{k}', \omega')\delta(\mathbf{k}')\delta(\omega') + \mathcal{A}_\sigma \quad (6.146)$$

where $\mathcal{A}_\sigma$ is a non-linearity, in FOURIER-space clumsily written as

$$\mathcal{A}_\sigma = \int đ^d k_1' đ\omega_1' đ^d k_2' đ\omega_2' \; \tilde{\phi}(\mathbf{k}_1', \omega_1')\tilde{\phi}(\mathbf{k}_2', \omega_2')\phi(-(\mathbf{k}_1' + \mathbf{k}_2'), -(\omega_1' + \omega_2')) \quad (6.147)$$

The diagram associated with this term is  , which, to stress it again, is merely generating correlations. The vertex should be thought of as providing **internal fields** $\tilde{\phi}$, $\tilde{\phi}$ and ϕ for other fields (internal or external) to latch on to, by way of pairing to form a propagator. Anticipating some of the below, the diagram for $\langle \phi(\mathbf{k}_2, \omega_2)\phi(\mathbf{k}_1, \omega_1)\tilde{\phi}(\mathbf{k}_0, \omega_0)\rangle$ is

$\langle \phi(\mathbf{k}_2, \omega_2)\phi(\mathbf{k}_1, \omega_1)\tilde{\phi}(\mathbf{k}_0, \omega_0)\rangle =$ 

(6.148)

where again $\phi_i = \phi(\mathbf{k}_i, \omega_i)$ *etc.*.

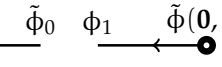
Just like for the source term discussed above, the very structure of the branching vertex means that it does not lead to a change of the normalisation, *i.e.* the argument $\mathcal{N}_1 = \mathcal{N}$ still holds, although obviously the bilinear norm $\mathcal{N}_1$ has changed, as the mass has been shifted by σ. As far as that shift is concerned, $\epsilon - \sigma < 0$ renders the path-integral Eq. (6.119) divergent. At this stage, we will therefore assume $\epsilon - \sigma > 0$.

From Eq. (6.146) we can simply read off the bare propagator by comparison to Eq. (6.117),

$$\langle \phi(\mathbf{k}, \omega)\tilde{\phi}(\mathbf{k}_0, \omega_0)\rangle = \frac{\delta(\mathbf{k} + \mathbf{k}_0)\delta(\omega + \omega_0)}{-\mathring{\imath}\omega + D\mathbf{k}^2 + \epsilon - \sigma} \,, \quad (6.149)$$

which is Eq. (6.119) with ϵ replaced by $\epsilon - \sigma$.

We may now proceed to calculate $\langle \phi_2\phi_1\tilde{\phi}_0\rangle$ of Eq. (6.148). Firstly, there are two identical sets of **disconnected diagrams**,

$\langle \phi_2\tilde{\phi}_0\rangle_{\text{bil.}} \, \beta \, \langle \phi_1\tilde{\phi}(\mathbf{0}, 0)\rangle_{\text{bil.}} \mathrel{\hat{=}}$ 

(6.150)

and

$$\left\langle \phi_1 \tilde{\phi}_0 \right\rangle_{\text{bil.}} \beta \left\langle \phi_2 \tilde{\phi}(\mathbf{0},0) \right\rangle_{\text{bil.}} \mathrel{\hat{=}} \phi_1 \longleftarrow \tilde{\phi}_0 \quad \phi_2 \longleftarrow\!\bullet\, \tilde{\phi}(\mathbf{0},0) \tag{6.151}$$

where $\tilde{\phi}(\mathbf{0},0)$ as the internal field represents the continuous uniform source.

Noticeably, the terms and diagrams above are both disconnected.[31] In the presence of momentum and frequency conservation (translational invariance and time homogeneity), each diagram comes with a δ-function that enforces vanishing net momentum and frequency, so that the disconnected diagrams come with *two* sets of δ-functions:

$$\left\langle \phi_1 \tilde{\phi}_0 \right\rangle_{\text{bil.}} \beta \left\langle \phi_2 \tilde{\phi}(\mathbf{0},0) \right\rangle_{\text{bil.}} = \frac{\delta(\mathbf{k}_1+\mathbf{k}_0)\delta(\omega_1+\omega_0)}{-\mathring{\imath}\omega_1 + D\mathbf{k}_1^2 + \epsilon - \sigma} \beta \frac{\delta(\mathbf{k}_2)\delta(\omega_2)}{-\mathring{\imath}\omega_2 + D\mathbf{k}_2^2 + \epsilon - \sigma} \,. \tag{6.152}$$

Taking the FOURIER-transform we have in real space and time

$$\left\langle \phi_1 \tilde{\phi}_0 \right\rangle_{\text{bil.}} \beta \left\langle \phi_2 \tilde{\phi}(\mathbf{0},0) \right\rangle_{\text{bil.}} = \frac{\beta}{\epsilon-\sigma} G_0(\mathbf{x}_1 - \mathbf{x}_0, t_1 - t_0; \epsilon - \sigma) = \frac{\beta}{\epsilon-\sigma} I_1(\mathbf{x}_1 - \mathbf{x}_0, t_1 - t_0) \tag{6.153}$$

using G_0 of Eq. (6.126), with ϵ replaced by $\epsilon - \sigma$. The expression above describes the contribution to the density-density correlation due to particles being placed somewhere, $(\mathbf{x}_0, t_0)$, appearing somewhere else (by diffusion), say $(\mathbf{x}_1, t_1)$ times the density in the background.

The second term contributing to $\left\langle \phi_2 \phi_1 \tilde{\phi}_0 \right\rangle$ is, diagrammatically,

ϕ_2, ϕ_1 $\longleftarrow$ $\tilde{\phi}_0$

$$\begin{aligned} &\mathrel{\hat{=}} \int \text{đ}^d k_1' \text{đ}\omega_1' \text{đ}^d k_2' \text{đ}\omega_2' \text{đ}^d k_3' \text{đ}\omega_3' \left\langle \phi_1 \tilde{\phi}_{1'} \right\rangle_{\text{bil.}} \left\langle \phi_2 \tilde{\phi}_{2'} \right\rangle_{\text{bil.}} \left\langle \phi_{3'} \tilde{\phi}_0 \right\rangle_{\text{bil.}} \\ &\qquad \times \sigma\delta(\mathbf{k}_1' + \mathbf{k}_2' + \mathbf{k}_3')\delta(\omega_1' + \omega_2' + \omega_3') \\ &= \sigma \frac{\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_0)\delta(\omega_1 + \omega_2 + \omega_0)}{(-\mathring{\imath}\omega_1 + D\mathbf{k}_1^2 + \epsilon - \sigma)(-\mathring{\imath}\omega_2 + D\mathbf{k}_2^2 + \epsilon - \sigma)(\mathring{\imath}\omega_0 + D\mathbf{k}_0^2 + \epsilon - \sigma)} \end{aligned} \tag{6.154}$$

where internal fields have dashed indeces, corresponding to (internal) $\mathbf{k}$ and ω that are integrated over. After the integration only one set of δ-functions is left, again enforcing overall conservation of momentum and frequency.[32] Writing out all terms explicitly, we have not included the second term with labels 1 and 2 swapped, resulting in exactly the same expression. The term $\bowtie$ is therefore said to have a multiplicative factor of 2.

It is very instructive to take the inverse FOURIER transform of the last expression. We may expect it to correspond to a single particle propagating from $\mathbf{x}_0$ at t_0 to some

[31] The notion of "disconnected" applied to expectations of products as the set of terms that is subtracted as "the background" in cumulants. To avoid confusion between disconnected diagrams that involve the non-linearity nevertheless and disconnected diagrams without the non-linearity, it is best understood diagrammatically, as for example in the presence of coagulation we have $\left\langle \phi_2 \tilde{\phi}_0 \right\rangle_{\text{bil.}} \neq \left\langle \phi_2 \tilde{\phi}_0 \right\rangle$.

[32] But notice the sign of ω_0 in the last propagator.

Exercise 14: *How come?*

intermediate point in time and space, where it branches with POISSONian rate σ. From there two particles emanate, finally arriving at $\mathbf{x}_1$ at time t_1 and the other one[33] at $\mathbf{x}_2$ and t_2.

To simplify notation, we should make use of the bare propagator Eq. (6.126), however according to Eq. (6.149) with ϵ replaced by $\epsilon - \sigma$. However, after consulting Eq. (6.126) it is clear that our bare propagator does not describe a lone particle happily hoping along until it happens to branch, but a flood and a cloud of particles whose population is drained by extinction and replenished by branching. This can be seen in the pre-factor $\exp(-(t-t_0)(\epsilon-\sigma))$ which contains some attenuation for the extinction, but amplification for the branching.[34]

What we need to calculate is

$$I_2 = \int đ^d k_1' đ\omega_1' đ^d k_2' đ\omega_2' đ^d k_0' đ\omega_0' \, \exp(-ı(\omega_1 t_1 - \mathbf{k}_1\mathbf{x}_1 + \omega_2 t_2 - \mathbf{k}_2\mathbf{x}_2 + \omega_0 t_0 - \mathbf{k}_0\mathbf{x}_0)) \cdot \sigma \frac{\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_0)\delta(\omega_1+\omega_2+\omega_0)}{(-ı\omega_1 + D\mathbf{k}_1^2 + \epsilon - \sigma)(-ı\omega_2 + D\mathbf{k}_2^2 + \epsilon - \sigma)(ı\omega_0 + D\mathbf{k}_0^2 + \epsilon - \sigma)}, \quad (6.155)$$

which is obviously an awful mess and therefore a matter of an

Exercise 15: *Determine Eq. (6.155). It will help to perform the inverse* FOURIER*-transform firstly only in ω (not $\mathbf{k}$), writing the three propagators in the form $G_i(\omega_i)$, or maybe more explicitly as $1/(-ı\omega_i + r_i)$. The (sole?) challenge is to get the arguments of the resulting convolution right. Translational invariance is obviously established by $\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_0)\delta(\omega_1+\omega_2+\omega_0)$, but after using that, one has to perform a* FOURIER*-transform of the form*

$$\int đ\omega_2 \, \frac{\exp(-ı\omega_2(t_2-t_0))}{(-ı\omega_2 + r_2)(-ı(\omega_1+\omega_2)+r_0)} \quad (6.156)$$

which becomes *a convolution, even though it looks a bit like one already, because of the $\omega_1 + \omega_2$ in the second propagator. However, there is a $\exp(-ı\omega_2(t_2-t_0))$ pre-factor. As for the* FOURIER*-transform of $1/(-ı(\omega_1+\omega_2)+r_0)$, we have*

$$\frac{1}{-ı(\omega_1+\omega_2)+r_0} = \int dt \, \exp(ı(\omega_1+\omega_2)t) \, G_0(t) \quad (6.157)$$

[33] Life is more complicated, see below. What about the second one going anywhere and the first one making its way from $(\mathbf{x}_1, t_1)$ to $(\mathbf{x}_2, t_2)$ all by itself? Firstly, yes, that's possible, but it isn't covered by the observable that we are currently considering. We should have used $\langle \phi_2 \phi_2^* \phi_1 \phi_1^* \phi_0^* \rangle$, which allows for all sorts of processes, because $\phi^* = 1 + \tilde{\phi}$, discussed below. Secondly, the vertex we are looking at provides correlation by branching, not particles, so we shouldn't really think of the legs indicating particles propagating.

[34] ...whereas pinning the branching to a particular intermediate time and space would have suggested that we disallow branching otherwise, $\exp(-(t-t_0)(\epsilon+\sigma))$.

and therefore

$$\int đ\omega_2 \, \frac{\exp(-\mathring{\imath}\omega_2 t)}{-\mathring{\imath}(\omega_1+\omega_2)+r_0} = G_0(t)\exp(\mathring{\imath}\omega_1 t) \ . \tag{6.158}$$

After some algebra and analysis we have from (6.155)

$$\begin{aligned} I_2 &= \sigma \int dt' d^d x' \, G_0(t_1-t', \mathbf{x}_1-\mathbf{x}') G_0(t_2-t', \mathbf{x}_2-\mathbf{x}') G_0(t'-t_0, \mathbf{x}'-\mathbf{x}_0) \\ &= I_2(\mathbf{x}_1-\mathbf{x}_0, \mathbf{x}_2-\mathbf{x}_0, t_1-t_0, t_2-t+0) \end{aligned} \tag{6.159}$$

where we have dropped $\epsilon-\sigma$ from the argument of G_0, and t' and x' are intermediate points (by shifting the integration limits of t' and x', one can see that the expression is in fact a function only of $\mathbf{x}_1-\mathbf{x}_0, \mathbf{x}_2-\mathbf{x}_0, t_1-t_0, t_2-t_0$, as indicated). Because of the causal nature of the propagator (remember its HEAVISIDE θ-function) we may constrain the integral over t' to $t' \in [t_0, \min(t_1-t_0, t_2-t_0)]$. This integral does indeed describe an (ever branching, ever evolving) particle that branches at $t', \mathbf{x}'$ and whose offspring are later found at $t_1, \mathbf{x}_1$ and $t_2, \mathbf{x}_2$.

To summarise, we have

$\langle \phi_2 \phi_1 \tilde{\phi}_0 \rangle =$  $+$

$$\hat{=} \frac{\beta}{\epsilon-\sigma}\Big(I_1(\mathbf{x}_1-\mathbf{x}_0, t_1-t_0) + I_1(\mathbf{x}_2-\mathbf{x}_0, t_2-t_0)\Big) + 2I_2(\mathbf{x}_1-\mathbf{x}_0, \mathbf{x}_2-\mathbf{x}_0, t_1-t_0, t_2-t+0) \ . \tag{6.160}$$

Let's make a "Gedankencomputerexperiment" (a nice exercise, see below). Lots of random walkers, occasionally they branch, occasionally they disappear, but there is a background "seeding" mechanism. What $\langle \phi_2 \phi_1 \tilde{\phi}_0 \rangle$ seems to describe is the correlations we see if we create an extra particle somewhere, and then measure the "repercussions" in terms of correlations at two points $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$. This is *almost* true, *i.e.* it is wrong. There are at least two different contributions missing. Above we have *enforced* that the particle created must end up being detected somewhere, somehow. What about densities being seen *not* as a result of the initial particle, but of the plain background, $(\beta/(\epsilon-\sigma))^2$? What about their correlated production by the background? We could construct them term by term and then wonder why we didn't capture then, or we could construct the relevant observable right from the start: The observable $\langle \phi_2 \phi_1 \tilde{\phi}_0 \rangle$ is not quite what it seems to be.

If we create a particle we must allow for it to just disappear and the following measurement being unrelated to that initial creation. In fact a particle creation operator is $a^\dagger$, not $\tilde{a}$, and thus $\phi^* = 1+\tilde{\phi}$, not just $\tilde{\phi}$ corresponding to $\tilde{a}$. The additional 1 is what allows the effect of the creation to be neglected. As for measuring the density, we must use the particle number (density) operator $a^\dagger a$, corresponding to $(\phi^*+1)\phi$. So, the physical

observable of our experiment is in fact

$$\begin{aligned}\langle\phi_2^*\phi_2\,\phi_1^*\phi_1\,\phi_0^*\rangle &= \big\langle(\tilde\phi_2+1)\phi_2\,(\tilde\phi_1+1)\phi_1\,(\tilde\phi_0+1)\big\rangle \\ &= \big\langle\phi_2\phi_1\tilde\phi_0\big\rangle + \big\langle\tilde\phi_2\phi_2\phi_1\tilde\phi_0\big\rangle + \big\langle\phi_2\tilde\phi_1\phi_1\tilde\phi_0\big\rangle + \big\langle\tilde\phi_2\phi_2\tilde\phi_1\phi_1\tilde\phi_0\big\rangle + \langle\phi_2\phi_1\rangle + \big\langle\tilde\phi_2\phi_2\phi_1\big\rangle \\ &\qquad + \big\langle\phi_2\tilde\phi_1\phi_1\big\rangle + \big\langle\tilde\phi_2\phi_2\tilde\phi_1\phi_1\big\rangle \quad (6.161)\end{aligned}$$

So far we have dealt only with the first term, Eq. (6.160). If we assume that $t_2 > t_1$, then any term involving $\tilde\phi_2$ will vanish because there is no annihilator with a a time $t_i > t_2$ to match it with (and those with $t_i < t_2$ give 0 because of the HEAVISIDE θ-function). That leaves us with three non-vanishing terms to consider, $\big\langle\phi_2\tilde\phi_1\phi_1\big\rangle$, $\big\langle\phi_2\tilde\phi_1\phi_1\tilde\phi_0\big\rangle$, and $\langle\phi_2\phi_1\rangle$.

Of those the first is the most straight-forward, because we have already determined $\big\langle\phi_2\tilde\phi_0\phi_1\big\rangle$, so all we need to do now is take $t_0 \to t_1^+$ in Eq. (6.160). The limit is to be taken from above because the recreation of the particle occurs after its annihilation. We thus have

$$\big\langle\phi_2\tilde\phi_1\phi_1\big\rangle = \frac{\beta}{\epsilon-\sigma} I_1(\mathbf{x}_2-\mathbf{x}_1, t_2-t_1)\,, \quad (6.162)$$

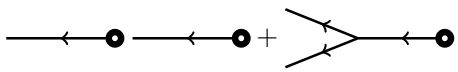
describing the background density at $(\mathbf{x}_1, t_1)$ and the propagation of a particle from there to $(\mathbf{x}_2, t_2)$.

The second term, $\big\langle\phi_2\tilde\phi_1\phi_1\tilde\phi_0\big\rangle$, is also pretty straight-forward, because there is no internal ϕ available to pair up with $\tilde\phi_1$. Therefore, we must pair them up with external fields. The pairing $\big\langle\tilde\phi_1\phi_1\big\rangle$ vanishes, and so

$$\begin{aligned}\big\langle\phi_2\tilde\phi_1\phi_1\tilde\phi_0\big\rangle = \big\langle\phi_2\tilde\phi_1\big\rangle_{\text{bil.}}\big\langle\phi_1\tilde\phi_0\big\rangle_{\text{bil.}} &= G_0(\mathbf{x}_2-\mathbf{x}_1, t_2-t_1)G_0(\mathbf{x}_1-\mathbf{x}_0, t_1-t_0) \\ &= I_3(\mathbf{x}_2-\mathbf{x}_0, \mathbf{x}_1-\mathbf{x}_0, t_2-t_0, t_1-t_0) \quad (6.163)\end{aligned}$$

describing the propagation of a particle from $(\mathbf{x}_0, t_0)$ to $(\mathbf{x}_1, t_1)$ and from there to $(\mathbf{x}_2, t_2)$.

The third term, $\langle\phi_2\phi_1\rangle$, is the most painful one. There are two contributions,[35]

$\langle\phi_2\phi_1\rangle =$  (6.164)

one by connecting both external fields to the source and one by connecting them to a single source via the branching vertex. The first one leads to the square average density,

$$I_4 = \left(\frac{\beta}{\epsilon-\sigma}\right)^2 \quad (6.165)$$

[35]These are almost the two terms of $\langle\phi_2\phi_1\rangle$, Eq. (6.160), with the right creator replaced by a source, but noticeably the terms corresponding to I_1 do not feature twice, as they do in Eq. (6.160), because the symmetry factor of this diagram is 1, not 2 (because of the symmetry of the two individual diagrams, so that a swap of external fields corresponds to a swap of internal sources, which is accounted for by the $1/n!$ in the expansion of the exponential).

the second one produces "correlations by branching of spontaneously created particles". The latter, with vertex  , corresponds to I_2 of Eq. (6.159), however with the external creator at $(\mathbf{x}_0, t_0)$ replaced by a source. Equivalently, one may replace $\mathbf{k}_0, \omega_0$ in Eq. (6.155) by $\mathbf{0}, 0$ and omit the integral over $\mathbf{k}_0', \omega_0'$:

$$I_5 = \frac{\beta\sigma}{\epsilon-\sigma}\int dt' d^d x'\, G_0(t_1-t', \mathbf{x}_1-\mathbf{x}'; \epsilon-\sigma) G_0(t_2-t', \mathbf{x}_2-\mathbf{x}'; \epsilon-\sigma) = I_5(\mathbf{x}_1-\mathbf{x}_2, t_1-t_2) \tag{6.166}$$

Collecting all terms we have for $t_2 > t_1$

$$\begin{aligned}
&\langle \phi_2^* \phi_2\, \phi_1^* \phi_1\, \phi_0^* \rangle = \left\langle (\tilde\phi_2 + 1)\phi_2\, (\tilde\phi_1 + 1)\phi_1\, (\tilde\phi_0 + 1) \right\rangle \\
&= \quad + \quad + \quad + \quad + \\
&= \frac{\beta}{\epsilon-\sigma}\Big(G_0(\mathbf{x}_1-\mathbf{x}_0, t_1-t_0) + G_0(\mathbf{x}_2-\mathbf{x}_0, t_2-t_0) \Big) \\
&+ 2I_2(\mathbf{x}_1-\mathbf{x}_0, \mathbf{x}_2-\mathbf{x}_0, t_1-t_0, t_2-t+0) + \frac{\beta}{\epsilon-\sigma} I_1(\mathbf{x}_2-\mathbf{x}_1, t_2-t_1) \\
&+ G_0(\mathbf{x}_2-\mathbf{x}_1, t_2-t_1) G_0(\mathbf{x}_1-\mathbf{x}_0, t_1-t_0) + \left(\frac{\beta}{\epsilon-\sigma}\right)^2 + I_5(\mathbf{x}_1-\mathbf{x}_2, t_1-t_2)
\end{aligned} \tag{6.167}$$

using Eq. (6.153), (6.159), (6.160), (6.162), (6.163), (6.165) and (6.166). This is, as expected, a function only of $\mathbf{x}_2-\mathbf{x}_0$,$t_2-t_0$,$\mathbf{x}_1-\mathbf{x}_0$ and t_1-t_0.

Exercise 16: *Test the analytical results for "induced correlations", Eq. (6.167) in a computer simulation.*

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