Bifurcations of two-dimensional dynamical systems close to a system with two separatrix loops

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We consider a two-parameter family of smooth dynamical systems $S(\mu)$ on a two-dimensional smooth manifold. We assume that S depends smoothly on $\mu = (\mu_1, \mu_2)$ and that $S(0)$ has the isolated equilibrium state 0 of saddle point type with two separatrix loops denoted by Γ_1 and Γ_2.

We also assume that the saddle point value $\sigma = \lambda_1 + \lambda_2$, where λ_1 and λ_2 are the roots of the characteristic equation of the system at 0 when $\mu = 0$, is non-zero and negative.

There exists a neighbourhood of 0 such that for all sufficiently small μ the equations of the vector field in this neighbourhood have the form

$$\begin{cases} \dot{\zeta} = \lambda_1(\mu)\zeta + f(\zeta, \eta; \mu)\cdot\zeta, \\ \dot{\eta} = \lambda_2(\mu)\eta + g(\zeta, \eta; \mu)\eta, \end{cases}$$

where $\lambda_1(\mu) < 0$ and $\lambda_2(\mu) > 0$ are the roots of the characteristic equation at 0. The equations of the stable separatrices in this neighbourhood are $\eta = 0$, and those of the unstable ones are $\xi = 0$. We choose a sufficiently small $d > 0$ and construct secants to the stable separatrices: $\pi_1 - \xi = d$ and $\pi_2 - \xi = -d$, and to the unstable separatrices: $\pi_3 - \eta = d$ and $\pi_4 - \eta = -d$. By assumption, the separatrices form loops for $\mu = 0$. This implies that for small μ and small ξ the trajectories emanating from the points (ξ, d) of the secant π_3 (or from the points $(\xi, -d)$ of the secant π_4) return to the neighbourhood and intersect the secant π_1 (or π_2). Thus, succession maps T_1 and T_2 are defined: $T_1 : \pi_3 \to \pi_1$ and $T_2 : \pi_4 \to \pi_2$. We assume that T_1 and T_2 have the form $T_1 : \eta = \mu_1 + A_1(\mu)\xi + \ldots$ and $T_2 : \eta_2 = \mu_2 + A_2(\mu)\xi + \ldots$. The quantities $A_1(\mu)$ and $A_2(\mu)$ are non-zero and are called the separatrix values. Various combinations of the signs of A_1 and A_2 are possible: 1) $A_1 > 0, A_2 > 0$; 2) $A_1 < 0, A_2 > 0$; 3) $A_1 < 0, A_2 < 0$. The case 1) always holds on an orientable manifold.

It is known that from one separatrix loop Γ_i with $\sigma < 0$ one can generate only one periodic trajectory homotopic to Γ_i ([1], [2]). The situation is richer in the case of two loops. A cycle of type $j_1, \ldots, j_n$ is defined to be a limit cycle homotopic to the product $\Gamma_{j_1}\Gamma_{j_2} \ldots \Gamma_{j_n}$ of loops ($j_k = 1$ or 2).

Theorem. *There exist a small neighbourhood V of the separatrix contour $\Gamma_1 \cup \Gamma_2 \cup O$ and a small neighbourhood U of variation of the parameters μ such that for $\mu \in U$ the system $S(\mu)$ has at most two limit cycles in V.*

Only cycles of type 1, 2, or 12 can occur in the case 1).

Only cycles of type 1, 2, 12, or 112 can occur in the case 2).

Only cycles of type 1, 2, 12, $(12)^r1$, or $(21)^r2$ ($1 \leqslant r < \infty$) can occur in the case 3). Bifurcation diagrams are constructed for each of these three cases:

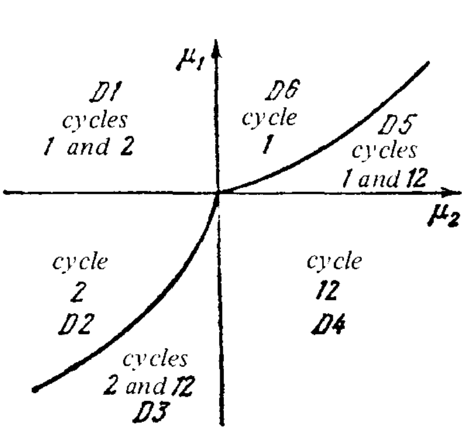

Fig. 1

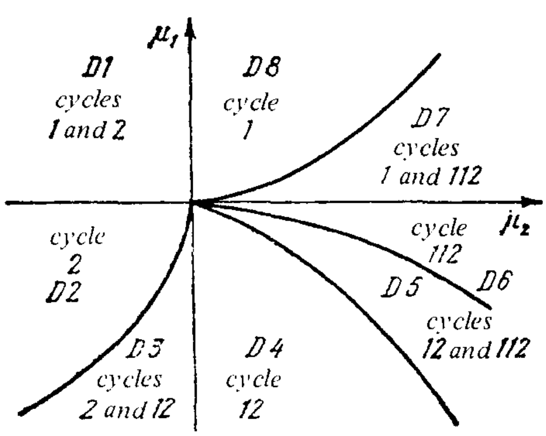

Fig. 2

1) The plane of the parameters (μ_1, μ_2) (Fig. 1) is partitioned into 6 domains: $D1$–$D6$. For each domain there are 1 or 2 cycles, as indicated directly on the diagram. Adjacent domains necessarily have a common limit cycle[(1)].

2) The plane is partitioned into 8 domains: $D1$–$D8$ (Fig. 2). For each domain there are 1 or 2 cycles, as indicated directly on the diagram (Fig. 3).

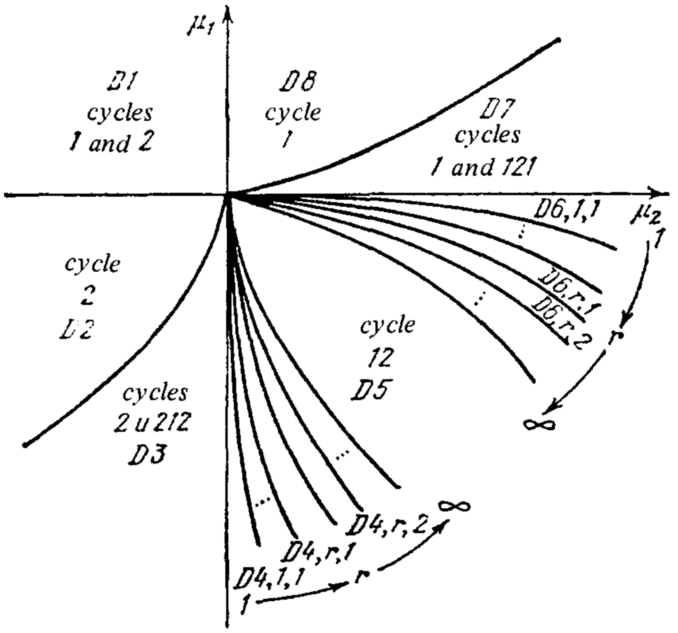

Fig. 3

3) The plane is partitioned into countably many domains: $D1$, $D2$, $D3$, $D5$, $D7$, $D8$; $D6, 1, 1$; $D6, 1, 2$; $D6, 2, 1$; $D6, 2, 2$; ..., $D6, r, 1$; $D6, r, 2$; ...; $D4, 1, 1$; $D4, 1, 2$; ...; $D4, r, 1$; $D4, r, 2$; ...
(here r can vary from 1 to ∞).

The cycles of the domains $D1$–$D3$, $D5$, $D7$, and $D8$ are indicated on the diagram. In a domain of the form $D4, r, 1$ there is the single cycle $(21)^r 2$. The cycle $(21)^{r+1}2$ is added to it in passing to the domain $D4, r, 2$. The value of r grows to infinity on approaching the boundary of the domain $D5$. Domains of the form $D6, r, 1$ have the single cycle $(12)^r 1$, and the cycle $(12)^{r+1}1$ is added to it in passing to the domain $D6, r, 2$. The value of r grows to infinity on approaching the boundary of the domain $D5$.

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(1)For tunnel diode problems this case was considered in [3], and for near-Hamiltonian systems in [4].