ISSN 1560-3547, Regular and Chaotic Dynamics, 2014, Vol. 19, No. 4, pp. 513–522. © Pleiades Publishing, Ltd., 2014.

Energy Growth for a Nonlinear Oscillator Coupled to a Monochromatic Wave

Dmitry V. Turaev*, **Christopher Warner****, and **Sergey Zelik*****

Imperial College, SW7 2 AZ London, UK;
University of Surrey, Guildford, Surrey GU2 7XH, UK

Received April 4, 2014; accepted May 17, 2014

Abstract—A system consisting of a chaotic (billiard-like) oscillator coupled to a linear wave equation in the three-dimensional space is considered. It is shown that the chaotic behavior of the oscillator can cause the transfer of energy from a monochromatic wave to the oscillator, whose energy can grow without bound.

MSC2010 numbers: 35B05, 35B42, 37K49

DOI: 10.1134/S1560354714040078

Keywords: delayed equation, invariant manifold, normal hyperbolicity, billiard

1. SETTING THE PROBLEM AND RESULT

The system we consider (a linear wave coupled to an oscillator) is formally defined by the Hamiltonian

$$H = \frac{1}{2}\left(p_y^2 + p_z^2\right) + V(y,z) + \epsilon k(y,z)\int_{\|\mathbf{x}\|\leqslant 1} u(\mathbf{x},t)\ \mathrm{d}^3\mathbf{x} + \frac{1}{2}\int \left(u_t^2 + (\nabla_x u)^2\right)\ \mathrm{d}^3\mathbf{x}, \tag{1.1}$$

where $u(\mathbf{x},t)$, the massless Klein – Gordon field, is a scalar function on $\mathbb{R}^3\times\mathbb{R}^1$, and $(y,z)\in\mathbb{R}^2$ are coordinates in the configuration space of the oscillator ($(p_y=\dot y, p_z=\dot z)\in\mathbb{R}^2$ are the corresponding momenta). The smooth potential $V(y,z)$ is bounded from below and tends to infinity as $\|y,z\|\to\infty$. To avoid technicalities, we assume that V equals to infinity outside a bounded domain in the (y,z) plane. The interaction coefficient $k(y,z)$ is smooth and bounded along with the first derivatives, and ϵ is small.

The corresponding equations of motion are

$$u_{tt} - \Delta u = -\epsilon k(y(t),z(t))\xi(\mathbf{x}), \tag{1.2}$$

where $\xi(\mathbf{x}) = \begin{cases} 1 & \|\mathbf{x}\|\leqslant 1 \\ 0 & \|\mathbf{x}\|>1 \end{cases}$ (the characteristic function of the unit ball in the $\mathbf{x}$-space), and

$$\begin{aligned} \ddot y + \frac{\partial}{\partial y}\left(V(y,z)\right) &= -\epsilon k'_y \int_{\|\mathbf{x}\|\leqslant 1} u(\mathbf{x},t)\ \mathrm{d}^3\mathbf{x}, \\ \ddot z + \frac{\partial}{\partial z}\left(V(y,z)\right) &= -\epsilon k'_z \int_{\|\mathbf{x}\|\leqslant 1} u(\mathbf{x},t)\ \mathrm{d}^3\mathbf{x}. \end{aligned} \tag{1.3}$$

We may think of the oscillator as being located in the unit ball in the $\mathbf{x}$-space and emitting/receiving the wave $u(\mathbf{x},t)$. It can be shown (similar to [1–3]) that if the initial energy of the wave is finite

*E-mail: d.turaev@imperial.ac.uk
**E-mail: c.warner08@imperial.ac.uk
***E-mail: S.Zelik@surrey.ac.uk

(i.e., $\int \left(u_t^2 + (\nabla_x u)^2\right)\ \mathrm{d}^3\mathbf{x} < \infty$ at $t = 0$), then the field u tends to a constant and, for a typical choice of the interaction coefficient $k(y, z)$,[1)] the energy of the oscillator decreases, and it comes to a rest at some stationary point of the potential function as $t \to +\infty$, i.e., the energy flows from the oscillator to the field and is carried away to infinity. In this paper we show that if the wave has infinite energy, then an opposite process may take place. Namely, when the oscillator operates in a chaotic regime, the energy can be pumped from the field to the oscillator, and the oscillator's kinetic energy can increase up to any given value.

The field equation (1.2) can be explicitly resolved:

$$u(\mathbf{x}, t) = u_0(\mathbf{x}, t) + \frac{\epsilon}{4\pi} \int \frac{k(z(t - \|\mathbf{s}\|), y(t - \|\mathbf{s}\|))}{\|s\|} \xi(\mathbf{x} - \mathbf{s})\ \mathrm{d}^3\mathbf{s}, \tag{1.4}$$

where u_0 is the solution to the homogeneous wave equation. We take as u_0 a monochromatic standing wave of infinite energy:

$$u_0(\mathbf{x}, t) = K \sin(\omega t) \int \alpha_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x})\ \delta(\omega - \|\mathbf{k}\|) d^3\mathbf{k} \tag{1.5}$$

(the fact that it is a standing wave is not very important; the fact that the frequency spectrum is discrete and finite is used in an essential way). By placing (1.4) into (1.3) we obtain a system of delayed differential equations

$$\begin{aligned} \ddot{y} + \frac{\partial}{\partial y}\left(V(y, z)\right) &= - A_\omega \epsilon k'_y(y, z) \sin(\omega t) - \epsilon^2 k'_y \int_0^2 k(z(t - s), y(t - s)) \mathcal{P}(s)\ \mathrm{d}s, \\ \ddot{z} + \frac{\partial}{\partial z}\left(V(y, z)\right) &= - A_\omega \epsilon k'_z(y, z) \sin(\omega t) - \epsilon^2 k'_z \int_0^2 k(z(t - s), y(t - s)) \mathcal{P}(s)\ \mathrm{d}s, \end{aligned} \tag{1.6}$$

where

$$A_\omega = K \int \alpha_{\mathbf{k}} \xi(\mathbf{x}) \cos(\mathbf{k} \cdot \mathbf{x})\ \delta(\omega - \|\mathbf{k}\|) d^3\mathbf{k} d^3\mathbf{x}, \tag{1.7}$$

and $\mathcal{P}(s) = \dfrac{\pi}{12}(16 - 12s + s^3)s$ (we denote here $s = \|\mathbf{s}\|$). This is a Hamiltonian system subject to a conservative periodic perturbation of order ϵ and a dissipative[2)] correction of order ϵ^2. Namely, if we drop the $O(\epsilon^2)$-terms (those including the delay), system (1.6) will be defined by a time-dependent Hamiltonian function

$$H = \frac{1}{2}\left(p_y^2 + p_z^2\right) + V(y, z) + \epsilon A_\omega k(y, z) \sin(\omega t). \tag{1.8}$$

In general, the systems with a time-dependent Hamiltonian do not preserve energy, and the periodic forcing may, in fact, lead to an unbounded growth of energy. Importantly, as the kinetic energy grows, the forcing becomes effectively slow. It is known [4–7] that if a Hamiltonian system behaves chaotically at all sufficiently large energies, then adding a slow periodic forcing creates orbits of unbounded energy growth. Similar to [7], we will show that in the system of type (1.8) this growth is linear in time; namely, one may construct orbits for which the energy gain per period of the force is bounded from below by a nonzero constant of order ϵ. Using the fact that the delay terms which can lead to a dissipation of energy are of order ε^2, i.e., they are much smaller than the energy gain, we show below that the full system (1.6) has, for sufficiently small ϵ, solutions for which the energy grows up to any given value, linearly in time.

Our method does not allow us to show the existence of orbits for which the energy tends to infinity (the larger the energy value we want to achieve, the smaller value of ϵ we have to take). Similarly, our

[1)] For a general choice of $k(y, z)$ the state of oscillator may approach an invariant set on which the function $k(y, z)$ stays constant, see more in [1, 3].

[2)] We are not showing here that the delay term always leads to a dissipation of energy. In order to get some insight, one can check that in the linear case (i.e., quadratic potential V and linear coefficient of interaction k) the addition of this term shifts the spectrum to the left of the imaginary axis, i.e., creates dissipation (however, as we consider bounded functions k and the potential V that is infinite outside a bounded domain, the linear case is not the subject of this paper).

approach is not applicable to the case of nonzero mass in the wave equation. However, we conjecture that such orbits do exist in system (1.6), like they do in the time-dependent Hamiltonian system defined by (1.8) (see [7]). We also think that our approach (based on a reduction to an invariant manifold) is interesting in its own right. We stress that the mechanism of the sufficiently fast (linear) energy growth we obtain here is based on the chaotic behavior of the oscillator. Would the oscillator be integrable, the existence of adiabatic invariants [8, 9] would impede a possible energy growth in the shortened system (1.8), and it seems plausible that the $O(\epsilon^2)$-dissipation due to delayed terms should arrest the energy growth completely in this case. So, the picture we propose is the following: while an integrable oscillator that interacts with a monochromatic wave (in $\mathbb{R}^3$) should lose energy, a chaotic oscillator may take the energy from the wave. Note that the reported effect cannot be immediately interpreted as a resonant phenomenon: since the potential is infinite outside a bounded domain, the region of allowed motions (the Hill's region) in the configuration plane (y, z) is always bounded, and since the velocities p_y and p_z tend to infinity as the energy grows, it follows that the characteristic return times tend to zero and become much smaller than the period of external force.

In order to formulate the result precisely, we need to define what we mean by the "chaotic oscillator". Consider a system of ODE's defined by the Hamiltonian

$$H = \frac{1}{2}\left(p_y^2 + p_z^2\right) + V(y, z) \tag{1.9}$$

(the Hamiltonian (1.8) at $\epsilon = 0$). As we mentioned, we assume that V is infinite outside a certain bounded region D. Following [7], assume that for each sufficiently large h, the system defined by (1.9) has, on the energy level $H = h$, a pair of hyperbolic periodic orbits L_a and L_b such that the unstable manifold of L_a has an orbit Γ_{ab} of transverse intersection with the stable manifold of L_b, and the unstable manifold of L_b has an orbit Γ_{ba} of transverse intersection with the stable manifold of L_a. Moreover, we assume that the orbits L_a, L_b, Γ_{ab} and Γ_{ba} depend continuously on h. When these conditions are fulfilled, we call system (1.9) a chaotic oscillator. It is well known [10, 11] that the existence of the transverse heteroclinic cycle implies a chaotic behavior indeed. Namely, fix a value of h, and take a sufficiently small neighborhood U of the heteroclinic cycle $L_a \cup L_b \cup \Gamma_{ab} \cup \Gamma_{ba}$ on the energy level $H = h$. Then the set of all orbits that stay in $U \cap \{H = h\}$ is in one-to-one correspondence with arbitrary sequences of the symbols a and b (one round made by the orbit near L_a is coded by a, and a round near L_b is coded by b).

The basic example of chaotic oscillators in our setting is given by systems with billiard-like potentials. Namely, let $D \subset \mathbb{R}^2$ be the bounded domain such that $V(y, z)$ is finite inside D and infinite outside of D. Let the boundary of D consist of a finite number of smooth arcs, $S_1, \ldots, S_n$, joined at corner points. Let C^r-smooth functions $Q_1(x, y), \ldots, Q_n(x, y)$ be such that, for each $j = 1, \ldots, n$, the function Q_j is defined in a neighborhood of the boundary arc S_j, the arc S_j is a level line of Q_j (i.e., $Q_j(x, y)|_{(x,y)\in S_j} = \text{const}$), and $\nabla Q_j \neq 0$ in the neighborhood of S_j. We will call the potential V that equals to infinity outside of D a billiard-like potential if there exist C^r-smooth, strictly monotonic functions $W_1(Q), \ldots, W_n(Q)$ such that for each $j = 1, \ldots, n$ the potential V in a small neighborhood of the arc S_j is given by

$$V(x, y) = W_j(Q_j(x, y)) \tag{1.10}$$

(we do not include the corner points into the arcs S_j, i.e., they are open intervals, so their small open neighborhoods do not need to contain the corner points, hence even if two arcs join at a corner point, their small neighborhoods where (1.10) holds do not need to intersect, i.e., no relation between the corresponding functions W_j arises). By scaling the momenta $p_{y,z}$ to $\sqrt{h}$, system (1.9) on the energy level $H = h$ transforms into the system

$$H = \frac{1}{2}\left(p_y^2 + p_z^2\right) + \frac{1}{h}V(y, z) \tag{1.11}$$

on the energy level $H = 1$. One can easily check that if V is a billiard-like potential (i.e., it is defined by (1.10) near the boundary arcs), then the family of Hamiltonians (1.11) (with h^{-1} being a small parameter) satisfies the conditions of [14] which guarantee that the flow defined by such a Hamiltonian is an approximation, at h large enough, to the billiard flow in D. Namely, the billiard

in D is a mechanical dynamical system which describes the following motion of a point mass [18]: the particle moves inertially (with a constant velocity) inside the region D in a plane until it hits the boundary of D, then the particle is reflected according to the elastic reflection law, the angle of reflection is the angle of incidence, and so on. As it follows from [14], the time-shift maps by the billiard flow in D and by the Hamiltonian flow defined by (1.11) with a billiard-like potential V and h large enough are C^r-close outside the set of singular billiard orbits (i.e., such billiard orbits which enter a corner point or are tangent to the boundary of the billiard domain D). Therefore, if the billiard in D has a heteroclinic cycle of two hyperbolic periodic orbits L_a and L_b and two transverse heteroclinic orbits Γ_{ab} and Γ_{ba} and neither of these orbits is singular, then system (1.11) on the energy level $H = 1$ (and hence system (1.9) on the energy level $H = h$) also has such a heteroclinic cycle for all h large enough, i.e., system (1.9) with a billiard-like potential V is a chaotic oscillator according to our definition. In particular, the oscillator defined by a billiard-like potential is chaotic when the underlying billiard is dispersive: all the boundary arcs are concave and meet each other at nonzero angles at the corner points [14–18].

Let L_a and L_b be the two nonsingular hyperbolic periodic orbits of the billiard in D that are connected by the transverse nonsingular heteroclinics Γ_{ab} and Γ_{ba}. Fix the speed of the particle in the billiard to be equal to 1, and let T_c (where $c = a, b$) be the period of L_c, and $(y_c(t), z_c(t))|_{t\in[0,T_c]}$ be the equation of the orbit L_c. Denote

$$v_c = \frac{1}{T_c}\int_0^{T_c} k(y_c(t), z_c(t))dt. \tag{1.12}$$

Assume

$$v_a \neq v_b. \tag{1.13}$$

Theorem 1. *Consider a system* (1.2), (1.3) *that describes a linear massless scalar field interacting with a chaotic oscillator, with a billiard-like potential V and the interaction coefficient $k(y, z)$ bounded with first derivatives. Let condition* (1.13) *hold. Then there exists h_0 such that for each ω for which $A_\omega \neq 0$ (see* (1.7)*) and for any $h_1 > h_0$ there exists $\epsilon_0(h_1) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the system has a solution with the wave component u given by* (1.4), (1.5) *and the oscillator component $(y(t), z(t), p_y(t), p_z(t))$ reaching from the energy h_0 at $t = 0$ the energy h_1 at some finite t (the energy of the oscillator is given by* (1.9)*).*

The proof of the theorem occupies the rest of the paper. In Section 2 we prove an invariant manifold theorem which allows us to reduce the system of delayed differential equations (1.6) to a four-dimensional nonautonomous system of ODE's, which is $O(\epsilon^2)$-close to the Hamiltonian system given by (1.8). This reduction is possible on any bounded set of values of (y, z, p_y, p_z) for sufficiently small ϵ. Note that the maximal value of ϵ for which we can guarantee the reduction increases as p_y and p_z grow, which is one of the reasons why we show only bounded energy growth in Theorem 1. In Section 3 we apply the construction of [7] (modified for the non-Hamiltonian case) to the reduced system, and finish the proof.

2. INVARIANT MANIFOLD THEOREM

In this section we reduce the infinite-dimensional system of differential equations with delay to a finite-dimensional system of ordinary differential equations. Consider the system

$$\dot{X}(t) = F(X(t)) - \delta\int_0^\tau G(X(t-s), s)\ \mathrm{d}s, \tag{2.1}$$

where X belongs to an n-dimensional smooth manifold $\mathbb{M}$, and the functions F and G are C^r-smooth. Let F be such that the differential equation

$$\dot{X} = F(X) \tag{2.2}$$

has, for any initial condition, a solution defined for all $t \in (-\infty, +\infty)$ (for example, this differential equation is Hamiltonian with compact energy levels). The system (1.6) can be represented in this form with $\delta = \epsilon^2$. Namely, we introduce a variable $\theta \in \mathbb{S}^1$ and rewrite (1.6) as

$$\begin{aligned} &\dot{y} = p_y, \qquad \dot{z} = p_z, \qquad \dot{\theta} = \omega, \\ &\dot{p}_y = -\frac{\partial}{\partial y}\left(V(y,z)\right) - A_\omega \epsilon k_y'(y,z)\sin(\theta) - \epsilon^2 k_y' \int_0^2 k(z(t-s), y(t-s))\mathcal{P}(s)\ \mathrm{d}s, \\ &\dot{p}_z = -\frac{\partial}{\partial z}\left(V(y,z)\right) - A_\omega \epsilon k_z'(y,z)\sin(\theta) - \epsilon^2 k_z' \int_0^2 k(z(t-s), y(t-s))\mathcal{P}(s)\ \mathrm{d}s, \end{aligned} \tag{2.3}$$

so $X = (y, z, p_y, p_z, \theta = \omega t) \in \mathbb{R}^4 \times \mathbb{S}^1$ here.

It is well known that given any continuous function $\hat{X} : [-\tau, 0] \to R^n$, there exists a unique solution $X(t)$ of Eq. (2.1) such that $X(t) \equiv \hat{X}(t)$ at $t \in [-\tau, 0]$. One can therefore view the evolution defined by Eq. (2.1) as a semiflow in the space C of continuous functions that act from $[-\tau, 0]$ to R^n: the time-s map of the semiflow takes the initial condition $X(s)|_{[-\tau,0]}$ to the segment $X(s)|_{[t-\tau,t]}$ of the corresponding solution. A smooth function $\mu : R^n \times [0, \tau] \to R^n$ defines a map $R^n \to C$ by the rule $X(s) = \mu(x, -s)$; the graph of such a map is an invariant manifold for the semiflow defined by (2.1) if the solution with the initial condition $\mu(X(0), -s)|_{s\in[-\tau,0]}$ satisfies

$$\mu(X(t), s) = X(t - s) \tag{2.4}$$

for all $t \geqslant 0$ and $s \in [0, \tau]$. When such a manifold exists, the restriction of system (2.1) onto it is a system of ordinary differential equations

$$\dot{X} = F(X) - \delta \int_0^\tau G(\mu(X, s), s)\ \mathrm{d}s. \tag{2.5}$$

In other words, the existence of a smooth function μ which satisfies the invariance condition (2.4) implies the existence of an n-parameter family of solutions (parameterized by $X(0)$) to the delayed differential equation (2.1) which also solve the ordinary differential equation (2.5).

Lemma 1. *Given any compact subset K of the X-space $\mathbb{M}$, for all sufficiently small δ there exists a C^{r-1}-smooth function $\mu : K \times [0, \tau] \to R^n$ such that for any $X(0) \in K$ the solution $X(t)$ of* (2.1) *which starts with the initial condition $\mu(X(0), -s)|_{s\in[-\tau,0]}$ satisfies the invariance condition* (2.4) *for the interval of t values for which the solution stays in K.*

Proof. By the Cauchy – Picard – Lindelöf theorem [19], given any smooth functions μ, F and G, the ordinary differential equation (2.5) generates a uniquely defined (on any given finite time interval) solution for any initial condition from K if δ is small enough. This solution depends smoothly on any parameter on which the system depends smoothly. In particular, the solution depends smoothly on the function μ. Thus, we will show below that the solution is a C^1-function of μ which is considered as an element of the space of C^{r-1}-smooth functions.

Given μ and δ, take any $X(t) \in K$, consider its backward orbit by Eq. (2.5), and let $X(t - s)$ be the point on this orbit which corresponds to the (backward) shift to time s. Denote as $\phi(\mu, \delta)$ the map $K \times [0, \tau] \to R^n$ which (for given μ and δ) sends $X(t) \in K$ and $s \in [0, \tau]$ to $X(t - s)$, i.e., $\phi(\mu, \delta)$ is the backward flow of the ordinary differential equation (2.5). The flow has the same smoothness as the equation, so if $\mu \in C^{r-1}$, then $\phi(\mu, \delta)$ is C^{r-1} with respect to X and s. At $\delta = 0$ the flow $\phi(\mu, \delta)$ is generated by Eq. (2.2), so it is independent of μ. We will show in a moment that ϕ depends C^1-smoothly on μ and δ. As $\phi(\mu, 0)$ does not depend on μ, it follows that the Frechet derivative $\dfrac{\partial \phi}{\partial \mu}$ vanishes at $\delta = 0$. Hence, by the implicit function theorem, the equation

$$\mu = \phi(\mu, \delta) \tag{2.6}$$

has, for every small δ, a unique solution $\mu \in C^{r-1}$. If we plug this particular μ into the right-hand side of (2.5), then Eq. (2.6) will exactly mean that condition (2.4) is satisfied by the solutions

of (2.5). Thus, each of these solutions will also solve the original delayed equation (2.1), which is the statement of the lemma.

Thus, to finish the proof, it remains to show the smooth dependence of the flow of (2.5) on μ. It is well known that the solutions of ordinary differential equations depend smoothly on the function on the right-hand side of the equation. So we are left to show that the so-called Nemytsky operator (or substitution operator) $\mathcal{N}$ which takes the function $\mu(X,s)$ to the function $G(\mu(X,s),s)$ is of class C^1 on the space of C^{r-1}-functions μ, provided G is C^r as a function of (X,s). In order to do this, it is enough to check that the Frechet derivative of $\mathcal{N}$ at a given function μ is the operator of multiplication to $\dfrac{\partial G}{\partial X}(\mu(X,s),s)$. For this, one needs to check that

$$\|G(\mu+\Delta\mu)-G(\mu)-G'(\mu)\cdot\Delta\mu\|_{C^{r-1}}=o(\|\Delta\mu\|_{C^{r-1}})$$

(we suppress, notationally, the dependence of s, so G' denotes here the derivative of G with respect to its first argument). This relation is rewritten as

$$\left\|\int_0^1 (G'(\mu+\xi\Delta\mu)-G'(\mu))\,\mathrm{d}\xi\cdot\Delta\mu\right\|_{C^{r-1}}=o(\|\Delta\mu\|_{C^{r-1}}),$$

which reduces to the obvious (since G is C^r) claim that $G'(\mu+\xi\Delta\mu)\underset{C^{r-1}}{\longrightarrow}G'(\mu)$ as $\Delta\mu\underset{C^{r-1}}{\longrightarrow}0$, uniformly for all $\xi\in[0,1]$.

Applying this lemma to system (2.3), we find that for all sufficiently small ϵ it has a family of solutions which satisfy the system of ordinary differential equations

$$\begin{aligned}
\ddot y+\frac{\partial}{\partial y}(V(y,z)) &= -A_\omega\epsilon k_y'(y,z)\sin(\theta)+\epsilon^2\mathcal{F}_1(y,z,\dot y,\dot z,\theta,\epsilon),\\
\ddot z+\frac{\partial}{\partial z}(V(y,z)) &= -A_\omega\epsilon k_z'(y,z)\sin(\theta)+\epsilon^2\mathcal{F}_2(y,z,\dot y,\dot z,\theta,\epsilon),\\
\dot\theta &= \omega,
\end{aligned}\tag{2.7}$$

where the smooth functions $\mathcal{F}_{1,2}$ incorporate the delay terms. Moreover, for all sufficiently small ϵ, every solution of this system satisfies the original delayed equations (1.6) for the interval of time for which the solution stays in a bounded ball in the phase space (we may take this ball as large as we want; however, to increase the radius of the ball, we might need to take ϵ smaller).

3. A PARTIALLY HYPERBOLIC SET AND THE ENERGY DRIFT

As we have just shown, it is enough to establish the existence of the solutions of growing energy in system (2.7). By taking a sufficiently large h_0 and scaling time to $\sqrt{h_0}$, we rewrite this system as

$$\begin{aligned}
\ddot y+\frac{1}{h_0}\frac{\partial}{\partial y}(V(y,z)) &= -A_\omega\frac{\epsilon}{h_0}k_y'(y,z)\sin(\theta)+\frac{\epsilon^2}{h_0}\mathcal{F}_1(y,z,\sqrt{h_0}\dot y,\sqrt{h_0}\dot z,\theta,\epsilon),\\
\ddot z+\frac{1}{h_0}\frac{\partial}{\partial z}(V(y,z)) &= -A_\omega\frac{\epsilon}{h_0}k_z'(y,z)\sin(\theta)+\frac{\epsilon^2}{h_0}\mathcal{F}_2(y,z,\sqrt{h_0}\dot y,\sqrt{h_0}\dot z,\theta,\epsilon),\\
\dot\theta &= \frac{\omega}{\sqrt{h_0}}.
\end{aligned}\tag{3.1}$$

At $\epsilon=0$ the first two equations are independent of θ:

$$\begin{aligned}
\ddot y+\frac{1}{h_0}\frac{\partial}{\partial y}(V(y,z)) &=0,\\
\ddot z+\frac{1}{h_0}\frac{\partial}{\partial z}(V(y,z)) &=0.
\end{aligned}\tag{3.2}$$

As we mentioned, the assumption that our billiard-like oscillator is chaotic means that this system has, at sufficiently large h_0, a uniformly hyperbolic set Λ on every energy level $H\geqslant 1$ (where the

rescaled energy H is given by (1.11) with $h = h_0$). Namely, we take the two nonsingular hyperbolic periodic orbits L_a and L_b of the billiard in D and the two transverse nonsingular heteroclinics Γ_{ab} and Γ_{ba} that connect them. As we mentioned, by virtue of [14], the hyperbolic heteroclinic cycle persists in the smooth Hamiltonian approximation (3.2) of the billiard, provided h_0 is large enough. This means that system (3.2) has on every energy level $H = h \geqslant 1$ a pair of hyperbolic periodic orbits $L_a(h)$ and $L_b(h)$ and the heteroclinic orbits, $\Gamma_{ab}(h)$ and $\Gamma_{ba}(h)$ where Γ_{ab} is the transverse intersection of the unstable manifold of L_a with the stable manifold of L_b, and similarly for Γ_{ba}:

$$\Gamma_{ab} \subseteq W^u(L_a) \cap W^s(L_b), \quad \Gamma_{ba} \subseteq W^u(L_b) \cap W^s(L_a),$$

and these four orbits are close (at h_0 large enough) to the corresponding orbits of the billiard on the same energy level.

By [10], the set $\Lambda(h)$ of all orbits which stay in a small neighborhood (on the level set $H = h$) of $L_a(h) \cup L_b(h) \cup \Gamma_{ab}(h) \cup \Gamma_{ba}(h)$ is a uniformly hyperbolic set which is in one-to-one correspondence with the set of all sequences of a's and b's. More precisely, on each energy level we take two small smooth cross-sections, Σ_a and Σ_b, to L_a and L_b, respectively (we assume that the cross-sections depend on h smoothly). Every orbit from Λ must intersect $\Sigma_a \cup \Sigma_b$ infinitely many times. The sequence of a's and b's denoting the corresponding cross-sections that the trajectory passes through gives the code, $\{\xi_i\}_{i=-\infty}^{+\infty}$ ($\xi_i \in \{a, b\}$), of the trajectory. Namely, if M_i is the sequence of points at which an orbit from Λ intersects $\Sigma_a \cup \Sigma_b$, then $M_i \in \Sigma_{\xi_i}$.

The flow induced by (3.2) on any given energy level $H = h$ defines Poicaré maps on these cross-sections, denoted by

$$\Pi_{cc'} : \Sigma_c \to \Sigma_{c'}, \quad c \in \{a, b\}. \tag{3.3}$$

These maps are such that for a trajectory of (3.2) that intersects $\Sigma_a \cup \Sigma_b$ at a point $M_i \in \Sigma_c$ and then at a point $M_{i+1} \in \Sigma_{c'}$ we have

$$M_{i+1} = \Pi_{cc'} M_i. \tag{3.4}$$

In other words, Π_{aa} and Π_{bb} are the Poincaré maps near the periodic orbits L_a and, respectively, L_b, while Π_{ab} and Π_{ba} correspond to a passage near the heteroclinics Γ_{ab} and Γ_{ba}. The theory of billiard-like potentials built in [14] implies that since the billiard orbits L_a, L_b, Γ_{ab}, Γ_{ba} are nonsingular, the Poincaré maps $\Pi_{cc'}$ for system (3.2) are C^r-close, at h_0 large enough, to the corresponding Poincaré maps defined by the billiard flow. In particular, the hyperbolicity of these maps for the billiard flow (which follows from the fact that the periodic orbits L_a and L_b are hyperbolic and the heteroclinic orbits Γ_{ab} and Γ_{ba} are transverse) is inherited in system (3.2) for all h_0 large enough.

In fact, since each periodic orbit is a saddle, one can represent the two-dimensional cross-sections $\Sigma_{a,b}$ as the cross product of two certain small intervals

$$\Sigma_a = U_a \times W_a, \quad \Sigma_b = U_b \times W_b,$$

where $U_{a,b} \in \mathbb{R}$ correspond to contracting directions and $W_{a,b} \in \mathbb{R}$ correspond to expanding directions. By virtue of [11], since the heteroclinic orbits Γ_{ab} and Γ_{ba} are transverse, one can write the Poincaré maps (3.3) in the so-called cross form (see [12, 13]). Namely, there exist smooth functions

$$f_{cc'} : U_c \times W_{c'} \to U'_c, \quad g_{cc'} : U_c \times W_{c'} \to W_c, \tag{3.5}$$

(where $c, c' \in \{a, b\}$) such that a point $M_i = (u_i, w_i) \in \Sigma_c$ is mapped to $M_{i+1} = (u_{i+1}, w_{i+1}) \in \Sigma_{c'}$ by the map $\Pi_{cc'}$ if and only if

$$f_{cc'}(u_i, w_{i+1}) = u_{i+1}, \quad g_{cc'}(u_i, w_{i+1}) = w_i. \tag{3.6}$$

Moreover, there exists $\lambda > 0$ such that

$$\left\| \frac{\partial(f_{cc'}, g_{cc'})}{\partial(u, w)} \right\| \leqslant \lambda < 1. \tag{3.7}$$

The latter inequality means essentially that the Poincaré maps $\Pi_{cc'}$ are contracting in the u-coordinate and expanding in the v-coordinate.

From [7] it follows that the operator

$$\{(u_i, w_i)\}_{i=-\infty}^{+\infty} \to \{(f_{\xi_{i-1}\xi_i}(u_{i-1}, w_i), g_{\xi_i\xi_{i+1}}(u_i, w_{i+1}))\}_{i=-\infty}^{+\infty} \tag{3.8}$$

is a contraction mapping for any code sequence $\xi = \{\xi_i\}_{i=-\infty}^{+\infty}$, which gives us the existence and uniqueness of the orbit $L_\xi(h) \in \Lambda(h)$ with the given code ξ (see [10, 11]). By fixing a code ξ, the family of the orbits with this code parameterized by h, i.e., the manifold

$$\mathcal{L}_\xi^0 = \bigcup_{1 \leqslant h \leqslant h_1/h_0} L_\xi(h)$$

for any given $h_1 > h_1$, is a normally hyperbolic invariant manifold of system (3.2). The system (3.1) at $\epsilon = 0$ is obtained from (3.2) simply by adding an equation $\dot\theta = \omega/\sqrt{h_0}$ for the phase $\theta \in \mathbb{S}^1$. As this equation is decoupled from the first two, and the evolution of θ is nonhyperbolic (it is just a linear rotation), the manifold $\mathcal{L}_\xi^0 \times \mathbb{S}^1$ is the invariant normally hyperbolic manifold for system (3.1) at $\epsilon = 0$. The normally hyperbolic invariant manifolds are known to persist at small perturbations [20], which implies that (3.1) has an invariant manifold $\mathcal{L}_\xi(\epsilon)$ close to $\mathcal{L}_\xi \times \mathbb{S}^1$ for all small ϵ (one might need to take smaller ϵ to make h_1 larger). Because the normal hyperbolicity is uniform, these manifolds exist for all codes ξ for the same range of ϵ values. A formal proof of the existence of the invariant manifolds $\mathcal{L}_\xi(\epsilon)$ is achieved as follows. Note that the Poincaré maps $\Pi_{cc'}$ are defined for all ϵ sufficiently small and depend smoothly on ϵ, so they can be written in the form

$$\begin{aligned} &u_{i+1} = f_{cc'}(u_i, w_{i+1}) + O(\epsilon), \quad w_i = g_{cc'}(u_i, w_{i+1}) + O(\epsilon), \\ &h_{i+1} = h_i + O(\epsilon), \quad \theta_{i+1} = \theta_i + O\left(\epsilon + \frac{1}{\sqrt{h_0}}\right), \end{aligned} \tag{3.9}$$

where the energy h is defined by (1.8); the $O(\cdot)$-terms are functions of $u_i, w_{i+1}, h_i, \theta_i$. Now, Lemma 1 of [7] is applied to these maps, which immediately gives the existence of the invariant manifolds $\mathcal{L}_\xi(\epsilon)$ close to $\mathcal{L}_\xi \times \mathbb{S}^1$ provided ϵ is small and h_0 is large enough.

By differentiating (1.8), we find that the rate of change of $h = H(y, z, p_y, p_z, \theta)$ along an orbit of (3.1) is given by

$$\frac{dh}{dt} = \epsilon(\omega A_\omega k(y, z) \cos\theta + \mathcal{O}(\epsilon)). \tag{3.10}$$

Since the Poincaré map (3.9) at $c = c'$ corresponds to one round near the billiard periodic orbit L_c, it follows from (3.10) that given an arbitrarily small δ, one can choose ϵ and h_0^{-1} small enough such that

$$\frac{h_{i+1} - h_i}{\theta_{i+1} - \theta_i} > \epsilon(A_\omega v_c \cos\theta - \delta) \tag{3.11}$$

in the map Π_{cc} ($c = a$ or b). In other words, as long as an orbit of system (3.1) stays near L_c, the change in h can be estimated by the inequality

$$\frac{dh}{d\theta} > \epsilon(A_\omega v_c \cos\theta - \delta). \tag{3.12}$$

Recall that the constants v_a and v_b are given by (1.12). By (1.13) we may fix the choice of L_a and L_b such that

$$v_a > v_b. \tag{3.13}$$

Formally, system (3.1) is not of the form studied in [7]. However, the study in [7] (see Theorem 1 there) is reduced to the study of Poincaré maps $\Pi_{cc'}$ of the form which includes (3.9) as a partial case. Therefore, we may apply the results of [7] to the study of the behavior of the orbits of system (3.1) which belong to the invariant manifolds $\mathcal{L}_\xi(\epsilon)$ (these orbits intersect the cross-sections $\Sigma_a \cup \Sigma_b$, so they are defined by the Poincaré maps $\Pi_{cc'}$). Namely, we have two main conclusions. First, we consider the codes ξ such that the symbols a and b go always in blocks of some fixed

size, large enough for the transitions from a neighborhood of L_a to a neighborhood of L_b or from a neighborhood of L_b to a neighborhood of L_a to happen relatively rarely, so the contribution of these transitions to the change in h can be neglected (more precisely, it is absorbed in the small δ term in (3.14)). Then, by (3.12), we find that for the orbits that stay on the invariant manifolds $\mathcal{L}_\xi(\epsilon)$ with such codes ξ the evolution of h is estimated by

$$\frac{dh}{d\theta} > \epsilon(A_\omega v_{c(t)} \cos\theta - \delta), \tag{3.14}$$

where $c(t) = a$ or b indicates where the orbit finds itself at the moment t, near L_a or near L_b. Second, by repeating the corresponding construction in [7], for each small ϵ we can find a code ξ and an initial condition on the corresponding manifold $\mathcal{L}_\xi(\epsilon)$ such that for the corresponding orbit

$$c(t) = a \quad \text{when} \quad A_\omega \cos\theta > 0, \qquad \text{and} \qquad c(t) = b \quad \text{when} \quad A_\omega \cos\theta < 0.$$

For such an orbit, we can rewrite (3.14) as

$$\frac{dh}{d\theta} > \epsilon \left(A_\omega \cos\theta \frac{v_a + v_b}{2} + |A_\omega \cos\theta| \frac{v_a - v_b}{2} - \delta \right).$$

By integrating over one period of $\theta = \omega t$, we find that for this particular orbit the change of the energy h over each consecutive $2\pi/\omega$ interval of time is estimated by

$$\Delta h > \epsilon(|A_\omega|(v_a - v_b) - 2\pi\delta) > 0$$

(recall that δ can be taken as small as we need and $v_a > v_b$ by assumption of the theorem). Thus, the energy of the chaotic oscillator steadily grows along the chosen orbit (at a linear rate), which completes the proof of the theorem.

ACKNOWLEDGMENTS

We are grateful to V. Gelfreich, A. Vladimirov, and V. Rom-Kedar for useful discussions. This work was supported by the Leverhulme Trust grant RPG-279 and the Russian Ministry of Education and Science grants 14.B37.21.0862.

REFERENCES

1. Komech, A., Spohn, H., and Kunze, M., Long-Time Asymptotics for a Classical Particle Interacting with a Scalar Wave Field, *Comm. Partial Differential Equations,* 1997, vol. 22, nos. 1–2, pp. 307–335.
2. Imaikin, V., Komech, A., and Spohn, H., Scattering Theory for a Particle Coupled to a Scalar Field: Partial Differential Equations and Applications, *Discrete Contin. Dyn. Syst.,* 2004, vol. 10, nos. 1–2, pp. 387–396.
3. Treschev, D. V., Oscillator and Thermostat, *Discrete Contin. Dyn. Syst.,* 2010, vol. 28, no. 4, pp. 1693–1712.
4. Bolotin, S. and Treschev, D., Unbounded Growth of Energy in Nonautonomous Hamiltonian Systems, *Nonlinearity,* 1999, vol. 12, no. 2, pp. 365–388.
5. Delshams, A., de la Llave, R., and Seara, T. M., A Geometric Approach to the Existence of Orbits with Unbounded Energy in Generic Periodic Perturbations by a Potential of Generic Geodesic Flows of $\mathbf{T}^2$, *Comm. Math. Phys.,* 2000, vol. 209, no. 2, pp. 353–392.
6. Delshams, A., de la Llave, R., and Seara, T. M., Orbits of Unbounded Energy in Quasi-Periodic Perturbations of Geodesic Flows, *Adv. Math.,* 2006, vol. 202, no. 1, pp. 64–188.
7. Gelfreich, V. and Turaev, D., Unbounded Energy Growth in Hamiltonian Systems with a Slowly Varying Parameter, *Comm. Math. Phys.,* 2008, vol. 283, no. 3, pp. 769–794.
8. Arnol'd, V. I., Kozlov, V. V., and Neĭshtadt, A. I., *Mathematical Aspects of Classical and Celestial Mechanics,* 3rd ed., Encyclopaedia Math. Sci., vol. 3, Berlin: Springer, 2006.
9. Arnol'd, V. I., *Mathematical Methods of Classical Mechanics,* 2nd ed., Grad. Texts in Math., vol. 60, New York: Springer, 1989.
10. Shilnikov, L. P., On a Poincaré – Birkhoff Problem, *Math. USSR-Sb.,* 1967, vol. 3, no. 3, pp. 353–371; see also: *Mat. Sb. (N. S.),* 1967, vol. 74(116), no. 3, pp. 378–397.
11. Afraimovich, V. S. and Shilnikov, L. P., On Critical Sets of Morse – Smale Systems, *Trans. Moscow Math. Soc.,* 1973, vol. 28, pp. 179–212.

12. Shilnikov, L. P., Shilnikov, A. L., Turaev, D., and Chua, L. O., *Methods of Qualitative Theory in Nonlinear Dynamics: Part 1,* World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises, vol. 4, River Edge, N.J.: World Sci., 1998.
13. Shilnikov, L. P., Shilnikov, A. L., Turaev, D., and Chua, L. O., *Methods of Qualitative Theory in Nonlinear Dynamics: Part 2,* World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises, vol. 5, River Edge, N.J.: World Sci., 2001.
14. Turaev, D. and Rom-Kedar, V., Elliptic Islands Appearing in Near-Ergodic Flows, *Nonlinearity,* 1998, vol. 11, no. 3, pp. 575–600.
15. Sinai, Ya. G., Dynamical Systems with Elastic Reflections: Ergodic Properties of Dispersing Billiards, *Russian Math. Surveys,* 1970, vol. 25, no. 2, pp. 137–189; see also: *Uspekhi Mat. Nauk,* 1970, vol. 25, no. 2(152), pp. 141–192.
16. Bunimovich, L. A., Sinai, Ya. G., and Chernov, N. I., Markov Partitions for Two-Dimensional Hyperbolic Billiards, *Russian Math. Surveys,* 1990, vol. 45, no. 3, pp. 105–152; see also: *Uspekhi Mat. Nauk,* 1990, vol. 45, no. 3, pp. 97–134.
17. Katok, A., Strelcyn, J.-M., Ledrappier, F., and Przytycki, F., *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities,* Lecture Notes in Math., vol. 1222, Berlin: Springer, 1986.
18. Chernov, N. and Markarian, R., *Chaotic Billiards,* Math. Surveys Monogr., vol. 127, Providence, R.I.: AMS, 2006.
19. Hirsch, M. W., Smale, S., and Devaney, R. L., *Differential Equations, Dynamical Systems, and an Introduction to Chaos,* 2nd ed., Pure Appl. Math. (Amst.), vol. 60, Amsterdam: Elsevier/Academic Press, 2004.
20. Fenichel, N., Persistence and Smoothness of Invariant Manifolds for Flows, *Indiana Univ. Math. J.,* 1971/1972, vol. 21, pp. 193–226.