Using Lin's method to solve Bykov's Problems

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Abstract

We consider nonwandering dynamics near heteroclinic cycles between two hyperbolic equilibria. The constituting heteroclinic connections are assumed to be such that one of them is transverse and isolated. Such heteroclinic cycles are associated with the termination of a branch of homoclinic solutions, and called *T-points* in this context. We study codimension-two T-points and their unfoldings in $\mathbb{R}^n$. In our consideration we distinguish between cases with real and complex leading eigenvalues of the equilibria. In doing so we establish Lin's method as a unified approach to (re)gain and extend results of Bykov's seminal studies and related works. To a large extent our approach reduces the study to the discussion of intersections of lines and spirals in the plane.

Contents

1 Introduction

Homoclinic bifurcations lie at the heart of our understanding of complicated (chaotic) recurrent behaviour in dynamical systems. The history goes back to Poincaré, with major subsequent contributions by the schools of Andronov, Shilnikov, Smale and Palis. The successes of the latter schools has been founded on a combination of analytical and geometrical tools; typical for the field of dynamical systems.

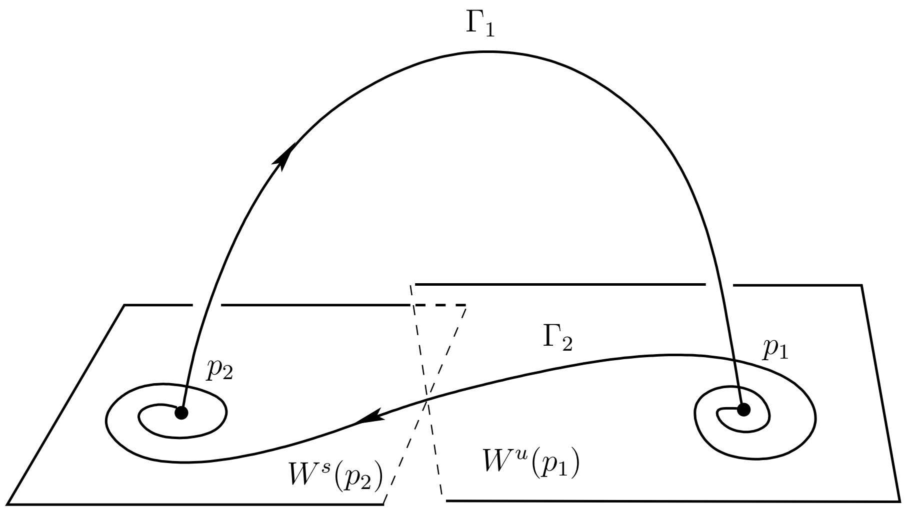


Figure 1: Sketch of an example of a T-point heteroclinic cycle in $\mathbb{R}^3$ between two saddle-foci, with robust heteroclinic orbit Γ_2 and non-robust heteroclinic orbit Γ_1.

We consider a parameter family of vector fields $f : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ ($n \geq 3$), f smooth:

$$\dot{x} = f(x, \mu). \tag{1.1}$$

We study the nonwandering dynamics in the neighbourhood of a *T-point* [15]: a heteroclinic cycle between two hyperbolic equilibria of saddle type p_1 and p_2, where one of the connections is transverse and isolated. See Figure 1 for a sketch of a T-point heteroclinic cycle in $\mathbb{R}^3$. T-points have been found to appear in many applications of interest, ranging from the Lorenz [15] and Kuramoto-Sivashinsky [25] systems, to electronic oscillators [1, 2, 10, 11, 12], semiconductor lasers [16, 38], magnetoconvection [32] and travelling waves in reaction-diffusion dynamics [24, 36, 31, 18].

We identify three cases according to the nature of the eigenvalue spectrum at each of the two saddle points p_1 and p_2. The three cases correspond to certain leading eigenvalues being both real **(RR)**, one real and one complex **(RC)**, or both complex **(CC)**. For a precise description of the setting and assumptions see section 2. We summarize our findings as follows:

- Case **(RR)**: Under open conditions on the eigenvalues, there exist open sets in parameter space for which there exists periodic orbits close to the heteroclinic cycle. In addition, there exist two one-parameter families of homoclinic orbits to each of the saddle points p_1 and p_2.

See Theorem 2.1 and Lemma 2.2 for precise statements and Figure 2 for bifurcation diagrams.

- Case **(RC)** and **(CC)**: At the bifurcation point $\mu = 0$ and for each $N \geq 2$, there exists an invariant set $\mathcal{S}_0^N$ close to the heteroclinic cycle on which the first return map is topologically conjugated to a full shift on N symbols. For any fixed $N \geq 2$, the invariant set $\mathcal{S}_\mu^N$ persists for $|\mu|$ sufficiently small.

  In addition, there exist infinitely many transversal and non-transversal heteroclinic orbits connecting the saddle points p_1 and p_2 in a neighbourhood of $\mu = 0$, as well as well as infinitely many one-parameter families of homoclinic orbits to each of the saddle points.

  For full statements of the results see Theorem 2.3 and Lemmas 2.4, 2.5 and Figure 3 for bifurcation diagrams.

The dynamics near T-points has been studied previously by Bykov [3, 4, 5], Glendinning and Sparrow [15] and Kokubu [23, 22]. See also the surveys by Homburg and Sandstede [19], Turaev et al. [35] and Fiedler [13]. T-points in the context of reversible systems have been discussed by Lamb et. al. [25]. These studies all consider dynamics in $\mathbb{R}^3$ using a geometric return map approach, and their results reflect the description of types of nonwandering dynamics described above.

Further related studies concerning T-points can be found in [27] and [29], where inclination flips were considered in this context. See also [30], where the asymptotic behaviour of typical trajectories near a T-point cycle is studied for C^1 vector fields.

The main aim of this paper is to present a comprehensive study of dynamics near T-points, including detailed proofs of all results, employing a unified functional-analytic approach, without making any assumption on the dimension of the phase space. In the process, we recover and generalise to higher dimensional settings all previously reported results for T-points in $\mathbb{R}^3$. In addition, we reveal the existence of richer dynamics in the **(RC)** and **(CC)** cases. A detailed discussion of our results is contained in section 2.

The functional analytic approach we follow is commonly referred to as *Lin's method*, after the seminal paper by Lin [26], and employs a reduction on an appropriate Banach space of piecewise continuous functions approximating the initial heteroclinic cycle to yield bifurcation equations whose solutions represent orbits of the nonwandering set. The development of such an approach is typical for the school of Hale, and is in contrast to the analysis contained in previous T-point studies, which relies on the construction of a first return map. Our choice of analytical framework is motivated by the fact that Lin's method provides a unified approach to study global bifurcations in arbitrary dimension, and has been shown to extend to a larger class of settings, such as delay and advance-delay equations [14, 26].

2 Statement of the main results

Without loss, we assume that the cycle exists at $\mu = 0$ and that the hyperbolic equilibria do not change position while μ is close to 0. Throughout this paper we denote the stable manifold of the equilibrium p_i at parameter(s) μ by $W^s(p_i,\mu)$. For brevity, we also denote $W^s(p_i,0)$ by $W^s(p_i)$. In the same manner we use $W^u(p_i,\mu)$ and $W^u(p_i)$ to denote the corresponding unstable manifolds.

Let q_j, $j \in \{1,2\}$, denote the heteroclinic solution connecting the equilibria p_{j+1} and p_j; throughout the term "$j+1$" is computed *modulo* 2,

$$\lim_{t\to-\infty} q_j(t) = p_{j+1}, \quad \lim_{t\to\infty} q_j(t) = p_j.$$

The corresponding orbits we denote by Γ_j, $\Gamma_j = \{q_j(t) \mid t \in \mathbb{R}\}$. The corresponding heteroclinic cycle is a T-point if Γ_2 is an isolated transversal intersection and Γ_1 is nondegenerate. We assume the following:

(H 1) (i) $W^s(p_2) \pitchfork_{q_2(0)} W^u(p_1)$ and $\dim T_{q_2(0)}W^s(p_2) + \dim T_{q_2(0)}W^u(p_1) = n+1$.
(ii) $T_{q_1(0)}W^s(p_1) \cap T_{q_1(0)}W^u(p_2) = \operatorname{span}\{f(q_1(0),0)\}$.

Figure 1 displays one possible scenario in $\mathbb{R}^3$. Note that due to this assumption the hyperbolic equilibria p_1 and p_2 have different saddle point indices (dimensions of the unstable manifolds) – more precisely

$$\operatorname{ind} p_1 = \operatorname{ind} p_2 + 1.$$

The above conditions specify the simplest case of a heteroclinic cycle between two fixed points where one heteroclinic orbit is structurally stable and the other is not.

The nonrobust connection Γ_1 has codimension two in general. In order to make this precise we first discuss the persistence of the heteroclinic orbit Γ_1. Let Z be the two-dimensional subspace which is perpendicular to $T_{q_1(0)}W^s(p_1) + T_{q_1(0)}W^u(p_2)$. Similar to [19, Lemma 2.1] the distance between $W^s(p_1,\mu)$ and $W^u(p_2,\mu)$ can be measured in this Z-direction (cf. also Lemma 3.1 below). Let $\xi^\infty(\mu)$ denote this distance. The following Hypothesis (H 2) is a transversality condition which ensures that these manifolds split with positive speed by varying μ.

(H 2) $D\xi^\infty(0)$ is non-singular.

This justifies our setting $\mu \in \mathbb{R}^2$, $\mu := (\mu_1,\mu_2)$. We make some further hypotheses ensuring that the T-point has codimension two:

(H 3) $\Gamma_i \not\subset W^{ss}(p_i), \quad \Gamma_i \not\subset W^{uu}(p_{i+1}), \quad i = 1,2.$

Here $W^{ss}(p)$ and $W^{uu}(p)$ denotes the strong stable manifold and the strong unstable manifold, respectively, of the equilibrium p. This is a standard non-orbit flip condition. Further we assume a slight modification of the standard non-inclination flip condition for Γ_1. To this end we introduce the local extended-unstable manifold $W^{eu}(p_2)$ of p_2 (this is an invariant manifold whose tangent space at p_2 comprises the unstable and weakest stable directions), and correspondingly the extended-stable manifold $W^{es}(p_1)$ of p_1. Note that these manifolds are not uniquely defined. However their tangent spaces along q_1 are well defined. With this notation the non-inclination flip condition reads

(H 4) $W^{eu}(p_2) \pitchfork_{q_1(0)} W^{es}(p_1)$.

For each fixed point p_i, we call the eigenvalue which is the closest one to the imaginary axis the *leading (un)stable eigenvalue*. Similarly we define the leading strong (un)stable eigenvalues.

We further assume:

(H 5) Both the leading unstable eigenvalue λ_1^u of p_1 and the leading stable eigenvalue λ_2^s of p_2 are simple.

We refer again to Figure 1 for a visualisation of the situation in $\mathbb{R}^3$ in case the addressed eigenvalues λ_1^u and λ_2^s are complex.

In our analysis we distinguish three cases:

(RR) Both $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$ are real.

(RC) One of $\lambda_1^u(\mu)$, $\lambda_2^s(\mu)$ is real and the other is complex,

(CC) Both $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$ are complex,

Before we describe the nearby dynamics of the heteroclinic cycle in each case we define some terminology. For that we introduce hyperplanes Σ_j which are transversal to Γ_j at $q_j(0)$ (which may be located "in the middle" of Γ_j), $j = 1, 2$. Let $\mathcal{U}$ be a sufficiently small neighbourhood of the primary heteroclinic cycle. All orbits which we describe in the following are assumed to be subsets of $\mathcal{U}$.

- A periodic orbit is called k-periodic, and a homoclinic orbit (to p_1 or p_2) is called k-homoclinic, if it passes through Σ_1 and Σ_2 k times in each case.
- A heteroclinic orbit connecting p_i to p_j (in forward time) is called an (i,j)-heteroclinic orbit. An (i,j)-heteroclinic orbit that passes k times through Σ_j is called a k-(i,j)-heteroclinic orbit.

First we describe the dynamics in the (RR) case. It turns out that the dynamics depends on the leading eigenvalues of the single equilibrium. More precisely we have to distinguish whether the corresponding eigenvectors point in direction of Γ_2 or not. The case where these eigenvectors do so is described in Theorem 2.1(i). The other possibilities are treated in Theorem 2.1(ii) and (iii). For these cases we need to make the following further assumptions. To this end we denote the leading strong stable eigenvalue of p_2 by λ_2^{ss} and the leading strong unstable eigenvalue of p_1 by λ_1^{uu}.

(H 6) Both λ_1^s and λ_2^{ss} are real and simple.

(H 7) Both λ_2^u and λ_1^{uu} are real and simple.

The dynamics depends on the global behaviour of $W^s(p_2)$ and $W^u(p_1)$. Following these manifolds along the primary cycle Γ we find that both manifolds are either *nontwisted* or *twisted* – in other words, topologically they are either orientable or nonorientable. For a more detailed definition we refer to [6], where global (un)stable manifolds along a homoclinic orbit are considered. Based on these considerations we present a definition of twisted or nontwisted T-point cycles related to the context of Lin's method at the beginning of Section 5.

Our main result for the (RR) case under consideration is the following.

Theorem 2.1. *Consider system* (1.1)*. Assume the case (RR) and Hypotheses (H 1)–(H 7). For the bifurcations of periodic and 1-homoclinic orbits the following cases have to be distinguished:*

(i) Let $\lambda_1^u(\mu) < |\lambda_1^s(\mu)|$, $|\lambda_2^s(\mu)| < \lambda_2^u(\mu)$, i.e. $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$ are the leading eigenvalues of p_1 and p_2, respectively. Then the T-point cycle Γ may be twisted or nontwisted. In either case there exists an open wedge Q_1 in parameter space such that for $\mu \in Q_1$ there exists one 1-periodic orbit. These bifurcate from 1-homoclinic orbits to either p_1 or p_2, cf. bifurcation diagram in Figure 2(a).

(ii) Let $|\lambda_1^s(\mu)| < \lambda_1^u(\mu)$, $|\lambda_2^s(\mu)| < \lambda_2^u(\mu)$, i.e. $\lambda_1^s(\mu)$ and $\lambda_2^s(\mu)$ are the leading eigenvalues of p_1 and p_2, respectively.

(a) If Γ is nontwisted, then there exist open wedges Q_1 and Q_2 in parameter space such that for $\mu \in Q_1$ there exists one 1-periodic orbit. For $\mu \in Q_2$ there exist two 1-periodic orbits. $\mathcal{S}_{sn}$ is a saddle-node curve for 1-periodic orbits. The relative position of the wedges Q_1 and Q_2 and the saddle-node curve $\mathcal{S}_{sn}$ is depicted in the left panel of Figure 2(b).

(b) If Γ is twisted, then there exist open wedges Q_1 and Q^2 in parameter space such that for $\mu \in Q_1$ there exists one 1-periodic orbit. In $\mu \in Q^2$ there also exists one 2-periodic orbit. $\mathcal{S}_{pd}$ is a period-doubling curve; $\mathcal{S}_{2\text{-}hom\text{-}p_1}$ is a curve of 2-homoclinic orbits to p_1. The relative position of the wedges Q_1 and Q^2 and the curves $\mathcal{S}_{pd}$ and $\mathcal{S}_{2\text{-}hom\text{-}p_1}$ is depicted in the right panel of Figure 2(b).

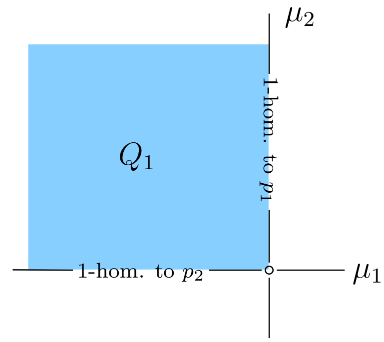
(iii) Let $|\lambda_1^s(\mu)| < \lambda_1^u(\mu)$, $\lambda_2^u(\mu) < |\lambda_2^s(\mu)|$, i.e. $\lambda_1^s(\mu)$ and $\lambda_2^u(\mu)$ are the leading eigenvalues.

(a) If Γ is nontwisted, then there exists an open wedge Q_1 and two open wedges Q_2 in parameter space such that for $\mu \in Q_1$ there exists one 1-periodic orbit. For $\mu \in Q_2$ there exist two 1-periodic orbits. $\mathcal{S}_{sn}$ are saddle-node curves for 1-periodic orbits. The relative position of the wedges Q_1 and Q_2 and the saddle-node curves $\mathcal{S}_{sn}$ are depicted in the left panel of Figure 2(c).

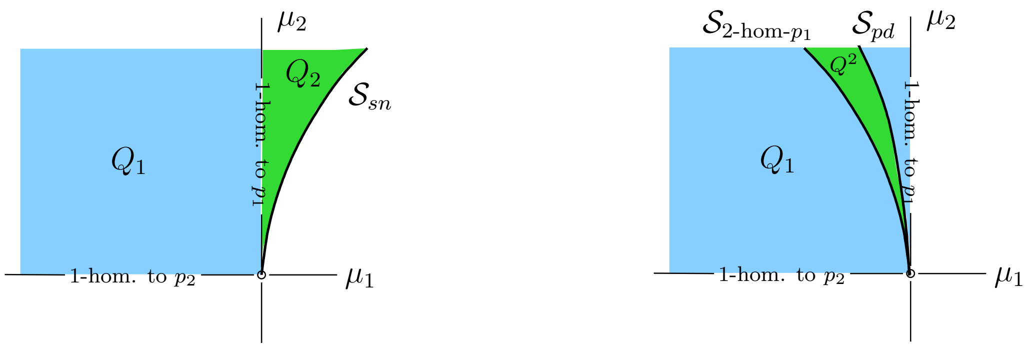
(b) If Γ is twisted, then there exists an open wedge Q_1 and and two open wedges Q^2 in parameter space such that for $\mu \in Q_1$ there exists one 1-periodic orbit. For $\mu \in Q^2$ there also exists one 2-periodic orbit. $\mathcal{S}_{pd}$ are period-doubling curves; $\mathcal{S}_{2\text{-}hom\text{-}p_1}$ is a curve of 2-homoclinic orbits to p_1, $\mathcal{S}_{2\text{-}hom\text{-}p_2}$ is a curve of 2-homoclinic orbits to p_2. The relative positions of the wedges Q_1 and Q^2 and the curves $\mathcal{S}_{pd}$, $\mathcal{S}_{2\text{-}hom\text{-}p_1}$ and $\mathcal{S}_{2\text{-}hom\text{-}p_2}$ are depicted in the right panel of Figure 2(c).

This theorem generalizes results of [3], cf. also [19, Theorem 5.31], to higher phase space dimensions. We note that Hypotheses (H 6) and (H 7) are automatically satisfied if $n = 3$, i.e. $x \in \mathbb{R}^3$.

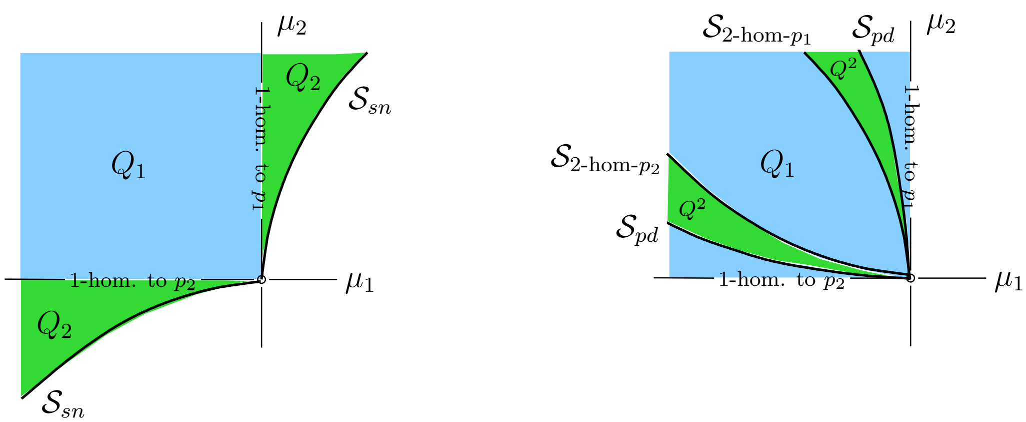
The actual proof of Theorem 2.1 is given in Sections 5.1–5.3. In these sections we discuss the corresponding bifurcation equations. However, Lemma 4.4 is the core element, where we provide the necessary estimate for the bifurcation equations. In its proof we assume the Hypotheses (H 1)–(H 7). Actually, Theorem 2.1(i) and (ii) require only somewhat weaker estimates. So, for Theorem 2.1(i) Hypotheses (H 6)–(H 7) are not necessary. Omitting these hypotheses the conditions in Theorem 2.1(i) read $\lambda_1^u(\mu) < |\mathrm{Re}\,\lambda_1^s(\mu)|$, $|\lambda_2^s(\mu)| < \mathrm{Re}\,\lambda_2^u(\mu)$. Theorem 2.1(ii) can be proved only assuming Hypotheses (H 1)–(H 6). Omitting (H 7) the condition of this part of the theorem reads $|\lambda_1^s(\mu)| < \lambda_1^u(\mu)$, $|\lambda_2^s(\mu)| < \mathrm{Re}\,\lambda_2^u(\mu)$. Theorem 2.1(ii) is true in equal mea-



(a) Statement of Theorem 2.1(i).



(b) Statement of Theorem 2.1(ii). Left panel: Γ nontwisted; right panel: Γ twisted.



(c) Statement of Theorem 2.1(iii). Left panel: Γ nontwisted; right panel: Γ twisted.

Figure 2: The bifurcation diagrams for Theorem 2.1. For $\mu \in Q_1$ there exists one 1-periodic orbit, for $\mu \in Q_2$ there exist two 1-periodic orbits, and for $\mu \in Q^2$ there is one additional 2-periodic orbit. $\mathcal{S}_{pd}$ is a saddle-node curve for 1-periodic orbits, $\mathcal{S}_{pd}$ is a period-doubling curve and $\mathcal{S}_{2\text{-hom-}p_i}$ is a curve of 2-homoclinic orbits to p_i.

sure under the assumption that $\lambda_1^u(\mu)$ and $\lambda_2^u(\mu)$ are the leading eigenvalues, and that (H 7) holds true. Note that in this case the wedges Q_2 and Q^2 in Figure 2(b) have to be placed correspondingly.

The existence of one 1-periodic in Q_1 in each case can be explained simply by considering intersections of lines in the plane. In Remark 5.1 we give an idea how this can be done and how these lines correspond to truncated bifurcation equations.

Lemma 2.2. *Consider system* (1.1). *Assume the case (RR) and Hypotheses (H 1)–(H 7).*

(i) There are no k-(2,1)-heteroclinic orbits for each $k \geq 2$, $k \in \mathbb{N}$, for all $|\mu|$ sufficiently small.

(ii) Assume the eigenvalue conditions of Theorem 2.1(i), $\lambda_1^u(\mu) < |\lambda_1^s(\mu)|$, $|\lambda_2^s(\mu)| < \lambda_2^u(\mu)$. At $\mu = 0$ there are no k-(1,2)-heteroclinic orbits for $k \geq 2$. But for each $\mu \in Q_1$ there exists a k-(1,2)-heteroclinic orbit for $k \geq 2$.

The proof of Lemma 2.2 is given in Section 5.4. We remark that Hypothesis (H 2) which prescribes that the stable manifold of p_1 and the unstable manifold of p_2 split with positive speed, implies that there exists a 1-(2,1)-heteroclinic orbit (the one which belongs to the original T-point cycle) only at $\mu = 0$. In $\mathbb{R}^3$ it is clear that there is at most one k-(2,1)-heteroclinic orbit for each parameter value, because in this case the stable manifold of p_1 is one-dimensional.

Next we turn to the complex eigenvalue cases. Here we mainly focus on the existence of shift dynamics in the neighborhood of Γ. Our main result is the following.

Theorem 2.3. *Consider the system* (1.1) *with the eigenvalue conditions (RC) or (CC) under Hypotheses (H 1)–(H 5). In the case (RC) let $\lambda_1^u(\mu)$ be real, and let $\lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu)$ be complex, and additionally assume Hypothesis (H 7). Then the following is true (typically in the case (CC)):*

(i) When $\mu = 0$, for each $N \geq 2$ there exists a set $\mathcal{S}_0^N \subset \Sigma_1$ which is invariant under the first-return-map $\Pi : \Sigma_1 \to \Sigma_1$ (defined by the flow), and $(\mathcal{S}_0^N, \Pi)$ is topologically conjugated to a full shift on N symbols.

(ii) Moreover, for fixed $N \geq 2$ there exists $\mu_N > 0$, such that if $|\mu| < \mu_N$ there is a set $\mathcal{S}_\mu^N$ such that $(\mathcal{S}_\mu^N, \Pi(\mu))$ is topologically conjugated to a full shift on N symbols, and $\mathcal{S}_\mu^N \to \mathcal{S}_0^N$ in the Hausdorff-metric as $\mu \to 0$.

The statements in (RC) case remain true if $\lambda_2^s(\mu)$ is real and $\lambda_1^u(\mu)$ is complex and Hypothesis (H 6) instead of (H 7) is assumed. This theorem covers results of Bykov [3] and Glendinning and Sparrow [15] concerning shift dynamics in the (RC) case, cf. also [19, Theorem 5.32]. In contrast to these works, our statement is not restricted to $n = 3$. Further we prove the existence of shifts on more than two symbols.

The proof of Theorem 2.3, which is also based on Lin's method, is given in Section 6.1 for the (RC) case and Section 6.2 for the (CC) case. The proof relies on the existence of infinitely many transversal intersections (for $\mu = 0$) of a spiral and a line or two spirals, cf. Figure 11. These geometrical objects correspond again to a truncated bifurcation equation. The addressed transversal intersections correspond to 1-periodic orbits. Roughly speaking these 1-periodic orbits serve as the symbols that arise in the formulation of the theorem.

In our analysis we exploit that the "angular degree" of the transversal intersections is bounded away from zero. In analytical terms this is described in the proof of Lemma 6.2 for the (RC) case and in Lemma 6.5 for the (CC) case.

The phrase "typically" in the (CC) case refers to an open and dense set $\mathcal{D}$ of vector fields which is discussed more closely in Section 6.2 as well. Geometrically, the 1-periodic orbits correspond to transversal intersections of two spirals, cf. right panel in Figure 11. Those intersections serve as symbols in the addressed shift dynamics.

The dynamics referred to in Theorem 2.3 does not concern l-homoclinic or l-heteroclinic orbits.

Lemma 2.4. *Assume the eigenvalue case (RC) or (CC), and let $f(\cdot,\mu))$ be a two parameter family of vector fields for which Theorem 2.3 holds. Then for each $l \geq 2$, $l \in \mathbb{N}$ there is a countable set $\mathcal{T}_l$ of parameter values accumulating at $\mu = 0$, for which there exists a l-(2,1)-heteroclinic orbit.*

In other words the lemma states that for each $\mu \in \mathcal{T}_l$ there is a T-point cycle built up by the l-(2,1)-heteroclinic orbit and Γ_2 (see Figure 3). For $n = 3$ this confirms results by Bykov [3] and Glendinning and Sparrow [15], cf. also [19, Theorem 5.32. Figure 5.16(i)]. The proof of Lemma 2.4 is given in Section 6.3.

Lemma 2.5. *Assume the eigenvalue case (RC) or (CC), and let $f(\cdot,\mu))$ be a two parameter family of vector fields for which Theorem 2.3 holds. Then we have the following:*

(i) *At $\mu = 0$ there exist a countable infinity of l-(1,2)-heteroclinic orbits for each $l \geq 2$. Moreover, for fixed l, $q_1(0)$ is an accumulation point of the intersections of the l-(1,2)-heteroclinic orbits with Σ_1. Each such l-(1,2)-heteroclinic orbit can be continued for $\mu \neq 0$ sufficiently small, but for fixed $\mu \neq 0$ and fixed l only finitely many l-(1,2)-heteroclinic orbits might exist.*

(ii) *There are curves $\mathcal{L}^{hom}_{1,j}$, $j \in \{1,2\}$, in μ-space for which each point on these curves there exists a* 1*-homoclinic orbit to the fixed point p_j. The curve $\mathcal{L}^{hom}_{1,1}$ is either a logarithmic spiral centred at $(0,0)$, if λ^s_2 is complex, or a line terminating at $(0,0)$, if λ^s_2 is real. An analogous statement applies for $\mathcal{L}^{hom}_{1,2}$ with λ^s_2 replaced by λ^u_1 accordingly.*

For each $l \geq 2$, $l \in \mathbb{N}$, and each $j \in \{1,2\}$, there are countably many curves $\mathcal{L}^{hom}_{l,j}$ in μ-space for which each point on these curves there exists a l-homoclinic orbit to the fixed point p_j. The curves $\mathcal{L}^{hom}_{l,1}$ are either logarithmic spirals centred at $(0,0)$, if λ^s_2 is complex, or lines terminating at $(0,0)$, if λ^s_2 is real. An analogous statement applies for $\mathcal{L}^{hom}_{1,2}$ with λ^s_2 replaced by λ^u_1 accordingly.

Along $\mathcal{L}^{hom}_{l,j}$, towards $(0,0)$, the homoclinic orbits approach heteroclinic cycles built up by Γ_1 and the l-(1,2)-heteroclinic orbits stated in (i).

The proof of Lemma 2.5 is given in Section 6.4. See again Figure 3 for schematic bifurcation diagrams.
Statement (i) of the lemma can also be read in terms of intersections of the corresponding unstable and stable manifolds $W^u(p_1)$ and $W^s(p_2)$. In the (CC) case this confirms results by Bykov [4, 5] cf. also [19, Theorem 5.33].
For $l = 1$ the statement (ii) of the lemma can be found in Bykov [3, 4, 5] or Glendinning and

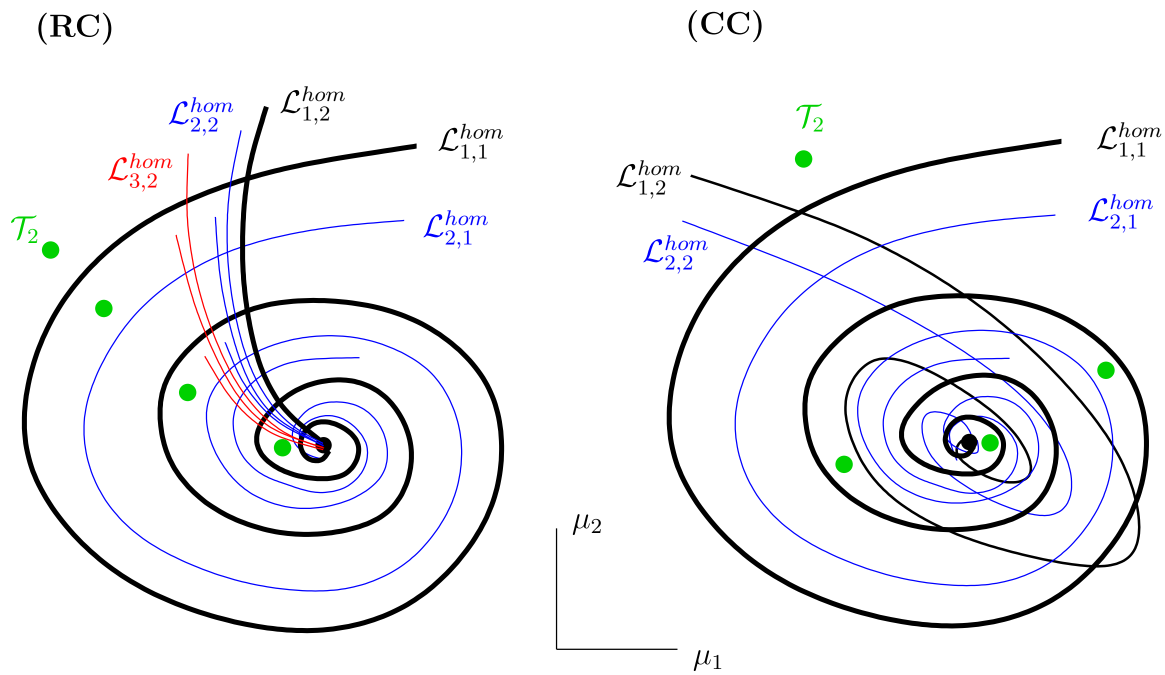


Figure 3: Schematic bifurcation diagrams showing a small part of the dynamics found in Lemmas 2.4 and 2.5. The (RC) case is shown for λ_1^u and λ_2^s complex. For $\mu \in \mathcal{L}^{hom}_{l,j}$ there exists a l-homoclinic orbit to p_j, and for $\mu \in \mathcal{T}_l$ there exists a T-point cycle consisting of an l-(2,1) heteroclinic orbit and Γ_2. One may think of the bifurcation diagram in the (CC) case as being obtained from the one in the (RC) case by replacing the lines $\mathcal{L}^{hom}_{l,2}$ by corresponding spirals which may have arbitrary orientation, similarly for the points $\mathcal{T}_l$.
A consequence of the proof is that the curves belonging to each family $\mathcal{L}^{hom}_{l,j}$, for fixed (l, j), have shortening length.
Note that as each point in the sets $\mathcal{T}_l$ corresponds to a distinct T-point, the bifurcation diagram in each case will be reproduced at these points and therefore the full bifurcation diagrams show some self-similarity.
Finally we note that the above diagrams are purely schematic, and the precise arrangement of the curves $\mathcal{L}^{hom}_{l,j}$ may depend on details.

Sparrow [15] (only (RC) case) cf. also [19, Theorems 5.32 and 5.33, Figure 5.16]. Unlike these theorems we do not consider the character of possible intersections of these lines.
Bifurcation diagrams may also be found in [13] and [19]. The drawings presented in [19, Figure 5.16] can be seen as a first approximation of bifurcation diagrams. In [13], based on Bykovs work, a more detailed bifurcation diagram for the (RC) case is given. This diagram is valid under more restrictive eigenvalue conditions – it is assumed that $n = 3$, p_2 is a real saddle with leading eigenvalue λ_2^s and that λ_1^s is the leading eigenvalue of the focus p_1. This bifurcation diagram also shows “our” curves $\mathcal{L}^{hom}_{2,1}$. Moreover it contains T-points which are related to 2-(1,2)-heteroclinic orbits (cf. Lemma 2.4 and the subsequent comment). The spirals of corresponding homoclinic orbits are also drawn attached to those T-points. Altogether this diagram conveys a kind of self-similarity in the bifurcation diagram. However, the statements in

Lemma 2.4 and Lemma 2.5 reveal that a complete bifurcation diagram is much more involved. Further, we emphasise that we do not make use of the restrictive eigenvalue condition of [13].

Finally we mention related works by Kokubu [22, 23]. There also the (RC) case has been considered, allowing also that λ_1^u is the leading eigenvalune of the focus p_1. A bifurcation diagram indicating the 1-homoclinic orbits is given.

We now outline the organisation of this paper. We recall the essentials of Lin's method in Section 3. The problem of finding particular orbits in the local nonwandering set reduces to that of solving a (infinite) set of bifurcation equations.

In Section 4 we derive an expression for the ith jump of a Lin orbit, depending on μ and the sequence of transition times $\boldsymbol{\omega}$. In particular we determine their leading order terms.

As previously mentioned, Sections 5 and 6 contain the actual proofs of our statements concerning the dynamics near T-points.

3 Lin's method

In this section we outline Lin's method, and explain the properties of Lin orbits corresponding to the system under consideration. Lin's method centres around the existence of 'piecewise continuous orbits' X of (1.1) which we call *Lin orbits*, after [20]. In the present context such orbits consist of pieces of actual orbits X_i, $X := (X_i)_{i\in\mathbb{Z}}$; the orbit X_i starts in Σ_1, follows Γ_1 until it reaches a neighbourhood of p_1 follows then Γ_2, meets Σ_2, stays further close to Γ_2 until it reaches a neighbourhood of p_2, follows then Γ_1 again, and terminates finally in Σ_1. Between two consecutive orbits X_{i-1} and X_i there may be a jump Ξ_i in a distinguished direction Z. We refer to Figure 4 for a visualisation. Note that in the present context all jumps in Σ_2 are equal to zero. This is due to the transversal intersection of the unstable manifold of p_1 and the stable manifold of p_2 in Γ_2.

Let $2\omega_{1,i}$ and $2\omega_{2,i}$ be (prescribed) transition times of X_i from Σ_1 to Σ_2 and Σ_2 to Σ_1, respectively. It can be proved that for each μ which is sufficiently close to 0, and each sequence $\boldsymbol{\omega} := ((\omega_{1,i}, \omega_{2,i}))_{i\in\mathbb{Z}}$, where $\omega_{j,i}$ are sufficiently large, there exists a unique Lin orbit $X(\boldsymbol{\omega}, \mu)$, see Theorem 3.2. By equating the jumps Ξ_i to zero one finds real orbits staying for all time close to the heteroclinic cycle Γ. Therefore the bifurcation equation for orbits staying close to Γ reads

$$\boldsymbol{\Xi} := (\Xi_i(\boldsymbol{\omega}, \mu))_{i\in\mathbb{Z}} = 0. \tag{3.1}$$

To begin the actual analysis, fix an inner product $\langle\cdot,\cdot\rangle$. Let, with respect to $\langle\cdot,\cdot\rangle$,

$$Y_i := \{f(q_i(0), 0)\}^\perp, \quad i = 1, 2.$$

With that we construct the cross-sections Σ_1 and Σ_2 as follows

$$\Sigma_i := q_i(0) + Y_i, \quad i = 1, 2.$$

Also, consistent with the standard theory, we define a subspace $Z \subset Y_1$:

$$Z := (T_{q_1(0)}W^s(p_1) + T_{q_1(0)}W^u(p_2))^\perp. \tag{3.2}$$

By construction we have $\dim Z = 2$. Assigned to $q_1(0)$ and $q_2(0)$ we consider the following orthogonal direct sum decomposition of $\mathbb{R}^n$:

$$\begin{aligned}\mathbb{R}^n &= \operatorname{span}\{f(q_1(0),0)\} \oplus W_1^+ \oplus W_1^- \oplus Z,\\ \mathbb{R}^n &= \operatorname{span}\{f(q_2(0),0)\} \oplus W_2^+ \oplus W_2^-,\end{aligned}$$

where $W_i^+ = T_{q_i(0)}W^s(p_i) \cap Y_i$ and $W_i^- = T_{q_i(0)}W^u(p_j) \cap Y_i$, $i = 1, 2$, $j \neq i$.

A further assumption, that simplifies the analysis, is that the local stable/unstable manifolds of the fixed points p_1, p_2 are flat; that is:

$$W^s_{loc}(p_i, \mu) \subset T_{p_i}W^s(p_i), \quad W^u_{loc}(p_i, \mu) \subset T_{p_i}W^u(p_i).$$

We can bring the local stable/unstable manifolds into this form by means of local transformations based around each of the fixed points p_i.

3.1 Splitting of the stable and unstable Manifolds

The first step of Lin's method is to study the splitting of the stable and unstable manifolds in Σ_i with respect to the parameter μ. Because of Hypothesis (H 1) the situation in Σ_2 is clear. For each μ which is sufficiently close to 0 there is exactly one point $q_{2,\mu} \in \Sigma_2 \cap W^u(p_1, \mu) \cap W^s(p_2, \mu)$ such that the orbit $\Gamma_{2,\mu}$ through $q_{2,\mu}$ is a 1-(1,2)-heteroclinic orbit. The corresponding solution with $q_2(\mu)(0) = q_{2,\mu}$ we denote by $q_2(\mu)(\cdot)$; we denote its restriction on $\mathbb{R}^\pm$ by $q_2^\pm(\mu)(\cdot)$.

In $\mathbb{R}^3$ also the situation in Σ_1 is rather simple. In this case both the stable manifold of p_1 and the unstable manifold of p_2 are one-dimensional, and the intersection with the two-dimensional hyperplane Σ_1 consists of single points in each case. The heteroclinic connection Γ_1 generally splits up under perturbation. Let $q^+_{1,\mu}$ and $q^-_{1,\mu}$ be determined by the 'first hit' of the stable manifold of p_1 and the unstable manifold of p_2, respectively. Of course $q^+_{1,\mu} - q^-_{1,\mu} \in Z$, recall that $Z = Y_1$ in this case. So, in a trivial way, for each μ we find a unique pair of orbits in the stable manifold of p_1 and the unstable manifold of p_2, respectively, such that the difference of their first hits in Σ_1 is in Z.

The main goal of this section is to show that this property persists in higher dimensions. More precisely we prove the following lemma:

Lemma 3.1. *For each sufficiently small μ there is a unique pair $(q_1^+(\mu)(\cdot), (q_1^-(\mu)(\cdot))$ of solutions of (1.1) such that:*

(i) $q_1^+(\mu)(0) \in \Sigma_1 \cap W^s(p_1, \mu)$, $\quad q_1^-(\mu)(0) \in \Sigma_1 \cap W^u(p_2, \mu)$,
(ii) $|q_1^+(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^+$ *and* $|q_1^-(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^-$,
(iii) $q_1^+(\mu)(0) - q_1^-(\mu)(0) \in Z$.

Proof. To investigate the splitting of Γ_1 we set

$$q_1^\pm(t) = q_1(t) + v^\pm(t).$$

This gives the following equations for $v^\pm$

$$\dot{v}^\pm = A(t)v^\pm + g(t, v^\pm, \mu), \tag{3.3}$$

which we consider on $\mathbb{R}^+$ or $\mathbb{R}^-$ respectively. Here $A(t) = D_x f(q_1(t), 0)$, and

$$g(t, v, \mu) = f(q_1(t) + v, \mu) - f(q_1(t), 0) - A(t)v.$$

With the above setting, we are looking for bounded solutions $v^{\pm}(\cdot)$ of (3.3). Hence $v^{\pm}(\cdot) \in C_b(\mathbb{R}^{\pm})$, the space of continuous bounded functions on $\mathbb{R}^{\pm}$ equipped with the norm $\|v^{\pm}\|_\infty := \sup_{t\in\mathbb{R}^{\pm}} \|v^{\pm}(t)\|$. If $\|v^{\pm}\|_\infty$ is close to zero, then by the theory of stable and unstable manifolds, $q^{\pm}(t) := q_1(t) + v^{\pm}(t)$ is in the desired stable/unstable manifold. Further, if $v^{\pm}(0) \in Y_1$ and $v^+(0) - v^-(0) \in Z$, then (i) - (iii) of the lemma holds true.

In what follows we show that (3.3) has unique solutions $v^{\pm}$ with the outlined boundary conditions. An important role is played by the properties of the linear nonautonomous equations

$$\dot{v} = A(t)v, \tag{3.4}$$

which we consider either on $\mathbb{R}^+$ or on $\mathbb{R}^-$. Denote by $\Phi(t, s)$ the transition matrix for (3.4). We have that $\lim_{t\to\infty} q_1(t) = p_1$, so then $\lim_{t\to\infty} A(t) = D_x f(p_1, 0)$. Similarly $\lim_{t\to-\infty} A(t) = D_x f(p_2, 0)$.

By the theory of exponential dichotomies [8], due to the fact that p_1, p_2 are hyperbolic, equation (3.4) has an exponential dichotomy on both $\mathbb{R}^+$ and $\mathbb{R}^-$. For $t \in \mathbb{R}_0^{\pm}$ we define projections $P^{\pm}(t)$ and $Q^{\pm}(t)$ as follows

$$\operatorname{im} P^+(0) = T_{q_1(0)} W^s(p_1), \quad \ker P^+(0) = W_1^- \oplus Z,$$

and

$$\ker P^-(0) = T_{q_1(0)} W^u(p_2) \quad \operatorname{im} P^-(0) = W_1^+ \oplus Z,$$

and for $t, s \geq 0$ or $t, s \leq 0$ let

$$P^{\pm}(t)\Phi(t, s) = \Phi(t, s)P^{\pm}(s).$$

Finally we define $Q^+ := I - P^+$ and $Q^- := I - P^-$.

Using the properties of exponential dichotomies we get that for each function $g(\cdot)$ bounded on $\mathbb{R}^+$:

$$\int_0^t \Phi(t, s)P^+(s)g(s)ds - \int_t^\infty \Phi(t, s)Q^+(s)g(s)ds$$

is well defined on $\mathbb{R}^+$ and solves $\dot{v} = A(t)v + g(t)$. Therefore we see that the solutions $v^+(\cdot)$ of (3.3) that are bounded on $\mathbb{R}^+$ solve the following fixed point problem in $C_b^0(\mathbb{R}^+, \mathbb{R}^n)$, and vice versa:

$$v^+(t) = \Phi(t, 0)\nu + \int_0^t \Phi(t, s)P^+(s)g(s, v^+(s), \mu)ds - \int_t^\infty \Phi(t, s)Q^+(s)g(s, v^+(s), \mu)ds,$$

where $\nu = P^+(0)v^+(0)$. In a similar way it can be shown that the solutions $v^-(\cdot)$ of (3.3) that are bounded on $\mathbb{R}^-$ solve the following fixed point problem in $C_b^0(\mathbb{R}^-, \mathbb{R}^n)$, and vice versa:

$$v^-(t) = \Phi(t, 0)\eta + \int_{-\infty}^t \Phi(t, s)P^-(s)g(s, v^-(s), \mu)ds - \int_t^0 \Phi(t, s)Q^-(s)g(s, v^-(s), \mu)ds,$$

where $\eta = Q^-(0)v^-(0)$. Both of these fixed point equations can be solved for $v^+ = v^+(\nu,\mu)$ and $v^- = v^-(\eta,\mu)$, near $(v^+,\nu,\mu) = (0,0,0)$ and $(v^-,\eta,\mu) = (0,0,0)$ in each case. We find that $v^\pm(0)$ can be written in the form

$$v^+(\nu,\mu)(0) = \nu + w^-(\nu,\mu) + z^+(\nu,\mu), \quad v^-(\eta,\mu)(0) = \eta + w^+(\eta,\mu) + z^-(\eta,\mu),$$

where $w^\pm \in W_1^\pm$ and $z^\pm \in Z$, and moreover $w^\pm(0,0) = 0$, $D_1 w^\pm(0,0) = 0$. In view of (iii) of the lemma we consider

$$\nu = w^+(\eta,\mu), \quad \eta = w^-(\nu,\mu).$$

This system can, near $(\nu,\eta,\mu) = (0,0,0)$, be solved for $\nu = \nu(\mu)$, $\eta = \eta(\mu)$.

The functions $q_1^+(\mu)(\cdot) := q_1(\cdot) + v^+(\nu(\mu),\mu)(\cdot)$ and $q_1^-(\mu)(\cdot) := q_1(\cdot) + v^-(\eta(\mu),\mu)(\cdot)$ are the desired solutions of (1.1). ∎

3.2 Construction of Lin orbits

The next step in the method is to search for orbits $X_{j,i}$, $j = 1,2$, $i \in \mathbb{Z}$, composing the Lin orbits $X = (X_i)_{i\in\mathbb{Z}}$ which we introduced in Section 1; more precisely $X_i = X_{1,i} \cup X_{2,i}$. Here $X_{j,i}$ is an orbit of the vector field starting in a point in Σ_j, staying close to Γ_j until it reaches a neighbourhood of p_j, and continuing close to Γ_{j+1} until it reaches Σ_{j+1}. By $x_{j,i}(\cdot)$ we denote solutions of (1.1) corresponding to the orbits $X_{j,i}$ with $x_{j,i}(0) \in \Sigma_j$ and $x_{j,i}(2\omega_{j,i}) \in \Sigma_{j+1}$. Actually $x_{1,i}(\cdot)$ is composed of solutions $x_{1,i}^+(\cdot)$ and $x_{2,i}^-(\cdot)$ which are defined on $[0,\omega_{1,i}]$ and $[-\omega_{1,i},0]$, respectively. Similarly $x_{2,i}(\cdot)$ is composed of solutions $x_{2,i}^+(\cdot)$ and $x_{1,i}^-(\cdot)$ which are defined on $[0,\omega_{2,i}]$ and $[-\omega_{2,i},0]$, respectively. This demands coupling conditions

$$x_{j,i}^+(\omega_{j,i}) = x_{j+1,i}^-(-\omega_{j,i}),\ j = 1,2, \tag{3.5}$$

and the jump conditions

$$\Xi_i := x_{1,i+1}^+(0) - x_{1,i}^-(0) \in Z, \quad x_{2,i}^+(0) = x_{2,i}^-(0), \tag{3.6}$$

for $i \in \mathbb{Z}$ in each case. Figure 4 visualises this situation.

It is a specific feature of T-points that there is no jump in Σ_2, $x_{2,i}^+(0) - x_{2,i}^-(0) = 0$. However, large parts of the procedure are similar to the exposition in [34], [21], [26] and [37]. In our presentation we confine to explain only those parts of Lin's method in more detail which differ from the standard scheme.

To begin, we look for solutions of the form

$$x_{j,i}^\pm(t) = q_j^\pm(\mu)(t) + v_{j,i}^\pm(t). \tag{3.7}$$

Then

$$\dot v_{j,i}^\pm = A_j^\pm(t,\mu)v_{j,i}^\pm + g_j^\pm(t, v_{j,i}^\pm,\mu) \tag{3.8}$$

where $A_j^\pm(t,\mu) = D_x f(q_j^\pm(\mu)(t),\mu)$, and

$$g_j^\pm(t,v,\mu) = f(q_j^\pm(\mu)(t) + v,\mu) - f(q_j^\pm(\mu)(t),\mu) - A_j^\pm(t,\mu)v.$$

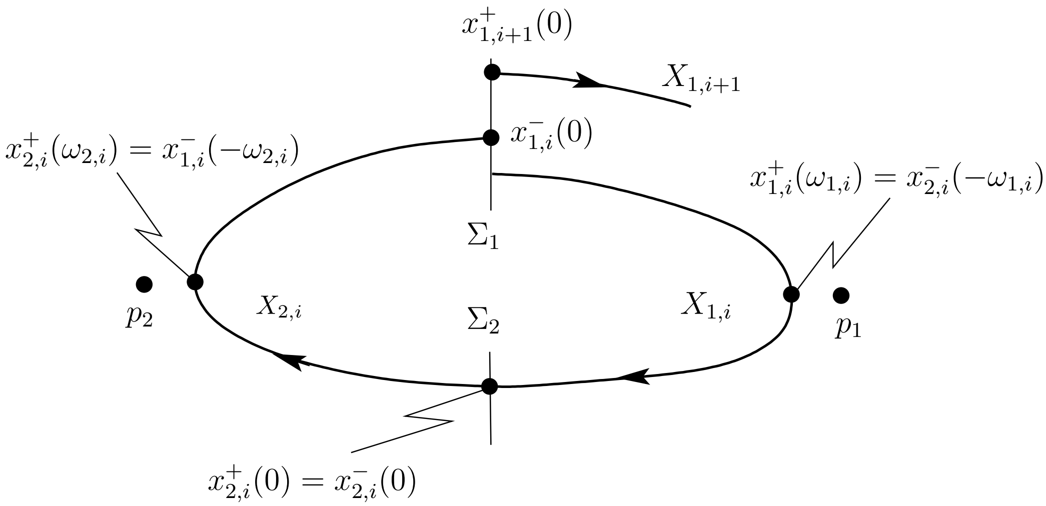


Figure 4: Ingredients of Lin orbits; $X_i = X_{1,i} \cup X_{2,i}$.

Let $\Phi^{\pm}_j(\mu, t, s)$ be the transition matrix for the equation

$$\dot{v} = A^{\pm}_j(t, \mu)v.$$

As before, these equations have an exponential dichotomy on $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively, with corresponding projections $P^+_j(\mu, t)$, $Q^+_j(\mu, t) = I - P^+_j(\mu, t)$, and $Q^-_j(\mu, t)$, $P^-_j(\mu, t) = I - Q^-_j(\mu, t)$; therefore we have for $j = 1, 2$

$$\operatorname{im} P^+_j(\mu, 0) = T_{q^+_j(\mu)(0)} W^s(p_j), \quad \operatorname{im} Q^-_j(\mu, 0) = T_{q^-_j(\mu)(0)} W^u(p_{j+1}).$$

Moreover we commit to

$$\ker P^+_1(\mu, 0) = W^-_1 \oplus Z, \quad \ker P^+_2(\mu, 0) = W^-_2$$

and

$$\ker Q^-_1(\mu, 0) = W^+_1 \oplus Z, \quad \ker Q^-_2(\mu, 0) = W^+_2.$$

For $\mu = 0$ the projections $P^{\pm}_1$ coincide with the projections $P^{\pm}$ introduced in the proof of Lemma 3.1.

Our main existence result in this respect is the following:

Theorem 3.2. *Consider the system (1.1) in the present setting. Then there are constants c, Ω such that for each μ with $|\mu| < c$ and each $\boldsymbol{\omega}$ with $\omega_{j,i} > \Omega$, $j = 1, 2$, $i \in \mathbb{Z}$, there is a unique sequence of solutions $x^{\pm}_{j,i}(\boldsymbol{\omega}, \mu)(\cdot)$, $j = 1, 2$, $i \in \mathbb{Z}$, of (1.1) satisfying the coupling condition (3.5) and the jump condition (3.6).*

In other words this theorem says that for each μ and for each sequence $\boldsymbol{\omega}$ there exists a unique Lin orbit $X(\boldsymbol{\omega}, \mu)$.

Next we explain the steps leading to the proof of this theorem. We consider (instead of the original equation (1.1) with boundary conditions (3.5) and (3.6)) the differential equation (3.8) with corresponding boundary conditions (according to (3.7))

$$v^+_{j,i}(\omega_{j,i}) - v^-_{j+1,i}(-\omega_{j,i}) = q^-_{j+1}(-\omega_{j,i}) - q^+_j(\omega_{j,i}), \; j = 1, 2, \tag{3.9}$$

$$v^{\pm}_{j,i}(0) \in Y_j, \quad v^{+}_{1,i+1}(0) - v^{-}_{1,i}(0) \in Z, \quad v^{+}_{2,i}(0) = v^{-}_{2,i}(0), \tag{3.10}$$

for $i \in \mathbb{Z}$ in each case. We fix a sequence $\boldsymbol{\omega}$ with sufficiently large $\omega_{j,i}$. Assigned to this sequence we denote by $\mathcal{V}_{\boldsymbol{\omega}}$ the space of all sequences $\mathbf{v} := (v^{+}_{1,i}, v^{-}_{2,i}, v^{+}_{2,i}, v^{-}_{1,i})_{i\in\mathbb{Z}}$, where $v^{+}_{j,i} \in C([0, \omega_{j,i}], \mathbb{R}^n)$ and $v^{-}_{j,i} \in C([-\omega_{j+1,i}, 0], \mathbb{R}^n)$.

In a first step we consider a "linearised" equation

$$\dot{v}^{\pm}_{j,i} = A^{\pm}_{j}(t, \mu)v^{\pm}_{j,i} + h^{\pm}_{j,i}(t), \tag{3.11}$$

with continuous $h^{\pm}_{j,i}(\cdot)$, and with boundary conditions (3.10) and

$$Q^{+}_{j}(\mu, \omega_{j,i})v^{+}_{j,i}(\omega_{j,i}) = a^{+}_{j,i}, \quad P^{-}_{j}(\mu, -\omega_{j+1,i})v^{-}_{j,i}(-\omega_{j+1,i}) = a^{-}_{j+1,i}, \tag{3.12}$$

for any given $a^{+}_{j,i} \in \operatorname{Im} Q^{+}_{j}(\mu, \omega_{j,i})$, $a^{-}_{j+1,i} \in \operatorname{Im} P^{-}_{j}(\mu, -\omega_{j+1,i})$.

Lemma 3.3. *The boundary value problem ((3.11), (3.10), (3.12)) has a unique solution $\mathbf{v}_{\boldsymbol{\omega}} \in \mathcal{V}_{\boldsymbol{\omega}}$.*

Proof. Solutions of (3.11) can be written in the form

$$v^{\pm}_{j,i}(t) = \Phi^{\pm}_{j}(\mu, t, 0)v^{\pm}_{j,i}(0) + \int_0^t \Phi^{\pm}_{j}(\mu, t, s)h^{\pm}_{j,i}(s)ds. \tag{3.13}$$

Incorporating the boundary condition (3.12) gives for $i \in \mathbb{Z}$

$$\begin{aligned} Q^{+}_{j}(\mu, 0)v^{+}_{j,i}(0) &= \Phi^{+}_{j}(\mu, 0, \omega_{j,i})a^{+}_{j,i} - \int_0^{\omega_{j,i}} \Phi^{+}_{j}(\mu, 0, s)Q^{+}_{j}(\mu, s)h^{+}_{j,i}(s)ds, \\ P^{-}_{j}(\mu, 0)v^{-}_{j,i}(0) &= \Phi^{-}_{j}(\mu, 0, -\omega_{j+1,i})a^{-}_{j+1,i} + \int_{-\omega_{j+1,i}}^{0} \Phi^{-}_{j}(\mu, 0, s)P^{-}_{j}(\mu, s)h^{-}_{j,i}(s)ds, \end{aligned} \tag{3.14}$$

Because of (3.10) we have

$$v^{+}_{1,i+1}(0) = w^{+}_{1,i} + w^{-}_{1,i} + z^{+}_{i},\ v^{-}_{1,i}(0) = w^{+}_{1,i} + w^{-}_{1,i} + z^{-}_{i} \quad \text{and} \quad v^{\pm}_{2,i}(0) = w^{+}_{2,i} + w^{-}_{2,i},$$

where $w^{\pm}_{j,i} \in W^{\pm}_{j}$ and $z^{\pm}_{i} \in Z$. For fixed j the left-hand sides of these equations are decoupled over i. Moreover, for each single $i \in \mathbb{Z}$ the left-hand side of (3.14) can be seen as a linear mapping

$$W^{+}_{1} \times W^{-}_{1} \times Z \times Z \to (W^{-}_{1} \oplus Z) \times (W^{+}_{1} \oplus Z) \quad \text{or} \quad W^{+}_{2} \times W^{-}_{2} \to W^{-}_{2} \times W^{+}_{2}$$

depending on $j = 1$ or $j = 2$, respectively. These mappings are invertible. So equation (3.14) can be solved for

$$(w^{\pm}_{1,i}, z^{\pm}_{i}) = (w^{\pm}_{1,i}, z^{\pm}_{i})(\mu, h^{+}_{1,i+1}, h^{-}_{2,i}, a^{+}_{1,i+1}, a^{-}_{2,i}), \quad w^{\pm}_{2,i} = w^{\pm}_{2,i}(\mu, h^{-}_{1,i}, h^{+}_{2,i}, a^{-}_{1,i}, a^{+}_{2,i}).$$

With (3.13) we get eventually the lemma. ∎

In the next step we "replace" the boundary conditions (3.12) by

$$v^+_{j,i}(\omega_{j,i}) - v^-_{j+1,i}(-\omega_{j,i}) = d_{j,i}, \quad d_{j,i} \in \mathbb{R}^n, \quad j = 1, 2, \tag{3.15}$$

and consider the boundary value problem ((3.11), (3.10), (3.15)).

Lemma 3.4. *The boundary value problem ((3.11), (3.10), (3.15)) has a unique solution $\hat{\mathbf{v}}_{\boldsymbol{\omega}} \in \mathcal{V}_{\boldsymbol{\omega}}$.*

Proof. First we claim that for sufficiently large $\omega > 0$ and sufficiently small μ it holds that $\operatorname{im} Q^+_j(\mu, \omega) \oplus \operatorname{im} P^-_{j+1}(\mu, -\omega) = \mathbb{R}^n$, $j = 1, 2$. This is due to asymptotic behaviour (as $\omega \to \infty$) of the involved projections and due to the hyperbolicity of the equilibria p_1 and p_2; we refer to [37] for more details.

The rest of the proof proceeds along the lines of the corresponding assertions in [34] or [21]: For each given sequence $(d_{j,i})$ one proves the existence of sequences $(a^+_{j,i})$ and $(a^-_{j,i})$ such that the corresponding solutions of the boundary value problem ((3.11), (3.10), (3.12)) solve the boundary value problem ((3.11), (3.10), (3.15)). ∎

Altogether we find $\hat{\mathbf{v}}_{\boldsymbol{\omega}} = \hat{\mathbf{v}}_{\boldsymbol{\omega}}(\mu, \mathbf{h}, \mathbf{d})$, where $\mathbf{h} := (h^+_{1,i}, h^-_{2,i}, h^+_{2,i}, h^-_{1,i})_{i\in\mathbb{Z}}$ and $\mathbf{d} := (d_{1,i}, d_{2,i})_{i\in\mathbb{Z}}$. It is worth to mention that actually each entry of $\hat{\mathbf{v}}_{\boldsymbol{\omega}}$ depends only on a finite part of the sequences $\mathbf{h}$ and $\mathbf{d}$.

A coupling of the obtained solutions can be achieved by setting

$$d_{j,i} = d_{\boldsymbol{\omega}_{j,i}}(\mu) := q^-_{j+1}(\mu)(-\omega_{j,i}) - q^+_j(\mu)(\omega_{j,i}),\ j = 1, 2. \tag{3.16}$$

Now, the nonlinear boundary value problem ((3.8),(3.9),(3.10)) is equivalent to the following fixed point equation in $\mathcal{V}_{\boldsymbol{\omega}}$:

$$\mathbf{v} = \hat{\mathbf{v}}_{\boldsymbol{\omega}}(\mu, \mathcal{G}(\mathbf{v}, \mu), \mathbf{d}_{\boldsymbol{\omega}}(\mu)), \tag{3.17}$$

where

$$\begin{array}{rccl} \mathcal{G}: & \mathcal{V}_{\boldsymbol{\omega}} \times \mathbb{R} & \to & \mathcal{V}_{\boldsymbol{\omega}} \\ & (\mathbf{v}, \mu) & \mapsto & (h^+_{1,i}, h^-_{2,i}, h^+_{2,i}, h^-_{1,i})_{i\in\mathbb{Z}}, \quad h^\pm_{j,i}(\cdot) := g^\pm_j(\cdot, v^\pm_{j,i}(\cdot), \mu). \end{array}$$

The following Lemma provides solutions to the fixed point problem (3.17).

Lemma 3.5. *For fixed $\boldsymbol{\omega}$, $\omega_{j,i}$ sufficiently large, and $|\mu|$ sufficiently small, the fixed point problem (3.17) has a unique solution $\bar{\mathbf{v}}_{\boldsymbol{\omega}}$ in a sufficiently small neighbourhood of $0 \in \mathcal{V}_{\boldsymbol{\omega}}$. Moreover, the mapping $\mu \mapsto \bar{\mathbf{v}}_{\boldsymbol{\omega}}(\mu)$ is smooth.*

Note that the necessary considerations have been done for fixed $\boldsymbol{\omega}$ in spaces $\mathcal{V}_{\boldsymbol{\omega}}$. Define

$$\bar{\mathbf{v}}(\boldsymbol{\omega}, \mu) = (\bar{v}^+_{1,i}, \bar{v}^-_{2,i}, \bar{v}^+_{2,i}, \bar{v}^-_{1,i})_{i\in\mathbb{Z}} := \bar{\mathbf{v}}_{\boldsymbol{\omega}}(\mu), \quad \bar{v}^\pm_{j,i} = \bar{v}^\pm_{j,i}(\boldsymbol{\omega}, \mu). \tag{3.18}$$

Lemma 3.6. *The mappings $l^\infty_{\mathbb{R}^2} \times \mathbb{R} \to l^\infty_{\mathbb{R}^n}$, $(\boldsymbol{\omega}, \mu) \mapsto \bar{v}^\pm_{j,i}(\boldsymbol{\omega}, \mu)(0)$ are smooth.*

For both Lemma 3.5 and Lemma 3.6 the proof of the corresponding statements in [34] or [21] can easily be adapted for the present situation.

4 The Bifurcation Equations

For given $\boldsymbol{\omega}$ and μ we find a unique Lin orbit $X(\boldsymbol{\omega},\mu)$, see Theorem 3.2, and a corresponding sequence $(\Xi_i(\boldsymbol{\omega},\mu))_{i\in\mathbb{Z}}$ of jumps, see (3.6). The Lin orbit becomes a real orbit if all jumps are equal to zero; this leads to the set of bifurcation equations

$$\Xi_i(\boldsymbol{\omega},\mu)=0,\quad i\in\mathbb{Z}, \tag{4.1}$$

where $\boldsymbol{\Xi}:=(\Xi_i)_{i\in\mathbb{Z}}$ can be read as a mapping

$$\boldsymbol{\Xi}:(l^\infty\times l^\infty)\times\mathbb{R}^2\to l^\infty\times l^\infty,$$

bearing in mind that $\boldsymbol{\omega}=(\omega_{1,i},\omega_{2,i})_{i\in\mathbb{Z}}$ and $\dim Z=2$. As in [34] or [21] we find (as a consequence of Lemma 3.6):

Lemma 4.1. $\boldsymbol{\Xi}$ *depends smoothly on* $\boldsymbol{\omega}$ *and* μ. ∎

By solving the bifurcation equations (4.1) we find all kinds of orbits staying close to the primary heteroclinic cycle Γ. For instance, a k-periodic orbit can be seen as a k-periodic Lin orbit where all jumps are zero. A Lin orbit is k-periodic if and only if $\boldsymbol{\omega}$ is k-periodic. So there arise only k different pairs $(\omega_{1,i},\omega_{2,i})$, $i\in\{1,\dots,k\}$. Therefore the bifurcation equation for detecting k-periodic orbits consists only of k equations

$$\Xi_i(\boldsymbol{\omega},\mu)=0,\quad i\in\{1,\dots,k\}, \tag{4.2}$$

and the corresponding $\boldsymbol{\Xi}$ can be read as a mapping $\mathbb{R}^{2k}\times\mathbb{R}^2\to\mathbb{R}^{2k}$.

Further it is worth to mention that our considerations remain true if some $\omega_{j,i}$ are formally put to infinity. If in addition the sequence $\boldsymbol{\omega}$ is periodic then this leads to the bifurcation equation for k-homoclinic orbits or heteroclinic cycles which are composed of k-heteroclinic connections. for instance a k-homoclinic orbit to p_1 corresponds to a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1}=\infty$.

From (3.7) we see that the jumps Ξ_i have the form

$$\Xi_i(\boldsymbol{\omega},\mu):=\xi^\infty(\mu)+\xi_i(\boldsymbol{\omega},\mu),$$

where

$$\xi^\infty(\mu):=q_1^+(\mu)(0)-q_1^-(\mu)(0),$$

and

$$\xi_i(\boldsymbol{\omega},\mu)=\bar{v}^+_{1,i+1}(\boldsymbol{\omega},\mu)(0)-\bar{v}^-_{1,i}(\boldsymbol{\omega},\mu)(0). \tag{4.3}$$

First we consider $\xi^\infty(\mu)$. By introducing appropriate coordinates, $\xi^\infty(\cdot)$ can be seen as a mapping

$$\xi^\infty(\cdot):\mathbb{R}^2\to\mathbb{R}^2.$$

Of course $\xi^\infty(0)=0$, which represents that the unstable manifold of p_2 and the stable manifold of p_1 intersect (in the primary heteroclinic orbit Γ_1). Due to Hypothesis (H 2) we may assume

$$\xi^\infty(\mu)=\mu.$$

During the further procedure we use the following representation of $\xi_i(\boldsymbol{\omega},\mu)$:

$$\xi_i(\boldsymbol{\omega},\mu) = \sum_{j=1}^{2} \langle \psi_j, \xi_i(\boldsymbol{\omega},\mu)\rangle \psi_j, \tag{4.4}$$

where $\{\psi_1, \psi_2\}$ is a orthonormal basis of Z. We can write, cf. (4.4), (4.3) and (3.2) and the definitions of Q_1^+ and P_1^-

$$\xi_i(\boldsymbol{\omega},\mu) = \sum_{j=1}^{2} \Big(\langle \psi_j, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle - \langle \psi_j, P_1^-(\mu,0)\bar{v}_{1,i}^-(\boldsymbol{\omega},\mu)(0)\rangle \Big)\psi_j. \tag{4.5}$$

4.1 Real leading eigenvalues

Before we turn to the actual estimates of the bifurcation equation, we consider nonautonomous perturbations of linear equations. Based on existence of exponential dichotomies we consider the asymptotics of solutions in the strong stable eigenspace of those equations.

Lemma 4.2. *Let both $A : \mathbb{R}^2 \to \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ and, for all t, $B(t,\cdot) : \mathbb{R}^2 \to \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ be smooth, and assume further:*

(i) *$x = 0$ is a hyperbolic equilibrium of $\dot{x} = A(0)x$.*

(ii) *The spectrum $\sigma(A(\mu))$ reads: $\sigma(A(\mu)) = \sigma^{sss} \cup \{\lambda^{ss}(\mu)\} \cup \{\lambda^s(\mu)\} \cup \sigma^u$, where there exist constants α^{sss}, α^{ss}, α^s and α^u, such that $\lambda < \alpha^{sss} < \lambda^{ss}(\mu) < \alpha^{ss} < \lambda^s(\mu) < \alpha^s < 0 < \alpha^u < \bar{\lambda}$, for all $\lambda \in \sigma^{sss}$ and all $\bar{\lambda} \in \sigma^u$.*

(iii) *The leading stable and strong stable eigenvalues λ^s and λ^{ss} of $A(0)$ are real and simple.*

(iv) *There is a $\beta > 0$ such that $|B(t,\mu)| < e^{-t\beta}$, and $\alpha^s\beta < \lambda^s(\mu)$ and $\alpha^{ss}\beta < \lambda^{ss}(\mu)$.*

Let $e^{ss}(\mu)$ and $e^s(\mu)$ be the eigenvectors of λ^{ss} and λ^s, respectively. Further, let E_t^{ss} and E_t^s be the strong stable and stable eigenspaces, respectively, of $\dot{x} = \big(A(\mu)+B(t,\mu)\big)x$ at time t. Under these assumptions the following is true for solutions $x(t)$ of $\dot{x} = \big(A(\mu)+B(t,\mu)\big)x$:

1. *For $x(t) \in E_t^s$ there exists an $\eta^s(x(0),\mu) \in \operatorname{span}\{e^s(\mu)\}$ such that*
$$x(t) = e^{\lambda^s(\mu)t}\eta^s(x(0),\mu) + O\big(e^{\max\{\alpha^{ss},\alpha^s+\beta\}t}\big).$$
2. *For $x(t) \in E_t^{ss}$ there exists an $\eta^{ss}(x(0),\mu) \in \operatorname{span}\{e^{ss}(\mu)\}$ such that*
$$x(t) = e^{\lambda^{ss}(\mu)t}\eta^{ss}(x(0),\mu) + O\big(e^{\max\{\alpha^{sss},\alpha^{ss}+\beta\}t}\big).$$

Note that $\dot{x} = \big(A(\mu)+B(t,\mu)\big)x$ has exponential dichotomies (on $\mathbb{R}^+$) with respect to splittings of the spectrum $\sigma(A(\mu))$ at α^s or α^{ss}, respectively. By means of these exponential dichotomies the addressed stable and strong stable eigenspaces of this equation can be defined. We further remark that variational equations of (1.1) along solutions within the stable manifolds of the equilibria p_1 and p_2 can be written in a form satisfying the assumptions of the above lemma. The same holds true for the corresponding adjoint variational equation. Of course these statements remain true also for variational equations (and their adjoints) along solutions within the unstable manifolds - all considerations merely have to be made for $t \to -\infty$.

The statement 1. of the lemma has been proven in [34] and [21]. The statement 2. generalises the corresponding lemmas in [34] or [21], respectively. However, the proof is similar to that of the first statement.

Further we specify these considerations to (adjoint) variational equations along a solution in the (un)stable manifold of the equilibrum p_2 of (1.1). Throughout we denote by A^* the adjoint of an operator A.

Lemma 4.3. *Let q^+ or q^- be a solutions within the stable or unstable manifold of p_2 of (1.1). Let E_t^{ss} denote the strong stable subspace of $\dot{x} = D_1 f(q^+(t), \mu)$ at time t (for $t \to \infty$). Similarly, let $E_{\perp,t}^{ss}$ denote the strong stable subspace of $\dot{x} = -D_1 f(q^+(t), \mu)^*$ at time t for $t \to -\infty$.*

(i) $E_t^{ss} = T_{q^+(t)} W^{ss}(q^+(t))$.

(ii) $E_{\perp,t}^{ss} = \big(T_{q^-(t)} W^{eu}(p_2)\big)^{\perp}$.

In [28, 33] a corresponding statement for stable subspaces has been proven. The proof of the above lemma for the strong stable subspaces runs along the same lines.

In the representation (4.4) we used an orthonormal basis $\{\psi_1, \psi_2\}$ of Z. For our further considerations we specify a corresponding scalar product $\langle \cdot, \cdot \rangle$ as follows: With the notations introduced at the beginning of Section 3 we write, cf. also (H 4):

$$T_{q_1(0)} W^{eu}(p_2) \cap Y_1 = W_1^- \oplus \operatorname{span}\{\psi_2\}, \quad T_{q_1(0)} W^{es}(p_1) \cap Y_1 = W_1^+ \oplus \operatorname{span}\{\psi_1\}.$$

Within this setting we choose the scalar product $\langle \cdot, \cdot \rangle$ such that

$$\psi_1 \perp \psi_2, \quad W_1^+ \perp W_1^- \quad \text{and} \quad \psi_i \perp W_1^{\pm},\ i = 1, 2.$$

With that we obtain the following representation of $\xi_i(\boldsymbol{\omega}, \mu)$;

Lemma 4.4. *Assume the case (RR) and Hypotheses (H 1)–(H 7). Then the jump $\xi_i(\boldsymbol{\omega}, \mu)$ can be written in the form*

$$\begin{aligned}\xi_i(\boldsymbol{\omega}, \mu) = {} & \big(c_{11}(\mu) e^{-2\lambda_1^u \omega_{1,i+1}} - c_{21}(\mu) e^{2\lambda_2^{ss}\omega_{2,i}} e^{2\lambda_1^s \omega_{1,i}} + \mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu)\big)\, \psi_1 \\ & + \big(c_{12}(\mu) e^{-2\lambda_1^{uu}\omega_{1,i+1}} e^{-2\lambda_2^u \omega_{2,i+1}} - c_{22}(\mu) e^{2\lambda_2^s \omega_{2,i}} + \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu)\big)\, \psi_2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu) &= o(e^{-2\lambda_1^u \omega_{1,i+1}}) + o(e^{2\lambda_2^{ss}\omega_{2,i}} e^{2\lambda_1^s \omega_{1,i}}) \\ \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu) &= o(e^{-2\lambda_1^{uu}\omega_{1,i+1}} e^{-2\lambda_2^u \omega_{2,i+1}}) + o(e^{2\lambda_2^s \omega_{2,i}}).\end{aligned}$$

Proof. Before starting the actual proof we introduce some assumptions which can always be realized by performing appropriate transformations. In this context we recall that, cf. (H 1) and Section 3.2, that $q_2^{\pm}(\mu) = q_2(\mu)$. For our analysis we assume that we have carried out appropriate transformations so that:

(A 1) $\quad q_2(\mu)(\omega) \in W^{eu}_{loc}(p_2)$, if $\omega \gg 1$.

(A 2) $\quad W^{eu}_{loc}(p_2) = T_{p_2}W^{eu}(p_2)$.

(A 3) $\quad T_{q_2(\mu)(\omega)}W^u(p_1) \subset W^{eu}(p_2)$.

To justify the Assumptions (A 1)–(A 3) we consider the following: Recall that the extended-unstable manifold of p_2 is not unique. Due to the λ-lemma, $W^{eu}_{loc}(p_2)$ can be chosen such that $\{\phi^\omega\big(W^u(p_1)\cap\Sigma_2\big), \omega \gg 1\} \subset W^{eu}_{loc}(p_2)$, where $\{\phi^t\}$ is the flow of (1.1). Hence Assumption (A 1) is fulfilled. The remaining assumptions can be achieved by means of a transformation flattening $W^{eu}_{loc}(p_2)$.

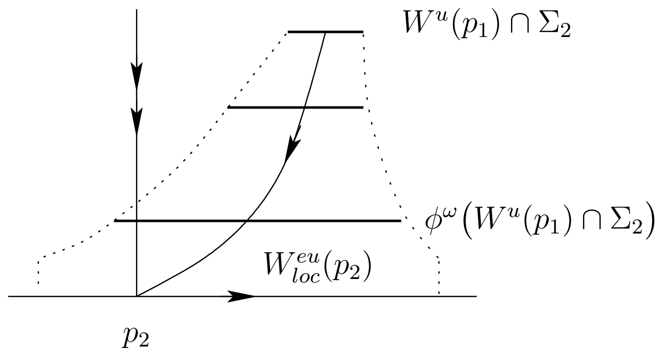


Figure 5: The manifold $W^{eu}(p_2)$ “generated” by $\phi^\omega\big(W^u(p_1)\cap\Sigma_2\big)$.

Further we assume

(A 4) $\quad T_{q_2(\mu)(0)}W^s(p_2)\cap\Sigma_2 = W_2^+$.

The situation required in Assumption (A 4) can always be achieved by appropriate transformations, cf. [34] or [21].

For the actual proof we proceed from the representation ξ_i given in (4.4). We confine ourselves to consider the jumps in ψ_1-direction. Recall that $\psi_1 \perp W^s(p_1)$, $\psi_1 \perp W^{eu}(p_2)$, and $\psi_1 \perp \psi_2$. In accordance with (4.5) we find

$$\langle\psi_1, \xi_i(\boldsymbol{\omega},\mu)\rangle = \langle\psi_1, Q_1^+(\mu,0)\bar{v}^+_{1,i+1}(\boldsymbol{\omega},\mu)(0)\rangle - \langle\psi_1, P_1^-(\mu,0)\bar{v}^-_{1,i}(\boldsymbol{\omega},\mu)(0)\rangle.$$

First we consider the term $\boldsymbol{\langle\psi_1, Q_1^+(\mu,0)\bar{v}^+_{1,i+1}(\omega,\mu)(0)\rangle}$:

Let $\Psi_1^+(\mu,\cdot,\cdot)$ be the transition matrix of the adjoint of the variational equation along q_1^+. Making use of (H 3)–(H 5) we find, in analogy to the considerations in [21, 34], that there is a quantity c_{11} depending on μ such that

$$\begin{aligned}\langle\psi_1, Q_1^+(\mu,0)\bar{v}^+_{1,i+1}(\boldsymbol{\omega},\mu)(0)\rangle &= \langle\Psi_1^+(\mu,\omega_{1,i+1},0)Q_1^{+*}(\mu,0)\psi_1, \tilde{Q}_1(\mu,\omega_{1,i+1})q_2^-(\mu)(-\omega_{1,i+1})\rangle \\ &\quad + o(e^{-2\lambda_1^u\omega_{1,i+1}}) \\ &= c_{11}(\mu)e^{-2\lambda_1^u\omega_{1,i+1}} + o(e^{-2\lambda_1^u\omega_{1,i+1}}). \end{aligned} \tag{4.6}$$

Here $\tilde{Q}_1(\mu,t)$ is the projection which projects on $\operatorname{im} Q_1^+(\mu,t)$ along $\operatorname{im} P_2^-(\mu,-t)$. For that we exploit that $\psi_1 \perp W^{eu}(p_2)$ together with Hypothesis (H 4).

Further, due to Hypothesis (H 4)

$$\operatorname{sign} c_{11}(\mu) = \operatorname{sign} \langle \Psi_1^+(\mu, \omega_{1,i+1}, 0) Q_1^{+*}(\mu, 0)\psi_1, \tilde{Q}_1(\mu, \omega_{1,i+1}) q_2^-(\mu)(-\omega_{1,i+1})\rangle \neq 0.$$

Indeed, roughly speaking, $\tilde{Q}_1(\mu, \omega_{1,i+1}) q_2^-(\mu)(-\omega_{1,i+1})$ points towards the eigenspace of λ_1^u, and $\Psi_1^+(\mu, \omega_{1,i+1}, 0) Q_1^{+*}(\mu, 0)\psi_1$ is perpendicular to the sum of the stable and strong unstable eigenspace. Further, both terms tend to zero exponentially fast with an exponential rate $-\lambda_1^u$.

According to (4.5) we next consider $\langle \boldsymbol{\psi_1}, \boldsymbol{P_1^-(\mu, 0)\bar{v}_{1,i}^-(\omega, \mu)(0)}\rangle$:

In principle we can proceed in a similar way as above, resulting in the fact that the corresponding $c_{21}(\mu)$ vanishes. This means that the corresponding "leading order term" in the representation of the jump vanishes. Hence we require a more sophisticated estimate.

Equations (3.12) and (3.14) yield

$$P_1^-(\mu, 0)\bar{v}_{1,i}^-(\boldsymbol{\omega}, \mu)(0) = P_1^-(\mu, 0)\Phi_1^-(\mu, 0, -\omega_{2,i}) a_{2,i}^- + \int_{-\omega_{2,i}}^{0} \Phi_1^-(\mu, 0, s) P_1^-(\mu, s) h_{1,i}^-(s) ds,$$

Therefore

$$\begin{aligned} \langle \psi_1, P_1^-(\mu, 0)\bar{v}_{1,i}^-(\boldsymbol{\omega}, \mu)(0)\rangle = &\langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, a_{2,i}^-\rangle \\ &+ \langle \psi_1, \int_{-\omega_{2,i}}^{0} \Phi_1^-(\mu, 0, s) P_1^-(\mu, s) h_{1,i}^-(s) ds\rangle, \end{aligned} \tag{4.7}$$

where $\Psi_1^-(\mu, \cdot, \cdot)$ denotes the transition matrix of the adjoint of the variational equation along q_1^-. Next we express $a_{2,i}^-$ in terms of v_2^+ or v_2^-, respectively. Due to (3.12), (3.14) and (3.16) we find

$$\begin{aligned} a_{2,i}^- &= P_1^-(\mu, -\omega_{2,i})\bar{v}_{1,i}^-(-\omega_{2,i}) = P_1^-(\mu, -\omega_{2,i})\big(\bar{v}_{2,i}^+(\omega_{2,i}) - d_{2,i}\big) \\ &= P_1^-(\mu, -\omega_{2,i})\big(\bar{v}_{2,i}^+(\omega_{2,i}) + q_2(\mu)(\omega_{2,i}) - q_1^-(\mu)(-\omega_{2,i})\big). \end{aligned}$$

Inserting in the first scalar product on the right-hand side in (4.7) and taking into consideration that $P_1^{-*}(\mu, -\omega_{2,i})$ commutes with $\Psi_1^-(\mu, -\omega_{2,i}, 0)$ yields:

$$\begin{aligned} \langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, a_{2,i}^-\rangle = &\langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, \bar{v}_{2,i}^+(\omega_{2,i})\rangle \\ &+ \langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, q_2(\mu)(\omega_{2,i})\rangle \\ &- \langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, q_1^-(\mu)(-\omega_{2,i})\rangle \end{aligned} \tag{4.8}$$

Let $\omega_{2,i}$ be sufficiently large, then $q_1^-(\mu)(-\omega_{2,i}) \in W_{loc}^{eu}(p_2)$, and due to Assumption (A 1) we also have $q_2(\mu)(\omega_{2,i}) \in W_{loc}^{eu}(p_2)$. Further, by construction $\Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1$ is perpendicular to $T_{q_1^-(-\omega_{2,i})} W_{loc}^{eu}(p_2)$. Hence, by Assumption (A 2) the last two scalar products in (4.8) vanish, and this equation reduces to

$$\langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, a_{2,i}^-\rangle = \langle \Psi_1^-(\mu, -\omega_{2,i}, 0) P_1^{-*}(\mu, 0)\psi_1, \bar{v}_{2,i}^+(\omega_{2,i})\rangle$$

Consider

$$\bar{v}_{2,i}^{+}(\omega_{2,i}) = (id - Q_2^-(\mu,\omega_{2,i}))\bar{v}_{2,i}^{+}(\omega_{2,i}) + Q_2^-(\mu,\omega_{2,i})\bar{v}_{2,i}^{+}(\omega_{2,i}).$$

Note that $Q_2^-(\mu,\omega_{2,i})\bar{v}_{2,i}^{+}(\omega_{2,i}) \in T_{q_2(\omega_{2,i})}W^u(p_1)$. Hence, due to Assumption (A 3)

$$\langle \Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1, Q_2^-(\mu,\omega_{2,i})\bar{v}_{2,i}^{+}(\omega_{2,i})\rangle = 0,$$

and (4.8) reduces further to

$$\langle \Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1, a_{2,i}^-\rangle = \langle \Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1, (id - Q_2^-(\mu,\omega_{2,i}))\bar{v}_{2,i}^{+}(\omega_{2,i})\rangle.$$

Consider

$$\begin{aligned}(id - Q_2^-(\mu,\omega_{2,i}))\bar{v}_{2,i}^{+}(\omega_{2,i}) = {} & \Phi_2^+(\mu,\omega_{2,i},0)(id - Q_2^-(\mu,0))\bar{v}_{2,i}^{+}(0) \\ & + \int_0^{\omega_{2,i}} \Phi_2^+(\mu,\omega_{2,i},s)(id - Q_2^-(\mu,s))h_{2,i}^+(s)ds.\end{aligned}$$

By the definition of Q_2^-, cf. Section 3.2, and Assumption (A 4) we have $(id - Q_2^-(\mu,0))\bar{v}_{2,i}^{+}(0) \in T_{q_2(\mu)(0)}W^s(p_2)$. (More precisely $(id - Q_2^-(\mu,0))\bar{v}_{2,i}^{+}(0) \in W_2^+$.) Note that $T_{q_2(\mu)(0)}W^s(p_2)$ is the stable eigenspace at $t = 0$ of the variational equation along q_2 on $\mathbb{R}^+$. Let $E_{t=0}^{ss}$ denote the corresponding strong stable eigenspace. Hence, cf. Figure 6

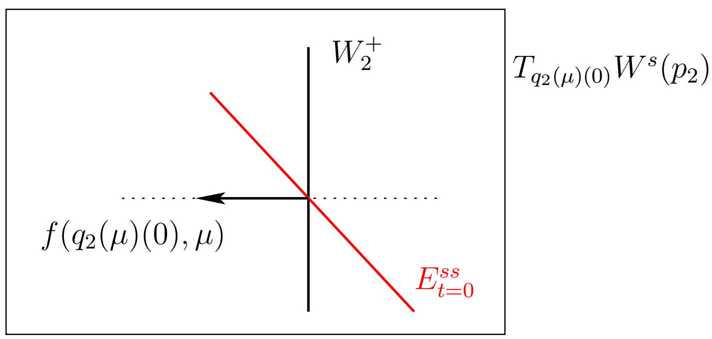


Figure 6: The space $E_{t=0}^{ss}$.

$$T_{q_2(\mu)(0)}W^s(p_2) = \operatorname{span}\{f(q_2(\mu)(0),\mu)\} \oplus E_{t=0}^{ss},$$

and correspondingly we decompose

$$(id - Q_2^-(\mu,0))\bar{v}_{2,i}^{+}(0) = v_{2,i}^s + v_{2,i}^{ss}, \qquad v_{2,i}^s \in \operatorname{span}\{f(q_2(\mu)(0),\mu)\},\ v_{2,i}^{ss} \in E_{t=0}^{ss}.$$

Using $\Phi_2^+(\mu,\omega_{2,i},0)v_{2,i}^s \in \operatorname{span}\{f(q_2(\mu)(\omega_{2,i}),\mu)$ and Assumptions (A 1) and (A 2) we find

$$\langle \Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1, \Phi_2^+(\mu,\omega_{2,i},0)v_{2,i}^s\rangle = 0.$$

Summarising our estimates regarding $\langle \psi_1, P_1^-(\mu,0)\bar{v}_{1,i}^-(\boldsymbol{\omega},\mu)(0)\rangle$ so far we find

$$\langle \psi_1, P_1^-(\mu,0)\bar{v}_{1,i}^-(\boldsymbol{\omega},\mu)(0)\rangle = \langle \Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1, \Phi_2^+(\mu,\omega_{2,i},0)v_{2,i}^{ss}\rangle + R_{1,i}(\boldsymbol{\omega},\mu), \quad (4.9)$$

where $R_{1,i}$ comprises all residual terms appearing in the course of our considerations.

By construction $\Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1$ is perpendicular to the extended-unstable manifold of p_2. Hence $P_1^{-*}(\mu,0)\psi_1$ is in the strong stable subspace of $\Psi_1^-(\mu,-t,0)$, cf. Lemma 4.3(ii). Further, according to Lemma 4.2 there is a $\eta_\perp^{ss}(\mu)\in\big(T_{p_2}W^{eu}(p_2)\big)^\perp$ such that

$$\Psi_1^-(\mu,-\omega_{2,i},0)P_1^{-*}(\mu,0)\psi_1 = e^{\lambda_2^{ss}\omega_{2,i}}\eta_\perp^{ss}(\mu)+o(e^{\lambda_2^{ss}\omega_{2,i}}).$$

More precisely, $\eta_\perp^{ss}(\mu)$ belongs to the eigenspace of the eigenvalue λ_2^{ss} of $D_1f(p_2,0)^*$. This eigenspace reads $(\mathbb{R}^n\ominus E_{\lambda_2^{ss}})^\perp$, where $E_{\lambda_2^{ss}}$ denotes the eigenspace of the eigenvalue λ_2^{ss} of $D_1f(p_2,0)$.

Similarly, $v_{2,i}^{ss}$ is in the strong stable subspace of $\Phi_2^+(\mu,t,0)$, cf. Lemma 4.3(i). Again according to Lemma 4.2 there is a $\eta^{ss}(\mu)\in E_{\lambda_2^{ss}}$ such that

$$\Phi_2^+(\mu,\omega_{2,i},0)v_{2,i}^{ss}/|v_{2,i}^{ss}| = e^{\lambda_2^{ss}\omega_{2,i}}\eta^{ss}(\mu)+o(e^{\lambda_2^{ss}\omega_{2,i}}).$$

Inserting in (4.9) yields

$$\langle\psi_j,P_1^-(\mu,0)\bar v_{1,i}^-(\boldsymbol{\omega},\mu)(0)\rangle = \big(e^{2\lambda_2^{ss}\omega_{2,i}}\langle\eta_\perp^{ss}(\mu),\eta^{ss}(\mu)\rangle+o(e^{2\lambda_2^{ss}\omega_{2,i}})\big)|v_{2,i}^{ss}|+R_{1,i}(\boldsymbol{\omega},\mu). \tag{4.10}$$

Due to Hypothesis (H 6) the vectors $\eta_\perp^{ss}(\mu)$ and $\eta^{ss}(\mu)$ cannot be perpendicular, and hence

$$\langle\eta_\perp^{ss}(\mu),\eta^{ss}(\mu)\rangle\neq 0.$$

Next we deduce estimates for $v_{2,i}^{ss}$. To this end we first note that $\bar v_{2,i}^+(0)=\bar v_{2,i}^-(0)$, and that $\dim W^+=\dim E_{t=0}^{ss}$. What is more, the projection $(id-Q_2^-(\mu,0))$ acts as an invertible mapping between these spaces. Hence there exist $c,C>0$ such that

$$c\,|(id-Q_2^-)\bar v_{2,i}^-(0)|\le|v_{2,i}^{ss}|\le C\,|(id-Q_2^-)\bar v_{2,i}^-(0)|.$$

The value $|(id-Q_2^-)\bar v_{2,i}^-(0)|$ can be estimated in the same way as the jump in the first part of this proof. Because λ_1^s is real and simple (cf. Hypothesis (H 6)) we get similar to the estimate (4.6)

$$|(id-Q_2^-)\bar v_{2,i}^-(0)| = \tilde c_{21}(\mu)e^{2\lambda_1^s\omega_{1,i}}+o(e^{2\lambda_1^s\omega_{1,i}}),\quad \tilde c_{21}(\mu)\neq 0.$$

Therefore

$$|v_{2,i}^{ss}| = \hat c_{21}(\mu,\omega_{2,i})e^{2\lambda_1^s\omega_{1,i}}+o(e^{2\lambda_1^s\omega_{1,i}}),\qquad c\,\tilde c_{21}(\mu)\le\hat c_{21}(\mu,\omega_{2,i})\le C\,\tilde c_{21}(\mu).$$

Inserting in (4.10) yields

$$\langle\psi_j,P_1^-(\mu,0)\bar v_{1,i}^-(\boldsymbol{\omega},\mu)(0)\rangle = e^{2\lambda_2^{ss}\omega_{2,i}}e^{2\lambda_1^s\omega_{1,i}}\hat c_{21}(\mu,\omega_{2,i})\langle\eta_\perp^{ss}(\mu),\eta^{ss}(\mu)\rangle+\hat R_{1,i}(\boldsymbol{\omega},\mu),$$

where $\hat R_{1,i}(\boldsymbol{\omega},\mu)$ comprises all remaining terms. Indeed estimates along the lines of [21, 34] in the particular situation which we exploited above, yield

$$\hat R_{1,i}(\boldsymbol{\omega},\mu)=o(e^{2\lambda_2^{ss}\omega_{2,i}}e^{2\lambda_1^s\omega_{1,i}}).$$

Finally we write

$$c_{21}(\mu,\omega_{2,i}):=\hat c_{21}(\mu,\omega_{2,i})\langle\eta_\perp^{ss}(\mu),\eta^{ss}(\mu)\rangle,$$

and note that

$$c\,\tilde{c}_{21}(\mu)\langle \eta_\perp^{ss}(\mu), \eta^{ss}(\mu)\rangle \le c_{21}(\mu,\omega_{2,i}) \le C\,\tilde{c}_{21}(\mu)\langle \eta_\perp^{ss}(\mu), \eta^{ss}(\mu)\rangle$$

and hence $c_{21}(\mu,\omega_{2,i}) \neq 0$. ■

Remark 4.5. If $n = 3$, i.e. $x \in \mathbb{R}$ the above estimate of $v_{2,i}^{ss}$ simplifies as follows. In this case we have $\dim W^+ = \dim E_{t=0}^{ss} = 1$ and hence there exists a $c > 0$ such that

$$c\,|(id - Q_2^-)\bar{v}_{2,i}^-(0)| = |v_{2,i}^{ss}|.$$

The value $|(id - Q_2^-)\bar{v}_{2,i}^-(0)|$ can be estimated in the same way as the jump in the first part of this proof. Similar to the estimate (4.6) we get

$$|(id - Q_2^-)\bar{v}_{2,i}^-(0)| = \tilde{c}_{21}(\mu)e^{2\lambda_1^s\omega_{1,i}} + o(e^{2\lambda_1^s\omega_{1,i}}), \quad \tilde{c}_{21}(\mu) \neq 0.$$

Therefore

$$|v_{2,i}^{ss}| = \hat{c}_{21}(\mu)e^{2\lambda_1^s\omega_{1,i}} + o(e^{2\lambda_1^s\omega_{1,i}}), \qquad \hat{c}_{21}(\mu) = c\,\tilde{c}_{21}(\mu).$$

□

Recall that in accordance with Lemma 4.1 the jumps depend smoothly on both $\boldsymbol{\omega}$ and μ. Let $D_k\xi_i(\boldsymbol{\omega},\mu)$, $k \in \{\omega_{j,i}, j = 1,2, i \in \mathbb{Z}\}$ or $k \in \{\mu_1,\mu_2\}$, denote the derivative with respect to the corresponding variable. Then, similarly to the corresponding estimates in [21, 34] we find the following Lemma.

Lemma 4.6. *The derivatives of the jumps ξ_i have the following form*

$$\begin{aligned} D_k\xi_i(\boldsymbol{\omega},\mu) = &\Big(D_k\big(c_{11}(\mu)e^{-2\lambda_1^u\omega_{1,i+1}} - c_{21}(\mu)e^{2\lambda_2^{ss}\omega_{2,i}}e^{2\lambda_1^s\omega_{1,i}}\big) + \tilde{\mathcal{R}}_{1,i}(\boldsymbol{\omega},\mu)\Big)\,\psi_1 \\ &+ \Big(D_k\big(c_{12}(\mu)e^{-2\lambda_1^{uu}\omega_{1,i+1}}e^{-2\lambda_2^u\omega_{2,i+1}} - c_{22}(\mu)e^{2\lambda_2^s\omega_{2,i}}\big) + \tilde{\mathcal{R}}_{2,i}(\boldsymbol{\omega},\mu)\Big)\,\psi_2, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{R}}_{1,i}(\boldsymbol{\omega},\mu) &= o(e^{-2\lambda_1^u\omega_{1,i+1}}) + o(e^{2\lambda_2^{ss}\omega_{2,i}}e^{2\lambda_1^s\omega_{1,i}}), \\ \tilde{\mathcal{R}}_{2,i}(\boldsymbol{\omega},\mu) &= o(e^{-2\lambda_1^{uu}\omega_{1,i+1}}e^{-2\lambda_2^u\omega_{2,i+1}}) + o(e^{2\lambda_2^s\omega_{2,i}}). \end{aligned}$$

■

Finally, adapted to our needs, we discuss the twist of Γ in terms of Lin's method. To this end we define the twist in the style of corresponding considerations in [6]. First we note that we can consider $W^u(p_1)$ as global manifold containing Γ. The part of this manifold containing Γ_1 "coincides" with $W^{eu}(p_2)$. Recall that $\psi_1 \perp T_{q_1(0)}W^{eu}(p_2)$. Similarly consider $\hat{\psi}_1 \perp T_{q_2(0)}W^u(p_1)$ such that for $\omega \gg 1$

$$\langle \Psi_1^-(0,-\omega,0)\psi_1, \Psi_2^+(0,\omega,0)\hat{\psi}_1\rangle > 0.$$

This conditions implies that both ψ_1 and $\hat{\psi}_1$ point to the same side of the addressed global manifold.

Further, we define

$$e_1^s := \lim_{t\to\infty}\frac{\dot{q}_1(t)}{|\dot{q}_1(t)|}, \quad e_1^u := \lim_{t\to-\infty}\frac{\dot{q}_2(t)}{|\dot{q}_2(t)|}.$$

Definition 4.7. *We call Γ nontwisted if for $\omega \gg 1$*

$$\operatorname{sign}\langle e_1^u, \Psi_1^+(0,\omega,0)\psi_1\rangle \neq \operatorname{sign}\langle e_1^s, \Psi_2^-(0,-\omega,0)\hat{\psi}_1\rangle,$$

whereas we call Γ twisted if for $\omega \gg 1$

$$\operatorname{sign}\langle e_1^u, \Psi_1^+(0,\omega,0)\psi_1\rangle = \operatorname{sign}\langle e_1^s, \Psi_2^-(0,-\omega,0)\hat{\psi}_1\rangle.$$

This definition means that "along Γ the global manifold $W^u(p_1)$" is nontwisted or twisted, respectively. Note that the corresponding global manifold $W^s(p_2)$ is nontwisted or twisted, respectively if and only if the global manifold $W^u(p_1)$ has this property.

Inspecting the proof of Lemma 4.4 we find similarly to the considerations in [21]

Lemma 4.8.

$$\begin{aligned}
\Gamma \text{ nontwisted} \quad &\Leftrightarrow \quad \operatorname{sgn} c_{11}(0) = \operatorname{sgn} c_{21}(0),\ \operatorname{sgn} c_{12}(0) = \operatorname{sgn} c_{22}(0),\\
\Gamma \text{ twisted} \quad &\Leftrightarrow \quad \operatorname{sgn} c_{11}(0) \neq \operatorname{sgn} c_{21}(0),\ \operatorname{sgn} c_{12}(0) \neq \operatorname{sgn} c_{22}(0).
\end{aligned}$$

■

4.2 Complex leading eigenvalues

We present equivalent statements to Lemma 4.4 for the cases (RC) and (CC). To establish the corresponding estimates we proceed again from the representations (4.4) and (4.5).

We assume that $\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu)$ is complex and consider $\langle \psi_1, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle$:

Lemma 4.9. *Assume Hypotheses (H1)-(H5). Assume further $\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu)$. Then there are constants $c_{1j}(\mu)$, $\phi_1(\mu)$ and φ_{1j} such that*

$$\langle \psi_j, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle = c_{1j}(\mu)e^{-2\rho_1(\mu)\omega_{1,i+1}}\sin(2\phi_1(\mu)\omega_{1,i+1} + \varphi_{1j}) + o(e^{-2\rho_1\omega_{1,i+1}}),$$

$j = 1,2$. Moreover $c_{1j}(0) \neq 0$ and $\varphi_{11} - \varphi_{12} \neq 0$ (mod π).

Proof. As in the proof of Lemma 4.4, again analogous to considerations in [21, 34], we find for both ψ_j, $j = 1,2$,

$$\begin{aligned}
\langle \psi_j, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle = &\langle \Psi_1^+(\mu,\omega_{1,i+1},0)Q_1^{+*}(\mu,0)\psi_j, \tilde{Q}_1(\mu,\omega_{1,i+1})q_2^-(\mu)(-\omega_{1,i+1})\rangle\\
&+ o(e^{-2\rho_1^u\omega_{1,i+1}}).
\end{aligned}$$

To estimate the first term on the right-hand side we make use of the complex counterpart of Lemma 4.2. This provides that $q_2^-(\mu)(t)$ behaves asymptotically, as $t \to -\infty$, like $e^{(D_1 f(p_1,\mu))t}\eta^u$, where $\eta^u = \eta^u(\mu)$ belongs to the generalised (real) eigenspace X^u of $\lambda_1^u(\mu)$. According to corresponding estimates in [34] or [21], the same holds true for $\tilde{Q}_1(\mu,\omega_{1,i+1})q_2^-(\mu)(-\omega_{1,i+1})$. Further, the term $\Psi_1^+(\mu,t,0)Q_1^{+*}(\mu,0)\psi_j$ behaves asymptotically, as $t \to \infty$, like $e^{-(D_1 f(p_1,\mu))^* t}\eta^+(\psi_j,\mu)$.

where $\eta^+(\psi_j,\mu)$ belongs to the generalised (real) eigenspace $(X^u)^\perp$ of the leading stable eigenvalue(s) $-\lambda_1^u(\mu)$, $(-\overline{\lambda}_1^u(\mu))$, of $-(D_1 f(p_1,\mu))^*$. Abbreviating, we write $\eta_j^+ := \eta^+(\psi_j,\mu)$. More precisely these terms can be estimated by:

$$\begin{aligned}\tilde{Q}_1(\mu,\omega_{1,i+1})q_2^-(\mu)(-\omega_{1,i+1}) &= e^{-(D_1 f(p_1,\mu))\omega_{1,i+1}}\eta^u + o(e^{-\rho_1^u\omega_{1,i+1}}),\\ \Psi_1^+(\mu,\omega_{1,i+1},0){Q_1^+}^*(\mu,0)\psi_j &= e^{-(D_1 f(p_1,\mu))^*\omega_{1,i+1}}\eta_j^+ + o(e^{-\rho_1^u\omega_{1,i+1}}).\end{aligned}$$

Therefore

$$\langle\psi_j, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle = \langle e^{-2(D_1 f(p_1,\mu))\omega_{1,i+1}}\eta^u, \eta_j^+\rangle + o(e^{-2\rho_1^u\omega_{1,i+1}}).$$

Since the generalised real eigenspace X^u has dimension two, the vector η^u has a coordinate representation $(\eta_1^u,\eta_2^u,0,\ldots,0)$ where $(\eta_1^u,\eta_2^u)\neq(0,0)$, and $e^{-D_1 f(p_1,\mu)2t}\eta^u$ acts like

$$e^{-2\rho_1(\mu)t}\begin{pmatrix}\begin{pmatrix}\cos(2\phi_1(\mu)t) & \sin(2\phi_1(\mu)t)\\ -\sin(2\phi_1(\mu)t) & \cos(2\phi_1(\mu)t)\end{pmatrix}\begin{pmatrix}\eta_1^u\\ \eta_2^u\end{pmatrix}\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

Let $(\eta_{1j}^+,\ldots,\eta_{nj}^+)$ be the coordinate representation of η_j^+ with respect to the chosen basis. Further, since $\eta_j^+\in(\mathbb{R}^n\ominus X^u)^\perp$ we have $(\eta_{1j}^+,\eta_{2j}^+)\neq(0,0)$. Therefore

$$\begin{aligned}\langle\psi_j, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle =& e^{-2\rho_1(\mu)\omega_{1,i+1}}\Big(\eta_{1j}\sin(2\phi_1(\mu)\omega_{1,i+1}) + \eta_{2j}\cos(2\phi_1(\mu)\omega_{1,i+1})\Big)\\ &+ o(e^{-2\rho_1^u\omega_{1,i+1}}),\end{aligned}$$

where

$$(\eta_{1j},\eta_{2j}) := (\eta_2^u\eta_{1j}^+ - \eta_1^u\eta_{2j}^+,\ \eta_1^u\eta_{1j}^+ + \eta_2^u\eta_{2j}^+)\neq(0,0).$$

The latter inequality is due to the fact that both (η_1^u,η_2^u) and $(\eta_{1j}^+,\eta_{2j}^+)$ are different from $(0,0)$. So there is an angle $\varphi_{1j}=\varphi_{1j}(\psi_j,\mu)$ such that

$$\sin\varphi_{1j} = \eta_{1j}\left(\eta_{1j}^2+\eta_{2j}^2\right)^{-1/2},\quad \cos\varphi_{1j} = \eta_{2j}\left(\eta_{1j}^2+\eta_{2j}^2\right)^{-1/2}.$$

Hence

$$\langle\psi_j, Q_1^+(\mu,0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)\rangle = c_{1j}(\mu)e^{-2\rho_1(\mu)\omega_{1,i+1}}\sin(2\phi_1(\mu)\omega_{1,i+1}+\varphi_{1j}) + o(e^{-2\rho_1^u\omega_{1,i+1}})$$

where $c_{1j}(\mu) = (\eta_1^2+\eta_2^2)^{1/2}$. By construction $c_{1j}(\cdot)$ is smooth and $c_{1j}(0)\neq 0$.

It remains to prove that $\varphi_{11}-\varphi_{12}\neq 0 \pmod{\pi}$: We show that $\sin(\varphi_{11}-\varphi_{12})\neq 0$. Note that the η_i^u does not depend on j. Exploiting $\sin(\varphi_{11}-\varphi_{12}) = \sin\varphi_{11}\cos\varphi_{12}-\sin\varphi_{12}\cos\varphi_{11}$ we find that $\sin(\varphi_{11}-\varphi_{12})=0$ if and only if (η_{11},η_{21}) and (η_{12},η_{22}) are linearly dependent. This however is the case if and only if $(\eta_1^+(\psi_1),\eta_2^+(\psi_1))$ and $(\eta_1^+(\psi_2),\eta_2^+(\psi_2))$ are linearly dependent. But these vectors are linearly independent because ψ_1 and ψ_2 are linearly independent. ∎

In the same way we find:

Lemma 4.10. *Assume Hypotheses (H 1)-(H 5). Assume further $\lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu)$, with $\rho_2 > 0$. Then there are constants $c_{2j}(\mu)$, $\phi_2(\mu)$ and φ_{2j} such that*

$$\langle \psi_j, P_1^-(\mu, 0)\bar{v}_{1,i}^-(\boldsymbol{\omega}, \mu)(0)\rangle = c_{2j}(\mu)e^{-2\rho_2(\mu)\omega_{2,i}} \sin(2\phi_2(\mu)\omega_{2,i} + \varphi_{2j}) + o(e^{-2\rho_2\omega_{2,i}}),$$

$j = 1, 2$. Moreover $c_{2j}(0) \neq 0$ and $\varphi_{21} - \varphi_{22} \neq 0$ (mod π). ■

Now we combine the statements of the previous estimates.

Lemma 4.11. *Assume Hypotheses (H 1)-(H 5). Assume further the (RC) case, more precisely that $\lambda_1^u(\mu)$ is real and $\lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu)$ is complex, and finally that Hypothesis (H 7) holds true. Then the jump $\xi_i(\boldsymbol{\omega}, \mu)$ can be written in the form*

$$\begin{aligned}\xi_i(\boldsymbol{\omega}, \mu) = \; & \big(c_{11}(\mu)e^{-2\lambda_1^u\omega_{1,i+1}} - c_{21}(\mu)e^{-2\rho_2(\mu)\omega_{2,i}} \sin(2\phi_2(\mu)\omega_{2,i} + \varphi_{21}) + \mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu)\big)\,\psi_1 \\ & + \big(c_{12}(\mu)e^{-2\lambda_1^{uu}\omega_{1,i+1}}e^{-2\lambda_2^u\omega_{2,i+1}} - c_{22}e^{-2\rho_2(\mu)\omega_{2,i}} \sin(2\phi_2(\mu)\omega_{2,i} + \varphi_{22}) + \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu)\big)\,\psi_2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu) &= o(e^{-2\lambda_1^u\omega_{1,i+1}}) + o(e^{-2\rho_2\omega_{2,i}}) \\ \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu) &= o(e^{-2\lambda_1^{uu}\omega_{1,i+1}}e^{-2\lambda_2^u\omega_{2,i+1}}) + o(e^{-2\rho_2\omega_{2,i}}).\end{aligned}$$

■

This lemma follows immediately from the Lemmas 4.4 and 4.10, while the next lemma follows from Lemma 4.9 together with Lemma 4.10.

Lemma 4.12. *Assume Hypotheses (H 1)-(H 5) and further the eigenvalue case (CC). Write $\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu)$ and $\lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu)$. Then the jump $\xi_i(\boldsymbol{\omega}, \mu)$ can be written in the form*

$$\begin{aligned}\xi_i(\boldsymbol{\omega}, \mu) = \big(&c_{11}(\mu)(\mu)e^{-2\rho_1(\mu)\omega_{1,i+1}} \sin(2\phi_1(\mu)\omega_{1,i+1} + \varphi_{11}) \\ & - c_{21}(\mu)e^{-2\rho_2(\mu)\omega_{2,i}} \sin(2\phi_2(\mu)\omega_{2,i} + \varphi_{21}) + \mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu)\big)\psi_1 \\ + \big(&c_{12}(\mu)(\mu)e^{-2\rho_1(\mu)\omega_{1,i+1}} \sin(2\phi_1(\mu)\omega_{1,i+1} + \varphi_{12}) \\ & - c_{22}e^{-2\rho_2(\mu)\omega_{2,i}} \sin(2\phi_2(\mu)\omega_{2,i} + \varphi_{22}) + \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu)\big)\psi_2,\end{aligned}$$

where

$$\mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu) = \; o(e^{-2\rho_1\omega_{1,i+1}}) + o(e^{-2\rho_2\omega_{2,i}}), \quad \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu) = \; o(e^{-2\rho_1\omega_{1,i+1}}) + o(e^{-2\rho_2\omega_{2,i}}).$$

Moreover $\varphi_{j1} - \varphi_{j2} \neq 0\,(\bmod \pi)$, $j = 1, 2$. ■

Remark 4.13. *As in the real case, cf. Lemma 4.6 we find that the derivatives of the residual term satisfy the estimates*

$$D_k\mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu) = \; o(e^{-2\rho_1\omega_{1,i+1}}) + o(e^{-2\rho_2\omega_{2,i}}), \quad D_k\mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu) = \; o(e^{-2\rho_1\omega_{1,i+1}}) + o(e^{-2\rho_2\omega_{2,i}}),$$

where $k \in \{\omega_{j,i}, j = 1, 2, i \in \mathbb{Z}\}$ or $k \in \{\mu_1, \mu_2\}$.

5 Dynamics in the case of real eigenvalues – (RR)

In the eigenvalue case (RR) we mainly focus on 1-periodic and 2-periodic orbits and their limiting 1-homoclinic and 2-homoclinic orbits. In our considerations we distinguish between twisted and nontwisted T-point cycles.

For the rest of this section we assume

(A 5) $\quad c_{11}(0),\ c_{22}(0) > 0.$

First we consider 1-periodic orbits. These are characterised by sequences $\boldsymbol{\omega} = (\omega_{1,i}, \omega_{2,i})_{i\in\mathbb{Z}}$ with

$$(\omega_{1,i}, \omega_{2,i}) =: (\omega_1, \omega_2), \quad i \in \mathbb{Z}.$$

Therefore, according to Lemma 4.4 the bifurcation equations for 1-periodic orbits are as follows:

$$\Xi(\boldsymbol{\omega}, \mu) = \begin{pmatrix} \mu_1 + c_{11}(\mu)e^{-2\lambda_1^u\omega_1} - c_{21}(\mu)e^{2\lambda_2^{ss}\omega_2}e^{2\lambda_1^s\omega_1} + \mathcal{R}_1(\boldsymbol{\omega}, \mu) \\ \mu_2 + c_{12}(\mu)e^{-2\lambda_1^{uu}\omega_1}e^{-2\lambda_2^u\omega_2} - c_{22}(\mu)e^{2\lambda_2^s\omega_2} + \mathcal{R}_2(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0, \tag{5.1}$$

where

$$\mathcal{R}_1(\boldsymbol{\omega}, \mu) = o(e^{-2\lambda_1^u\omega_1}) + o(e^{2\lambda_2^{ss}\omega_2}e^{2\lambda_1^s\omega_1}), \qquad \mathcal{R}_2(\boldsymbol{\omega}, \mu) = o(e^{-2\lambda_1^{uu}\omega_1}e^{-2\lambda_2^u\omega_2}) + o(e^{2\lambda_2^s\omega_2}).$$

Setting formally $\omega_1 = \infty$ or $\omega_2 = \infty$ in this representation we get the bifurcation equation for 1-homoclinic orbits to p_1 or p_2, respectively. Taking the structure of the residual term $\mathcal{R}_i$ into consideration we find for the 1-homoclinic orbits to p_1:

$$\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= c_{22}(\mu)e^{2\lambda_2^s\omega_2} + o(e^{2\lambda_2^s\omega_2}), \end{aligned}$$

and similarly for the 1-homoclinic orbits to p_2:

$$\begin{aligned} \mu_1 &= -c_{11}(\mu)e^{-2\lambda_1^u\omega_1} + o(e^{-2\lambda_1^u\omega_1}) \\ \mu_2 &= 0. \end{aligned}$$

According to Assumption (A 5) we find 1-homoclinic orbits p_1 or p_2 on the positive part of the μ_2-axis or the negative part of the μ_1-axis, respectively.

Next we formulate the bifurcation equations for 2-periodic orbits. Those orbits are characterised by sequences $\boldsymbol{\omega}$

$$(\omega_{1,i}, \omega_{2,i}) = \begin{cases} (\omega_{1,1}, \omega_{2,1}), & i \text{ odd} \\ (\omega_{1,2}, \omega_{2,2}), & i \text{ even.} \end{cases}$$

Hence the bifurcation equation for 2-periodic orbits reads:

$$\begin{aligned} \Xi_1(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} \mu_1 + c_{11}(\mu)e^{-2\lambda_1^u\omega_{1,2}} - c_{21}(\mu)e^{2\lambda_2^{ss}\omega_{2,1}}e^{2\lambda_1^s\omega_{1,1}} + \mathcal{R}_{1,1}(\boldsymbol{\omega}, \mu) \\ \mu_2 + c_{12}(\mu)e^{-2\lambda_1^{uu}\omega_{1,2}}e^{-2\lambda_2^u\omega_{2,2}} - c_{22}(\mu)e^{2\lambda_2^s\omega_{2,1}} + \mathcal{R}_{2,1}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \\ \Xi_2(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} \mu_1 + c_{11}(\mu)e^{-2\lambda_1^u\omega_{1,1}} - c_{21}(\mu)e^{2\lambda_2^{ss}\omega_{2,2}}e^{2\lambda_1^s\omega_{1,2}} + \mathcal{R}_{1,2}(\boldsymbol{\omega}, \mu) \\ \mu_2 + c_{12}(\mu)e^{-2\lambda_1^{uu}\omega_{1,1}}e^{-2\lambda_2^u\omega_{2,1}} - c_{22}(\mu)e^{2\lambda_2^s\omega_{2,2}} + \mathcal{R}_{2,2}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \end{aligned} \tag{5.2}$$

Again, setting formally $\omega_{1,1} = \infty$ or $\omega_{1,2} = \infty$ in (5.2) we get the bifurcation equation for 2-homoclinic orbits to p_1, and similarly setting $\omega_{2,1} = \infty$ or $\omega_{2,2} = \infty$ in (5.2) we get the bifurcation equation for 2-homoclinic orbits to p_2,

5.1 Proof of Theorem 2.1(i)

Recall that under the assumptions of Theorem 2.1(i) $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$ are the leading eigenvalues of the equilibria p_1 and p_2, respectively, i.e.

$$\lambda_1^u(\mu) < |\lambda_1^s(\mu)|, \quad |\lambda_2^s(\mu)| < \lambda_2^u(\mu) \tag{5.3}$$

5.1.1 1-periodic Orbits

Adapted to the present eigenvalue condition (5.3) we introduce new variables

$$\varpi_1 := \frac{\lambda_1^u(\mu)}{\lambda_1^u(0)}\omega_1, \qquad \varpi_2 := \frac{\lambda_2^s(\mu)}{\lambda_2^s(0)}\omega_2,$$

and subsequently

$$r_1 := e^{2\lambda_1^u(0)\varpi_1}, \qquad r_2 := e^{2\lambda_2^s(0)\varpi_2}.$$

Therefore there exist $\delta_1^s(\mu), \delta_2^{ss}(\mu), \delta_1^{uu}(\mu), \delta_2^u(\mu) > 1$ such that in terms of these new variables the bifurcation equation for 1-periodic orbits (5.1) reads

$$\begin{aligned} \mu_1 + c_{11}(\mu)r_1 - c_{21}(\mu)r_2^{\delta_2^{ss}} r_1^{\delta_1^s} + \tilde{\mathcal{R}}_1(r_1, r_2, \mu) &= 0 \\ \mu_2 + c_{12}(\mu)r_1^{\delta_1^{uu}} r_2^{\delta_2^u} - c_{22}(\mu)r_2 + \tilde{\mathcal{R}}_2(r_1, r_2, \mu) &= 0, \end{aligned}$$

where

$$\tilde{\mathcal{R}}_1(r_1, r_2, \mu) = o(r_1) + o(r_2^{\delta_2^{ss}} r_1^{\delta_1^s}), \qquad \tilde{\mathcal{R}}_2(r_1, r_2, \mu) = o(r_1^{\delta_1^{uu}} r_2^{\delta_2^u}) + o(r_2).$$

Taking into consideration that $c_{11}(0), c_{22}(0) \neq 0$, these equations can be solved for (r_1, r_2) depending on μ. Using in particular Assumption (A 5) we find 1-periodic orbits exactly for $\mu \in Q_1$,

$$Q_1 := \{(\mu_1, \mu_2) : \mu_1 < 0,\ \mu_2 > 0\}.$$

Altogether we get the bifurcation diagram depicted in Figure 2(a).

Remark 5.1. At this point we shortly discuss an idea which we seize on for the discussion of the complex eigenvalue cases in Section 6 below.

We define

$$L((\omega_1, \omega_2), \mu) := \begin{pmatrix} \mu_1 + c_{11}(\mu)e^{-2\lambda_1^u\omega_1} \\ \mu_2 - c_{22}(\mu)e^{2\lambda_2^s\omega_2} \end{pmatrix}.$$

Assuming the eigenvalue condition (5.3) we find that solutions $(\hat{\omega}_1(\mu), \hat{\omega}_2(\mu))$ of the truncated equation $L((\omega_1, \omega_2), \mu) = 0$ determine solutions of the full equation (5.1) which determines 1-periodic orbits.

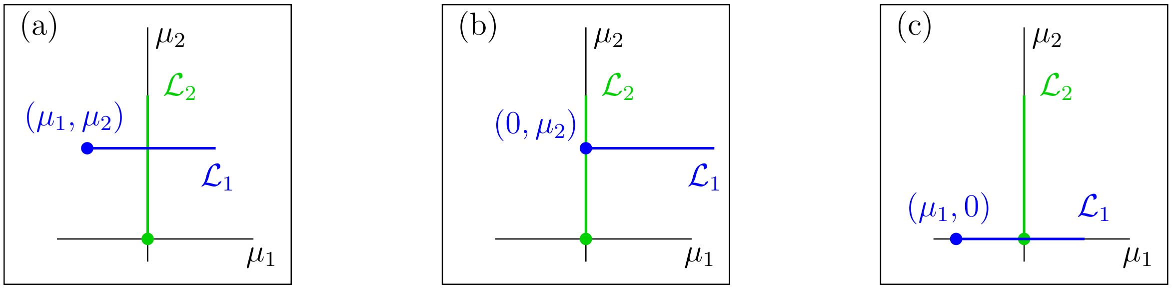


Figure 7: The lines $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to the truncated equation $L = 0$.

The solutions of the truncated equation $L((\omega_1, \omega_2), \mu) = 0$ corresponds to transversal intersections of the two lines $\mathcal{L}_1(\mu) = (\mu_1, \mu_2) + s_1(c_{11}(\mu), 0)$ and $\mathcal{L}_2(\mu) = s_2(0, c_{22}(\mu))$, where $s_i > 0$ in each case, see Figure 7(a).

The "border locations" where the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ just touch each other correspond to homoclinic orbits to p_1 or p_2, see Figure 7(b) or (c), respectively – compare also Figure 2(a).

These arguments remain valid under the assumptions of Theorem 2.1(ii) or (iii), respectively. However, in these cases the discussion of the truncated equation $L = 0$ and its visualisation displayed in Figure 7 only explains the existence of one 1-periodic orbit in Q_1 and the 1-homoclinic orbits on the boundary of Q_1. In order to describe the behaviour in Q_2 and Q^2 and on their boundaries we need to consider also higher order terms as we explain in Sections 5.2 and 5.3. $\square$

5.1.2 k-periodic Orbits

As stated in Section 4, k-periodic orbits correspond to sequences $\boldsymbol{\omega}$

$$(\omega_{1,i}, \omega_{2,i}) = (\omega_{1,i \bmod k}, \omega_{2,i \bmod k}),$$

and the bifurcation equation consists of k pairs of equations, cf. (4.2).

Due to the present eigenvalue condition (5.3) this set of equations can be treated in the same way as the bifurcation equation for 1-periodic orbits above. It turns out that for each $\mu \in Q_1$ there exist a unique set $(\omega_{1,i}, \omega_{2,i})$, $i = 1, \ldots, k$ solving this equations. On the other hand it is obvious that there is a solution

$$(\omega_{1,i}, \omega_{2,i}) = (\omega_1, \omega_2), \quad i = 1, \ldots, k, \tag{5.4}$$

where (ω_1, ω_2) solves the bifurcation equation for 1-periodic orbits (5.1). So, the solution given in (5.4) defines a 1-periodc orbit which is run through k-times. Altogether this shows that there are no k-periodic orbits for $k > 1$.

5.2 Proof of Theorem 2.1(ii)

We assume that $\lambda_1^s(\mu)$ and $\lambda_2^s(\mu)$ are the leading eigenvalues of the equilibria p_1 and p_2, respectively, i.e.

$$|\lambda_1^s(\mu)| < \lambda_1^u(\mu), \quad |\lambda_2^s(\mu)| < \lambda_2^u(\mu). \tag{5.5}$$

In our further discussion we distinguish whether the primary heteroclinic cycle Γ is twisted or not.

5.2.1 Γ nontwisted

1-periodic orbits. Adapted to the eigenvalue condition (5.5) we introduce new variables

$$\varpi_1 := \frac{\lambda_1^s(\mu)}{\lambda_1^s(0)}\omega_1, \qquad \varpi_2 := \frac{\lambda_2^s(\mu)}{\lambda_2^s(0)}\omega_2,$$

and subsequently

$$r_1 := e^{2\lambda_1^s(0)\varpi_1}, \qquad r_2 := e^{2\lambda_2^s(0)\varpi_2}. \tag{5.6}$$

Therefore there exist $\delta_1^u(\mu), \delta_2^{ss}(\mu), \delta_1^{uu}(\mu), \delta_2^u(\mu) > 1$ such that in terms of these new variables the bifurcation equation for 1-periodic orbits (5.1) reads

$$\begin{aligned} \mu_1 + c_{11}(\mu)r_1^{\delta_1^u} - c_{21}(\mu)r_2^{\delta_2^{ss}}r_1 + \tilde{\mathcal{R}}_1(r_1, r_2, \mu) &= 0 \\ \mu_2 + c_{12}(\mu)r_1^{\delta_1^{uu}}r_2^{\delta_2^u} - c_{22}(\mu)r_2 + \tilde{\mathcal{R}}_2(r_1, r_2, \mu) &= 0, \end{aligned} \tag{5.7}$$

where

$$\tilde{\mathcal{R}}_1(r_1, r_2, \mu) = o(r_1^{\delta_1^u}) + o(r_2^{\delta_2^{ss}}r_1), \qquad \tilde{\mathcal{R}}_2(r_1, r_2, \mu) = o(r_1^{\delta_1^{uu}}r_2^{\delta_2^u}) + o(r_2).$$

By Assumption (A 5) and Lemma 4.8 all $c_{ij}(0)$, $i, j = 1, 2$, are positive. Note also that $\delta_1^{uu} > \delta_1^u$.

Since $c_{22}(0) \neq 0$, the second equation in (5.7) can be solved for $r_2 = r_2^*(r_1, \mu)$. Note that this equation can be solved only for nonnegative μ_2. We find

$$r_2^*(r_1, \mu) = \frac{1}{c_{22}(0)}\mu_2 + O(r_1^{\delta_1^{uu}}).$$

Inserting in the first equation of (5.7) yields

$$\mu_1 + c_{11}(\mu)r_1^{\delta_1^u} - \tilde{c}_{21}(\mu)\mu_2^{\delta_2^{ss}}r_1 + o(r_1^{\delta_1^u}) + o(r_2^{*\delta_2^{ss}}r_1) = 0, \tag{5.8}$$

where both $c_{11}(\mu)$ and $\tilde{c}_{21}(\mu)$ are positive. For fixed μ the graph of the function

$$h(r_1, \mu) := -c_{11}(\mu)r_1^{\delta_1^u} + \tilde{c}_{21}(\mu)\mu_2^{\delta_2^{ss}}r_1 + o(r_1^{\delta_1^u}) + o(r_2^{*\delta_2^{ss}}r_1)$$

looks qualitatively as depicted in Figure 8.

For the only maximum point we find

$$r_{1,max}(\mu) \approx \mu_2^{\frac{\delta_2^{ss}}{\delta_1^u - 1}}, \qquad h_{max}(\mu) := h(r_{1,max}(\mu), \mu) \approx \mu_2^{\delta_2^{ss}\frac{\delta_1^u}{\delta_1^u - 1}}$$

Altogether this gives the following results for periodic orbits, cf. Figure 2(b):

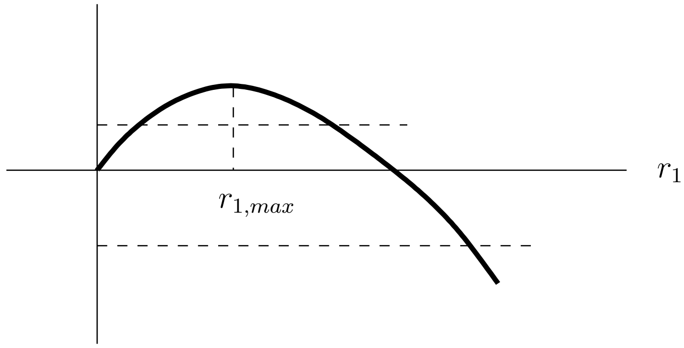


Figure 8: The graph of $h(\cdot, \mu)$.

- For $\mu \in Q_1$ there exists exactly one 1-periodic orbit.
- For $\mu \in Q_2 := \{\mu_2 > 0,\ \mu_1 \in (0, h_{max}(\mu))\}$ there are exactly two 1-periodic orbits.
- $\mathcal{S}_{sn} := \{\mu : \mu_2 > 0,\ \mu_1 = h_{max}(\mu)\}$ is a saddle-node line for 1-periodic orbits.

5.2.2 Γ twisted

According to Assumption (A 5) and Lemma 4.8 it holds

$$c_{11}(0), c_{22}(0) > 0, \qquad c_{12}(0), c_{21}(0) < 0. \tag{5.9}$$

1-periodic orbits. In order to detect 1-periodic orbits we proceed as in Section 5.2.1. However, due to (5.9) the counterpart of Equation (5.8) reads

$$\mu_1 + c_{11}(\mu) r_1^{\delta_1^u} + \hat{c}_{21}(\mu) \mu_2^{\delta_2^{ss}} r_1 + o(r_1^{\delta_1^u}) + o(r_2^{*\delta_2^{ss}} r_1) = 0,$$

where both $c_{11}(\mu)$ and $\hat{c}_{21}(\mu)$ are positive. Obviously this equation can only be solved for nonpositive μ_1. Indeed, this equation can (for all negative μ_1) be solved for $r_1(\mu)$. Together with our general considerations regarding 1-periodic orbits this shows that exactly for $\mu \in Q_1$ there exist 1-periodic orbits, in fact exactly one for each such μ.

2-periodic and 2-homoclinic orbits. Similarly to (5.6) we introduce variables $r_{j,i}$, $i, j = 1, 2$. In these variables the bifurcation equation for 2-periodic orbits, cf. (5.2), reads

$$\begin{aligned}
\mu_1 + c_{11}(\mu) r_{1,2}^{\delta_1^u} - c_{21}(\mu) r_{2,1}^{\delta_2^{ss}} r_{1,1} + \tilde{\mathcal{R}}_{1,1} &= 0 \\
\mu_2 + c_{12}(\mu) r_{1,2}^{\delta_1^{uu}} r_{2,2}^{\delta_2^u} - c_{22}(\mu) r_{2,1} + \tilde{\mathcal{R}}_{2,1} &= 0 \\
\mu_1 + c_{11}(\mu) r_{1,1}^{\delta_1^u} - c_{21}(\mu) r_{2,2}^{\delta_2^{ss}} r_{1,2} + \tilde{\mathcal{R}}_{1,2} &= 0 \\
\mu_2 + c_{12}(\mu) r_{1,1}^{\delta_1^{uu}} r_{2,1}^{\delta_2^u} - c_{22}(\mu) r_{2,2} + \tilde{\mathcal{R}}_{2,2} &= 0,
\end{aligned} \tag{5.10}$$

where

$$\tilde{\mathcal{R}}_{1,1} = o(r_{1,2}^{\delta_1^u}) + o(r_{2,1}^{\delta_2^{ss}} r_{1,1}), \quad \tilde{\mathcal{R}}_{2,1} = o(r_{1,2}^{\delta_1^{uu}} r_{2,2}^{\delta_2^u}) + o(r_{2,1}),$$
$$\tilde{\mathcal{R}}_{1,2} = o(r_{1,1}^{\delta_1^u}) + o(r_{2,2}^{\delta_2^{ss}} r_{1,2}), \quad \tilde{\mathcal{R}}_{2,2} = o(r_{1,1}^{\delta_1^{uu}} r_{2,1}^{\delta_2^u}) + o(r_{2,2}).$$

We write

$$r_{1,2} = a_1 r_{1,1} \quad \text{and} \quad r_{2,2} = a_2 r_{2,1}.$$

Note that $a_1 = 0$ corresponds to 2-homoclinic orbits to p_1, whereas $a_2 = 0$ corresponds to 2-homoclinic orbits to p_2. For $a_1 = a_2 = 1$ the system (5.10) reduces to the first two equations which describe 1-periodic orbits.

In our explanations we focus on the curve $\mathcal{S}_{\text{2-hom-}p_1}$ of 2-homoclinic orbits to p_1 and on the period-doubling curve $\mathcal{S}_{pd}$ which are depicted in Figure 2(b).

Subtracting the third equation from the first equation in (5.10) and subtracting the fourth equation from the second equation in (5.10) yields

$$\begin{aligned} c_{11}(\mu)\big(a_1^{\delta_1^u} - 1\big) r_{1,1}^{\delta_1^u} - c_{21}(\mu)\big(1 - a_2^{\delta_2^{ss}} a_1\big) r_{2,1}^{\delta_2^{ss}} r_{1,1} + \tilde{\mathcal{R}}_{1,1} - \tilde{\mathcal{R}}_{1,2} &= 0 \\ c_{12}(\mu)\big(a_1^{\delta_1^{uu}} a_2^{\delta_2^u} - 1\big) r_{1,1}^{\delta_1^{uu}} r_{2,1}^{\delta_2^u} - c_{22}(\mu)\big(1 - a_2\big) r_{2,1} + \tilde{\mathcal{R}}_{2,1} - \tilde{\mathcal{R}}_{2,2} &= 0. \end{aligned} \tag{5.11}$$

First we consider **2-homoclinic orbits to p_1**, i.e. we set $a_1 = 0$. Hence (5.11) reads

$$\begin{aligned} -c_{11}(\mu) r_{1,1}^{\delta_1^u} - c_{21}(\mu) r_{2,1}^{\delta_2^{ss}} r_{1,1} + \tilde{\mathcal{R}}_{1,1} - \tilde{\mathcal{R}}_{1,2} &= 0 \\ -c_{12}(\mu) r_{1,1}^{\delta_1^{uu}} r_{2,1}^{\delta_2^u} - c_{22}(\mu)\big(1 - a_2\big) r_{2,1} + \tilde{\mathcal{R}}_{2,1} - \tilde{\mathcal{R}}_{2,2} &= 0. \end{aligned} \tag{5.12}$$

The first equation of (5.12) is equivalent to

$$-c_{11}(\mu) r_{1,1}^{\delta_1^u - 1} - c_{21}(\mu) r_{2,1}^{\delta_2^{ss}} + o(r_{1,1}^{\delta_1^u - 1}) + o(r_{2,1}^{\delta_2^{ss}}) = 0.$$

This equation can be solved for $r_{1,1}$. Up to higher order terms we find that

$$r_{1,1} \approx \big(-c_{21}/c_{11}\big)^{\frac{1}{\delta_1^u - 1}} r_{2,1}^{\frac{\delta_2^{ss}}{\delta_1^u - 1}}. \tag{5.13}$$

We insert the result in the second equation of (5.12) where we factor out $r_{2,1}$. The remaining equation can be solved for $r_{2,1}$. We find with some appropriate $\hat{c}_{12} < 0$ that

$$r_{2,1}(a_2, \mu) \approx \left(-\frac{c_{22}}{\hat{c}_{12}}\big(1 - a_2\big)\right)^{1/\Delta_{12}}, \quad \Delta_{12} = \delta_2^u - 1 + \frac{\delta_2^{ss}\delta_1^{uu}}{\delta_1^u - 1}.$$

Now we insert these terms in the first two equations of (5.10) and solve these equations for $\mu = \mu(a_2)$. This defines in the μ-plane a curve $\mathcal{S}_{\text{2-hom-}p_1}$ of 2-homoclinic orbits to p_1. Indeed this curve can be considered as the graph of a function $\mu_1 = g(\mu_2)$. In order to detect the leading order term of g we consider the first two equations in (5.10) for $a_1 = 0$ or equivalently for $r_{1,2} = 0$. These equations can be solved for μ. Up to higher order terms we find

$$\mu_1 \approx c_{21} r_{2,1}^{\delta_2^{ss}} r_{1,1}, \qquad \mu_2 \approx c_{22} r_{2,1}.$$

Combining this with (5.13) yields

$$\mu_1 \approx c_{21}\big(-c_{21}/c_{11}\big)^{\frac{1}{\delta_1^u-1}}\big(1/c_{22}\big)^{\frac{\delta_1^u\delta_2^{ss}}{\delta_1^u-1}}\mu_2^{\frac{\delta_1^u\delta_2^{ss}}{\delta_1^u-1}}.$$

Next we show that there are **no 2-homoclinic orbits to p_2**. To this end we set $a_2 = 0$ and consider the second equation in (5.11). For $r_{2,1} \neq 0$ this equation is equivalent to

$$-c_{12}(\mu)r_{1,1}^{\delta_1^{uu}}r_{2,1}^{\delta_2^u-1} - c_{22}(\mu) + o(r_{1,1}^{\delta_1^{uu}}r_{2,1}^{\delta_2^u-1}) + o(1) = 0.$$

Since $c_{22}(0) \neq 0$ this equation has no solution for sufficiently small $r_{1,1}$, $r_{2,1}$.

Finally we consider **2-periodic orbits**. In the course of this we proceed in the same way as computing 2-homoclinic orbits to p_1 but just taking $a_1 \in (0,1)$. Instead of (5.13) we arrive at

$$r_{1,1} \approx \left(-\frac{c_{21}}{c_{11}}\cdot\frac{1-a_2^{\delta_2^{ss}}a_1}{1-a_1^{\delta_1^u}}\right)^{\frac{1}{\delta_1^u-1}} r_{2,1}^{\frac{\delta_2^{ss}}{\delta_1^u-1}}. \tag{5.14}$$

We insert this in the second equation in (5.11), factor out $r_{2,1}$, and solve the remaining for $r_{2,1}$. We find

$$r_{2,1} \approx \left(-\frac{c_{22}}{c_{12}}\cdot\frac{1-a_2}{1-a_1^{\delta_1^{uu}}a_2^{\delta_2^u}}\left(-\frac{c_{11}}{c_{21}}\cdot\frac{1-a_1^{\delta_1^u}}{1-a_2^{\delta_1^{ss}}a_1}\right)^{\frac{\delta_1^{uu}}{\delta_1^u-1}}\right)^{1/\Delta_{12}}.$$

Again we insert these terms in the first two equations of (5.10) and solve these equations for $\mu = \mu(a_1, a_2)$. This defines in the μ-plane a region of 2-periodic orbits. This region is bounded by the curve $\mathcal{S}_{2\text{-hom-}p_1}$ and a period-doubling curve $\mathcal{S}_{pd}$. In order to detect the period-doubling curve we consider the limit $a_1 \to 1$ along curves $a_2 = a_1^\alpha$, $\alpha \in \mathbb{R}^+$. We define

$$\lim_{\substack{a_2=a_1^\alpha\\ a_1\to 1}} r_{2,1} =: r_{2,1}(\alpha), \qquad \lim_{\substack{a_2=a_1^\alpha\\ a_1\to 1}} r_{1,1} =: r_{1,1}(\alpha).$$

According to (5.14) we find

$$r_{1,1}(\alpha) \approx \left(-\frac{c_{21}}{c_{11}\delta_1^u}\right)^{\frac{1}{\delta_1^u-1}} r_{2,1}(\alpha)^{\frac{\delta_2^{ss}}{\delta_1^u-1}}.$$

Again inspecting the first two equations in (5.10) we find that up to higher order terms

$$\mu_1 \approx c_{21}r_{2,1}^{\delta_2^{ss}}(\alpha)r_{1,1}(\alpha), \qquad \mu_2 \approx c_{22}r_{2,1}(\alpha).$$

This finally yields the following representation of the period-doubling curve $\mathcal{S}_{pd}$:

$$\mu_1 \approx c_{21}\left(-\frac{c_{21}}{c_{11}\delta_1^u}\right)^{\frac{1}{\delta_1^u-1}}\big(1/c_{22}\big)^{\frac{\delta_1^u\delta_2^{ss}}{\delta_1^u-1}}\mu_2^{\frac{\delta_1^u\delta_2^{ss}}{\delta_1^u-1}}.$$

5.3 Proof of Theorem 2.1(iii)

We assume that $\lambda_1^s(\mu)$ and $\lambda_2^u(\mu)$ are the leading eigenvalues of the equilibria p_1 and p_2, respectively, i.e.

$$|\lambda_1^s(\mu)| < \lambda_1^u(\mu), \quad \lambda_2^u(\mu) < |\lambda_2^s(\mu)|. \tag{5.15}$$

5.3.1 Γ nontwisted

1-periodic orbits. Corresponding to the eigenvalue condition (5.15) we introduce new variables

$$\varpi_1 := \frac{\lambda_1^s(\mu)}{\lambda_1^s(0)}\omega_1, \qquad \varpi_2 := \frac{\lambda_2^u(\mu)}{\lambda_2^u(0)}\omega_2,$$

and subsequently

$$r_1 := e^{2\lambda_1^s(0)\varpi_1}, \qquad r_2 := e^{2\lambda_2^u(0)\varpi_2}.$$

Therefore there exist $\delta_1^u(\mu), \delta_2^{ss}(\mu), \delta_1^{uu}(\mu), \delta_2^s(\mu) > 1$ such that in terms of these new variables the bifurcation equation for 1-periodic orbits (5.1) reads

$$\begin{aligned} \mu_1 + c_{11}(\mu) r_1^{\delta_1^u} - c_{21}(\mu) r_2^{\delta_2^{ss}} r_1 + \tilde{\mathcal{R}}_1(r_1, r_2, \mu) &= 0 \\ \mu_2 + c_{12}(\mu) r_1^{\delta_1^{uu}} r_2 - c_{22}(\mu) r_2^{\delta_2^s} + \tilde{\mathcal{R}}_2(r_1, r_2, \mu) &= 0, \end{aligned} \tag{5.16}$$

where

$$\tilde{\mathcal{R}}_1(r_1, r_2, \mu) = o(r_1^{\delta_1^u}) + o(r_2^{\delta_2^{ss}} r_1), \qquad \tilde{\mathcal{R}}_2(r_1, r_2, \mu) = o(r_1^{\delta_1^{uu}} r_2) + o(r_2^{\delta_2^s}).$$

By Assumption (A 5) and Lemma 4.8 all $c_{ij}(0)$, $i, j = 1, 2$, are positive.

We focus on the wedges Q_2 and the saddle-node curves $\mathcal{S}_{sn}$ marking part of the boundary of Q_2 as they are depicted in the left panel of Figure 2(c):

Equation (5.16) can be solved for

$$\begin{aligned} \mu_1 = \mu_1^*(r_1, r_2) = &\ -c_{11}(0) r_1^{\delta_1^u} + c_{21}(0) r_2^{\delta_2^{ss}} r_1 + o(r_1^{\delta_1^u}) + o(r_2^{\delta_2^{ss}} r_1) \\ \mu_2 = \mu_2^*(r_1, r_2) = &\ -c_{12}(0) r_1^{\delta_1^{uu}} r_2 + c_{22}(0) r_2^{\delta_2^s} + o(r_1^{\delta_1^{uu}} r_2) + o(r_2^{\delta_2^s}). \end{aligned}$$

We start with considering the part of Q_2 lying in the third quadrant. In order to show that for each point of Q_2 there exist two 1-periodic orbits we consider the contour lines of μ_1^* and μ_2^*. For fixed $r_1 > 0$ the function $\mu_2^*(r_2, \cdot)$ has two zeros

$$r_{2,1} = 0 \quad \text{and} \quad r_{2,2} = \frac{c_{12}(0)}{c_{22}(0)} r_1^{\frac{\delta_1^{uu}}{\delta_2^s - 1}} + o\big(r_1^{\frac{\delta_1^{uu}}{\delta_2^s - 1}}\big) > 0,$$

and in between $\mu_2^*(r_2, \cdot)$ has a unique critical point $r_2^*(r_1)$,

$$r_2^*(r_1) = \frac{c_{12}(0)}{\delta_2^s c_{22}(0)} r_1^{\frac{\delta_1^{uu}}{\delta_2^s - 1}} + o\big(r_1^{\frac{\delta_1^{uu}}{\delta_2^s - 1}}\big).$$

Since $\frac{\partial}{\partial r_2}\mu_2^*(r_1, 0) < 0$ this critical point is a minimum. Hence $\mu_2^*(r_1, r_2^*(r_1)) < 0$. We find, since $\delta_1^{uu} > \delta_1^u$, $\delta_2^{ss} > \delta_2^s > 1$ and $c_{11} > 0$, that

$$\mu_1^*(r_1, r_2^*(r_1)) = -c_{11}(0) r_1^{\delta_1^u} + o(r_1^{\delta_1^u}) < 0. \tag{5.17}$$

Further, $\mu_2^*(r_1, r_2^*(r_1))$ considered as a function of r_1 is monotonically decreasing. Altogether we refer to Figure 9 for a visualization.

Therefore, locally around the curve $\{(r_1, r_2^*(r_1)),\ r_1 > 0\}$, the contour lines of μ_2^* are parabola-like curves as depicted in Figure 10.

In Figure 10 also the contours of μ_2^* near the line of the nontrivial zeros are depicted. The same considerations apply to the contours of μ_1^* near its nontrivial zeros. Those yield that the contour lines $\mu_1^* = const < 0$ are shaped as depicted in Figure 10. Altogether, the intersections of the contours of μ_1^* and μ_2^* verify the existence of two 1-periodic orbits for each $(\mu_1, \mu_2) \in Q_2$.

The lower boundary of the wedge Q_2 under consideration is a saddle-node curve $\mathcal{S}_{sn}$ - see again right panel of Figure 2(c). This curve corresponds to the nontransversal intersections of the contours of μ_1^* and μ_2^*. Analytically the saddle-node curve is determined by the singularities of the Jacobian $\frac{\partial(\mu_1^*, \mu_2^*)}{\partial(r_1, r_2)}$,

$$\begin{aligned} 0 = \det \frac{\partial(\mu_1^*, \mu_2^*)}{\partial(r_1, r_2)} =& \big(- c_{11}(0)\delta_1^u r_1^{\delta_1^u - 1} + c_{21}(0) r_2^{\delta_2^{ss}} \big) \big(- c_{12}(0) r_1^{\delta_1^{uu}} + c_{22}(0)\delta_2^s r_2^{\delta_2^s - 1} \big) \\ &+ c_{12}(0) c_{21}(0) \delta_2^{ss} \delta_1^{uu} r_1^{\delta_1^{uu}} r_2^{\delta_2^{ss}} + hot\,, \end{aligned} \tag{5.18}$$

where "*hot*" represents terms of higher order. Near $(r_1, r_2^*(r_1))$ this equation can be solved for

$$r_2^{sn}(r_1) = \frac{c_{12}(0)}{\delta_2^s c_{22}(0)}\, r_1^{\frac{\delta_1^{uu}}{\delta_2^s - 1}} + o\big(r_1^{\frac{\delta_1^{uu}}{\delta_2^s - 1}}\big).$$

The addressed saddle-node curve reads

$$\mathcal{S}_{sn} = \big\{ \big(\mu_1^*(r_1, r_2^{sn}(r_1)), \mu_2^*(r_1, r_2^{sn}(r_1)) \big),\ r_1 > 0 \big\}\,.$$

Similar to (5.17) we find

$$\mu_1^*(r_1, r_2^{sn}(r_1)) = -c_{11}(0) r_1^{\delta_1^u} + o\big(r_1^{\delta_1^u}\big) \quad \text{and} \quad \mu_2^*(r_1, r_2^{sn}(r_1)) = O\big(r_1^{\frac{\delta_2^s \delta_1^{uu}}{\delta_2^s - 1}}\big).$$

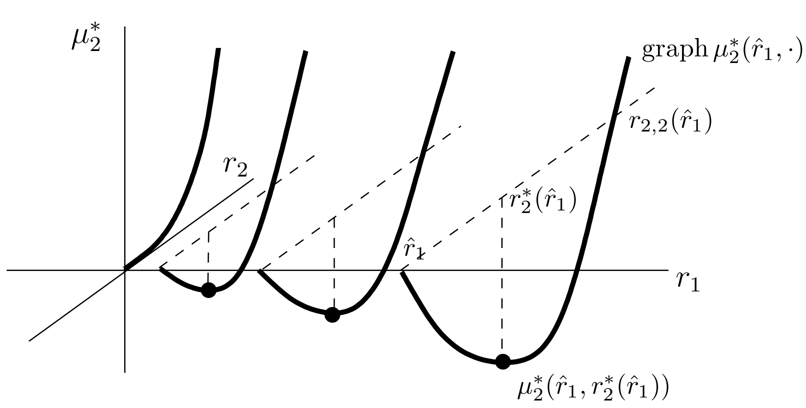


Figure 9: The graph of μ_2^*.

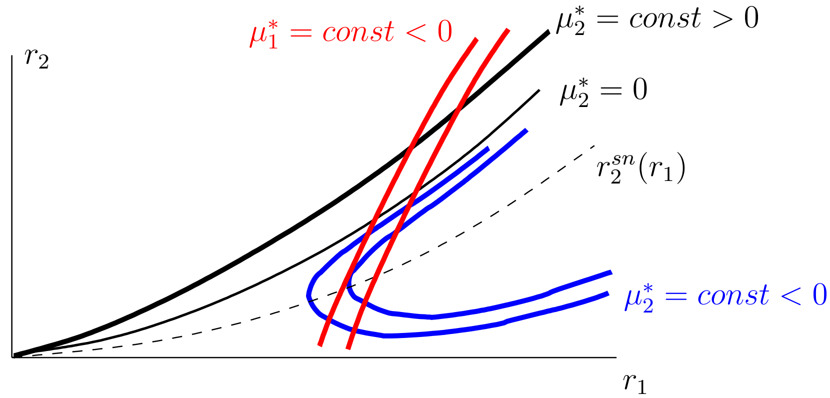


Figure 10: The contour lines of μ_1^* and μ_2^*.

This finally yields that there is an $\alpha > 1$ such that $\mathcal{S}_{sn}$ is the graph of

$$\mu_2 = -(-\mu_1)^\alpha + o((-\mu_1)^\alpha).$$

Similarly, the wedge Q_2 and its corresponding saddle-node curve lying in the first quadrant can be discussed by inspecting the contour lines of μ_1^* and μ_2^* near the counterpart of the line $(r_1, r_2^*(r_1))$, which is defined by the critical points of $\mu_1^*(\cdot, r_2)$.

5.3.2 Γ twisted

According to Assumption (A 5) and Lemma 4.8 it holds that

$$c_{11}(0), c_{22}(0) > 0, \qquad c_{12}(0), c_{21}(0) < 0. \tag{5.19}$$

1-periodic orbits. We adopt the notations introduced in Section 5.3.1. Consequently, equation (5.16) again determines 1-periodic orbits. But the sign condition (5.19) implies that those orbits only exists for (μ_1, μ_2) within the second quadrant.

In order to exclude saddle-node bifurcations of 1-periodic orbits we show simply that the Jacobian $\frac{\partial(\mu_1^*, \mu_2^*)}{(r_1, r_2)}$ has no singularities. To this end we consider the leading order term at the right-hand side of (5.18). The sign condition (5.19) implies that $c_{12}(0)c_{21}(0)\big(\delta_1^{uu} - 1\big)\big(\delta_2^{ss} - 1\big)r_1^{\delta_1^{uu}} r_2^{\delta_2^{ss}}$ is the only positive summand. This one, however, is dominated by the negative term $-c_{11}(0)c_{22}(0)\delta_1^u\delta_2^s r_1^{\delta_1^u - 1} r_2^{\delta_2^s - 1}$. Hence, for sufficiently small r_1, r_2, the addressed Jacobian is nonsingular.

2-periodic and 2-homoclinic orbits. We adopt the notations introduced in Section 5.2.2. The counterpart of (5.11) reads:

$$\begin{aligned} c_{11}(\mu)\big(a_1^{\delta_1^u} - 1\big)r_{1,1}^{\delta_1^u} - c_{21}(\mu)\big(1 - a_2^{\delta_2^{ss}} a_1\big)r_{2,1}^{\delta_2^{ss}} r_{1,1} + \tilde{\mathcal{R}}_{1,1} - \tilde{\mathcal{R}}_{1,2} &= 0 \\ c_{12}(\mu)\big(a_1^{\delta_1^{uu}} a_2 - 1\big)r_{1,1}^{\delta_1^{uu}} r_{2,1} - c_{22}(\mu)\big(1 - a_2^{\delta_2^s}\big)r_{2,1}^{\delta_2^s} + \tilde{\mathcal{R}}_{2,1} - \tilde{\mathcal{R}}_{2,2} &= 0, \end{aligned} \tag{5.20}$$

where

$$\tilde{\mathcal{R}}_{1,1} = o(r_{1,1}^{\delta_1^u}) + o(r_{2,1}^{\delta_2^{ss}} r_{1,1}), \quad \tilde{\mathcal{R}}_{2,1} = o(r_{1,1}^{\delta_1^{uu}} r_{2,1}) + o(r_{2,1}^{\delta_2^s}),$$
$$\tilde{\mathcal{R}}_{1,2} = o(r_{1,1}^{\delta_1^u}) + o(r_{2,1}^{\delta_2^{ss}} r_{1,1}), \quad \tilde{\mathcal{R}}_{2,2} = o(r_{1,1}^{\delta_1^{uu}} r_{2,1}) + o(r_{2,1}^{\delta_2^s}).$$

Again, as in Section 5.2.2, we first consider **2-homoclinic orbits to $\boldsymbol{p_1}$**. The counterpart to (5.12) reads

$$\begin{aligned} -c_{11}(\mu) r_{1,1}^{\delta_1^u} - c_{21}(\mu) r_{2,1}^{\delta_2^{ss}} r_{1,1} + \tilde{\mathcal{R}}_{1,1} - \tilde{\mathcal{R}}_{1,2} &= 0 \\ -c_{12}(\mu) r_{1,1}^{\delta_1^{uu}} r_{2,1} - c_{22}(\mu)\big(1 - a_2^{\delta_2^s}\big) r_{2,1}^{\delta_2^s} + \tilde{\mathcal{R}}_{2,1} - \tilde{\mathcal{R}}_{2,2} &= 0. \end{aligned} \tag{5.21}$$

Now we proceed along the corresponding lines of Section 5.2.2. Note that the first equations of (5.12) and (5.21) coincide. So we arrive at (5.13), and proceeding correspondingly we insert this in the second equation of (5.21). Since $\delta_1^{uu} > \delta_1^u - 1$ and $\delta_2^{ss} > \delta_2^s$ we can factor out $r_{2,1}^{\delta_2^s}$ in the remaining equation. Now we can repeat the considerations of Section 5.2.2 leading to the curve of 2-homoclinic orbits to p_1.

In order to analyse the **2-homoclinic orbits to $\boldsymbol{p_2}$** we set $a_2 = 0$ in (5.20). Due to the "symmetry" in the structure of (5.20) the remaining equation can be treated in the same way as (5.21). This finally leads to the curve of 2-homoclinic orbits to p_2 depicted in the right panel of Figure 2(c).

With our remarks given in the context of 2-homoclinic orbits, it becomes clear that also **2-periodic orbits** and the corresponding period-doubling curves can be discussed in the same way as in Section 5.2.2. This is true for both parts Q^2 depicted in Figure 2(c).

5.4 Proof of Lemma 2.2

5.4.1 k-(2,1) heteroclinic orbits

A k-(2,1) heteroclinic connection may be considered as part of a heteroclinic cycle, together with a 1-(1,2) heteroclinic orbit. In terms of the $\boldsymbol{\omega}$-sequence, this corresponds to a k-periodic sequence with $\omega_{1,1} = \omega_{2,1} = \infty$. Now, let $k > 1$. Consider $\Xi_1 = 0$, $\Xi_k = 0$, and assume that there is a solution $\boldsymbol{\omega}$, μ of these equations. Then this is also a solution of $\Xi_1 - \Xi_k = 0$, i.e.

$$\begin{aligned} c_{11}(\mu) e^{-2\lambda_1^u \omega_{1,2}} + c_{21}(\mu) e^{2\lambda_2^{ss}\omega_{2,k}} e^{2\lambda_1^s \omega_{1,k}} + o\big(e^{-2\lambda_1^u \omega_{1,2}}\big) + o\big(e^{2\lambda_2^{ss}\omega_{2,k}} e^{2\lambda_1^s \omega_{1,k}}\big) &= 0 \\ c_{12}(\mu) e^{-2\lambda_1^{uu}\omega_{1,2}} e^{-2\lambda_2^u \omega_{2,2}} + c_{22}(\mu) e^{2\lambda_2^s \omega_{2,k}} + o\big(e^{-2\lambda_1^{uu}\omega_{1,2}} e^{-2\lambda_2^u \omega_{2,2}}\big) + o\big(e^{2\lambda_2^s \omega_{2,k}}\big) &= 0. \end{aligned}$$

From the first equation it follows that $\omega_{1,2} > -\frac{\lambda_2^{ss}}{\lambda_1^u}\omega_{2,k}$ whereas from the second equation it follows that $\omega_{1,2} < -\frac{\lambda_2^s}{\lambda_1^{uu}}\omega_{2,k}$. But these conditions are mutually exclusive. Therefore the subsystem $\Xi_1 = 0$, $\Xi_k = 0$ has no solution, and hence there are no k-(2,1) heteroclinic orbits for $k > 1$.

5.4.2 k-(1,2) heteroclinic orbits

We now search for k-(1,2) heteroclinic orbits, for $k \geq 2$. We consider the concatenation of a k-(1,2) heteroclinic orbit with Γ_1, which corresponds to a k-periodic Lin orbit with a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1} = \omega_{2,k} = \infty$. Note that $\Xi_k(\boldsymbol{\omega}, \mu) = \xi^\infty(\mu) = 0$ determines the existence of Γ_1. So this equation needs not to be solved, here. Therefore the bifurcation equation for k-(1,2) heteroclinic orbits reads

$$\Xi_1(\boldsymbol{\omega}, \mu) = \ldots = \Xi_{k-1}(\boldsymbol{\omega}, \mu) = 0, \quad \boldsymbol{\omega} \text{ is } k\text{-periodic}, \quad \omega_{1,1} = \omega_{2,k} = \infty.$$

Due to the eigenvalue condition (5.3) these equations can be solved for $\boldsymbol{\omega}$ depending on $\mu \in Q$.

6 Dynamics in the presence of complex eigenvalues

This section is devoted to the proofs of the Theorem 2.3 and of the Lemmas 2.4 and 2.5.

We write the bifurcation equation for orbits staying for all time close to the primary cycle in the following form:

$$\Xi(\boldsymbol{\omega}, \mu) := (\Xi_i(\boldsymbol{\omega}, \mu))_{i \in \mathbb{Z}} = 0, \quad \Xi_i(\boldsymbol{\omega}, \mu) = L(\omega_{1,i+1}, \omega_{2,i}, \mu) + r_i(\boldsymbol{\omega}, \mu), \tag{6.1}$$

where $L(\omega_{1,i+1}, \omega_{2,i}, \mu)$ denotes leading order terms (which we determine below) and $r_i(\boldsymbol{\omega}, \mu)$ denotes higher order terms.

In order to construct the sets $\mathcal{S}_\mu^N$ we show that the truncated form of the bifurcation equation, $L(\omega_1, \omega_2, 0) = 0$, has an infinite set of nondegenerate (transversal) solutions $(\hat{\omega}_1(k), \hat{\omega}_2(k))$, ($k \in \mathbb{N}$) with $(\hat{\omega}_1(k), \hat{\omega}_2(k)) \to (\infty, \infty)$ as $k \to \infty$. It turns out that the solutions of $L(\omega_1, \omega_2, 0) = 0$ can be interpreted as (transversal) intersections of a line $\mathfrak{L}_1$ and a spiral $\mathfrak{S}_2$ ((RC) case) or two spirals $\mathfrak{S}_1$ and $\mathfrak{S}_2$ ((CC) case). We refer to Figure 11 for a visualisation.

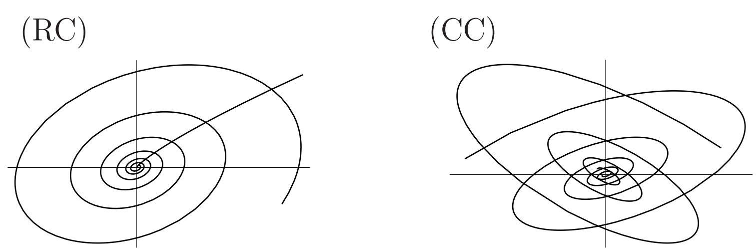


Figure 11: Graphical representations of the solutions of $L(\omega_1, \omega_2, 0) = 0$.

We show that each such solution corresponds to a periodic orbit near Γ. And what is more we show that for each sequence $\hat{\boldsymbol{\omega}}$ built of finitely many of these $(\hat{\omega}_1(k), \hat{\omega}_2(k))$ there exists a sequence $\boldsymbol{\omega}_{\hat{\omega}}$ which solves the full bifurcation equation.

6.1 Proof of Theorem 2.3 – the eigenvalue case (RC)

We assume that λ_1^u is real and write $\lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu)$, $\rho_2 > 0$, and consider the bifurcation equation in the form (6.1).

6.1.1 The set $\mathcal{S}_\mu^N$

We stipulate (cf. (6.1))

$$L(\omega_1, \omega_2, \mu) = \begin{pmatrix} \mu_1 + c_{11}e^{-2\lambda_1^u\omega_1} & -c_{21}e^{-2\rho_2\omega_2}\sin(2\phi_2\omega_2 + \varphi_{21}) \\ \mu_2 & -c_{22}e^{-2\rho_2\omega_2}\sin(2\phi_2\omega_2 + \varphi_{22}) \end{pmatrix}. \tag{6.2}$$

Hence, according to Lemma 4.11, we have

$$r_i(\boldsymbol{\omega}\mu) = \begin{pmatrix} \mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu) \\ c_{12}(\mu)e^{-2\lambda_1^{uu}\omega_{1,i+1}}e^{-2\lambda_2^u\omega_{2,i}} + \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix}. \tag{6.3}$$

The line $\mathfrak{L}_1(t, 0)$ and spiral $\mathfrak{S}_2(t, 0)$ which we addressed at the beginning of this section are defined by

$$\mathfrak{L}_1(t, \mu) := \begin{pmatrix} \mu_1 + c_{11}e^{-2\lambda_1^u t} \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \mathfrak{S}_2(t, \mu) := \begin{pmatrix} c_{21}e^{-2\rho_2 t}\sin(2\phi_2 t + \varphi_{21}) \\ c_{22}e^{-2\rho_2 t}\sin(2\phi_2 t + \varphi_{22}) \end{pmatrix}.$$

Due to $\varphi_{21} - \varphi_{22} \neq 0 \pmod{\pi}$, cf. Lemma 4.10, $\mathfrak{S}_2(t, \mu)$ is indeed a spiral. Note that the appearing “constants” may depend on μ.

Solutions $(\hat{\omega}_1, \hat{\omega}_2)$ of $L(\omega_1, \omega_2, 0) = 0$ correspond to the infinite set of transversal intersections of the line $\mathfrak{L}_1(t, 0)$ and the spiral $\mathfrak{S}_2(t, 0)$. These intersections accumulate to the origin, and therefore have times $(\hat{\omega}_1(k), \hat{\omega}_2(k))$, $k \in \mathbb{N}$, that tend to infinity as $k \to \infty$.

More precisely, let $\hat{\omega}_2^*$ be the “first” value at which $\mathfrak{S}_2(t, 0)$ intersects the line $\mathfrak{L}_1(t, 0)$. Then, inspecting $L(\omega_1, \omega_2, 0) = 0$ we find

$$\hat{\omega}_2(k) = \hat{\omega}_2^* + \frac{k\pi}{\phi_2} \quad \text{and} \quad \hat{\omega}_1(k) = \frac{\rho_2\hat{\omega}_2(k)}{\lambda_1^u} + \hat{C}_1, \tag{6.4}$$

where $\hat{C}_1$ is a constant, i.e. it does not depend on k. We define

$$\boldsymbol{\Omega}_{k_0} := \{(\hat{\omega}_1(k), \hat{\omega}_2(k)),\ k \geq k_0\}, \tag{6.5}$$

for a sufficiently large k_0, and for those k_0 and some $N \in \mathbb{N}$ we define further

$$\boldsymbol{\Omega}_{k_0,N} := \{(\hat{\omega}_1(k), \hat{\omega}_2(k)),\ k = k_0, \ldots, k_0 + N - 1\}. \tag{6.6}$$

With that we define finally the following set of sequences $\boldsymbol{\omega} = ((\omega_{1,i}, \omega_{2,i}))_{i\in\mathbb{Z}}$:

$$\boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}} := \{\boldsymbol{\omega} : (\omega_{1,i+1}, \omega_{2,i}) \in \boldsymbol{\Omega}_{k_0,N}\}.$$

We show that for any sequence $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}$ we may solve $\boldsymbol{\Xi}(\boldsymbol{\omega}, \mu) = 0$ near $(\hat{\boldsymbol{\omega}}, 0)$.

The full bifurcation equation $\Xi(\boldsymbol{\omega},\mu)=0$ is equivalent to the fixed point equation

$$\boldsymbol{\omega}=\boldsymbol{\omega}-\Big[D_1\mathbf{L}(\hat{\boldsymbol{\omega}},0)\Big]^{-1}\Xi(\boldsymbol{\omega},\mu)=:\boldsymbol{\mathcal{A}}(\boldsymbol{\omega},\mu), \tag{6.7}$$

where

$$\mathbf{L}(\boldsymbol{\omega},\mu):=(L(\omega_{1,i+1},\omega_{2,i},\mu))_{i\in\mathbb{Z}}.$$

In what follows we consider $\boldsymbol{\mathcal{A}}$ as a mapping $\boldsymbol{\mathcal{A}}:(l^\infty\times l^\infty)\times\mathbb{R}^2\to(l^\infty\times l^\infty)$. We show that $\boldsymbol{\mathcal{A}}(\cdot,\mu)$ is a contractive mapping of some closed neighbourhood of $\hat{\boldsymbol{\omega}}$ into itself. The contraction principle enures a unique solution $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}$ within this neighbourhood.

To simplify matters we first consider 1-periodic orbits near the primary T-point cycle. To this end we consider the fixed point equation (6.7) near one-periodic sequences $\hat{\boldsymbol{\omega}}$, i.e.

$$(\hat{\omega}_{1,i},\hat{\omega}_{2,i})=(\hat{\omega}_1(k),\hat{\omega}_2(k))\in\mathbf{\Omega}_{k_0}.$$

In this case (6.7) reduces to

$$(\omega_1,\omega_2)=(\omega_1,\omega_2)-\Big[D_1L(\hat{\omega}_1(k),\hat{\omega}_2(k),0)\Big]^{-1}\Xi((\omega_1,\omega_2),\mu)=:A_k((\omega_1,\omega_2),\mu),$$

and

$$\Xi((\omega_1,\omega_2),\mu)=L(\omega_1,\omega_2,\mu)+r((\omega_1,\omega_2),\mu).$$

Lemma 6.1. *There exist a $k_0\in\mathbb{N}$ and constants d, $0<d<\frac{\pi}{3\phi_2}$, and $d_\mu(k)$ such that for all $k\geq k_0$ and all μ with $|\mu|<d_\mu(k)$, the mapping $A_k(\cdot,\mu)$ has a unique fixed point in $B[(\hat{\omega}_1(k),\hat{\omega}_2(k)),d]$, the closed ball in $\mathbb{R}^2$ centered at $(\hat{\omega}_1(k),\hat{\omega}_2(k))$ with radius d.*

Proof. To prove the lemma we employ the contraction principle, cf. [7, Chapter 2, Theorem 2.2].

Throughout the proof we use the abbreviations $\omega:=(\omega_1,\omega_2)$ and $\hat{\omega}(k):=(\hat{\omega}_1(k),\hat{\omega}_2(k))$.

a) Preliminary estimates:

According to (6.2) and (6.4) we find the following estimate for D_1L^{-1}:

$$0<|(D_1L(\hat{\omega}(k),0))^{-1}|<C_{DL^{-1}}e^{2k(\rho_2/\phi_2)\pi}, \tag{6.8}$$

where the constant $C_{DL^{-1}}>1$ does not depend on k.

Similarly we find for the second derivative of L

$$0<|D_1^2L(\hat{\omega}(k),\mu)|<C_{D^2L}e^{-2k(\rho_2/\phi_2)\pi},$$

where the constant $C_{D^2L}>1$ can be chosen to be independent of k. Accordingly we find

$$0<\max_{\omega\in B[\hat{\omega}(k),d]}|D_1^2L(\omega,\mu)|<\hat{C}_{D^2L}(d)e^{-2k(\rho_2/\phi_2)\pi},$$

with $\hat{C}_{D^2L}(d)\to C_{D^2L}$ as $d\to 0$.

For the derivative of L with respect to μ we find (for sufficiently large k)

$$|D_2L(\hat{\omega}(k),\mu)|<2. \tag{6.9}$$

Further, according to (6.3), Lemma 4.11 and Remark 4.13 we find:

$$\max_{\omega \in B[\hat{\omega}(k),d]} |D_1 r(\hat{\omega}(k), \mu)| = o(e^{-2k(\rho_2/\phi_2)\pi}), \text{ as } \quad k \to \infty. \tag{6.10}$$

b) Setting of the constants d and $d_\mu(k)$:

With the above constants we choose the constant d such that

$$d := \min\left\{\frac{\pi}{3\phi_2}, \frac{1}{2(e^{2\pi} C_{D^2L} C_{DL^{-1}} - 1)}\right\}, \tag{6.11}$$

and the constant $d_\mu(k)$

$$d_\mu(k) := \frac{d}{8\, C_{DL^{-1}}} e^{-2k(\rho_2/\phi_2)\pi} \tag{6.12}$$

c) $A_k(\cdot,\mu)$ is a "mapping into":

$$|\hat{\omega}(k) - A_k(\omega,\mu)| \le |\hat{\omega}(k) - A_k(\hat{\omega}(k),\mu)| + |A_k(\hat{\omega}(k),\mu) - A_k(\omega,\mu)|.$$

First we consider the first addend on the right-hand side

$$|\hat{\omega}(k) - A_k(\hat{\omega}(k),\mu)| = |D_1 L(\hat{\omega}(k),0)^{-1}\big(L(\hat{\omega}(k),\mu) + r(\hat{\omega}(k),\mu)\big)|$$

Expanding $L(\hat{\omega}(k),\cdot)$ at $\mu = 0$ up to first order terms and further exploiting (6.9) and that $L(\hat{\omega}(k),0) = 0$ we find

$$|\hat{\omega}(k) - A_k(\hat{\omega}(k))| \le 2|\mu|\, |D_1 L(\hat{\omega}(k),0)^{-1}| + |D_1 L(\hat{\omega}(k),0)^{-1} r(\hat{\omega}(k),0)|.$$

Due to (6.8), (6.12) the first addend is less than $d/4$. Finally, since $|D_1 L(\hat{\omega}(k),0)^{-1} r(\hat{\omega}(k),0)| \to 0$, as $k \to \infty$, there is a k_0^1 such that for all $k \ge k_0^1$ the following estimate holds true:

$$|\hat{\omega}(k) - A_k(\hat{\omega}(k))| < \frac{d}{2}, \quad \forall |\mu| < d_\mu(k).$$

Next we consider

$$|A_k(\hat{\omega}(k),\mu) - A_k(\omega,\mu)| \le \max_{\tau \in B[\hat{\omega}(k),d]} |D_1 A(\tau,\mu)|\, |\omega - \hat{\omega}(k)|, \tag{6.13}$$

$$D_1 A(\tau,\mu) = D_1 L(\hat{\omega}(k),0)^{-1}\big(D_1 L(\hat{\omega}(k),\mu) - D_1 L(\tau,\mu)\big) - D_1 L(\hat{\omega}(k),0)^{-1} D_1 r(\tau,\mu). \tag{6.14}$$

Using the mean value theorem and the estimates of part a) of the proof we get

$$|D_1 L(\hat{\omega}(k),\mu) - D_1 L(\tau,\mu)| \le \max_{t \in B[\hat{\omega}(k),d]} |D_1^2 L(t,\mu)|\, |\hat{\omega}(k) - \tau| < C_{D^2L} e^{-2(k-1)(\rho_2/\phi_2)\pi} d.$$

Therefore, again invoking part a) of the proof, the first addend on the right-hand side of (6.14) can be estimated by

$$|D_1 L(\hat{\omega}(k),0)^{-1}\big(D_1 L(\hat{\omega}(k),\mu) - D_1 L(\tau,\mu)\big)| < e^{2(\rho_2/\phi_2)\pi} C_{DL^{-1}} C_{D^2L}\, d.$$

Further, in accordance with (6.8) and (6.10) we may choose a k_0^2 sufficiently large so that for all $k > k_0^2$

$$|(D_1 L(\hat{\omega}(k), 0))^{-1} D_1 r(\tau, \mu)| < d.$$

Plugging these estimates in (6.13) and taking into consideration (6.11) yields for all $k \geq k_0^2$

$$|A_k(\hat{\omega}(k), \mu) - A_k(\omega, \mu)| < \left(e^{2(\rho_2/\phi_2)\pi} C_{DL^{-1}} C_{D^2L} - 1\right) d^2 < \frac{d}{2}.$$

Set now

$$k_0 := \max\{k_0^1, k_0^2\}.$$

Then $A_k(\cdot, \mu)$ maps for all $k \geq k_0$ and all $|\mu| < d_\mu(k)$ the closed ball $B[\hat{\omega}(k), d]$ into itself.

d) $A_k(\cdot, \mu)$ is contractive:

In part c) of the proof, cf. analysis of (6.14), we have shown that for all $k \geq k_0$ and all $|\mu| < d_\mu(k)$

$$\max_{\tau \in B[\hat{\omega}(k), d]} |DA_k(\tau, \mu)| < \left(e^{2(\rho_2/\phi_2)\pi} C_{DL^{-1}} C_{D^2L} - 1\right) d.$$

With (6.11) we infer that for all those k and μ

$$\max_{\tau \in B[\hat{\omega}(k), d]} |DA_k(\tau, \mu)| < \frac{1}{2}.$$

■

Next we turn towards solutions of (6.7).

Lemma 6.2. *Assume the eigenvalue case (RC). Fix any $N \in \mathbb{N}$. There exist a $k_0 \in \mathbb{N}$ and constants d, $0 < d < \frac{\pi}{3\phi_2}$, and $d_\mu(k)$ such that for all $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}$ and for all μ with $|\mu| < d_\mu(k_0, N)$, equation (6.7) has a unique solution $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$ in $B[\hat{\boldsymbol{\omega}}, d]$.*

Proof. The proof follows along the lines of the proof of Lemma 6.1, with some modifications. We adopt the notations as introduced above.

According to (6.6) the building blocks of $\hat{\boldsymbol{\omega}}$ are N consecutive intersections of the line $\mathfrak{L}_1$ and the spiral $\mathfrak{S}_2$.

The key observation at the heart of the proof of Lemma 6.1 is that $\mathbf{L}(\boldsymbol{\omega}, \mu)$ decouples. We have $|\left(D_1 L(\omega_{1,i+1}, \omega_{2,i}, 0)\right)^{-1}| < C_{DL^{-1}} e^{2(k_0+N-1)(\rho_2/\phi_2)\pi}$ and hence

$$|\left(D_1 \mathbf{L}(\hat{\boldsymbol{\omega}}, 0)\right)^{-1}| < C_{DL^{-1}} e^{2(k_0+N-1)(\rho_2/\phi_2)\pi}. \tag{6.15}$$

Further we find

$$\max_{\boldsymbol{\omega} \in B[\hat{\boldsymbol{\omega}}, d]} |D_1^2 \mathbf{L}(\boldsymbol{\omega}, \mu)| < e^{2(\rho_2/\phi_2)\pi} C_{D^2L} e^{-2k_0(\rho_2/\phi_2)\pi}, \tag{6.16}$$

and

$$\max_{\boldsymbol{\omega} \in B[\hat{\boldsymbol{\omega}}, d]} D_1 \mathbf{r}(\boldsymbol{\omega}, \mu) = o(e^{2k_0(\rho_2/\phi_2)\pi}),$$

where $\mathbf{r} := (r_i)_{i\in\mathbb{Z}}$.

Finally we stipulate

$$d := \min\left\{\frac{\pi}{3\phi_2},\ \frac{1}{2(e^{2N(\rho_2/\phi_2)\pi}C_{D^2L}C_{DL^{-1}}-1)}\right\}.$$

and

$$d_\mu(k_0,N) := \frac{d}{8\,C_{DL^{-1}}}e^{-2(k_0+N-1)(\rho_2/\phi_2)\pi}.$$

The remainder of the proof is analogous to parts c) and d) of the proof of Lemma 6.1. ∎

Finally we define the sets $\mathcal{S}_\mu^N$ stated in Theorem 2.3. To this end we proceed from a set $\mathbf{\Omega}_{k_0,N}$ under the terms of Lemma 6.2. According to that lemma, for any sequence $\hat{\boldsymbol{\omega}} \in \mathbf{\Omega}_{k_0,N}^{\mathbb{Z}}$ we may solve $\Xi(\boldsymbol{\omega},\mu)=0$ near $(\hat{\boldsymbol{\omega}},0)$ to find $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$. Note that moreover $\operatorname{dom}\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot)$ does not depend on $\hat{\boldsymbol{\omega}} \in \mathbf{\Omega}_{k_0,N}^{\mathbb{Z}}$. By $x_{\hat{\boldsymbol{\omega}}}(\mu)(\cdot)$ we denote the corresponding solution of (1.1), see also (3.7) and (3.18),

$$x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = q_1^+(\mu)(0) + v_{1,1}^+(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu),\mu)(0),$$

and define

$$\mathcal{S}_\mu^N := \{x_{\hat{\boldsymbol{\omega}}}(\mu)(0) : \hat{\boldsymbol{\omega}} \in \mathbf{\Omega}_{k_0,N}^{\mathbb{Z}}\}.$$

6.1.2 Shift dynamics

The verification of topological conjugacy is mainly based on the continuous dependence of Ξ (see (3.1)) on sequences $\boldsymbol{\omega}$ in spaces of sequences equipped with the product topology. Those ideas go back to similar considerations in [21, 34].

The actual proof follows lines of argument similar to those developed in [17].

We introduce a shift operator on $\mathbf{\Omega}_{k_0,N}^{\mathbb{Z}}$:

$$\zeta : \mathbf{\Omega}_{k_0,N}^{\mathbb{Z}} \to \mathbf{\Omega}_{k_0,N}^{\mathbb{Z}}, \quad \boldsymbol{\omega} \mapsto \boldsymbol{\tau}, \quad (\tau_{1,i},\tau_{2,i}) := (\omega_{1,i+2},\omega_{2,i+1}).$$

Now the system $(\mathbf{\Omega}_{k_0,N}^{\mathbb{Z}},\zeta)$ is a full shift on N symbols. For the further analysis let $\mathbf{\Omega}_{k_0,N}^{\mathbb{Z}} \cong \{(\omega_1^k,\omega_2^k),\ k=1,\dots,N\}^{\mathbb{Z}}$ be equipped with the product topology.

There is a canonical one-to-one mapping

$$h_\mu : \mathbf{\Omega}_{k_0,N}^{\mathbb{Z}} \to \mathcal{S}_\mu^N, \quad \hat{\boldsymbol{\omega}} \mapsto x_{\hat{\boldsymbol{\omega}}}(\mu)(0). \tag{6.17}$$

Let $\mathcal{U}$ be a sufficiently small neighbourhood of the primary heteroclinic cycle. We denote the first-return map on Σ_1 by Π_μ. Then $x \in \operatorname{dom}\Pi_\mu$ if there is a $t^* > 0$ such that $\phi_\mu^{t^*}(x) \in \Sigma_1$, (where $\{\phi_\mu^t(\cdot)\}$ denotes the flow of (1.1)), and $\phi_\mu^t(x) \notin \Sigma_1$, $\phi_\mu^t(x) \in \mathcal{U}$, for all $t \in (0,t^*)$. We claim that $\mathcal{S}_\mu^N \subset \operatorname{dom}\Pi_\mu$, that $\mathcal{S}_\mu^N$ is Π_μ invariant, and moreover

$$\Pi_\mu \circ h_\mu = h_\mu \circ \zeta. \tag{6.18}$$

The latter equation can be concluded from the uniqueness part of Theorem 3.2 and the construction of $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}$ as follows. We have

$$\Pi_\mu x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\hat{\boldsymbol{\omega}}}(\mu)(2(\omega_{1,1}(\mu) + \omega_{2,1}(\mu))) = q_1^+(0) + \bar{v}_{1,2}^+(\boldsymbol{\omega}(\mu), \mu)(0),$$

and hence

$$\Pi_\mu x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\zeta\hat{\boldsymbol{\omega}}}(\mu)(0).$$

This is just another representation of (6.18).

Equation (6.18) means that $(\mathcal{S}_\mu^N, \Pi_\mu)$ is conjugated to $(\boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}, \zeta)$. So, in order to prove topological conjugacy, as claimed in Theorem 2.3, it remains to prove that h_μ is a homeomorphism.

Lemma 6.3. *The mapping h_μ, defined in* (6.17)*, is a homeomorphism.*

Proof. We consider h_μ as a composition of mappings

$$\mathfrak{H} : \boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}} \to \mathfrak{O} := \{\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu) : \hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}\}, \quad \hat{\boldsymbol{\omega}} \mapsto \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu),$$

and

$$\mathfrak{h} : \mathfrak{O} \to \mathcal{S}_\mu^N, \quad \boldsymbol{\omega} \mapsto q_1^+(\mu)(0) + v_{1,1}^+(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu), \mu)(0).$$

First we show that $\mathfrak{H}$ is a homeomorphism, where both $\boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}$ and $\mathfrak{O}$ are considered to be equipped with the product topology. We start with showing that $\mathfrak{O}$ is compact. Let ρ be small enough that for any two (different) elements of $\boldsymbol{\Omega}_{k_0,N}$ the closed balls with radius ρ centred at these elements do not intersect:

$$B[(\omega_1^i, \omega_2^i), \rho] \cap B[(\omega_1^j, \omega_2^j), \rho] = \emptyset, \quad (\omega_1^i, \omega_2^i), (\omega_1^j, \omega_2^j) \in \boldsymbol{\Omega}_{k_0,N}, \quad i \neq j.$$

By construction $\mathfrak{O} \subset \left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}}$. Since $\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]$ is compact, the set of sequences $\left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}}$ is also compact by the Tychonoff theorem, see [9].

So, in order to verify the compactness of $\mathfrak{O}$ it remains to show that $\mathfrak{O}$ is closed. The set $\mathfrak{O}$ however is the set of zeros of the mapping $\Xi(\cdot, \mu) : \left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}} \to l^\infty_{\mathbb{R}^2}$. By [26, Lemma 3.4] this mapping is continuous, where the spaces are equipped with the product topology. Altogether this proves that $\mathfrak{O}$ is compact.

Consider now the one-to-one map

$$\mathfrak{H}^{-1} : \mathfrak{O} \to \boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}.$$

Obviously $\mathfrak{H}^{-1}$ is continuous. Due to the compactness of $\mathfrak{O}$ also $\mathfrak{H}$ is continuous – hence $\mathfrak{H}$ is a homeomorphism.

Next we consider $\mathfrak{h}$. Again by invoking [26, Lemma 3.4] or its proof respectively, we find that $\mathfrak{h}$ is continuous. (Again both $\mathfrak{O}$ and $\boldsymbol{\Omega}_{k_0,N}^{\mathbb{Z}}$ should be equipped with the product topology.) Once more the compactness of $\mathfrak{O}$ gives that $\mathfrak{h}$ is a homeomorphism.

All in all $h_\mu = \mathfrak{h} \circ \mathfrak{H}$ is a homeomorphism and therefore $(\mathcal{S}_\mu^N, \Pi_\mu)$ is topologically conjugated to the full shift on N symbols. ■

To complete the proof of the (RC) case we remark that the sets $\mathcal{S}_\mu^N$ satisfy the conditions of the theorem.

6.2 Proof of Theorem 2.3 – the eigenvalue case (CC)

This section deals with the case of two complex eigenvalues λ_1^u and λ_2^s. We write

$$\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu), \quad \lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu),$$

where both, $\phi_1(\mu)$ and $\phi_2(\mu)$ are greater than zero.

6.2.1 Analysis of spirals

The leading order terms for the case (CC) are as follows, cf. Lemma 4.12 and (6.1):

$$L(\omega_1, \omega_2, \mu) = \begin{pmatrix} \mu_1 + c_{11}e^{-2\rho_1\omega_1}\sin(2\phi_1\omega_1 + \varphi_{11}) - c_{21}e^{-2\rho_2\omega_2}\sin(2\phi_2\omega_2 + \varphi_{21}) \\ \mu_2 + c_{12}e^{-2\rho_1\omega_1}\sin(2\phi_1\omega_1 + \varphi_{12}) - c_{22}e^{-2\rho_2\omega_2}\sin(2\phi_2\omega_2 + \varphi_{22}) \end{pmatrix}. \tag{6.19}$$

Further, the residual terms r_i, cf. (6.1), take the form

$$r_i(\boldsymbol{\omega}\mu) = \begin{pmatrix} \mathcal{R}_{1,i}(\boldsymbol{\omega}, \mu) \\ \mathcal{R}_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix},$$

with $\mathcal{R}_{j,i}$ in accordance with Lemma 4.12.

In this case $L(\omega_1, \omega_2, \mu) = 0$ can be interpreted in a geometric way as describing the set of intersections of the two logarithmic spirals $\mathfrak{S}_1(\cdot, \mu)$ and $\mathfrak{S}_2(\cdot, \mu)$:

$$\mathfrak{S}_1(t, \mu) := \begin{pmatrix} \mu_1 + c_{11}e^{-2\rho_1 t}\sin(2\phi_1 t + \varphi_{11}) \\ \mu_2 + c_{12}e^{-2\rho_1 t}\sin(2\phi_1 t + \varphi_{12}) \end{pmatrix}, \quad \mathfrak{S}_2(t, \mu) := \begin{pmatrix} c_{21}e^{-2\rho_2 t}\sin(2\phi_2 t + \varphi_{21}) \\ c_{22}e^{-2\rho_2 t}\sin(2\phi_2 t + \varphi_{22}) \end{pmatrix}.$$

Note that $\mathfrak{S}_i(\cdot, \mu)$, are indeed spirals since $\varphi_{j1} - \varphi_{j2} \neq 0 \,(\bmod\, \pi)$, $j = 1, 2$, cf. Lemma 4.12.

We continue with analysing the solutions of equations $L(\omega_1, \omega_2, \mu) = 0$. For ease of notation in this section, we drop the μ-dependence of variables inside $L = 0$. Then the equation reads

$$\begin{pmatrix} \mu_1 + e^{-2\rho_1\omega_1}c_{11}\sin(2\phi_1\omega_1 + \varphi_{11}) - e^{-2\rho_2\omega_2}c_{21}\sin(2\phi_2\omega_2 + \varphi_{21}) \\ \mu_2 + e^{-2\rho_1\omega_1}c_{12}\sin(2\phi_1\omega_1 + \varphi_{12}) - e^{-2\rho_2\omega_2}c_{22}\sin(2\phi_2\omega_2 + \varphi_{22}) \end{pmatrix} = 0 \tag{6.20}$$

The following Lemma defines a renormalisation of equations (6.20).

Lemma 6.4. *There exists an invertible change of coordinates such that equations* (6.20) *take the form*

$$\begin{pmatrix} \mu_1 + e^{-\rho_1\omega_1}\sin\omega_1 & -e^{-\rho_2\omega_2}c_{21}\sin\omega_2 \\ \mu_2 + e^{-\rho_1\omega_1}\sin(\omega_1 + \varphi_{12}) & -e^{-\rho_2\omega_2}c_{22}\sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0 \tag{6.21}$$

Proof. We define the variables

$$\begin{aligned} \hat{\omega}_1 &= 2\phi_1\omega_1 + \varphi_{11}, & \hat{c}_{11} &= e^{\rho_1\varphi_{11}/\phi_1}c_{11}, & \hat{c}_{21} &= e^{\rho_2\varphi_{21}/\phi_2}c_{21}, \\ \hat{\omega}_2 &= 2\phi_2\omega_2 + \varphi_{21}, & \hat{c}_{12} &= e^{\rho_1\varphi_{11}/\phi_1}c_{12}, & \hat{c}_{22} &= e^{\rho_2\varphi_{21}/\phi_2}c_{22}, \\ \hat{\varphi}_{12} &= \varphi_{12} - \varphi_{11}, & \hat{\varphi}_{22} &= \varphi_{22} - \varphi_{21}, & \hat{\rho}_1 &= \rho_1/\phi_1, \\ \hat{\rho}_2 &= \rho_2/\phi_2, \end{aligned}$$

under which (6.20) becomes

$$\begin{pmatrix} \mu_1 + e^{-\hat{\rho}_1\hat{\omega}_1}\hat{c}_{11}\sin\hat{\omega}_1 & -e^{-\hat{\rho}_2\hat{\omega}_2}\hat{c}_{21}\sin\hat{\omega}_2 \\ \mu_2 + e^{-\hat{\rho}_1\hat{\omega}_1}\hat{c}_{12}\sin(\hat{\omega}_1+\hat{\varphi}_{12}) & -e^{-\hat{\rho}_2\hat{\omega}_2}\hat{c}_{22}\sin(\hat{\omega}_2+\hat{\varphi}_{22}) \end{pmatrix} = 0$$

Now divide the first equation by $\hat{c}_{11}$ and the second equation by $\hat{c}_{12}$. Defining $\bar{c}_{21} = \hat{c}_{21}/\hat{c}_{11}$, $\bar{c}_{22} = \hat{c}_{22}/\hat{c}_{12}$, $\hat{\mu}_1 = \mu_1/\hat{c}_{11}$, $\hat{\mu}_2 = \mu_2/\hat{c}_{12}$ and dropping hats/bars provides equation (6.21).

Note that the new variables in equation (6.21) have the properties $\varphi_{12}, \varphi_{22} \neq 0 \,(\mathrm{mod}\ \pi)$ and $c_{21}, c_{22} \neq 0$. ■

Given the above change of variables, we define the spirals $\mathfrak{S}_1(t,\mu)$, $\mathfrak{S}_2(t,\mu)$ in the plane as before:

$$\mathfrak{S}_1(t,\mu) := \begin{pmatrix} \mu_1 + e^{-\rho_1 t}\sin t \\ \mu_2 + e^{-\rho_1 t}\sin(t+\varphi_{12}) \end{pmatrix}, \quad \mathfrak{S}_2(t,\mu) := \begin{pmatrix} e^{-\rho_2 t}c_{21}\sin t \\ e^{-\rho_2 t}c_{22}\sin(t+\varphi_{22}) \end{pmatrix}. \tag{6.22}$$

We consider the bifurcation equations in these new coordinates. We have

$$L(\omega_1,\omega_2,\mu) = \mathfrak{S}_1(\omega_1,\mu) - \mathfrak{S}_2(\omega_2,\mu) \tag{6.23}$$

with $\mathfrak{S}_i$, $i = 1,2$, as given in (6.22).

We first consider solutions to the equation (6.21) where $\mu_1 = \mu_2 = 0$, or in other words $L(\omega_1,\omega_2,0) = 0$.

We distinguish between the cases where the ratio of ρ_1/ρ_2 is irrational and where this ratio is rational.

Irrational ratio ρ_1/ρ_2. Consider the two spirals, where we suppose without loss of generality that $\rho_1 < \rho_2$

$$\mathfrak{S}_1(\omega_1,0) = e^{-\rho_1\omega_1}\begin{pmatrix} \sin\omega_1 \\ \sin(\omega_1+\varphi_{12}) \end{pmatrix}, \qquad \mathfrak{S}_2(\omega_2,0) = e^{-\rho_2\omega_2}\begin{pmatrix} c_{21}\sin\omega_2 \\ c_{22}\sin(\omega_2+\varphi_{22}) \end{pmatrix}. \tag{6.24}$$

The following lemma is at the core of our analysis of intersections of the spirals $\mathfrak{S}_1$ and $\mathfrak{S}_2$ with irrational ratio ρ_1/ρ_2.

Lemma 6.5. *For fixed $N \in \mathbb{N}$, there exists a sequence $(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k))_{k\in\mathbb{N}}$, $i = 1,\ldots,N$, with $\hat{\omega}_j^1(k) < \hat{\omega}_j^2(k) < \ldots < \hat{\omega}_j^N(k)$, $j = 1,2$, $k \in \mathbb{N}$, that has the following properties:*

(i) $\hat{\omega}_1^i(k), \hat{\omega}_2^i(k) \to \infty$ *as* $k \to \infty$, $i = 1,\ldots,N$,

(ii) $\mathfrak{S}_1(\hat{\omega}_1^i(k),0) = \mathfrak{S}_2(\hat{\omega}_2^i(k),0)$, $i = 1,\ldots,N$, $k \in \mathbb{N}$,

(iii) *There are positive constants W_1^N and W_2^N, such that $\hat{\omega}_1^N(k) - \hat{\omega}_1^1(k) < W_1^N$ and $\hat{\omega}_2^N(k) - \hat{\omega}_2^1(k) < W_2^N$ for all $k \in \mathbb{N}$,*

(iv) *The points* $(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k))$, $k \in \mathbb{N}$, $i = 1, \dots, N$ *correspond to transversal intersections of the two spirals* $\mathfrak{S}_1(\omega_1, 0)$ *and* $\mathfrak{S}_2(\omega_2, 0)$, *and there exists a constant* $\theta_N^* > 0$ *that is independent of* k *and* i, *such that the angle between the spiral tangents at each intersection point is greater than* θ_N^*.

To begin with, we need to prove some basic properties regarding the two spirals $\mathfrak{S}_1$ and $\mathfrak{S}_2$ and their tangencies. The following Lemma says that tangential intersections of the two spirals are rare; in particular, there are at most two straight lines through the origin on which a tangential intersection is possible.

Lemma 6.6. *Let* (ω_1, ω_2) *be such that* $\mathfrak{S}_1(\omega_1, 0)$ *and* $\mathfrak{S}_2(\omega_2, 0)$ *lie on a straight line that passes through the origin in the* (ψ_1, ψ_2)*-plane. There are at most two such straight lines through the origin for which the tangent vectors* $\frac{d\mathfrak{S}_1}{d\omega_1}$ *and* $\frac{d\mathfrak{S}_2}{d\omega_2}$ *are parallel.*

Proof. Our approach is to consider the two spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ as solutions of the linear differential equations

$$\frac{d}{d\omega_1}\begin{pmatrix} \psi_1(\omega_1) \\ \psi_2(\omega_1) \end{pmatrix} = \begin{pmatrix} -\rho_1 - \cot\varphi_{12} & \csc\varphi_{12} \\ -\sin\varphi_{12} - \cos\varphi_{12}\cot\varphi_{12} & -\rho_1 + \cot\varphi_{12} \end{pmatrix}\begin{pmatrix} \psi_1(\omega_1) \\ \psi_2(\omega_1) \end{pmatrix} \tag{6.25}$$

and

$$\frac{d}{d\omega_2}\begin{pmatrix} \psi_1(\omega_2) \\ \psi_2(\omega_2) \end{pmatrix} = \begin{pmatrix} -\rho_2 - \cot\varphi_{22} & \frac{c_{21}}{c_{22}}\csc\varphi_{22} \\ -\frac{c_{22}}{c_{21}}(\sin\varphi_{22} + \cos\varphi_{22}\cot\varphi_{22} & -\rho_2 + \cot\varphi_{22} \end{pmatrix}\begin{pmatrix} \psi_1(\omega_2) \\ \psi_2(\omega_2) \end{pmatrix}. \tag{6.26}$$

We use the same coordinate variables (ψ_1, ψ_2) for both ODEs as we wish to consider overlaying the two 'solutions' $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$.

To study tangencies of the two spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ we can consider studying points at which the two linear vectors fields (6.25) and (6.26) are collinear. In fact it is straightforward to consider this problem for two general linear planar vector fields:

$$\begin{aligned} \frac{d}{d\omega_1}\begin{pmatrix} \psi_1(\omega_1) \\ \psi_2(\omega_1) \end{pmatrix} &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} \psi_1(\omega_1) \\ \psi_2(\omega_1) \end{pmatrix}, \\ \frac{d}{d\omega_2}\begin{pmatrix} \psi_1(\omega_2) \\ \psi_2(\omega_2) \end{pmatrix} &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} \psi_1(\omega_2) \\ \psi_2(\omega_2) \end{pmatrix}. \end{aligned}$$

Assume that these two vector fields are not simply scalar multiples of each other. The above two linear vector fields are collinear at the same points where the following two vector fields are perpendicular:

$$\begin{aligned} \frac{d}{d\omega_1}\begin{pmatrix} \psi_1(\omega_1) \\ \psi_2(\omega_1) \end{pmatrix} &= \begin{pmatrix} c_1 & d_1 \\ -a_1 & -b_1 \end{pmatrix}\begin{pmatrix} \psi_1(\omega_1) \\ \psi_2(\omega_1) \end{pmatrix}, \\ \frac{d}{d\omega_2}\begin{pmatrix} \psi_1(\omega_2) \\ \psi_2(\omega_2) \end{pmatrix} &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} \psi_1(\omega_2) \\ \psi_2(\omega_2) \end{pmatrix}. \end{aligned}$$

Taking the standard inner product, these vector fields are perpendicular whenever the following equality holds:

$$(c_1a_2 - c_2a_1)\psi_1^2 + (c_1b_2 + d_1a_2 - c_2b_1 - d_2a_1)\psi_1\psi_2 + (b_2d_1 - d_2b_1)\psi_2^2 = 0.$$

The above equation is simply describing a degenerate conic section, and has three possible solution sets: either the single solution $\psi_1 = \psi_2 = 0$, a set of solutions corresponding to a straight line through the origin, or a set of solutions corresponding to a distinct pair of straight lines intersecting at the origin. These cases correspond to whether the discriminant

$$(c_1b_2 + d_1a_2 - c_2b_1 - d_2a_1)^2 - 4(c_1a_2 - c_2a_1)(b_2d_1 - d_2b_1)$$

is negative, zero, or positive respectively. Therefore these are also the three possible cases for where the vector fields (6.25) and (6.26) are tangent. Hence any tangencies between the two spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ can only occur on at most two straight lines intersecting at the origin. ■

Lemma 6.7. *Let $\mathscr{L}$ be any straight line through the origin in the (ψ_1, ψ_2)-plane. There is at most one intersection between the two spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ along $\mathscr{L}$.*

Proof. The proof follows from the fact that the ratio ρ_1/ρ_2 is irrational. Suppose we have two intersection points of the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ related to the pairs (ω_1', ω_2') and (ω_1'', ω_2'') that lie on $\mathscr{L}$, that is, $\mathfrak{S}_1(\omega_1', 0) = \mathfrak{S}_2(\omega_2', 0)$, $\mathfrak{S}_1(\omega_1'', 0) = \mathfrak{S}_2(\omega_2'', 0)$ and $\mathfrak{S}_1(\omega_1', 0), \mathfrak{S}_1(\omega_1'', 0) \in \mathscr{L}$.

Given that, it follows from (6.24) that there are integers n_1 and n_2 such that $\omega_1' = \omega_1'' + n_1\pi$ and $\omega_2' = \omega_2'' + n_2\pi$. Note that n_1 and n_2 are either both even or are both odd.

Then, again using (6.24), we have

$$\begin{aligned}
\mathfrak{S}_1(\omega_1', 0) &= (-1)^{(n_1 \bmod 2)} e^{-\rho_1 n_1 \pi} \mathfrak{S}_1(\omega_1'', 0) \\
&= (-1)^{(n_1 \bmod 2)} e^{-\rho_1 n_1 \pi} \mathfrak{S}_2(\omega_2'', 0) \\
&= (-1)^{(n_1 \bmod 2)} (-1)^{(n_2 \bmod 2)} e^{-\rho_1 n_1 \pi} e^{\rho_2 n_2 \pi} \mathfrak{S}_2(\omega_2', 0) \\
&= e^{(\rho_2 n_2 - \rho_1 n_1)\pi} \mathfrak{S}_1(\omega_1', 0),
\end{aligned}$$

and so $\rho_2 n_2 - \rho_1 n_1 = 0$, which is a contradiction as ρ_1/ρ_2 is irrational. ■

Remark 6.8. *Lemmas 6.6 and 6.7 together show that the two spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ can have at most two tangential intersections in total, in the case that ρ_1/ρ_2 is irrational.*

The following Lemma is useful in providing a lower bound for the angle between the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ at intersection points.

Lemma 6.9. *Let $(\hat{\omega}_1, \hat{\omega}_2)$ be such that $\mathfrak{S}_1(\hat{\omega}_1, 0)$ and $\mathfrak{S}_2(\hat{\omega}_2, 0)$ lie on a straight line that passes through the origin in the (ψ_1, ψ_2) plane. Let the constant M be defined by $\mathfrak{S}_1(\hat{\omega}_1, 0) = M\mathfrak{S}_2(\hat{\omega}_2, 0)$ and suppose $M \neq 1$, $M > 0$. Denote the angle in polar coordinates of $\mathfrak{S}_1(\omega_1, 0)$ by $\theta_1(\omega_1)$ and $\mathfrak{S}_2(\omega_2, 0)$ by $\theta_2(\omega_2)$. Define $\hat{\theta}$ by $\theta_1(\hat{\omega}_1) = \theta_2(\hat{\omega}_2) = \hat{\theta}$. Then there exists $\epsilon > 0$ such that $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ do not have any intersections for $|\omega_1 - \hat{\omega}_1|, |\omega_2 - \hat{\omega}_2| < \epsilon$.*

Moreover, within the ranges $|\omega_1 - \hat{\omega}_1|, |\omega_2 - \hat{\omega}_2| < \epsilon$ we may conclude that there exists a $\theta_\epsilon > 0$ such that the angles $\theta(\omega_1)$ and $\theta(\omega_2)$ cover the interval $[\hat{\theta} - \theta_\epsilon, \hat{\theta} + \theta_\epsilon]$. That is,

$$min\Big\{|\theta(\hat{\omega}_i + \epsilon) - \hat{\theta}|, |\theta(\hat{\omega}_i - \epsilon) - \hat{\theta}|\Big\} \geq \theta_\epsilon \qquad i = 1, 2. \tag{6.27}$$

Informally, the last statement says that there are no intersections with a polar coordinates angle in the range $[\hat{\theta} - \theta_\epsilon, \hat{\theta} + \theta_\epsilon]$ that are close to $(\hat{\omega}_1, \hat{\omega}_2)$.

Proof. This follows from the fact that the expression $|\mathfrak{S}_1(\omega_1, 0) - \mathfrak{S}_2(\omega_2, 0)|$ is a continuous function of ω_1 and ω_2, and as it is nonzero at $(\hat{\omega}_1, \hat{\omega}_2)$, there exists an $\epsilon > 0$ such that $|\mathfrak{S}_1(\omega_1, 0) - \mathfrak{S}_2(\omega_2, 0)| \neq 0$ for $|(\omega_1, \omega_2) - (\hat{\omega}_1, \hat{\omega}_2)| < \epsilon$. Note also that since $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ are 2π-invariant up to a multiplicative constant, that ϵ may be chosen independent of $(\hat{\omega}_1, \hat{\omega}_2)$.

Since $d\theta_1/d\omega_1$ and $d\theta_2/d\omega_2$ are nonzero for all ω_1, ω_2 (see (6.24), or alternatively consider (6.25) and (6.26)), we may choose $\theta_\epsilon > 0$ small enough so that (6.27) is satisfied. ∎

Remark 6.10. *Note in Lemma 6.9 that the angle θ_ϵ depends only on the multiplicative constant M. Also $\theta_\epsilon(M)$ can be chosen so that $\theta_\epsilon(M) \to 0$ monotonically as $(M-1) \to 0^+$ or $(M-1) \to 0^-$.*

The following Lemma is a particular consequence of the ratio ρ_1/ρ_2 being irrational.

Lemma 6.11. *The set of values of M for which there exists a reparameterization of the spirals given in (6.24) so that the equation $\mathfrak{S}_1(\omega_1, 0) = \mathfrak{S}_2(\omega_2, 0)$ becomes $\mathfrak{S}_1(\omega_1, 0) = M\mathfrak{S}_2(\omega_2, 0)$ are dense in $\mathbb{R}^+$.*

Proof. Central to the proof is the fact that numbers of the form $m_2\rho_2 - m_1\rho_1$ (where $m_1, m_2 \in \mathbb{N}$) are dense in the real line. This may easily be seen from the fact that numbers of the form

$$m_1 \frac{\rho_1}{\rho_2} \pmod 1, \qquad m_1 \in \mathbb{N}$$

are dense in the unit interval (since ρ_1/ρ_2 is irrational). Therefore numbers of the form $m_2 - m_1\frac{\rho_1}{\rho_2}$ (where both $m_1, m_2 \in \mathbb{N}$) are dense in the real line, and hence so are numbers of the form $m_2\rho_2 - m_1\rho_1$. We may make $m_2\rho_2 - m_1\rho_1$ arbitrarily close to any real number, by choosing appropriate m_1 and m_2 (which, if necessary, have to be chosen sufficiently large).

Now consider the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$. These spirals are centred on the origin in the plane. They also have the property that they are invariant under transformations $\omega \mapsto \omega - 2n_j\pi$ ($n_j \in \mathbb{N}$). Such a transformation has the effect of enlarging the spiral $\mathfrak{S}_i(\omega, 0)$ by a factor of $e^{2\rho_i n_j \pi}$, which recovers the original spiral. Therefore the change of variables $\omega_1 \mapsto \omega_1 - 2m_1\pi$ and $\omega_2 \mapsto \omega_2 - 2m_2\pi$ transforms the equation $\mathfrak{S}_1(\omega_1, 0) = \mathfrak{S}_2(\omega_2, 0)$ into $e^{2m_1\rho_1\pi}\mathfrak{S}_1(\omega_1, 0) = e^{2m_2\rho_2\pi}\mathfrak{S}_2(\omega_2, 0)$. As the numbers $m_2\rho_2 - m_1\rho_1$ are dense in the real line, $e^{(2m_2\rho_2\pi - 2m_1\rho_1\pi)}$ are dense in $\mathbb{R}^+$. ∎

The following Corollary is immediate from the proof of Lemma 6.11.

Corollary 6.12. *For any open interval $I \subset \mathbb{R}^+$, there exists a sequence $(m_1(k), m_2(k))_{k\in\mathbb{N}}$ with $m_1(k), m_2(k) \to \infty$ as $k \to \infty$ such that $e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)} \in I$.*

The following Lemma provides suitable (ω_1, ω_2)-intervals in which to look for intersections of the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$. Recall that we have assumed without loss of generality that $\rho_1 < \rho_2$.

Lemma 6.13. *Suppose $\rho_1 < \rho_2$. For fixed $N \in \mathbb{N}$, there exist integers $n_N > \ldots > n_2 > n_1 \geq 1$ such that*

$$e^{-2\pi(n_i-i)\rho_2} > e^{-2\pi n_i\rho_1} > e^{-2\pi(n_i+1)\rho_1} > e^{-2\pi(n_i-i+1)\rho_2}, \quad i = 1, \ldots, N \tag{6.28}$$

Proof. Let n_1 be the smallest integer such that

$$(n_1 + 1)\rho_1 < n_1\rho_2.$$

Clearly $n_1 \geq 1$. Then we have $n_1\rho_1 > (n_1 - 1)\rho_2$. In this case $(n_1 + 1)\rho_1 > (n_1 - 1)\rho_2$ and so there exists $n_2 > n_1$ such that n_2 is the smallest integer such that

$$(n_2 + 1)\rho_1 < (n_2 - 1)\rho_2.$$

Then we also have $n_2\rho_1 > (n_2 - 2)\rho_2$. Continuing in this way we obtain integers $n_N > \ldots > n_2 > n_1 \geq 1$ such that $(n_i + 1)\rho_1 < (n_i - i + 1)\rho_2$ and $n_i\rho_1 > (n_i - i)\rho_2$. Altogether this gives the inequalities (6.28). ∎

Remark 6.14. *Note that it is possible that $n_{i+1} = n_i + 1$ for one or more $i = 1, \ldots, N - 1$. In that case $e^{-2\pi(n_i-i)\rho_2} = e^{-2\pi(n_{i+1}-(i+1))\rho_2}$ and $e^{-2\pi(n_i-i+1)\rho_2} = e^{-2\pi(n_{i+1}-(i+1)+1)\rho_2}$ in (6.28). In the case that $(N + 1)\rho_1 < \rho_2$, we have $n_i = i$ for $i = 1, \ldots, N$.*

We now return to the proof of Lemma 6.5.

Proof of Lemma 6.5. From Lemma 6.6 we know that there are at most two straight lines through the origin for which the tangent vectors of the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ are parallel. Let us denote these lines l_0 and l_1. (In the case of there being just one (resp. no) such straight lines, l_1 (resp. l_1 and l_0) may be chosen arbitrarily.)

Now let ω_1^*, ω_2^* be such that there is $M_0 > 0$ with

$$\mathfrak{S}_1(\omega_1^*, 0) = M_0\mathfrak{S}_2(\omega_2^*, 0). \tag{6.29}$$

Using Lemma 6.13 and (6.29), we obtain for M sufficiently close to M_0 the inequalities:

$$\begin{aligned} |M\mathfrak{S}_2(\omega_2^* + 2(n_i - i)\pi)| &> |\mathfrak{S}_1(\omega_1^* + 2n_i\pi)| \\ &> |\mathfrak{S}_1(\omega_1^* + 2(n_i + 1)\pi)| > |M\mathfrak{S}_2(\omega_2^* + 2(n_i - i + 1)\pi)| \\ &\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, N \end{aligned} \tag{6.30}$$

Now consider the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $M_0\mathfrak{S}_2(\omega_2, 0)$ parameterised in the intervals

$$\begin{aligned} \omega_1 &\in [\omega_1^* + 2n_i\pi, \omega_1^* + 2(n_i + 1)\pi] := I_1^i \\ \omega_2 &\in [\omega_2^* + 2(n_i - i)\pi, \omega_2^* + 2(n_i - i + 1)\pi] := I_2^i \end{aligned}$$

for $i = 1, \ldots, N$. As mentioned in Lemma 6.13 it is possible that some or all of the intervals I_2^i are the same, in the case that $n_{i+1} = n_i + 1$ (for some or all of $i = 1, \ldots, N - 1$). However the intervals I_1^i are all different.

We may consider these i pairs of segments of $\mathfrak{S}_1(\omega_1, 0)$ and $M_0\mathfrak{S}_2(\omega_2, 0)$ (parameterised by I_2^i and I_1^i) represented in polar coordinates. Note that all of the points in the inequalities (6.30) have the same angle (modulo 2π), and that all the intervals I_2^i, I_1^i cover an angle of 2π. As the spirals are continuous functions of ω_1 and ω_2 respectively, we may conclude from the Intermediate Value Theorem and (6.30) that there exist intersection points $(\tilde{\omega}_1^i, \tilde{\omega}_2^i) \in I_1^i \times I_2^i$ for each i. Note that these are necessarily distinct intersection points.

Now it is clear that there exists a constant $M_\epsilon > 1$, with $|M_\epsilon - 1|$ sufficiently close to zero, such that for $M \in [M_0/M_\epsilon, M_0 M_\epsilon]$ the inequalities (6.30) still hold and hence $\mathfrak{S}_1(\omega_1, 0)$ and $M\mathfrak{S}_2(\omega_2, 0)$ have intersection points $(\tilde{\omega}_1^i(M), \tilde{\omega}_2^i(M)) \in I_1^i \times I_2^i$ for $M \in [M_0/M_\epsilon, M_0 M_\epsilon]$, $i = 1, \ldots, N$. It is possible however that there may be finitely many isolated values of M in the interval $[M_0/M_\epsilon, M_0 M_\epsilon]$ for which the intersection points lie on l_0 or l_1—in this case we may choose a subinterval $I_\epsilon^{M_0} \subset [M_0/M_\epsilon, M_0 M_\epsilon]$ such that the spirals have intersection points $(\tilde{\omega}_1^i(M), \tilde{\omega}_2^i(M))$, where no intersection point lies on l_0 or l_1 for $M \in I_\epsilon^{M_0}$.

Now, further shrinking $I_\epsilon^{M_0}$ if necessary we may apply Lemma 6.9 to show that there exists an open interval $I_\epsilon^{M_0}$, such that for $M \in I_\epsilon^{M_0}$, the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $M\mathfrak{S}_2(\omega_2, 0)$ have intersection points $(\tilde{\omega}_1^i(M), \tilde{\omega}_2^i(M))$, $i = 1, \ldots, N$, with

$$\begin{aligned} \hat{\omega}_1^N(M) - \hat{\omega}_1^1(M) &\leq 2(n_N - n_1 + 1)\pi \\ \hat{\omega}_2^N(M) - \hat{\omega}_2^1(M) &\leq 2(n_N - n_1 - N + 2)\pi \end{aligned}$$

and whose angle in polar coordinates is bounded away from l_0 and l_1 by some fixed angle $\hat{\theta}$. From (6.25) and (6.26) it is clear that this is equivalent to the statement that there exists a fixed angle θ^* such that the angle between the spiral tangents is greater than θ^* at the intersection points.

Thus we have found a set of N intersection points of the two spirals $\mathfrak{S}_1(\omega_1, 0)$ and $M\mathfrak{S}_2(\omega_2, 0)$ with the desired properties as given in Lemma 6.5, for each M in an open interval $I_\epsilon^{M_0} \subset \mathbb{R}^+$. By Corollary 6.12, we can find a sequence $(m_1(k), m_2(k))_{k\in\mathbb{N}}$ with $m_1(k), m_2(k) \to \infty$ as $k \to \infty$ with the property that $e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)} \in I_\epsilon^{M_0}$.

Therefore the sequence $(\tilde{\omega}_1^i(e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)}), \tilde{\omega}_2^i(e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)}))_{k\in\mathbb{N}}$ $(i = 1, \ldots, N)$ are solutions to the equation

$$\mathfrak{S}_1(\omega_1, 0) = e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)}\mathfrak{S}_2(\omega_2, 0).$$

and so the sequence $(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k))_{k\in\mathbb{N}}$ defined by

$$\hat{\omega}_j^i(k) := \tilde{\omega}_j^i(e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)}) + 2m_j(k)\pi, \quad j = 1, 2, k \in \mathbb{N}, i = 1, \ldots, N \tag{6.31}$$

are solutions to the equation

$$\mathfrak{S}_1(\omega_1, 0) = \mathfrak{S}_2(\omega_2, 0)$$

that satisfy the requirements of the Lemma. ∎

Using the notations introduced in the foregoing proof we define

$$M_k := e^{(2m_1(k)\rho_1\pi - 2m_2(k)\rho_2\pi)}.$$

With that we find

$$m_1(k) = \frac{\rho_2}{\rho_1} m_2(k) + \frac{1}{2\pi\rho_1} \ln M_k. \tag{6.32}$$

Note that $M_k \in I_\epsilon^{M_0}$, where $|I_\epsilon^{M_0}|$ is small.

In those terms the issue of Lemma 6.5 is:

Corollary 6.15. *There exist sequences (M_k) and $(m_j(k))$, $j = 1, 2$ with $M_k \in I_\epsilon^{M_0}$ and $m_j(k) \to \infty$ as $k \to \infty$ in each case such that*

$$L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0) = L(\tilde{\omega}_1^i(M_k) + 2m_1(k)\pi, \tilde{\omega}_2^i(M_k) + 2m_2(k)\pi, 0) = 0, \qquad i = 1, \ldots, N.$$

Moreover, there exist constants $C_{DL^{-1}}$ and C_{D^2L} such that

(i) $|(D_1 L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0))^{-1}| < C_{DL^{-1}} e^{(2\rho_2 m_2(k)\pi + \frac{1}{2}(\rho_1 W_1(N) + \rho_2 W_2(N)))}$,

(ii) $|D_1^2 L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0)| < C_{D^2L} e^{-2\rho_2 m_2(k)\pi}$.

Proof. The first part of the statement follows immediately by the above considerations. So we confine ourselves to prove the estimates (i) and (ii).

To verify (i) we consider $D_1 L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0)$ as a matrix. By means of the determinant $\det A$ and the adjugate matrix $\operatorname{adj}(A)$ the inverse of a matrix A can be written as

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

According to the stipulation of L, cf. (6.23), we find

$$|\det D_1 L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0)| = |\dot{\mathfrak{S}}_1(\hat{\omega}_1^i(k), 0)| \cdot |\dot{\mathfrak{S}}_2(\hat{\omega}_2^i(k), 0)| \cdot |\sin \sphericalangle(\dot{\mathfrak{S}}_1(\hat{\omega}_1^i(k), 0), \dot{\mathfrak{S}}_2(\hat{\omega}_2^i(k), 0))|,$$

where $\dot{\mathfrak{S}}_j$ denotes the derivative with respect to ω_j. In order to determine $|\dot{\mathfrak{S}}_j(\hat{\omega}_j^i(k), 0)|$ we exploit (6.25) or (6.26), respectively. Taking also (6.24), (6.31) and Lemma 6.5(iii) into consideration we find constants C_j such that

$$|\dot{\mathfrak{S}}_j(\hat{\omega}_j^i(k), 0)| > C_j e^{-\rho_j(2m_j(k)\pi + W_j(N))}.$$

By Lemma 6.5(iv) we find

$$|\sin \sphericalangle(\dot{\mathfrak{S}}_1(\hat{\omega}_1^i(k), 0), \dot{\mathfrak{S}}_2(\hat{\omega}_2^i(k), 0))| \geq \sin \theta_N^*.$$

Further, we write

$$\operatorname{adj}(D_1 L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0)) = e^{-2\rho_2 m_2(k)\pi} L_R.$$

Because of (6.32) the norm of L_R can be estimated (to above) independently of k.

Combining the estimates regarding the determinant and the adjugate of $D_1 L(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k), 0)$, and exploiting in the course of this again (6.32) we infer the estimate (i).

Next we verify estimate (ii). To this end we exploit that $D_1^2 L$ can be reduced to a 2×2-matrix $\mathfrak{L}$ consisting of the columns $\ddot{\mathfrak{S}}_j(\hat{\omega}_j^i(k), 0)$. Again exploiting (6.25) and (6.26) the matrix $\mathfrak{L}$ can be written as $\mathfrak{L} = e^{-2\rho_2 m_2(k)\pi} \hat{\mathfrak{L}}$. Finally, the norm of $\hat{\mathfrak{L}}$ can be estimated by a constant which does not depend on k and N. ∎

It is clear that under perturbation of μ, (6.21) retains only finitely many transversal intersections, but this number of transversal intersections may be arbitrarily large by taking μ sufficiently close to zero.

Rational ratio ρ_1/ρ_2. Next we consider the case where ρ_1/ρ_2 is rational. The following lemma states that in this case , the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ are together self-similar.

Lemma 6.16. *Assume that that $\mu_1 = \mu_2 = 0$ and ρ_1/ρ_2 is rational, $\rho_1/\rho_2 = p/q$, with $p, q \in \mathbb{N}$ (recall $\rho_1, \rho_2 > 0$). Then equation* (6.21) *is periodic in (ω_1, ω_2).*

Proof. Under the transformations

$$\begin{aligned} \omega_1 &\rightarrow \omega_1 + 2q\pi \\ \omega_2 &\rightarrow \omega_2 + 2p\pi \end{aligned}$$

equation (6.21) becomes

$$\begin{pmatrix} e^{-2\rho_1 q\pi} e^{-\rho_1\omega_1} \sin\omega_1 & -e^{-2\rho_2 p\pi} e^{-\rho_2\omega_2} c_{21} \sin\omega_2 \\ e^{-2\rho_1 q\pi} e^{-\rho_1\omega_1} \sin(\omega_1 + \varphi_{12}) & -e^{-2\rho_2 p\pi} e^{-\rho_2\omega_2} c_{22} \sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0 \tag{6.33}$$

Multiplying through both equations by $e^{2\rho_1 q\pi}$ and using $\rho_1 q - \rho_2 p = 0$ recovers the original equations (6.21). ■

Corollary 6.17. *Assume that that $\mu_1 = \mu_2 = 0$ and ρ_1/ρ_2 is rational, $\rho_1/\rho_2 = p/q$, with $p, q \in \mathbb{N}$. If $L(\omega_1^*, \omega_2^*, 0) = 0$ then, for all $k \in \mathbb{N}$ also $L(\omega_1^* + 2kp\pi, \omega_2^* + 2kq\pi, 0) = 0$. Further $e^{-2k\rho_1 q\pi} D_1 L(\omega_1^*, \omega_2^*, 0) = D_1 L(\omega_1^* + 2kp\pi, \omega_2^* + 2kq\pi, 0)$.*

Proof. Recall that the two spirals $\mathfrak{S}_1$ and $\mathfrak{S}_1$ intersect if (6.21) is satisfied. Plugging in the values $\omega_1^* + 2kp\pi$ and $\omega_2^* + 2kq\pi$ in (6.21) yields (6.33). Now, following the arguments in the proof of Lemma 6.16 gives the statement. ■

Let $\mathcal{H}$ be the (topological) space of two parameter vector fields containing a heteroclinic cycle under the assumptions stated above for $f(\cdot, 0)$, and let this space be endowed with the C^1 topology.

Lemma 6.18. *Consider the system* (1.1) *under the assumptions (H1)–(H5). Assume the eigenvalue case (CC). There exists an open and dense set $\mathcal{D} \subset \mathcal{H}$, such that for each $f(\cdot, 0) \in \mathcal{D}$, equation* (6.20) *has an infinite number of non-degenerate solutions, or equivalently the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ have an infinite number of transversal intersections.*

Proof. First we note that since $\mathcal{H}$ is endowed with the C^1 topology, the constants in equation (6.20) varies continuously in this topology, meaning that they depend continuously on the vector field. Therefore Lemma 6.5 implies that there is a dense set $\tilde{\mathcal{D}} \subset \mathcal{H}$ such that equation (6.20) has an infinite number of transversal intersections.

We now consider a vector field f in $\tilde{\mathcal{D}}$, and take a small neighbourhood $B(f; \epsilon)$ around it. It is clear that if ϵ is sufficiently small, the constants in (6.20) undergo an arbitrarily small perturbation, and at least one transversal intersection of $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ persists. By Lemma 6.5, for every point in $B(f; \epsilon)$ where ρ_1/ρ_2 is irrational, there exist infinitely many transversal intersections. Also, since there is as least one transversal intersection at every point in $B(f;\epsilon)$, we may use Lemma 6.16 to show that where ρ_1/ρ_2 is rational, there must also be infinitely many transversal intersections.

This provides an open and dense set in $\mathcal{H}$ where equation (6.20) has infinitely many non-degenerate solutions, and completes the Lemma. ■

Remark 6.19. *For each $f(\cdot,\mu) \in \mathcal{D}$, $\mu \neq 0$, equation (6.20) has only a finite number of non-degenerate solutions, or equivalently the spirals $\mathfrak{S}_1(\omega_1,\mu)$ and $\mathfrak{S}_2(\omega_2,\mu)$ have only a finite number of transversal intersections.*

6.2.2 Nonwandering dynamics

Again we distinguish the two cases where the ratio ρ_1/ρ_2 is irrational or rational, respectively.

Irrational ratio ρ_1/ρ_2. As the counterparts of (6.5) and (6.6) we define

$$\mathbf{\Omega}_{k_0} := \{(\hat{\omega}_1^i(k), \hat{\omega}_2^i(k)),\ k \geq k_0,\ i = 1,\ldots,N\}$$

and

$$\mathbf{\Omega}_{k_0,N} := \{(\hat{\omega}_1^i(k_0), \hat{\omega}_2^i(k_0)),\ i = 1,\ldots,N\}.$$

The estimates given in Corollary 6.15 enable corresponding counterparts to (6.15) and (6.16). This finally makes it possible to apply an equivalent to Lemma 6.2 to detect solutions which correspond to sequences of $\mathbf{\Omega}_{k_0,N}^{\mathbb{Z}}$. The verification of shift dynamics is analogous to the considerations in Section 6.1.2.

Rational ratio ρ_1/ρ_2. Next we deal with the case where the ratio ρ_1/ρ_2 is rational. Let $L(\omega_1^*, \omega_2^*, 0) = 0$ with nonsingular Jacobian, cf. Lemma 6.18. Then we employ Corollary 6.17 to find as the counterpart of (6.4)

$$\hat{\omega}_1(k) = \omega_1^* + 2kp\pi, \quad \hat{\omega}_2(k) = \omega_2^* + 2kq\pi, \tag{6.34}$$

Starting from that we define formally in the same way as we did in (6.5) and (6.6)

$$\mathbf{\Omega}_{k_0} := \{(\hat{\omega}_1(k), \hat{\omega}_2(k)),\ k \geq k_0\}$$

and

$$\mathbf{\Omega}_{k_0,N} := \{(\hat{\omega}_1(k), \hat{\omega}_2^i(k)),\ k = k_0,\ldots,k_0+N-1\}.$$

From that point on we proceed as above – along the lines of the proof of Lemma 6.1, Lemma 6.2 and Section 6.1.2. Note that the counterparts to (6.15) and (6.16) follow easily from (6.34).

6.3 Proof of Lemma 2.4

Similarly as in the proof of shift dynamics we distinguish the cases (RC) and (CC). Here we perform the proof for the eigenvalue case (RC). Afterwards we comment briefly on the necessary adjustments one has to make in the eigenvalue case (CC).

A l-(2,1) heteroclinic connection may be considered as part of a heteroclinic cycle, together with a 1-(1,2) heteroclinic orbit. In terms of the $\boldsymbol{\omega}$-sequence, this corresponds to a l-periodic sequence with $\omega_{1,1} = \omega_{2,1} = \infty$. In order to set up the bifurcation equation (6.1) for those orbits explicitly we define (adapted to the formal setting $\omega_{1,1} = \omega_{2,1} = \infty$)

$$L_1(\omega_1, \mu) = \begin{pmatrix} \mu_1 + c_{11}e^{-2\lambda_1^u \omega_1} \\ \mu_2 \end{pmatrix}, \; L_l(\omega_2, \mu) = \begin{pmatrix} \mu_1 & -c_{21}e^{-2\rho_2\omega_2} \sin(2\phi_2\omega_2 + \varphi_{21}) \\ \mu_2 & -c_{22}e^{-2\rho_2\omega_2} \sin(2\phi_2\omega_2 + \varphi_{22}) \end{pmatrix}. \tag{6.35}$$

Further we recall the notations which we introduced in Section 6.1.1. With (6.2) and (6.3) the bifurcation equation for l-(2,1) heteroclinic orbits is as follows.

$$\begin{aligned} \Xi_1(\boldsymbol{\omega}, \mu) = & \quad L_1(\omega_1, \mu) + r_1(\boldsymbol{\omega}, \mu) = 0, \\ \Xi_i(\boldsymbol{\omega}, \mu) = & \quad L(\omega_{1,i+1}, \omega_{2,i}, \mu) + r_i(\boldsymbol{\omega}, \mu) = 0, \quad i = 2, \dots, l-1, \\ \Xi_l(\boldsymbol{\omega}, \mu) = & \quad L_l(\omega_2, \mu) + r_l(\boldsymbol{\omega}, \mu) = 0. \end{aligned}$$

We begin by considering the truncated form of equations $\Xi_1(\boldsymbol{\omega}, \mu) = 0$ and $\Xi_l(\boldsymbol{\omega}, \mu) = 0$:

$$L_1(\omega_{1,2}, \mu) = 0, \quad L_l(\omega_{2,k}, \mu) = 0. \tag{6.36}$$

Solutions of (6.36) are related to intersections of the line $\mathfrak{L}_1(t, 0)$ and the spiral $-\mathfrak{S}_2(t, 0)$ which we discussed in Section 6.1.1 as follows: Similar to (6.4) we get

$$\hat{\omega}_2(k) = \hat{\omega}_2^* + \frac{k\pi}{\phi_2} \quad \text{and} \quad \hat{\omega}_1(k) = \frac{\rho_2 \hat{\omega}_2(k)}{\lambda_1^u} + \hat{C}_1.$$

as solutions of $L_1(\omega_1, 0) = L_l(\omega_2, 0)$. Further we assign μ-values

$$\hat{\mu}(k) := -L_1(\hat{\omega}_1(k), 0).$$

Now we solve (6.36) near $(\hat{\omega}_1(k), \hat{\omega}_2(k), \hat{\mu}(k))$ by means of the Banach fixed point theorem (note that the "constants" appearing in this equation may depend on μ) and get a solution $(\tilde{\omega}_1(k), \tilde{\omega}_2(k), \tilde{\mu}(k))$ with

$$\tilde{\mu}(k) = \begin{pmatrix} O\big(e^{-2k(\rho_2/\phi_2)\pi}\big) \\ 0 \end{pmatrix}.$$

Next we choose $m(k)$ such that

$$|\tilde{\mu}(k)| < d_\mu(m(k)) = Ce^{-2m(k)(\rho_2/\phi_2)\pi},$$

where the constant d_μ was defined in (6.12); C is an appropriate constant in accordance with (6.12). Therefore we might choose

$$m(k) = k + \tilde{m}$$

with an appropriate $\tilde{m} \in \mathbb{N}$.

With that we consider

$$L(\omega_1, \omega_2, \tilde{\mu}(k)) = 0, \tag{6.37}$$

near $\hat{\omega}_1(k + \tilde{m}), \hat{\omega}_2(k + \tilde{m})$. Similar considerations as made in Section 6.1.1 yield solutions $(\tilde{\omega}_1(k + \tilde{m}), \tilde{\omega}_2(k + \tilde{m}), \tilde{\mu}(k))$ of (6.37). Hence, the quantities

$$\begin{aligned}
&\tilde{\omega}_{1,2}(k) := \tilde{\omega}_1(k), \quad \tilde{\omega}_{2,l}(k) := \tilde{\omega}_2(k), \quad \tilde{\omega}_{1,i+1}(k) := \tilde{\omega}_1(k + \tilde{m}), \quad \tilde{\omega}_{2,i}(k) := \tilde{\omega}_2(k + \tilde{m}), \\
&i = 2, \ldots, l - 1, \quad \text{and} \quad \mu = \tilde{\mu}(k)
\end{aligned}$$

solve the truncated bifurcation equation

$$\begin{aligned}
L_1(\omega_{1,2}, \mu) &= 0, \\
L(\omega_{1,i+1}, \omega_{2,i}, \mu) &= 0, \quad i = 2, \ldots, l - 1, \\
L_l(\omega_{2,l}, \mu) &= 0.
\end{aligned}$$

With

$$\tilde{\boldsymbol{\omega}}(k) := \big(\tilde{\omega}_{1,i}(k), \tilde{\omega}_{2,i}(k)\big)_{i \in \{2, \ldots, l\}}$$

and

$$\mathbf{L}(\boldsymbol{\omega}, \mu) = (L_1(\omega_{1,2}, \mu), L(\omega_{1,3}, \omega_{2,2}, \mu), \ldots, L(\omega_{1,k}, \omega_{2,k-1}, \mu), L_k(\omega_{2,k}, \mu)),$$

we rewrite the (full) bifurcation equation as the fixed point equation

$$(\boldsymbol{\omega}, \mu) = (\boldsymbol{\omega}, \mu) - [D\mathbf{L}(\tilde{\boldsymbol{\omega}}(k), \tilde{\mu}(k)))]^{-1}\, \boldsymbol{\Xi}(\boldsymbol{\omega}, \mu). \tag{6.38}$$

The further procedure runs parallel to the one in the proof of Lemma 6.2. In doing so we rely on the following estimates

$$\begin{aligned}
&|\, [D\mathbf{L}(\tilde{\boldsymbol{\omega}}(k), \tilde{\mu}(k)))]^{-1}\, | < C_{-1} e^{2(k+\tilde{m})(\rho_2/\phi_2)\pi} \\
&\max_{(\boldsymbol{\omega},\mu) \in B[\tilde{\boldsymbol{\omega}}(k), d] \times B[\tilde{\mu}(k), d_{mu}(k)]} |D^2 \mathbf{L}(\boldsymbol{\omega}, \mu)| < C_2 e^{-2k(\rho_2/\phi_2)\pi}.
\end{aligned}$$

where C_{-1} and C_2 are appropriate constants.

Hence there is a $k(l)$ such that for each $k > k(l)$ the full bifurcation equation has a solution $(\boldsymbol{\omega}(k), \mu(k))$ near $(\tilde{\boldsymbol{\omega}}(k), \tilde{\mu}(k))$. Note that in particular the quantities $\mu(k))$ depends also on l (which is dropped in our notation). With that we find

$$\mathcal{T}_l := \{\mu(k),\ k > k(l)\}.$$

In the eigenvalue case (CC) essentially we proceed as in the (RC) case. However, now L_1 has the same structure as L_l, cf. (6.19) and (6.35). The quantities $\hat{\omega}_j(k)$, $j = 1, 2$ have to be replaced by $\hat{\omega}_j^i(k)$, $j = 1, 2$, $i = 1, \ldots, N$, cf. Lemma 6.5 and Section 6.2.2. For the residual the comments given in Section 6.2.2 apply.

6.4 Proof of Lemma 2.5

6.4.1 l-(1,2) heteroclinic orbits

We now search for l-(1,2) heteroclinic orbits, for $l \geq 2$. For that we consider the concatenation of a l-(1,2) heteroclinic orbit with Γ_1, which corresponds to a l-periodic Lin orbit with a l-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1} = \omega_{2,l} = \infty$. Since we are only interested in solving for l-(1,2) heteroclinic orbits, we do not need to solve the equation $\Xi_l(\boldsymbol{\omega}, \mu) = \xi^\infty(\mu) = 0$, which is only related to Γ_1.

The bifurcation equations to be solved are then as follows:

$$\Xi_i(\boldsymbol{\omega}, \mu) = L(\omega_{1,i+1}, \omega_{2,i}, \mu) + r_i(\boldsymbol{\omega}, \mu) = 0, \quad i = 1, \ldots, l-1.$$

Here, the leading order term L is defined by (6.2) in the eigenvalue case (RC), while in the eigenvalue case (CC) L is defined by (6.19) or (6.23), respectively. The residual terms r_i have to be chosen correspondingly.

Again we begin by considering the equation

$$L((\omega_1, \omega_2), 0) = 0.$$

As in Sections 6.1.1 or 6.2.2 respectively, we find solutions $(\hat{\omega}_1(k), \hat{\omega}_2(k))$ (or $(\hat{\omega}_1^1(k), \hat{\omega}_2^1(k))$, respectively), $k \in \mathbb{N}$. Now we define $\hat{\boldsymbol{\omega}}(k)$ by

$$(\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i}) = (\hat{\omega}_1(k), \hat{\omega}_2(k)), \quad i = 1, \ldots, l-1.$$

The main observation is that the corresponding $\mathbf{L}(\hat{\boldsymbol{\omega}}, 0)$ consists of copies of $L(\hat{\omega}_1(k), \hat{\omega}_2(k), 0)$. This allows us to solve the full bifurcation equation $\boldsymbol{\Xi}(\boldsymbol{\omega}, \mu) = 0$ in the same manner as in Section 6.1.1 - which works in this context also for the eigenvalue case (CC).

6.4.2 Homoclinic orbits

Here we consider only l-homoclinic orbits to p_1, since the proof is similar for homoclinic orbits to p_2. Such a homoclinic orbit corresponds to a l-periodic $\boldsymbol{\omega}$ sequence with $\omega_{1,1} = \infty$. The bifurcation equations then read as follows:

$$\begin{aligned} \Xi_i(\boldsymbol{\omega}, \mu) &= L(\omega_{1,i+1}, \omega_{2,i}, \mu) + r_i(\boldsymbol{\omega}, \mu) = 0, \quad i = 1, \ldots, l-1, \\ \Xi_l(\boldsymbol{\omega}, \mu) &= L_l(\omega_{2,l}, \mu) + r_l(\boldsymbol{\omega}, \mu) = 0. \end{aligned} \tag{6.39}$$

Here, the leading order term L is defined by (6.2) in the eigenvalue case (RC), while in the eigenvalue case (CC) L is defined by (6.19) or (6.23), respectively, and L_l is defined as in (6.35) if λ_2^s is complex or it takes a corresponding form as L_1 in (6.35) if λ_2^s is real (of course the c_{11}, λ_1^u and ω_1 have to be replaced accordingly).

The first equation in (6.39) does not apply if $l = 1$, i.e. if we search for 1-homoclinic orbits to p_1. In this case the remaining second equation in (6.39) only depends on ω_2 and μ, and can easily be solved for $\mu = \mu(\omega_2)$. This represents the stated spiral (if λ_2^s is complex) or line (if λ_2^s is real).

From now on we assume that $l > 1$. First we solve $L_l(\omega_{2,l}, \mu) = 0$ for $\mu = \hat{\mu}(\omega_{2,l})$. Note that $\hat{\mu}$ tends to zero as $\omega_{2,l} \to \infty$.

Next we consider $L(\omega_1, \omega_2, 0) = 0$. In accordance with our considerations in Section 6.1.1 (eigenvalue case (RC)) or Section 6.2.2 (eigenvalue case (CC)), respectively, we find infinitely many solutions $(\hat{\omega}_1(k), \hat{\omega}_2(k))$, $k \in \mathbb{N}$. With arguments given in the addressed sections it is clear that these solutions can be continued for sufficiently small μ.

Now fix $k \in \mathbb{N}$, and take $\omega_{2,l}$ large enough such that $L(\omega_1, \omega_2, \mu(\omega_{2,l})) = 0$ has near $(\hat{\omega}_1(k), \hat{\omega}_2(k))$ a solution $(\hat{\omega}_1(\omega_{2,l}; k), \hat{\omega}_2(\omega_{2,l}; k))$:

$$L(\hat{\omega}_1(\omega_{2,l}; k), \hat{\omega}_2(\omega_{2,l}; k), \hat{\mu}(\omega_{2,l})) = 0.$$

Set $(\hat{\omega}_{1,i+1}(k), \hat{\omega}_{2,i}(k)) := (\hat{\omega}_1(\omega_{2,l}; k), \hat{\omega}_2(\omega_{2,l}; k))$, $i = 1, \ldots, l-1$, and $\hat{\omega}_{2,l} := \omega_{2,l}$. As in Section 6.3 we rewrite (6.39) as a fixed point equation as in (6.38):

$$(\boldsymbol{\omega}, \mu) = (\boldsymbol{\omega}, \mu) - [D\mathbf{L}(\hat{\boldsymbol{\omega}}(k), \hat{\mu}(\omega_{2,l})))]^{-1}\, \Xi(\boldsymbol{\omega}, \mu).$$

This equation can be solved for $(\omega_{1,i+1}, \omega_{2,i})(\omega_{2,l})$ and $\mu = \hat{\mu}(\omega_{2,l}; k)$, where $\omega_{2,l}$ can be taken from an interval $(a(k), \infty)$, for some appropriate $a(k)$.

The curves $\mathcal{L}_{l,1}^{hom}$ as stated in Lemma 2.5 are given by $\mu = \hat{\mu}(\omega_{2,l}; k)$.

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