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# SUPER-HOMOCLINIC ORBITS AND MULTI-PULSE HOMOCLINIC LOOPS IN HAMILTONIAN SYSTEMS WITH DISCRETE SYMMETRIES

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4D-Hamiltonian systems with discrete symmetries are studied. The symmetries under consideration are such that the systems possess two invariant sub-planes which intersect each other transversally at an equilibrium state. The equilibrium state is supposed to be of saddle type; moreover, in each invariant sub-plane there are two homoclinic loops to the saddle. We establish the existence of stable and unstable invariant manifolds for the bouquet composed of four homoclinic trajectories at the Hamiltonian level corresponding to the saddle. These manifolds may intersect transversely along an orbit that we will call *super-homoclinic orbit*. We prove that the existence of a super-homoclinic orbit implies the existence of a countable set of multi-pulse homoclinic trajectories to the saddle.

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*Dedicated to V. Arnold  
on his 60-th birthday*

## 1. Introduction

The finding and classification of the soliton-like states of non-integrable equations is one of the important problems of the physics of non-linear phenomena. In many cases the problem reduces to the study of non-integrable conservative (Hamiltonian, in particular) dynamical systems with at least two degrees of freedom: the self-localized states appear as homoclinic loops of zero hyperbolic equilibrium of the dynamical system.

The present paper deals with a particular problem of description of infinite series of self-localized solutions. It is well known that in many cases there exist series of complex solitons obtained by concatenation of some single pulses. In the context of Hamiltonian dynamical system this phenomenon is modelled by the existence of an infinite series of multi-round homoclinic solutions in a small neighborhood of a few simple homoclinic loops. The corresponding result is well-known for the case where the zero equilibrium state is a saddle-focus [1, 2, 3, 4]. At the same time [5], generically there can not be multi-pulse homoclinic loops in a sufficiently small neighborhood of any given finite number of single-pulse homoclinic loops if the equilibrium state is a saddle (the leading characteristic exponents are real). Roughly speaking, the solitons with monotonically decreasing tails can not, in general, be concatenated to a complex soliton solution.

Nevertheless, the multi-pulse solutions with monotonic tails are seen in many applications. An adequate structure which describes the generic creation of such solutions was proposed in [5] where we established the existence of an infinite series of multi-pulse homoclinic loops near a homoclinic cycle involving a saddle, a periodic orbit and a pair of homoclinic ones.

In this paper we investigate another scenario of appearance of multi-pulse homoclinic loops to a saddle. Instead of generic Hamiltonian systems, we consider the case where the system is invariant with respect to a number of discrete symmetries. In these circumstances we introduce a new object for the theory of dynamical systems — *an orbit which is homoclinic to a set of homoclinic loops*.

Namely, we give conditions (theorem 1) under which a symmetric bunch composed of four homoclinic loops to a saddle equilibrium of a two degrees of freedom Hamiltonian system has smooth invariant stable and unstable manifolds (the orbits in these manifolds tend to the homoclinic loops as  $t \rightarrow +\infty$  or as  $t \rightarrow -\infty$ , never approaching finally the saddle). These invariant manifolds are two-dimensional and belong to the three-dimensional zero energy level. Thus, they may intersect in this level transversely along the orbit which is homoclinic to the primary bunch of the homoclinic loops (Fig. 1).

We call such an orbit *super-homoclinic*. It is not a homoclinic loop itself, but its presence implies the existence of a large number of multi-pulse homoclinic loops with a complicated structure described in theorem 4. Though, in our particular example, we impose many symmetries on the system which allow for the existence of the homoclinic bunch having stable and unstable manifolds, the established connection between the super-homoclinic orbits and the creation of infinite series of homoclinic loops seems to be quite general phenomenon.

Note that the object studied here was originally seen [6] in a system describing a plane high-frequency electro-magnetic field in non-linear non-dissipative media, the detection of a super-homoclinic orbit was based upon an analysis of the structure of a numerically obtained series of self-localized states (a good correspondence with the case  $\beta_1\theta_2 > 0, \beta_2\theta_1 < 0, \alpha > 0$  of theorem 2 was found).

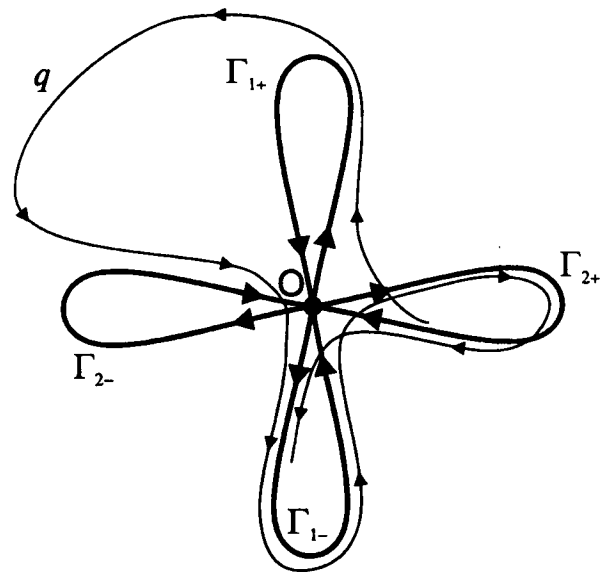


FIG. 1. A super-homoclinic orbit  $g$ : an orbit which is homoclinic to the bunch of four homoclinic loops  $\Gamma_{i\sigma}$  of a hyperbolic equilibrium  $O$

## 2. The saddle bunch of homoclinic loops

Consider a four-dimensional dynamical system

$$\dot{x} = X(x)$$

with a smooth first integral  $H$ , i. e.

$$H'(x)X(x) \equiv 0. \tag{1}$$

A Hamiltonian system with two degrees of freedom is a natural example but the symplectic structure is not important for our purposes.

Let  $X$  have a hyperbolic (i. e. not having pure imaginary characteristic exponents) equilibrium state  $O$  at the origin. By (1)

$$H'(0)X'(0) = 0$$

so, since  $X'(0)$  is non-degenerate by assumption, the linear part of  $H$  at  $O$  vanishes. Assume that the quadratic part of  $H$  at  $O$  is a *non-degenerate quadratic form*. One can compute that this non-degeneracy assumption implies that the system may be written near  $O$  as follows

$$\dot{u} = -Au + \dots, \quad \dot{v} = A^T v + \dots \tag{2}$$

where  $u \in \mathbb{R}^2$ ,  $v \in \mathbb{R}^2$ , the dots stand for nonlinearities and  $A$  is a matrix with positive eigenvalues. Moreover, the first integral takes the form

$$H = (v, Au) + \dots \quad (3)$$

where the dots stand for the third and higher order terms.

Assume that the system is *invariant with respect to the symmetries*

$$(u_1, v_1) \leftrightarrow (-u_1, -v_1) \quad \text{and} \quad (u_2, v_2) \leftrightarrow (-u_2, -v_2). \quad (4)$$

In particular, there exist two invariant planes:  $(u_1 = 0, v_1 = 0)$  and  $(u_2 = 0, v_2 = 0)$ .

The matrix  $A$  is therefore diagonal and we assume that its eigenvalues *are equal*. This appears naturally if, for example, the system is reversible (with respect to  $t \leftrightarrow -t$ ,  $(u_1, u_2) \leftrightarrow (v_2, v_1)$ ). By scaling time in (2), the matrix  $A$  is defined modulo an arbitrary scalar factor, therefore we assume that  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  from now on.

The equilibrium state  $O$  is a saddle with two-dimensional stable and unstable manifolds  $W_O^s$  and  $W_O^u$  which are tangent at  $O$  to the  $u$ -plane and  $v$ -plane respectively. Both the invariant manifolds lie in the three-dimensional level  $\{H = 0\}$  and they may intersect *transversely* in that level, producing a number of homoclinic loops. We consider a specific case where *in each of the invariant planes  $(u_1 = 0, v_1 = 0)$  and  $(u_2 = 0, v_2 = 0)$  there is a pair of homoclinic orbits of transverse intersection of  $W_O^s$  and  $W_O^u$ .*

Note that these four loops always exist if the level  $\{H = 0\}$  is compact (the system on an invariant plane is two-dimensional hence the compact zero level of the first integral must be a homoclinic figure-eight). The transversality is a genericity assumption.

Denote the homoclinic loops in  $(u_1 = 0, v_1 = 0)$  as  $\Gamma_{1+}$  and  $\Gamma_{1-}$  and the loops in  $(u_2 = 0, v_2 = 0)$  as  $\Gamma_{2+}$  and  $\Gamma_{2-}$ . Let  $\Gamma_{i+}$  leave  $O$  towards positive  $v_i$  and return to  $O$  from the side of positive  $u_i$ . Correspondingly, the loops  $\Gamma_{i-}$  lie to the negative side. This convention may require inversion of signs of some of the variables  $u_i$  and  $v_i$  which does not change the form of system (2) (recall that  $A$  is the identity matrix) but the first integral (3) may change.

We have

$$H = u_1 v_1 - \alpha u_2 v_2 + \dots \quad (5)$$

where  $\alpha$  must be  $-1$  by (3) but when the signs of some of the coordinates change,  $\alpha$  may become equal to  $1$ . In any case

$$|\alpha| = 1.$$

The dots in (5) stand for higher order terms which are invariant with respect to the symmetries (4). In particular, there are no odd-order terms in  $H$ .

Let us put the system near  $O$  in the normal form up to the third order terms (the second order terms vanish due to the symmetry). The result is

$$\begin{cases} \dot{v}_1 = v_1 + \beta_1 u_1 v_2^2 + \gamma_{11} u_1 v_1^2 + \gamma_{12} u_2 v_1 v_2 + \dots, \\ \dot{v}_2 = v_2 + \beta_2 u_2 v_1^2 + \gamma_{21} u_2 v_2^2 + \gamma_{22} u_1 v_1 v_2 + \dots, \\ \dot{u}_1 = -u_1 + \alpha \beta_2 u_2^2 v_1 - \gamma_{11} u_1^2 v_1 + (\alpha \gamma_0 - \gamma_{12}) u_1 u_2 v_2 + \dots, \\ \dot{u}_2 = -u_2 + \alpha \beta_1 u_1^2 v_2 - \gamma_{21} u_2^2 v_2 + (\gamma_0 - \gamma_{22}) u_1 u_2 v_1 + \dots, \end{cases} \quad (6)$$

where the dots stand for the fifth and higher order terms. The system (6) contains the resonant third order monomials agreed with the symmetries (4); relations between the coefficients are due to the conservation of the first integral (5).

**THEOREM 1.** *If  $\beta_1 \neq 0$ ,  $\beta_2 \neq 0$ , then the homoclinic bunch  $B = \Gamma_{1+} \cup \Gamma_{1-} \cup \Gamma_{2+} \cup \Gamma_{2-}$  has two-dimensional smooth invariant stable and unstable manifolds  $W^s(B)$  and  $W^u(B)$  which are tangent*

at each point of  $B$  to  $W_O^s$  and  $W_O^u$  respectively. The orbits from  $W^s(B)$  tend to  $B$  as  $t \rightarrow +\infty$  and the orbits from  $W^u(B)$  tend to  $B$  as  $t \rightarrow -\infty$ . All the other orbits leave a small neighborhood of  $B$  both as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$ .

The manifolds  $W_B^u \setminus B$  and  $W_B^s \setminus B$  consist of a number of connected components. On each of the components the orbits approach some periodic sequence of the homoclinic loops from  $B$ . The corresponding sequence of symbols  $\Gamma_{i\sigma}$  which forms the least period will be called the coding of the component. The number of the components and their codings depend on the signs of  $\beta_1$  and  $\beta_2$  and on the signs of the angles between  $W_O^s$  and  $W_O^u$  at the points of the homoclinic loops  $\Gamma_{i\sigma}$ .

Namely, take two small two-dimensional cross-sections to the loop  $\Gamma_{1+}$ :  $\Pi^u = \{v_1 = \delta\} \cap \{H = 0\}$  and  $\Pi^s = \{u_1 = \delta\} \cap \{H = 0\}$  where  $\delta$  is some small positive number. Since  $\Gamma_{1+}$  is an intersection of  $W_O^u$  and  $W_O^s$ , the orbits close to  $\Gamma_{1+}$  define the Poincaré map  $T_{1+}$  from  $\Pi^u$  to  $\Pi^s$ . By (5) the coordinates on  $\Pi^{u,s}$  are  $(u_2, v_2)$  (the values of  $u_1$  or  $v_1$  are found from the condition  $H = 0$ ). We locally straighten the manifolds  $W_O^s$  and  $W_O^u$ , thus the intersection line  $W_{O,loc}^s \cap \Pi^s$  is  $\{v_2 = 0\}$  and the intersection line  $W_{O,loc}^u \cap \Pi^u$  is  $\{u_2 = 0\}$ . By assumption, the image  $T_{1+}(W_{O,loc}^s \cap \Pi^s)$  intersects  $W_{O,loc}^u \cap \Pi^u$  transversely at the point  $(u_2 = 0, v_2 = 0) = \Gamma_{1+} \cap \Pi^s$ . We define the sign  $\theta_1$  of the angle between these two curves in the following way: if the positive (corresponding to  $v_2 > 0$ ) part of  $\{u_2 = 0\}_{\Pi^u}$  is mapped to the region  $v_2 > 0$  on  $\Pi^s$ , then  $\theta_1 = 1$ , and  $\theta_1 = -1$  otherwise. Analogously, the map  $T_{2+}$  near the loop  $\Gamma_{2+}$  defines the sign  $\theta_2$ . The loops  $\Gamma_{i-}$  are symmetric to  $\Gamma_{i+}$ , therefore the corresponding maps  $T_{i-}$  do not carry a new information.

- THEOREM 2.** *a) If  $\theta_2\beta_1 > 0, \theta_1\beta_2 > 0$ , then  $W^u(B) \setminus B$  consists of four connected components with the codings  $\Gamma_{1+}\Gamma_{2+}, \Gamma_{1+}\Gamma_{2-}, \Gamma_{1-}\Gamma_{2+}$  and  $\Gamma_{1-}\Gamma_{2-}$ .*  
*b) If  $\theta_2\beta_1 > 0, \theta_1\beta_2 < 0$ , then  $W^u(B) \setminus B$  consists of two connected components with the codings  $\Gamma_{1+}\Gamma_{2+}\Gamma_{1+}\Gamma_{2-}$  and  $\Gamma_{1-}\Gamma_{2+}\Gamma_{1-}\Gamma_{2-}$ .*  
*c) If  $\theta_2\beta_1 < 0, \theta_1\beta_2 > 0$ , then  $W^u(B) \setminus B$  consists of two connected components with the codings  $\Gamma_{1+}\Gamma_{2+}\Gamma_{1-}\Gamma_{2+}$  and  $\Gamma_{1-}\Gamma_{2-}\Gamma_{1+}\Gamma_{2-}$ .*  
*d) If  $\theta_2\beta_1 < 0, \theta_1\beta_2 < 0$ , then  $W^u(B) \setminus B$  consists of two connected components with the codings  $\Gamma_{1+}\Gamma_{2+}\Gamma_{1-}\Gamma_{2-}$  and  $\Gamma_{1+}\Gamma_{2-}\Gamma_{1-}\Gamma_{2+}$ .*

The structure of the decomposition of  $W^s(B) \setminus B$  into connected components may be obtained from the same theorem applied to the system obtained from  $X$  by reversion of time. The new values of  $\beta_i$  are found from (6):

$$\beta'_1 = -\alpha\beta_2, \quad \beta'_2 = -\alpha\beta_1.$$

For the system in  $R^4$ , the maps  $T_{i\sigma}$  preserve orientation, therefore the values of  $\theta_i$  does not change when the time is reversed. However, when the system  $X$  is defined on a non-orientable manifold, the maps  $T_{i\sigma}$  are not necessary orientation preserving so the values of  $\theta_i$  for the system  $X$  and for the reversed system may be different in that case. The latter situation appears naturally if one considers the homoclinic bunch in a system with more degrees of freedom. In this case, generically, a non-local four-dimensional invariant manifold exists (see conditions in [7]) which may be non-orientable as well.

Let us now prove the theorems above. Take a small positive  $\delta$  and consider a pair of small two-dimensional cross-sections  $\Pi_{i\sigma}^s$  and  $\Pi_{i\sigma}^u$  to each of the homoclinic loops  $\Gamma_{i\sigma}$  ( $i = 1, 2, \sigma = \pm$ ):  $\Pi_{i\sigma}^s = \{u_i = \sigma\delta\} \cap \{H = 0\}$  and  $\Pi_{i\sigma}^u = \{v_i = \sigma\delta\} \cap \{H = 0\}$ .

Orbits which lie in the level  $\{H = 0\}$  in a small neighborhood of the homoclinic bunch  $B$  must intersect  $\Pi_{i\sigma}^{s,u}$  so the problem reduces to the study of the Poincaré map on the set of these cross-sections. As we have already mentioned, the flow near the global piece of the loop  $\Gamma_{i\sigma}$  outside the  $\delta$ -neighborhood of the saddle defines the global map  $T_{i\sigma}$  from  $\Pi_{i\sigma}^u$  to  $\Pi_{i\sigma}^s$ . Since the corresponding flight time is bounded, this map is a diffeomorphism and it is well approximated by its Taylor expansion at the point  $\Gamma_{i\sigma} \cap \Pi_{i\sigma}^u$ .

Thus, the map  $T_{i\sigma}$  is written as

$$\begin{cases} \bar{v} = a_i v + b_i u + \dots, \\ \bar{u} = c_i v + d_i u + \dots, \end{cases} \quad (7)$$

where the dots stand for non-linear (cubic and higher order) terms. Here,  $(u, v)$  are the coordinates on  $\Pi_{i\sigma}^{u,s}$ : by (5) we may take  $(u_2, v_2)$  as the coordinates on  $\Pi_{1\sigma}^{u,s}$  — the values of  $u_1$  or  $v_1$  are found from the condition  $H = 0$ , and analogously, the coordinates on  $\Pi_{2\sigma}^{u,s}$  are  $(u_1, v_1)$ . Note that the right-hand sides of (7) do not depend on  $\sigma$ : due to the symmetry of the system with respect to (4) the map  $T_{i+}$  coincides with  $T_{i-}$  for such choice of the coordinates. We will therefore denote this map just as  $T_i$ . Note that these maps are symmetric themselves:  $T_i(-u, -v) = -T_i(u, v)$ .

We may always locally straighten the stable and unstable manifolds of  $O$ , so the intersection  $W_{O,\text{loc}}^u \cap \Pi_{i\sigma}^u$  is  $\{u = 0\}$  and  $W_{O,\text{loc}}^s \cap \Pi_{i\sigma}^s$  is  $\{v = 0\}$  (the point  $\Gamma_{i\sigma} \cap \Pi_{i\sigma}^u$  is  $(u = 0, v = 0)$ ). Thus, the transversality of  $W_O^u$  and  $W_O^s$  along  $\Gamma_{i\pm}$  means that

$$a_i \neq 0. \quad (8)$$

By definition, the sign of  $a_i$  is the value  $\theta_i$  of theorem 2.

The evaluation of the *local map* from the cross-sections  $\Pi_{i\sigma}^s$  to  $\Pi_{j\sigma'}^u$  which is defined by the orbits in the  $\delta$ -neighborhood of the saddle  $O$  is much less trivial because an orbit starting on  $\Pi_{i\sigma}^s$  may stay near  $O$  for an unboundedly large time before reaching one of the cross-sections  $\Pi_{j\sigma'}^u$ .

The regular method which allows for resolving this difficulty is based upon the study of a specific boundary value problem considered in [9]. Namely, as it follows from [9] for our particular case, if an orbit in a small neighborhood of a saddle starts at  $t = 0$  with some point  $M_0(u_{10}, u_{20}, v_{10}, v_{20})$  and reaches a point  $M_\tau(u_{1\tau}, u_{2\tau}, v_{1\tau}, v_{2\tau})$  at the moment  $t = \tau$ , then the values of  $(v_{10}, v_{20})$  and  $(u_{1\tau}, u_{2\tau})$  are uniquely defined by  $(u_{10}, u_{20})$ ,  $(v_{1\tau}, v_{2\tau})$  and  $\tau$ . Moreover, such  $M_0$  and  $M_\tau$  exist for any given  $\tau \geq 0$  and small  $(u_{10}, u_{20})$ ,  $(v_{1\tau}, v_{2\tau})$ ; the corresponding piece of the orbit is found as the unique solution of the following system of integral equations

$$\left\{ \begin{array}{l} v_1(t) = e^{-(\tau-t)}v_{1\tau} - \int_t^\tau e^{t-s} \left( \beta_1 u_1(s)v_2(s)^2 + \gamma_{11} u_1(s)v_1(s)^2 + \gamma_{12} u_2(s)v_1(s)v_2(s) \right) ds - \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad - \int_t^\tau e^{t-s} V_1(u(s), v(s)) ds, \\ v_2(t) = e^{-(\tau-t)}v_{2\tau} - \int_t^\tau e^{t-s} \left( \beta_2 u_2(s)v_1(s)^2 + \gamma_{21} u_2(s)v_2(s)^2 + \gamma_{22} u_1(s)v_1(s)v_2(s) \right) ds - \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad - \int_t^\tau e^{t-s} V_2(u(s), v(s)) ds, \\ u_1(t) = e^{-t}u_{10} + \int_0^t e^{s-t} \left( \alpha\beta_2 u_2(s)^2 v_1(s) - \gamma_{11} u_1(s)^2 v_1(s) + (\alpha\gamma_0 - \gamma_{12}) u_1(s)u_2(s)v_2(s) \right) ds + \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \int_0^t e^{s-t} U_1(u(s), v(s)) ds, \\ u_2(t) = e^{-t}u_{20} + \int_0^t e^{s-t} \left( \alpha\beta_1 u_1(s)^2 v_2(s) - \gamma_{21} u_2(s)^2 v_2(s) + (\gamma_0 - \gamma_{22}) u_1(s)u_2(s)v_1(s) \right) ds + \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \int_0^t e^{s-t} U_2(u(s), v(s)) ds. \end{array} \right. \quad (9)$$

This system is obtained by integration of (6); here,  $V_i$  and  $U_i$  replace the higher order terms denoted in (6) by dots. Recall that we straighten the local stable and unstable manifolds: they are the planes  $\{v = 0\}$  and  $\{u = 0\}$  respectively. The local invariance of these planes means that

$$U_i|_{u=0} \equiv 0, \quad V_i|_{v=0} \equiv 0.$$

Moreover, like in [10], one can show (see [8]) that all terms in  $U_i$  and  $V_i$  which are linear in  $u$  or in  $v$  can be “killed” simultaneously by a smooth local coordinate transformation, close to identity at  $O$ . Thus, in (9)

$$U_i, V_i = O(u^2v^2) \tag{10}$$

at the appropriate choice of coordinates which does not destroy the local form (6) of the system.

According to [9], the solution of (9) on the interval  $t \in [0, \tau]$  is found by successive approximations. The first approximation is

$$u_i(t) = e^{-t}u_{i0}, \quad v_i(t) = e^{-(\tau-t)}v_{i\tau}.$$

Using (10), one can immediately see that the second and all the further approximations have the form

$$\begin{cases} v_1(t) = e^{-(\tau-t)}v_{1\tau} - (\tau-t)e^{-(\tau-t)}e^{-\tau}(\beta_1u_{10}v_{2\tau}^2 + \gamma_{11}u_{10}v_{1\tau}^2 + \gamma_{12}u_{20}v_{1\tau}v_{2\tau}) + e^{-2\tau}O(1), \\ v_2(t) = e^{-(\tau-t)}v_{2\tau} - (\tau-t)e^{-(\tau-t)}e^{-\tau}(\beta_2u_{20}v_{1\tau}^2 + \gamma_{21}u_{20}v_{2\tau}^2 + \gamma_{22}u_{10}v_{1\tau}v_{2\tau}) + e^{-2\tau}O(1), \\ u_1(t) = e^{-t}u_{10} + te^{-t}e^{-\tau}(\alpha\beta_2u_{20}^2v_{1\tau} - \gamma_{11}u_{10}^2v_{1\tau} + (\alpha\gamma_0 - \gamma_{12})u_{10}u_{20}v_{2\tau}) + e^{-2\tau}O(1), \\ u_2(t) = e^{-t}u_{20} + te^{-t}e^{-\tau}(\alpha\beta_1u_{10}^2v_{2\tau} - \gamma_{21}u_{20}^2v_{2\tau} + (\gamma_0 - \gamma_{22})u_{10}u_{20}v_{1\tau}) + e^{-2\tau}O(1), \end{cases} \tag{11}$$

where the  $O(1)$ -terms are bounded uniformly, for all successive approximations. Hence, the solution of (9) has the same form. In particular, the following relation holds for the point  $M_0$  and its time  $\tau$  shift  $M_\tau$ :

$$\begin{cases} v_{10} = e^{-\tau}v_{1\tau} - \tau e^{-2\tau}(\beta_1u_{10}v_{2\tau}^2 + \gamma_{11}u_{10}v_{1\tau}^2 + \gamma_{12}u_{20}v_{1\tau}v_{2\tau}) + O(e^{-2\tau}), \\ v_{20} = e^{-\tau}v_{2\tau} - \tau e^{-2\tau}(\beta_2u_{20}v_{1\tau}^2 + \gamma_{21}u_{20}v_{2\tau}^2 + \gamma_{22}u_{10}v_{1\tau}v_{2\tau}) + O(e^{-2\tau}), \\ u_{1\tau} = e^{-\tau}u_{10} + \tau e^{-2\tau}(\alpha\beta_2u_{20}^2v_{1\tau} - \gamma_{11}u_{10}^2v_{1\tau} + (\alpha\gamma_0 - \gamma_{12})u_{10}u_{20}v_{2\tau}) + O(e^{-2\tau}), \\ u_{2\tau} = e^{-\tau}u_{20} + \tau e^{-2\tau}(\alpha\beta_1u_{10}^2v_{2\tau} - \gamma_{21}u_{20}^2v_{2\tau} + (\gamma_0 - \gamma_{22})u_{10}u_{20}v_{1\tau}) + O(e^{-2\tau}). \end{cases} \tag{12}$$

Suppose now that  $M_0 \in \Pi_{1+}^s$  and  $M_\tau \in \Pi_{1+}^u$ . It means that  $u_{10} = \delta$  and  $v_{1\tau} = \delta$ . The value of  $\tau$  can be found from the first equation of (12):

$$e^{-\tau} = \frac{v_{10}}{\delta}(1 + \dots).$$

Plugging this in the second equation of (12) gives

$$v_{20} = v_{10} \cdot o(1)$$

where  $o(1)$  denotes terms which tend to zero as the size of cross-sections  $\Pi$  diminishes (for a fixed small  $\delta$ ). From the other hand, the point  $M_0$  lies in  $\{H = 0\}$  whence (see (5))

$$v_{10} = v_{20} \cdot o(1)$$

at  $u_{10} = \delta$ . We got that if the size of cross-sections  $\Pi$  is small with respect to  $\delta$ , then  $v_{20} = v_{10} = 0$ , hence  $\tau = +\infty$ . Thus, a local orbit starting on  $\Pi_{1+}^s$  can not reach  $\Pi_{1+}^u$  in a finite time, provided the cross-sections are sufficiently small.

Obviously, the same is true for any pair of cross-sections  $\Pi_{i\sigma}^s$  and  $\Pi_{j\sigma'}^u$  if  $i = j$ . Therefore, orbits in a small neighborhood of the homoclinic bunch  $B$  after a round near a loop  $\Gamma_{i\sigma}$  must make the next round near one of the loops  $\Gamma_{j\pm}$  where  $j \neq i$  (in agreement with theorem 2). To study such orbits we now derive from (12) an expression for the local maps  $T_{012\sigma\sigma'}$  and  $T_{021\sigma'\sigma}$  from  $\Pi_{1\sigma}^s$  to  $\Pi_{2\sigma'}^u$  and from  $\Pi_{2\sigma'}^u$  to  $\Pi_{1\sigma}^s$ , respectively (due to the symmetries, the dependence on the signs  $\sigma$  and  $\sigma'$  is not so much essential).

Let  $M_0 \in \Pi_{1\sigma}^s$  and  $M_\tau \in \Pi_{2\sigma'}^u$ . It means that  $u_{10} = \sigma\delta$  and  $v_{2\tau} = \sigma'\delta$ . The value of  $\tau$  is found from the second equation of (12):

$$e^{-\tau} = \sigma' \frac{v_{20}}{\delta} (1 + O(v_{20} \ln |v_{20}|)).$$

Note that  $e^{-\tau}$  must be positive, thus to get to  $\Pi_{2+}^u$  in a finite time the coordinate  $v_{20}$  of the initial point must be positive whereas to get to  $\Pi_{2-}^u$  the coordinate  $v_{20}$  must be negative.

Plugging the expression for  $\tau$  in the first and third equations of (12) gives (taking into account that  $M_0 \in \{H = 0\}$  whence, by (5),  $v_{10} = \sigma\alpha u_{20}v_{20}(1 + O(|u_{20}| + |v_{20}|))/\delta$ ) the following formula for the local map  $T_{012\sigma\sigma'} : (u_{20}, v_{20}) \mapsto (u_{1\tau}, v_{1\tau})$ :

$$\begin{cases} \sigma\sigma'v_1 = \alpha u_2 + \beta_1\delta^2v_2|\ln|v_2|| + o(u_2, v_2 \ln|v_2|), \\ \sigma\sigma'u_1 = v_2(1 + o(1)), \end{cases} \quad (13)$$

(we drop extra indices). The right-hand sides here are defined at all small  $u_2$  and  $v_2$ : positive  $v_2$  give the map  $T_{012\sigma+}$  and the negative  $v_2$  give the map  $T_{012\sigma-}$ . Due to the symmetries (4), the right-hand sides do not depend on  $\sigma$  and  $\sigma'$ , moreover, they are odd (they change the sign when both  $u_2$  and  $v_2$  change the sign).

Analogously, the local map  $T_{021\sigma'\sigma} : \Pi_{2\sigma'}^s \rightarrow \Pi_{1\sigma}^u$  is written as:

$$\begin{cases} \sigma\sigma'v_2 = \alpha u_1 + \beta_2\delta^2v_1|\ln|v_1|| + o(u_1, v_1 \ln|v_1|), \\ \sigma\sigma'u_2 = v_1(1 + o(1)). \end{cases} \quad (14)$$

Here the right-hand sides are odd and do not depend on  $\sigma$  and  $\sigma'$ ; the positive  $v_1$  give the map  $T_{021\sigma'+}$  and the negative  $v_1$  give the map  $T_{021\sigma'-}$ .

Consider a pair of auxiliary maps  $T_{012}$  and  $T_{021}$  obtained, respectively, from (13) and (14) by dropping the factor  $\sigma\sigma'$  to the left. Let us take the compositions with the corresponding global maps:  $T_2T_{012} : (u_2, v_2) \mapsto (\bar{u}_1, \bar{v}_1)$  and  $T_1T_{021} : (u_1, v_1) \mapsto (\bar{u}_2, \bar{v}_2)$ , see (7). Denote by  $D_1$  a neighborhood of zero in  $(u_2, v_2)$ -plane, by  $D_2$  a neighborhood of zero in  $(u_1, v_1)$ -plane and consider the map  $T : T = T_2T_{012}$  on  $D_1$  and  $T = T_1T_{021}$  on  $D_2$ .

The map  $T$  carries all information on the dynamics near the homoclinic bunch  $B$ . Indeed, it is a symmetrized Poincaré map for the system  $X$  — take an orbit  $(u^1, v^1), (u^2, v^2), (u^3, v^3), \dots$  of  $T$ :

$$(u^{2k}, v^{2k}) = T_2T_{012}(u^{2k-1}, v^{2k-1}), \quad (u^{2k+1}, v^{2k+1}) = T_1T_{021}(u^{2k}, v^{2k}), \quad (15)$$

define

$$\sigma_1 = +1, \quad \sigma_{m+1} = \text{sign}(v^1 \cdot \dots \cdot v^m), \quad (16)$$

then, by construction, the sequence of points

$$\begin{aligned} M_+^{2k-1} \left\{ (u_2, v_2) = \sigma_{2k-1} (u^{2k-1}, v^{2k-1}) \right\} &\in \Pi_{1\sigma_{2k-1}}^s, \\ M_+^{2k} \left\{ (u_1, v_1) = \sigma_{2k} (u^{2k}, v^{2k}) \right\} &\in \Pi_{2\sigma_{2k}}^s \end{aligned} \quad (17)$$

is the consecutive points of intersection with the cross-sections  $\Pi^s$  for the orbit of the system  $X$  starting with  $M_+^1$ , and the sequence

$$\begin{aligned} M_-^{2k-1} \left\{ (u_2, v_2) = -\sigma_{2k-1} (u^{2k-1}, v^{2k-1}) \right\} &\in \Pi_{1, -\sigma_{2k-1}}^s, \\ M_-^{2k} \left\{ (u_1, v_1) = -\sigma_{2k} (u^{2k}, v^{2k}) \right\} &\in \Pi_{2, -\sigma_{2k}}^s \end{aligned} \quad (18)$$

is the consecutive points of intersection with the cross-sections for the orbit of the symmetric point  $M_-^1$ . Note also the exclusive case where one of the coordinates  $v^m$  is zero. It corresponds to entering the  $W_{O, \text{loc}}^s$ : the orbit of the system  $X$  tends to  $O$  and does not intersect the cross-sections anymore, though the corresponding orbit of the map  $T_1T_{021}T_2T_{012}$  is formally continued after that point (formulas (13), (14) are well defined at  $v = 0$ ).

The origin is the period two point for the map  $T : (0, 0)_{D_1} \xrightarrow{T} (0, 0)_{D_2} \xrightarrow{T} (0, 0)_{D_1}$ . These points correspond to the intersection of the homoclinic loops  $\Gamma_{i\sigma}$  with the corresponding cross-sections  $\Pi_{i\sigma}^s$ . Due

to the correspondence between the orbits of the system  $X$  and of the Poincaré map  $T$  it suffices to prove the existence of the stable and unstable manifolds for this period two point in order to prove theorem 1.

Note that the local maps  $T_{0ij}$  (see (13), (14)) are saddle maps in terms of [9]: since  $\frac{dv \ln |v|}{dv} = \infty$  at  $v = 0$ , there is a strong expansion in  $v$ -direction and, correspondingly, a strong contraction in  $u$ -direction at small  $(u, v)$ . The expansion/contraction factors are infinite at  $(0, 0)$ , therefore the compositions with the global maps (7)

$$T_2 T_{012} : \begin{cases} \bar{v}_1 = a_2 (\beta_1 \delta^2 v_2 |\ln |v_2|| + \alpha u_2 + o(u_2, v_2 \ln |v_2|)) + b_2 v_2 + \dots, \\ \bar{u}_1 = c_2 (\beta_1 \delta^2 v_2 |\ln |v_2|| + \alpha u_2 + o(u_2, v_2 \ln |v_2|)) + d_2 v_2 + \dots, \end{cases} \quad (19)$$

and

$$T_1 T_{021} : \begin{cases} \bar{v}_2 = a_1 (\beta_2 \delta^2 v_1 |\ln |v_1|| + \alpha u_1 + o(u_1, v_1 \ln |v_1|)) + b_1 v_1 + \dots, \\ \bar{u}_2 = c_1 (\beta_2 \delta^2 v_1 |\ln |v_1|| + \alpha u_1 + o(u_1, v_1 \ln |v_1|)) + d_1 v_1 + \dots, \end{cases} \quad (20)$$

are also saddle in a small neighborhood of  $(0, 0)$ , provided  $a_1 \neq 0$  and  $a_2 \neq 0$ . The latter inequalities are the transversality conditions (8), thus the Poincaré map  $T$  is a saddle map indeed: it is strongly expanding in  $v$  and  $T^{-1}$  is strongly expanding in  $u$ .

More precisely, the map  $T$  can be represented in the cross-form where the values of  $(\bar{u}, v)$  appear as uniquely defined functions of  $(u, \bar{v})$  and this cross-map is smooth ( $C^1$ ) and strongly contracting. The existence of the  $C^1$  stable and unstable manifolds for the periodic point of the map which admits such representation is a routine fact (see [11]).

The unstable manifold is the limit of iterations of any line transverse to  $\{u = 0\}$  at  $(0, 0)$ . It is seen from (19), (20) that all the forward images of any of such line are tangent at  $(0, 0)$  to  $u_2 = \frac{c_1}{a_1} v_2$  on  $\Pi_1^s$  and to  $u_1 = \frac{c_2}{a_2} v_1$  on  $\Pi_2^s$ . By (7), the intersections  $T_i W_{O,loc}^u \cap \Pi_i^s$  of the unstable manifold of  $O$  with the cross-sections have the same tangents at  $(0, 0)$ . Thus, the unstable manifold of the homoclinic bunch  $B$  is indeed tangent to  $W_O^u$  at the points of  $B$ . By the same arguments applied to the system obtained from  $X$  by reversion of time, the stable manifold of  $B$  is tangent to  $W_O^s$ . This completes the proof of theorem 1.

To prove theorem 2, it remains to study the codings of the orbits on  $W^u(B)$ . The coding is defined by the sequence of signs  $\sigma_m$  corresponding to the consecutive intersections of the orbit with the cross-sections  $\Pi_{i\sigma}^s$ . By (16)–(18) this sequence is recovered by the signs of the  $v$ -coordinate of the points of the corresponding orbit of the map  $T$ . It is seen immediately from (19) that when  $a_2 \beta_1 > 0$  the positive component of the unstable manifold in  $D_1$  is mapped into positive component of the unstable manifold in  $D_2$  and the negative component is mapped into the negative component. If  $a_2 \beta_1 < 0$ , the positive component is mapped into negative component and vice versa. The same rule is for the map from  $D_2$  to  $D_1$  in dependence on the sign of  $a_1 \beta_2$ . Thus, in case a) of theorem 2, the orbits starting on  $D_1$  on the positive component of the unstable manifold generate the positive sequence of the signs of the  $v$ -coordinate:  $++ \dots$ , and the orbits starting on the negative component generate the sequence  $-- \dots$ . The corresponding sequences of  $\sigma$ 's in (16) are  $+++ \dots$  and  $+-+ \dots$  which give, by (17), that the orbits of  $X$  starting on  $\Pi_{1+}^s$  on the positive component of the intersection of  $W^u(B)$  with the cross-section intersect consequently  $\Pi_{1+}^s, \Pi_{2+}^s, \Pi_{1+}^s, \Pi_{2+}^s, \dots$ , whereas the orbits starting on the negative component intersect  $\Pi_{1-}^s, \Pi_{2-}^s, \Pi_{1+}^s, \Pi_{2-}^s, \dots$ . At the same time, by (18), the orbits of  $X$  starting on the intersection of  $W^u(B)$  with  $\Pi_{1-}^s$  intersect consequently either  $\Pi_{1-}^s, \Pi_{2-}^s, \Pi_{1-}^s, \Pi_{2-}^s, \dots$ , or  $\Pi_{1-}^s, \Pi_{2+}^s, \Pi_{1-}^s, \Pi_{2+}^s, \dots$ . This proves the theorem in case a). The other cases are considered absolutely analogously; the corresponding sequences of the signs of the  $v$ -coordinate are: in case b)  $-++ \dots$  and  $--++$ , in case c)  $-+-+ \dots$  and  $-+-+$ , in case d)  $-+-+ \dots$  and  $-+-+$ .



### 3. Multi-pulse homoclinic loops

The local stable and unstable manifolds of the homoclinic bunch  $B$  are continued by the orbits of the system outside a small neighborhood of  $B$ . The obtained global two-dimensional stable and unstable invariant manifolds lie in the three-dimensional level  $\{H = 0\}$ , therefore they may have a common orbit at the points of which they intersect transversely. Such orbit is homoclinic to  $B$ . Due to the symmetries, the existence of one orbit homoclinic to  $B$  implies three more such orbits.

**THEOREM 3.** *Let  $G$  be the set composed of the homoclinic bunch  $B$  and of a finite number of homoclinic to  $B$  orbits along which the stable and unstable manifolds of  $B$  intersect transversely. Suppose that for any super-homoclinic orbit  $g$  in  $G$  its three images by the symmetries (4) are also included in  $G$ . Then, in a small neighborhood of  $G$  there exist infinitely many homoclinic loops of the equilibrium state  $O$ .*

Before we give a proof of the theorem, let us describe a typical structure of the set of the multi-pulse homoclinic loops near the super-homoclinic set  $G$ . Let us enumerate the connected components of  $W^s(B)$  and  $W^u(B)$  in an arbitrary way. Let us enumerate the super-homoclinic orbits  $g_{n\sigma}$  where  $g_{n+}$  is mapped to  $g_{n-}$  by the symmetry  $(u_1, v_1) \leftrightarrow -(u_1, v_1)$ . For each indices  $n$  and  $\sigma$  there are uniquely defined integers  $p(n, \sigma)$  and  $q(n, \sigma)$  such that  $g_{n\sigma}$  is an intersection of the components  $W_p^u(B)$  and  $W_q^s(B)$ .

By definition, the orbits  $g_{n\sigma}$  spend only finite time outside a small neighborhood of  $B$ : a piece of  $g_{n\sigma}$  which corresponds to  $t \rightarrow +\infty$  lies in  $W_{loc}^s(B)$  and a piece which corresponds to  $t \rightarrow -\infty$  lies in  $W_{loc}^u(B)$ . By theorem 2, each orbit in  $W_{loc}^s(B)$  and  $W_{loc}^u(B)$  intersects the local cross-section  $\Pi_{1+}^s \cup \Pi_{1-}^s$ . Let us denote the first points where the positive and negative semi-orbits of  $g_{n\sigma}$  intersect the local cross-section as  $M_{n\sigma}^+$  and  $M_{n\sigma}^-$ .

The following notation will also be used:  $C_{n\sigma}^+$  - the infinite to the right coding of the forward orbit of  $M_{n\sigma}^+$ , and  $C_{n\sigma}^-$  - the infinite to the left coding of the backward orbit of  $M_{n\sigma}^-$  (the coding is a string of symbols  $\Gamma_{i\sigma'}$  denoting the consequent rounds of the orbit near the corresponding homoclinic loops from  $B$ ). The possible variants of  $C_{n\sigma}^\pm$  can be extracted from theorem 2: since  $M_{n\sigma}^+ \in W_{q,loc}^s(B)$  and  $M_{n\sigma}^- \in W_{p,loc}^u(B)$ , the sequences  $C_{n\sigma}^\pm$  are obtained by the infinite repetition of (shifted, may be) codings of the components  $W_q^s(B)$  and  $W_p^u(B)$ , respectively; moreover,  $C_{n\sigma}^\pm$  must start with either  $\Gamma_{1+}$  or  $\Gamma_{1-}$  because the points  $M_{n\sigma}^\pm$  are chosen on  $\Pi_1^s$ .

The orbits which stay in a small neighborhood of the set  $G$  behave in the following way: they intersect  $\Pi_1^s$  near one of the points  $M_{n\sigma}^+$ , then spend some time in a small neighborhood of the bunch  $B$  until, after some number  $k$  of intersections with the local cross-sections  $\Pi_1^s$  and  $\Pi_2^s$ , they reach a small neighborhood of some point  $M_{n'\sigma'}^-$  (the number  $k$  must be even because all the points  $M^+$  and  $M^-$  are chosen on  $\Pi_1^s$  but the intersections with  $\Pi_1^s$  and  $\Pi_2^s$  alternate; see the proof of theorem 1). After that, the orbit moves along a piece of  $g_{n'\sigma'}$  and reach a small neighborhood of the point  $M_{n'\sigma'}^+$ , etc.. Thus, we may code the orbits by the sequences

$$\dots B_{k_{m-1}} g_{n_m \sigma_m} B_{k_m} g_{n_{m+1} \sigma_{m+1}} B_{k_{m+1}} \dots$$

where the symbol  $g_{n\sigma}$  corresponds to a passage near the corresponding super-homoclinic orbit and  $B_k$  is a length  $k$  sequence of the symbols  $\Gamma_{i\sigma'}$  denoting the consequent rounds near the corresponding homoclinic loops from the bunch  $B$ . The coding sequence is infinite for the orbits which do not lie in the invariant manifolds of  $B$  or  $O$ . If the orbit belongs to  $W^s(B)$  or  $W_O^s$ , then the coding is finite to the right and the last value of  $k$  is infinite for the orbits in  $W^s(B)$  and finite for the orbits in  $W_O^s$ . For the orbits in the unstable manifolds, the codings are finite to the left. In particular, the homoclinic loops of  $O$  have codings of finite length.

**THEOREM 4.** *In the class of systems under consideration, for a typical (belonging to a subset of the second Baire category) system there exists sufficiently large  $\bar{k}$  such that for any  $L > 0$ , for any sequence  $c = \{k_0 n_1 k_1 \dots n_L k_L\}$  where  $k_m \geq \bar{k}$  and  $k_1, \dots, k_{L-1}$  are even, there exists a unique sequence of signs*

$\sigma_1 = +1, \sigma_2, \dots, \sigma_L$  such that in a small neighborhood of the super-homoclinic set  $G$  there exists a unique homoclinic loop of the equilibrium  $O$  with the coding of type

$$B_{k_0} g_{n_1 \sigma_1} B_{k_1} \dots g_{n_L \sigma_L} B_{k_L} \tag{1}$$

and a unique homoclinic loop (symmetric to the previous one) with the coding of type

$$B_{k_0} g_{n_1, -\sigma_1} B_{k_1} \dots g_{n_L, -\sigma_L} B_{k_L}. \tag{2}$$

In the first coding (the second coding is symmetric with respect to reversion of all signs  $\sigma$  in symbols  $\Gamma$ ),  $B_{k_0}$  stands for the initial segment of  $C_{n_1+}^-$  of the length  $k_0$  and  $B_{k_L}$  corresponds to the initial segment of  $C_{n_L \sigma_L}^+$  of the length  $k_L$ ; the intermediate segments  $B_{k_m}$  ( $0 < m < L$ ) have the following structure: there exist (uniquely defined by the sequence  $c$ ) integers  $s_1, \dots, s_{L-1}$  (where  $s_m \rightarrow +\infty$  and  $k_m - s_m \rightarrow +\infty$  as  $k_m \rightarrow +\infty$ ) such that  $B_{k_m}$  is the segment of  $C_{n_m \sigma_m}^+$  of the length  $s_m$  continued by the segment of  $C_{n_{m+1} \sigma_{m+1}}^-$  of the length  $(k_m - s_m + 1)$  (the first symbol of the second segment coincides with the last symbol of the first one).

In other words, the homoclinic loops near the super-homoclinic set  $G$  include a number of long multi-pulse segments composed of pieces close to the primary homoclinic loops and separated by short pieces which are not close to the primary loops (the passages near the super-homoclinic orbits). Essentially, there is no correlations between the lengths and the structure of different multi-pulse segments in the same homoclinic loop. However, the structure of any given segment itself must obey a number of regularities.

The most general observation is that the pieces close to  $\Gamma_{1\pm}$  (these correspond to almost zero  $(u_2, v_2)$ ) must strictly alternate with the pieces close to  $\Gamma_{2\pm}$  (almost zero  $(u_1, v_1)$ ). The alternation of positive and negative pulses is also regular: they form periodic (of period two or four) sequences. The periodicity may be broken only once, somewhere in the middle of the multi-pulse segment: the periodic sequence to the right falls to one of the cases described in theorem 2 in dependence on the signs of the quantities  $\beta_i$  and  $\theta_i$  and the periodic sequence to the left is defined by the values of the quantities  $\beta'_i$  and  $\theta'_i$  computed for the system obtained by the reversion of time. Since the last symbol of the left sequence must coincide with the first symbol of the right sequence, it implies some additional restrictions.

For example (Fig. 2), in case a) of theorem 2 (both for the system  $X$  itself and for the reversed system:  $\beta_1 \theta_2 > 0, \beta_2 \theta_1 > 0, \beta'_1 \theta'_2 > 0, \beta'_2 \theta'_1 > 0$ ) the multi-pulse segment is forbidden to include the following quadruplets of positive and negative pulses:  $++--$ ,  $--++$ ,  $+- -+$  and  $-+ +-$ . In other words, in this case, if a multi-pulse segment starts with a sequence of positive pulses, it must either be all composed from the positive pulses or be finished by a sign-alternating sequence of pulses and it can not be finished by the sequence of negative pulses.

Let us now probe theorems 3 and 4. As we did in the previous section, one may identify the cross-sections  $\Pi_{1+}^s$  and  $\Pi_{1-}^s$ . The points  $M_{n+}^\pm$  and  $M_{n-}^\pm$  will coincide, so we use the notation  $M_n^\pm$ . The points  $M_n^+$  and  $M_n^-$  lie on the stable and, respectively, unstable manifolds  $w^s$  and  $w^u$  of the period two point  $(0, 0)$  of the symmetrized Poincaré map  $T$  defined by the flow near  $B$ . The period two point corresponds to the intersections of the homoclinic loops in  $B$  with the local cross-sections  $\Pi^s$ , the curves  $w^s$  and  $w^u$  correspond to the intersections of, respectively,  $W_{loc}^s(B)$  and  $W_{loc}^u(B)$  with the cross-sections.

The flow near the piece of  $g_n$  outside a small neighborhood of  $B$  defines a map  $\tilde{T}_n$  from a small neighborhood  $\mathcal{D}_n^-$  of  $M_n^-$  to a small neighborhood  $\mathcal{D}_n^+$  of  $M_n^+$ . The map  $T$  is a saddle map (its cross-map is smooth and strongly contracting). By construction, the image of  $M_n^-$  by the map  $T^2$  lies outside the domain of definition of this map. Therefore, we replace the map  $T$  by the map  $\tilde{T}_n$  in  $\mathcal{D}_n^-$  (denote the result as  $T'$ ). The maps  $\tilde{T}_n$  are smooth and each takes a small piece of  $w^u$  through the corresponding point  $M_n^-$  into a curve transverse to  $w^s$  at  $M_n^+$ . This picture is absolutely analogous to that near an orbit homoclinic to a hyperbolic periodic orbit [9] so the dynamics here may be treated in the same way.

The orbits in a small neighborhood of the set  $G$  are obtained from the orbits of the map  $T'$  by the same rule (16)–(18) as the orbits in a small neighborhood of the bunch  $B$  are recovered from the map  $T$ .

The homoclinic loops to the equilibrium state  $O$  are found as finite orbits of  $T'$  starting on  $l_1^u$  or  $l_2^u$  and finishing on  $l_1^s$  or on  $l_2^s$ , where the curves  $l_j^u$  and  $l_j^s$  are the intersections of, respectively, the unstable and stable manifolds of  $O$  with the cross-section:  $l_j^u = T_j^u(W_{O,\text{loc}}^u \cap \Pi_j^s)$  and  $l_j^s = W_{O,\text{loc}}^s \cap \Pi_j^s$ . We will call such orbits *the*  $(u, s)$ -orbits.

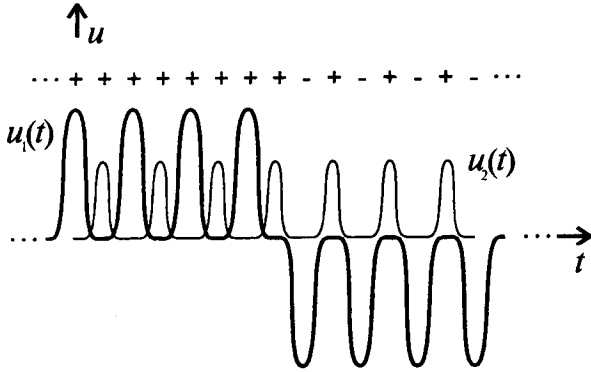


FIG. 2. An example of a multi-pulse segment (case a) of theorem 2). Pulses of  $u_1$  and  $u_2$  strictly alternate. The sequence of positive pulses changes to sign-alternating sequence, theorem 4 does not allow a switch to strictly negative sequence

images accumulate at  $w^u \cap \mathcal{D}_n^-$  as  $k' \rightarrow +\infty$ . By the map  $\tilde{T}_n$  they are taken in  $\mathcal{D}_n^+$  and we get one more sequence  $l_{k_n k'_n}$  of the images of  $l^u$ .

This procedure may be repeated arbitrarily many times. As a result we get the following statement: for some large enough  $\bar{k}$ , for any sequence

$$c = \{k_0 n_1 k_1 \dots n_L k_L\}$$

where  $k_m \geq \bar{k}$  and  $k_1, \dots, k_{L-1}$  are even, there exists a uniquely defined image  $l_c$  of  $l^u$ :

$$l_{k_0 n_1 k_1 \dots n_L k_L} = T^{k_L} \circ \tilde{T}_{n_L} (l_{k_0 n_1 \dots n_{L-1} k_{L-1}} \cap \mathcal{D}_{n_L}^-).$$

These images are uniformly  $C^1$ -close to  $w^u$  and their size is bounded away from zero. Thus, for sufficiently large  $\bar{k}$  each of the curves  $l_c$  have a unique point of transverse intersection with  $l^s$ .

We have found the set of  $(u, s)$ -orbits of the map  $T'$ . Each of them corresponds to a pair of symmetric homoclinic loops of system  $X$  (the signs  $\sigma_m$  are found from (16)), provided there is no intermediate point which falls on  $l^s$  (in this case the correspondence between the orbits of the map  $T$  and the system  $X$  fails, see explanations in the previous section). As we showed, given a sequence  $c$  the corresponding  $(u, s)$ -orbit is uniquely defined by the map  $T'$ . Falling an extra point of this orbit onto  $l^s$  (the line  $v = 0$ ) is an event of codimension one and it can be cancelled by a small smooth perturbation of the system localized near any point of the orbit (the specific structure we require for the system  $X$  is not destroyed by the perturbation). Therefore, for typical system we have a one-to-one correspondence between the  $(u, s)$  orbits of  $T'$  and the pairs of homoclinic loops of  $X$ .

In the situation of theorem 3 we can not apply small perturbations. In this case, if some intermediate point of an  $(u, s)$  orbit lies on  $l^s$ , the image of this point by the map  $T$  lies on  $l^u$  (see (7), (19), (20)). Thus, such  $(u, s)$  orbit is a concatenation of shorter  $(u, s)$  orbits which do not have intermediate points on  $l^s$  hence they do correspond to homoclinic loops of  $X$ . The number of all such indecomposable  $(u, s)$  orbits can not be finite (otherwise, all the other  $(u, s)$  orbits would be concatenations of some  $(u, s)$  orbits of finite length but this can not give  $(u, s)$  orbits whose coding sequence  $c$  contains unboundedly large values of  $k_m$ ). Hence, theorem 3 is proved.

To finish the proof of theorem 4 it remains to establish that the segments  $B_{k_m}$  in the coding (1) or (2) of the obtained homoclinic loops have indeed the structure described in the theorem. The internal structure of  $B_{k_m}$  is determined by the sequence of signs  $\sigma$  computed by the rule (16) for that piece of the corresponding  $(u, s)$  orbit which connects the neighborhoods  $\mathcal{D}_{n_m}^+$  and  $\mathcal{D}_{n_{m+1}}^-$  of the points  $M_{n_m}^+$  and  $M_{n_{m+1}}^-$  respectively. Since this piece is close to the orbit of  $M_{n_m}^+$  on initial stage and to the orbit of  $M_{n_{m+1}}^-$  on the final stage, it follows that indeed  $B_{k_m}$  coincides with  $C_{n_m}^+$  on the left end and with  $C_{n_{m+1}}^-$  on the right end.

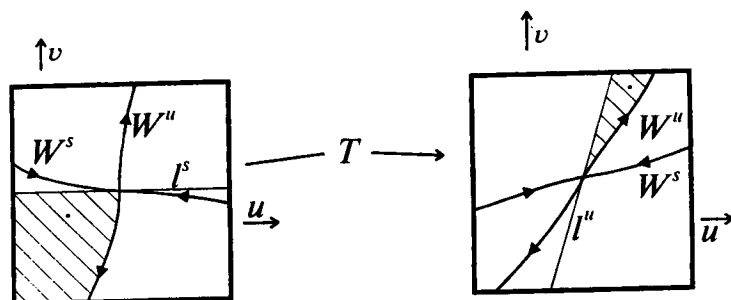


FIG. 3. A point in the dashed region on one of the cross-sections (to the left) is mapped by the symmetrized Poincaré map to a point in the dashed wedge on the other cross-section (to the right)

point of the orbit under consideration belongs to a region bounded by this ray of  $l^s$  and one of the halves of  $w^u$  (Fig. 3). The sign of  $v$ -coordinate is now the same as for any point on this half of  $w^u$ . The image of the ray of  $l^s$  (see (19), (20)) is the half of the line  $l^u$  which is tangent to a component of  $w^u$ ; the image of the point under consideration lies in a thin wedge between these two curves. Obviously, for all further iterations, the signs of the  $v$ -coordinate of the points of the orbit under consideration are the same as the orbit would lie on the corresponding component of  $w^u$ .

In particular, it means that for the piece of the  $(u, s)$  orbit under consideration there exists some  $s_m$  that for the first  $s_m$  points the signs of the  $v$ -coordinate coincide with those of the orbit of  $M_{n_m}^+$  and for the next points they coincide with the signs of the  $v$ -coordinate of the points of the orbit of  $M_{n_{m+1}}^-$ . Now, recovering the sequence of signs  $\sigma$  by (16) gives exactly the rule of theorem 4.

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## References

- [1] Devaney R. "Homoclinic orbits in Hamiltonian systems", J. Diff. Equat., v. 21 (1976), p. 431-438.
- [2] Belyakov L. A., Shilnikov L. P. "Homoclinic curves and complex solitary waves", Selecta Math. Sov., v. 9 (1990), p. 219-229.
- [3] Shilnikov L. P. "Multidimensional Hamiltonian chaos", Chaos, v. 1 (1991), № 2, p. 134-136.
- [4] Lerman L. M. "Complex dynamics and bifurcations in a Hamiltonian system having a transversal homoclinic orbit to a saddle focus". Chaos, v. 1, (1991), № 2.
- [5] Turaev D. V., Shilnikov L. P. "On Hamiltonian systems with homoclinic saddle curves", Soviet Math. Dokl., v. 39, 1989, № 1, p. 212-216.
- [6] V. M. Eleonsky V. M., Kulagin N. E., Turaev D. V., Shilnikov L. P. "On the classification of self-localized states of electro-magnetic field within nonlinear medium", DAN SSSR, 1989, v. 309, № 4, p. 848-851.
- [7] Turaev D. V. "On dimension of non-local bifurcational problems", Bifurcation and Chaos, v. 6 (1996), № 6, p. 919-948.
- [8] D. V. Turaev. "On the bifurcations of dynamical systems with two homoclinic curves of a saddle". Dissertation (1991), Nizhny Novgorod.
- [9] Shilnikov L. P. "A problem of Poincaré-Birkhoff". Math. USSR Sbornik, v. 74 (1967), № 3, p. 378-397.
- [10] Ovsyannikov I. M. and Shilnikov L. P. "Systems with a homoclinic curve of a multidimensional saddle-focus and spiral chaos". Math. USSR Sbornik v. 2 (1992), p. 415-443.
- [11] Afraimovich V. S. and Shilnikov L. P. "On attainable transitions from Morse-Smale systems to systems with many periodic motions". Math. USSR Izvestia, v. 8 (1974), № 6, p. 1235-1270.

**Л. П. ШИЛЬНИКОВ, Д. В. ТУРАЕВ**

**СУПЕРГОМОКЛИНИЧЕСКИЕ ОРБИТЫ И МНОГООБХОДНЫЕ ГОМОКЛИНИЧЕСКИЕ ПЕТЛИ В ГАМИЛЬТОНОВЫХ СИСТЕМАХ С ДИСКРЕТНЫМИ СИММЕТРИЯМИ**

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Рассматриваются четырехмерные гамильтоновы системы с дискретными симметриями, для которых две двумерные плоскости являются инвариантными. Предполагается, что в каждой из этих двумерных плоскостей система имеет по две гомоклинические траектории к седловому состоянию равновесия. Для такого гомоклинического букета вводятся понятия двумерных устойчивых и неустойчивых многообразий. Эти многообразия могут трансверсально пересекаться по некоторой траектории, которую мы называем супергомоклинической орбитой.

В работе доказывается, что существование такой траектории влечет за собой существование счетного множества гомоклинических орбит к седловому состоянию равновесия.