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Scientific Heritage of L. P. Shilnikov

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Abstract—This is the first part of a review of the scientific works of L.P. Shilnikov. We group his papers according to 7 major research topics: bifurcations of homoclinic loops; the loop of a saddle-focus and spiral chaos; Poincare homoclinics to periodic orbits and invariant tori, homoclinic in noautonous and infinite-dimensional systems; Homoclinic tangency; Saddle-node bifurcation — quasiperiodicity-to-chaos transition, blue-sky catastrophe; Lorenz attractor; Hamiltonian dynamics. The first two topics are covered in this part. The review will be continued in the further issues of the journal.

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Our dear friend, mentor and fellow researcher, Leonid Pavlovich Shilnikov was a creator of the theory of global bifurcations of high-dimensional systems and one of the founders of the mathematical theory of dynamical chaos. He built a profound research school in the city of Nizhny Novgorod (Gorky formerly) – the Shilnikov School that continues to this day. His works greatly influenced the overall development of the mathematical theory of dynamical systems as well as nonlinear dynamics in general. Shilnikov's findings have been included in most text- and reference books, and are used worldwide by mathematics students and nonlinear dynamists to study the qualitative theory of dynamical systems and chaos. The elegance and completeness of his results let them reach "the heart of the matter", and provide applied researchers with an in-depth mathematical understanding of the outcomes of natural experiments. The popularity and appreciation were reflected by the "living classic" status attained by Professor Shilnikov over several decades of his life through continuous hard work on bifurcation theory of multidimensional dynamical systems, mathematical chaos theory, and theory of strange attractors.

In this article we would like to overview the scientific works of Leonid Pavlovich. As the material is very large to fit in one journal publication, we will proceed its publication in further issues of this journal. We group LP's works into 7 major topics: Bifurcations of homoclinic loops, The loop of a saddle-focus, Poincare's homoclinic, Homoclinic tangency, Destruction of a torus, Lorenz attractor, Hamiltonian dynamics. We start with the first two topics.

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1. BIFURCATIONS OF HOMOCLINIC LOOPS

L. P. Shilnikov started his research with studying (as a PhD student under the guidance of Yu. I. Neimark) dynamics of systems of automatic control. In particular, he proposed a method of developing asymptotic expansions for piece-wise smooth systems [1]. He, however, quickly became disillusioned with the whole field, which was too crowded and oriented to narrow engineering applications. He chose a different problem for his PhD thesis — bifurcations of separatrix loops in multidimensional systems. The Poincaré map near such loop can become singular, so the analysis would essentially reduce to a study of a non-smooth map, which bears a similarity to systems of automatic control. However, the homoclinic bifurcation problem is of a theoretical and fundamental nature, i.e. more about ideas than computations or specific applications. Besides, nobody else in the world seemed to be looking at it at that time. "I immediately knew that this problem is for me", as L. P. Shilnikov later recalled.

Bifurcation theory was founded in the end of 30s by A. A. Andronov and E. A. Leontovich who, with their co-authors, described and analyzed main bifurcations of systems of differential equations on a plane [40–43]. They studied local bifurcations (those in a small neighborhood of an equilibrium state or a periodic orbit), as well as global bifurcations, i.e. those near homoclinic loops. In the end of 50s, the question whether one can generalize the Andronov – Leontovich theory to higher dimensions became actual, and several local bifurcations of multidimensional systems were studied [46, 47]. Global bifurcations became Shilnikov's choice of his research theme. His first results [2] on the subject were quite similar to the original results by Andronov and Leontovich, however his approach was different. Andronov and Leontovich worked within the framework of the Poincaré – Bendixson theory of systems on a plane. As this theory does not hold for multidimensional systems, Shilnikov based his considerations in [2] entirely on the Banach contraction mapping principle. Below we review the two cases of homoclinic bifurcations considered in [2].

1.1. Birth of a Stable Periodic Orbit at the Disappearance of a Saddle-node Equilibrium

For a one-parameter family X_μ of sufficiently smooth (at least C^{2k}, where k is determined by the order of degeneracy of the saddle-node, as explained below) systems

$$\dot{z} = Q(z, \mu), \quad z \in \mathbb{R}^{n+1}, \tag{1.1}$$

which depends continuously on the parameter $\mu \in [-\mu_0, \mu_0]$, assume that

- at $\mu = 0$ the system has an equilibrium $O : \{z = 0\}$ and an orbit Γ_0 homoclinic to it (Fig. 1);
- the equilibrium O is a saddle-node of a finite order of degeneracy.

The latter condition means, in particular, that the linearization matrix $\dfrac{\partial Q}{\partial z}(0,0)$ has one zero eigenvalue, while all other n eigenvalues have negative real parts. It is well-known now that under this condition, in some local coordinates $(x_1, \ldots, x_n, y)$ in a small neighborhood U of the equilibrium state O the system (1.1) can be written in the following form[1)]

$$\dot{x} = Ax + f(x, y, \mu)x, \quad \dot{y} = R(y, \mu), \tag{1.2}$$

where the matrix A has eigenvalues $\lambda_i(\mu)$ with $\mathrm{Re}\,\lambda_i < 0$, $i = 1, \ldots, n$, $f(0, 0, \mu) \equiv 0$ and $R(y, 0) = l_p y^p + O(y^{p+1})$ for some $p \geqslant 2$. The point O is said to have a finite order of degeneracy if $l_p \neq 0$, and O is called a saddle-node if p is even, $p = 2k$ (the case $k = 1$ corresponds to a non-degenerate saddle-node). It is known that the change of the parameter μ can lead to the disappearance of the equilibrium in the case of even p. Assume that

- the parameter dependence is such that no equilibria exist in U at $\mu > 0$.

1) See e.g. [38, 49]; this convenient form of equations uses the center manifold theory which was not known at the time, so Shilnikov based his formulation and derivations on the theory from [47].

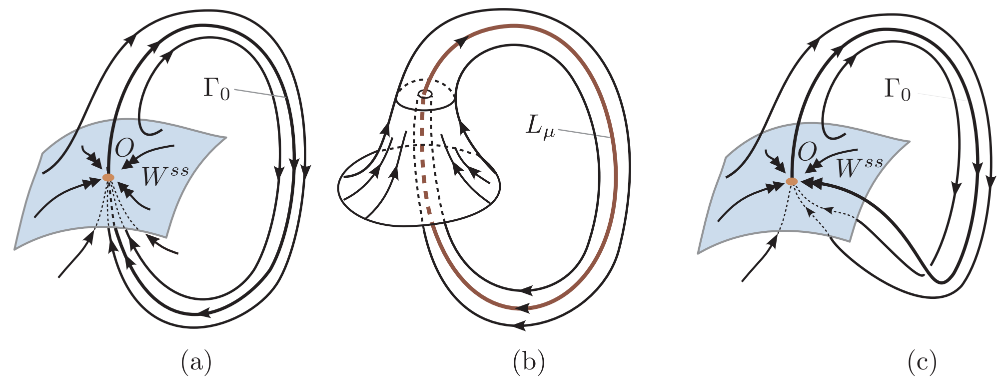

Fig. 1. a) Nongenerate homoclinic loop Γ_0 to a saddle-node O at $\mu = 0$ – the orbit Γ_0 lies in $W^s \backslash W^{ss}$ while entering O; b) Birth of a stable limit cycle in a neighborhood of Γ_0 at $\mu > 0$ after the saddle-node has disappeared; c) Degenerate homoclinic loop Γ_0 belongsto W^{ss}.

It is seen from (1.2) that the manifold $y = 0$ is, at $\mu = 0$, locally invariant and every orbit in it tends to O exponentially. This is the so-called strongly stable manifold $W^{ss}(O)$. It divides the neighborhood U into two parts; the orbits that start at $l_{2k}y < 0$ tend (non-exponentially) to O as $t \to +\infty$, while the orbits that start at $l_{2k}y > 0$ leave U. Thus the region $l_{2k}y \leqslant 0$ in U is the local stable manifold W^s of the saddle-node O. Assume that

- the homoclinic orbit Γ_0 does not belong to W^{ss} (Fig. 1a)[2)].

Theorem 1 ([2]). *There is a small neighborhood $V = V(O \cup \Gamma_0)$, the same for all small μ, such that for all small $\mu > 0$ the system* (1.1) *has a unique limit cycle L_μ in V. The limit cycle L_μ is asymptotically stable; its topological limit as $\mu \to +0$ coincides with $O \cup \Gamma_0$.*

Note that Shilnikov proved this result for any finite k, i.e. the result was new even in the two-dimensional case, cf. [40, 42]. In the further works (in collaboration with V. Afraimovich, V. Lukyanov, D. Turaev, A. Shilnikov [15, 16, 18, 20, 22, 30, 33, 34]) he considered bifurcations of homoclinics to saddle-node periodic orbits which led to several ground-breaking discoveries, including a sudden transition from quasi-periodicity to chaos, and the blue-sky catastrophe, see Section “Saddle-Node Bifurcation” in [85]..

1.2. Birth of a Stable Periodic Orbit from a Homoclinic Loop to a Saddle Equilibrium with a Negative Saddle Value

The next theorem of [2] deals with the case of a hyperbolic (saddle) equilibrium. Such equilibria persist as the bifurcation parameter μ varies. Without loss of generality we may assume that O_μ stays at the origin for all small μ. We make the following assumption about the characteristic exponents (the eigenvalues of the linearization matrix) of O.

- The saddle O has only one positive characteristic exponent γ, while all the other, $\lambda_1, \ldots, \lambda_n$, have negative real parts:

$$\gamma > 0 > \operatorname{Re} \lambda_i, \quad i = 1, \ldots, n.$$

By this assumption, the unstable manifold W^u of the saddle O is one-dimensional, while the stable manifold W^s of O is n-dimensional. The unstable manifold consists of three orbits: the saddle O itself, and two separatrices, Γ_1 and Γ_2, which tend to O as $t \to -\infty$. We suppose that

[2)] Bifurcations in the case where this genericity condition is violated, i.e. $\Gamma_0 \in W^{ss}$, as in Fig. 1c, were later considered by Shilnikov's student V. Lukyanov [38, 50]

- at $\mu = 0$ the system has a separatrix loop to the saddle O; i.e. Γ_1 tends to O as $t \to +\infty$. Thus,

$$\Gamma_1 \subset W^s(O) \text{ at } \mu = 0;$$

- at $\mu \neq 0$ the loop splits: inwards (above W^s) at $\mu > 0$ and outwards (below W^s) at $\mu < 0$ (see Fig. 2).

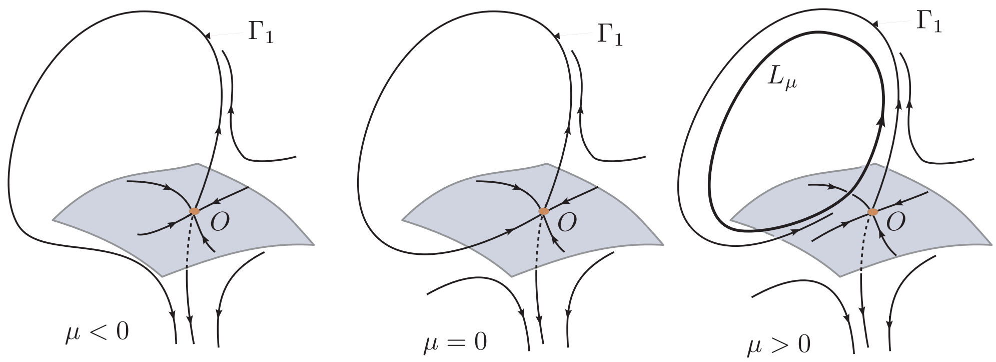

Fig. 2. Birth of a stable periodic orbit from a separatrix loop of a saddle with $\sigma < 0$.

Introduce an important quantity, the so-called *saddle value* σ, which is defined as follows

$$\sigma = \gamma + \max \operatorname{Re} \lambda_i.$$

Let us assume that

- the saddle value is negative: $\sigma < 0$.

Theorem 2 ([2][3]). *A single stable limit cycle L_μ is born from the homoclinic loop at $\mu > 0$ (see Fig. 2). The separatrix Γ_1 tends to L_μ as $t \to +\infty$ (the topological limit of L_μ as $\mu \to +0$ coincides with $O \cup \Gamma_1$). At $\mu \leqslant 0$ there are no limit cycles in a small neighborhood V of the homoclinic loop. Moreover, at $\mu < 0$ all orbits that do not lie in W^s leave V as t grows.*

The method of the proof of both Theorems 1 and 2 was based on showing that the Poincaré map of a small cross-section to the homoclinic loop is a contraction (the contracting Poincaré map has a unique stable fixed point, which corresponds to the stable limit cycles of the system). Shilnikov takes two cross-sections to the loop, Π_0 and Π_1, such that the orbits on the way from Π_0 to Π_1 stay in the small neighborhood U of the equilibrium O, and the orbits that start on Π_1 follow $\Gamma_1 \backslash U$ until they hit Π_0. The dwelling time from Π_1 to Π_0 is bounded, so possible expansion or contraction here is bounded as well. The time a trajectory needs to travel from Π_0 to Π_1 can be as large as we want (as it passes close to the saddle), which implies an arbitrarily strong contraction. Shilnikov indeed proves that under the conditions of Theorems 1 and 2 the map from Π_0 and Π_1 is a contraction, which can be made arbitrarily strong as $\mu \to 0$. Thus, the composite map from Π_0 to Π_1 and then back to Π_0 by the orbits of the system is a strong contraction as well. The proof of contraction in [2] is based on the reduction of the system in U to a system of integral equations and a subsequent analysis of these equations. Without computations, one can establish the contraction by noticing that the strong contraction of distances by the Poincaré map from Π_0 to Π_1 is equivalent to the contraction of two-dimensional areas by the linearized flow in U. This area-contraction property is ensured whenever the sum of the real parts of each pair of different eigenvalues of the linearization matrix at O is strictly negative (see more details in [38, 55]). This condition is automatically fulfilled when O is a saddle-node. When O is a saddle, the area-contraction at O is equivalent to the negativity of the saddle value, i.e. the $\sigma < 0$ condition is crucial for Theorem 2 to hold true.

[3] Originally, this theorem was proved for C^2-smooth systems; the proof for the C^1-case was given in [38, 57].

The natural wonder about the case $\sigma > 0$ led Shilnikov to his first great discovery that determined his further life, and made him one of the founding fathers of "Chaos Theory". In [3, 6, 11], just a few years after S. Smale devised his horseshoe example [48], Shilnikov found that if O is a saddle-focus, i.e. the leading eigenvalue of the linearization matrix is complex, then the Poincaré map near the homoclinic loop has infinitely many Smale horseshoes; i.e. dynamics is chaotic and has nothing in common with dynamics of plane systems whatsoever. We discuss the theory of the saddle-focus loop in Section 2. Before that, we review works of 1967-69 where the case of a real leading eigenvalue was considered and the birth of saddle limit cycles from homoclinic loops was studied. Later, these results laid the foundation for the theory of the Lorenz attractor, see Section "Lorenz Attractor" in [85]. .

1.3. Birth of a Saddle Periodic Orbit from a Homoclinic Loop to a Saddle Equilibrium

We start with recapitulating the results of [9]. Consider a continuous one-parameter family X_μ of smooth systems of differential equations in $\mathbb{R}^{m+n}$ which have a saddle equilibrium state O_μ with n-dimensional unstable manifold W^u and m-dimensional stable manifold W^s. Such equilibrium has n characteristic exponents $\gamma_1, \ldots, \gamma_n$ with positive real parts and m characteristic exponents $\lambda_1, \ldots, \lambda_m$ with negative real parts. We assume that

- the nearest to the imaginary axis characteristic exponent is simple and *real*, and all the other characteristic exponents lie further away from the imaginary axis.

This means that O_μ is not a saddle-focus in the sense of [3]. It is enough to consider the case where this real characteristic exponent is positive (if not, we reverse the time, thus changing the sign of all characteristic exponents). Therefore, we assume that

- the characteristic exponents satisfy
$$\min_{j>1} \mathrm{Re}\, \gamma_j \; > \; \gamma_1 \; > \; 0 \; > \max_{i=1,\ldots,m} \mathrm{Re}\, \lambda_i, \quad \text{and} \quad \sigma = \gamma_1 + \max \mathrm{Re}\, \lambda_i \; < \; 0.$$

Because of the gap between the exponent γ_1 and the rest of γ's, a smooth $(n-1)$-dimensional invariant strong-unstable manifold W^{uu} exists in W^u. It divides W^u into two parts, which we denote as W^u_+ and W^u_- (see e.g. [38]). Every orbit from W^u tends to O as $t \to -\infty$. The orbits belonging to $W^u_+ \cup W^u_-$ are all tangent, at $t = -\infty$, to the same one-dimensional eigenspace of the linearization matrix which corresponds to the eigenvalue γ_1 (the orbits from W^u_+ and W^u_- leave O in the opposite directions).

Assume that at $\mu = 0$

i) the stable and unstable manifolds intersect along a homoclinic orbit Γ;

ii) the intersection of W^s and W^u along Γ has the least possible degeneracy, i.e.
$$\dim(T_M W^u \cap T_M W^s) = 1$$
for any point $M \in \Gamma$ (we denote as $T_M W$ the tangent space to a manifold W at a point M);

iii) $\Gamma \notin W^{uu}$, so we may assume $\Gamma \in W^u_+ \cap W^s$;

iv) the limit, as $t \to +\infty$, of the tangent space to the unstable manifold at the points of Γ contains the tangent space to W^{uu} at O

(the last three assumptions are conditions of a general position for systems with homoclinic loops, i.e. they hold true for an open and dense subset of the set of such systems).

We will consider bifurcations in a small neighborhood V of the homoclinic loop $\Gamma \cup O$ as the parameter μ varies. Assume that

- the homoclinic loop splits at $\mu \neq 0$, inwards if $\mu > 0$ and outwards if $\mu < 0$, i.e. the limit of the piece of W^u_+ that passes close to Γ is contained in W^u_+ if $\mu \to +0$, and in W^u_- if $\mu \to -0$.

Theorem 3 ([9]). *A single periodic orbit L_μ is born in V at $\mu > 0$, and no periodic orbits exist in V at $\mu \leqslant 0$. The orbit L_μ has an n-dimensional unstable manifold and $(m+1)$-dimensional stable manifold.*

The proof was based on the derivation and examination of the Poincaré map of a certain cross-section to Γ chosen near O, which was performed via the analysis of a system of integral equations to which the original system was reduced near O. Next, bifurcations of periodic orbits in this map were studied. This map is no longer a contraction, because it expands in the directions that correspond to the positive characteristic exponents $\gamma_2, \dots, \gamma_n$. To deal effectively with the saddle character of the Poincaré map, it was represented in the so-called cross-form, following the method developed in [3] (see Section "Homoclinic Chaos" in [85]). A geometric method of the proof was described e.g. in [38]. It follows from [56, 62] that the non-degeneracy conditions iii and iv imply the existence, for all small μ, of an $(m+1)$-dimensional C^1-smooth invariant manifold which is tangent at O to the eigenspace that corresponds to the characteristic exponents $\gamma_1, \lambda_1, \dots, \lambda_m$ and, most importantly, contains all the orbits that do not leave V as $t \to +\infty$ (the rest of the orbits is repelled from this manifold due to the expansion in the directions corresponding to $\gamma_2, \dots, \gamma_n$). The system restricted to this manifold satisfies the assumptions of Theorem 2; in this way Theorem 3 is inferred from Theorem 2. It also follows that except for the limit cycle L_μ at $\mu > 0$, the homoclinic orbit Γ at $\mu = 0$, and the equilibrium state O_μ, there are no other orbits that stay in V for all times t.

As we mentioned, the case of positive saddle value σ is reduced to the previous one by the time reversal. To elucidate the results, let us consider only the case of one-dimensional unstable manifold. Assume that

- the eigenvalues $\gamma, \lambda_1, \dots, \lambda_m$ of the linearization matrix at O_μ satisfy
$$\gamma \;>\; 0 \;>\; \lambda_1 > \max_{j>1} \mathrm{Re}\lambda_j, \qquad \sigma = \gamma + \lambda_1 > 0.$$

The non-degeneracy assumptions of Theorem 3 transform to

- $\Gamma \not\subset W^{ss}$;
- the extended unstable manifold W^{ue} is transverse to the stable manifold W^s at the points of Γ (see Fig. 3).

The strong stable manifold W^{ss} is a smooth invariant $(m-1)$-dimensional submanifold of W^s tangent at O to the eigenspace that corresponds to the characteristic exponents $\lambda_2, \dots, \lambda_m$; the extended unstable manifold W^{ue} is a C^1-smooth two-dimensional invariant manifold that contains W^u and is tangent at O to the eigenspace that corresponds to the characteristic exponents λ_1 and γ. It is easy to see that the transversality of W^{ue} to W^s is equivalent to the condition that the limit, as $t \to -\infty$, of the tangent space to W^s at the points of Γ contains the tangent space to W^{ss} at O (the time-reversed version of condition iv).

Let us introduce coordinates (x, u, y) near the origin at O such that the unstable manifold locally (i.e. at small x, u, y) coincides with the y-axis, the stable manifold is locally given by $y = 0$, the strong stable manifold is locally given by $(x, y) = 0$, and the extended unstable manifold is tangent to the plane $u = 0$ at the points of W^u_{loc}. The system near O takes the form
$$\dot{y} = \gamma y + \dots, \qquad \dot{x} = \lambda_1 x + \dots, \qquad \dot{u} = Bu + \dots,$$
where the dots stand for nonlinear terms, and B is a matrix with the eigenvalues $\lambda_2, \dots, \lambda_m$. Thus, the system near O expands in the y-direction, contracts in the x- and u- directions, and the contraction in u is stronger than the contraction in x.

Let the homoclinic orbit Γ coincide, when it leaves O at $t = -\infty$, with the positive part of the y-axis, and assume it is tangent to the positive part of the x-axis when it enters O as $t \to +\infty$ (as $\Gamma \notin W^{ss}$, it must be tangent to the x-axis when entering O). Take a small $d > 0$ and consider two small cross-sections, $\Pi_0 : \{x = d\}$ and $\Pi_1 : \{y = d\}$, to the homoclinic loop Γ. The orbits that lie near Γ define two maps, $T_0 : \Pi_0 \cap \{y > 0\} \to \Pi_1$ and $T_1 : \Pi_1 \to \Pi_0$; the composition of these

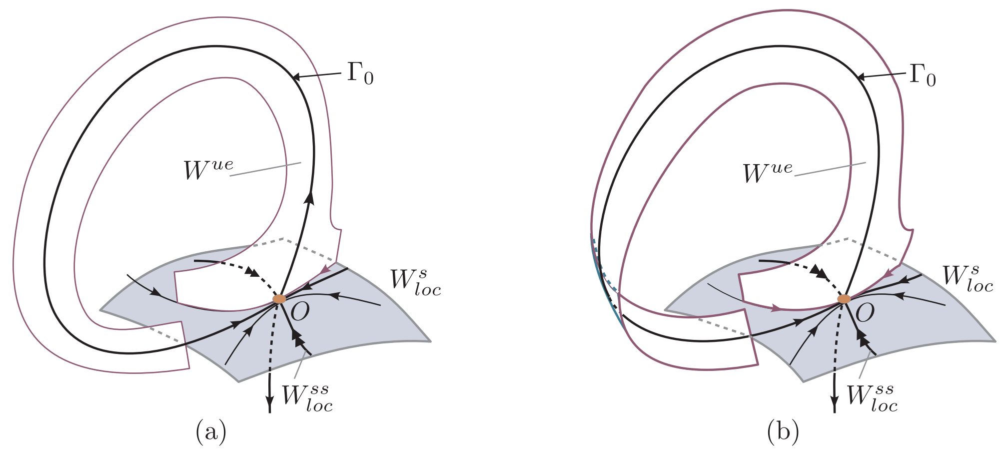

Fig. 3. Non-degeneracy conditions for the three-dimensional case: the homoclinic loop enters the saddle along the leading direction (corresponding to λ_1); the manifolds W^{ue} and W^s intersect transversely — (a) the case $A > 0$, and (b) the case $A < 0$.

maps is the Poincaré map $T = T_1T_0$. The orbits between Π_0 and Π_1 stay in a small neighborhood of O where the contraction in the u-directions is stronger than in the x-direction. The flight time from Π_0 to Π_1 tends to infinity as $y \to +0$, so the overall contraction in u is getting infinitely stronger than in x. This means that the image of $\Pi_0 \cap \{y > 0\}$ by T_0 is a thin wedge tangent to the line $W^{ue} \cap \Pi_1 : \{u = 0\}$ at the apex at the point $(x = 0, u = 0)$ in Π_1. The map T_1 corresponds to a finite-time travel between Π_1 and Π_0, so it is a regular diffeomorphism. Thus, the image of $\Pi_0 \cap \{y > 0\}$ by the Poincaré map $T = T_1T_0$ is also a wedge which is tangent to $W^{ue} \cap \Pi_0$ at the apex $W^u \cap \Pi_0$, see Fig. 4.

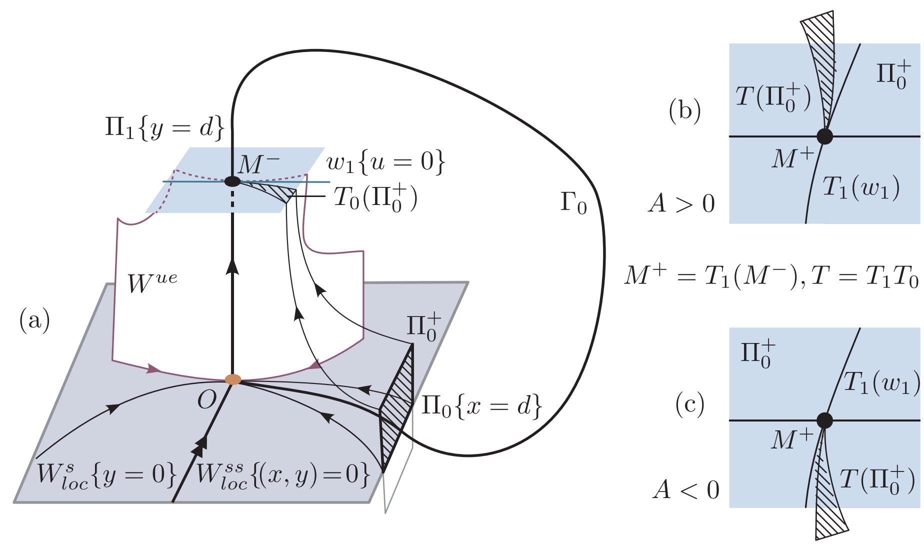

Fig. 4. (a) The local map T_0 takes the upper part Π_0^+ of the cross-section Π_0 into a curvilinear triangle on the cross-section Π_1. The points in $\Pi_0 \cap W^s_{loc}$ are mapped to a single point M^-. (b) The image of Π_0^+ by the Poincaré map T in the case $A > 0$. (c) The case $A < 0$.

The assumed transversality of W^{ue} and W^s at the points of Γ is equivalent to the transversality of the line $T_1(W^{ue} \cap \Pi_1)$ and $W^s_{loc} : \{y = 0\}$, so if we denote $T_1 : (x, u) \mapsto (\bar{y}, \bar{v})$, then this transversality condition reads as

$$A = \left.\frac{\partial \bar{y}}{\partial x}\right|_{(x,u)=0} \neq 0.$$

The number A is called the *separatrix value*. It plays an important role in the further study of homoclinic bifurcations and, particularly, the theory of the Lorenz attractor (see Section "Lorenz Attractor" in [85]).

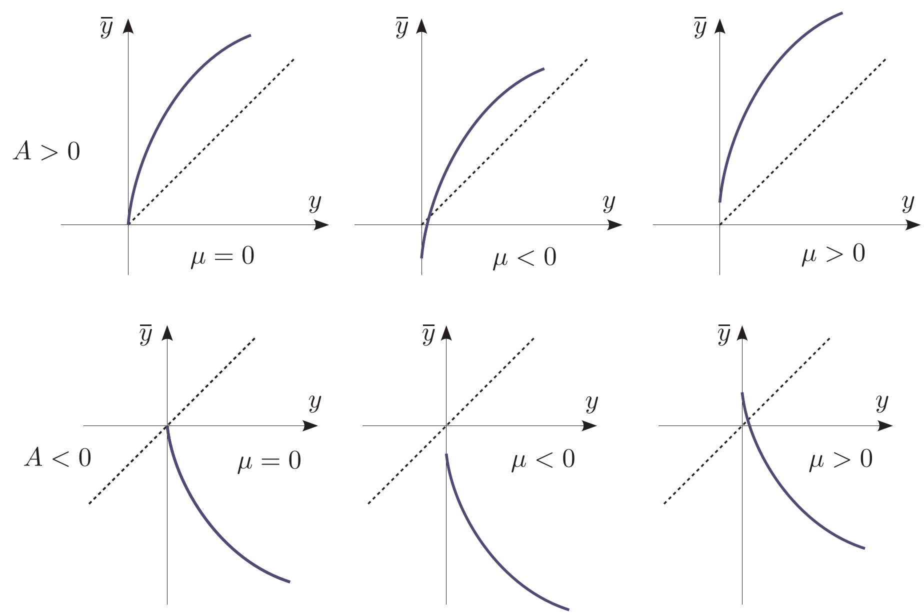

Fig. 5. The birth of an unstable fixed point of map (1.3) with $0 < \nu < 1$ at $A > 0$ and $A < 0$.

If $A > 0$, the wedge $T_1(\Pi_0 \cap \{y > 0\})$ is oriented towards positive values of y, and if $A < 0$, the orientation is reversed, see Fig. 4c. If we approximate the motion near the saddle O by $y(t) \sim y(0)e^{\gamma t}$, $x(t) \sim x(0)e^{\lambda_1 t}$, then the orbit starting at a point on $\Pi_0 : \{x = d\}$ with the coordinate $y > 0$ will hit $\Pi_1 : \{y = d\}$ at a point with the coordinate $x \sim d(y/d)^\nu$ where

$$\nu = |\lambda_1/\gamma|$$

is the so-called *saddle index* (the exact derivation can be found e.g. in [38]). By scaling the coordinates on Π_0 and Π_1 one can make $d = 1$. As the action of the map T_1 on the coordinate y is, essentially, a multiplication by A, it follows that the Poincaré map $T = T_1 T_0$ transforms the y-coordinate as follows:

$$\bar{y} = \mu + Ay^\nu + o(y^\nu), \qquad y > 0, \tag{1.3}$$

where we assume that the parameter μ that splits the homoclinic loop Γ is chosen to be equal to the y-coordinate of the intersection point $W^u \cap \Pi_0$. The action in the u-direction is just a strong contraction, so one can, for a model, ignore the variables u and focus on the study of the one-dimensional map (1.3). The dynamics and bifurcations of this map can be investigated easily (see Fig. 5); the crucial point is that $0 < \nu < 1$, as it follows from the saddle value condition $\gamma + \lambda_1 > 0$. Thus, one finds that this map has a single unstable fixed point at $A\mu < 0$, in agreement with the result of Theorem 3 (after the time reversal) which, in the case under consideration, gives that

a single saddle periodic orbit L_μ is born when the loop Γ is split inwards if $A < 0$ or outwards if $A > 0$.

Shilnikov remarked in [9] that Theorem 3 (along with the results on the homoclinic loop to a saddle-node and a saddle-saddle, Theorems 1 and 4) completes the study of main cases of bifurcations of periodic orbits from homoclinic loops. Indeed, homoclinic loops to a saddle-focus correspond to chaotic dynamics and, therefore, have to be considered separately. Other cases of bifurcations of homoclinic loops to a saddle correspond to the violation of one of the non-degeneracy assumptions of Theorem 3, i.e. to bifurcations of a higher codimension in modern terminology.

There are three main possibilities for the violation of these non-degeneracy assumptions. The first corresponds to zero saddle value; this case was known for systems on a plane and was exhaustively studied in the planar case by E. A. Leontovich-Andronova [44, 45]. The other two codimension-2 cases are not planar and stem from [9]. These are the bifurcation of a homoclinic loop that belongs to W^{uu} (in the case $\sigma < 0$) or W^{ss} (in the case $\sigma > 0$), which is nowadays called the *orbit-flip* bifurcation (see [38, 61, 63, 65]), and the bifurcation of a homoclinic loop with zero separatrix value A, the so-called *inclination-flip* bifurcation (see [38, 51, 52, 63–65]). The interest to the codimension-2 bifurcations of a homoclinic loop to a saddle, as well as to the bifurcations of pairs of homoclinic loops [25, 54, 71], emerged in the early 80s, especially in connection with the study of scenarios of the birth and destruction of the Lorenz attractor (see Section "Lorenz Attractor" in [85]).

1.4. The Birth of Saddle Periodic Orbits at the Disappearance of a Saddle-Saddle Equilibrium

The last codimension-1 case of the birth of periodic orbits from homoclinic loops is related to the disappearance of a non-hyperbolic saddle-saddle equilibrium (or a Shilnikov saddle-node) [4, 10]. Such an equilibrium state in a dynamical system (flow) of dimension $n+m+1 \geqslant 3$ is the result of the merger of two saddles O_1 and O_2 of different topological types, i.e., $\dim W^s(O_1) = m$, $\dim W^u(O_1) = n+1$ and $\dim W^s(O_2) = m+1$, $\dim W^u(O_1) = n$ for some m and n, see Fig. 6a. Before the merger, $W^u(O_1)$ intersects transversely with $W^s(O_2)$ along a heteroclinic orbit that connects O_1 and O_2. The manifold $W^u(O_1)$ has $W^u(O_2)$ as its boundary; and $W^s(O_2)$ has $W^s(O_1)$ as its boundary, see Fig. 6a. At the moment of merge, the saddle-saddle O becomes a non-hyperbolic equilibrium whose stable set W^s (the set of all points whose orbits tend to O as $t \to +\infty$) is an $(m+1)$-dimensional manifold with the boundary W^{ss}, which consists of the orbits converging to O exponentially fast, and the unstable set W^u (the union of the orbits that tend to O as $t \to -\infty$) is an $(n+1)$-dimensional manifold with the boundary W^{uu}, which consists of the orbits converging to O exponentially fast as $t \to -\infty$. Both W^s and W^u are diffeomorphic to an $(m+1)$-dimensional and, respectively, $(n+1)$-dimensional closed half-spaces, see Fig. 6b.

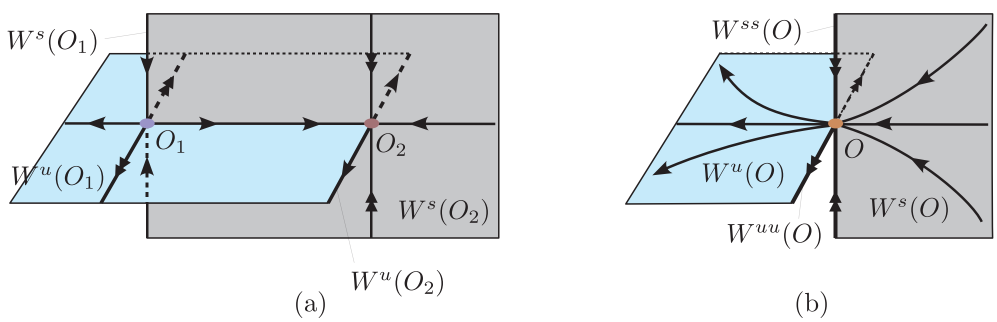

Fig. 6. The three-dimensional case: a) two saddles with $\dim W^s(O_1) = 1$, $\dim W^u(O_1) = 2$ and $\dim W^s(O_2) = 2$, $\dim W^u(O_2) = 1$; b) a saddle-saddle equilibrium state and its invariant manifolds.

Since the dimension of the phase space is one less than the sum of dimensions of $W^u(O)$ and $W^s(O)$, they can intersect *transversely* along a homoclinic orbit Γ_0, see Fig. 7a. The transversality of the intersection implies that it cannot be removed by a small perturbation unless the equilibrium O disappears. The bifurcations in a small neighborhood V of $\Gamma_0 \cup O$ at the disappearance of the saddle-saddle were studied in [4]. The result is as follows.

Let X_μ be a one parameter family of C^r-smooth ($r \geqslant 2$) flows in $\mathbb{R}^{n+m+1}$. Assume that the following conditions hold:

- the system X_0 has an equilibrium O of the saddle-saddle type;
- the manifolds $W^u(O)$ and $W^s(O)$ intersect transversely along a homoclinic orbit Γ_0;

- Γ_0 does not belong to $W^{uu}(O) \cup W^{ss}(O)$;

- the equilibrium disappears at $\mu > 0$.

Theorem 4 ([4]). *Under the assumptions above, there exists a small fixed neighborhood $V = V(O \cup \Gamma_0)$ and a number $\mu^* > 0$ such that for all $0 < \mu < \mu^*$ the system X_μ has in V a unique saddle limit cycle L_μ whose the stable and unstable invariant manifolds have dimensions $(m+1)$ and $(n+1)$, respectively. The topological limit of L_μ as $\mu \to +0$ coincides with $O \cup \Gamma_0$.*

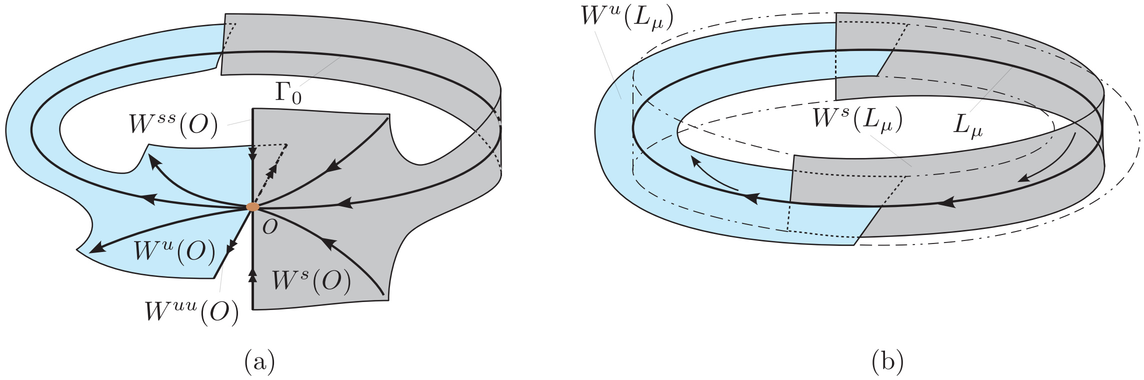

Fig. 7. Illustration of Theorem 4 in the three-dimensional case: (a) $\mu = 0$, the two-dimensional manifolds $W^u(O)$ and $W^s(O)$ have a transverse intersection along the homoclinic loop Γ_0; (b) $\mu > 0$, the saddle limit cycle L_μ with $\dim W^s(L_\mu) = 2$, $\dim W^u(L_\mu) = 2$ replaces Γ_0.

This result is a direct generalization of Theorem 1. However, Shilnikov realized that the saddle-saddle case could offer a new possibility, unavailable in the saddle-node case. Namely, since the stable and unstable manifolds of the saddle-saddle have dimensions higher than 1, their intersection can contain more than a single orbit. As so, let us assume that

- the manifolds $W^u(O)$ and $W^s(O)$ intersect transversely along a number of isolated homoclinic loops $\Gamma_1, \dots, \Gamma_k$, $k \geqslant 2$, none of which lies in $W^{uu}(O) \cup W^{ss}(O)$ (so they are all tangent to each other as they leave and enter the saddle-saddle).

Then, as the saddle-saddle disappears at $\mu > 0$, Theorem 4 implies that k saddle periodic orbits are born, one from each of the homoclinic loops. These saddle periodic orbits all pass close by the phantom of the disappeared saddle-saddle, i.e. quite close to each other (see Fig. 8b). Therefore, their stable and unstable manifolds may intersect. Such intersections correspond to Poincaré homoclinic orbits, hence — to shift dynamics, and chaos (see Section "Homoclinic Chaos" in [85]). More precisely, since all homoclinic loops Γ_j are tangent to each other as they tend to O as $t \to +\infty$, we may take a small cross-section Π_0 to $W^s(O)$ such that all the homoclinic loops $\Gamma_1, \dots, \Gamma_k$ intersect it. At small $\mu > 0$ the orbits that start near $\Gamma_j \cap \Pi_0$ define the Poincaré map $T_j : \Pi_0 \to \Pi_0$. This is a saddle map: since the orbits spend a long time in a small neighborhood of O, we have a very strong contraction in directions parallel to $W^s(O)$ and a very strong expansion in directions parallel to $W^u(O)$. The set $\sigma_j = T_j^{-1}\Pi_0$ is a thin strip parallel to W^s, and its image $T\sigma_j$ is a thin strip parallel to W^u. The size of the strips σ_j and $T\sigma_j$ is the same for all $j = 1, \dots, k$, so $T\sigma_i \cap \sigma_j \neq \emptyset$ for each pair i, j. Thus, at all small $\mu > 0$ we have, in the cross-section Π_0, a Smale horseshoe with k branches (see Fig. 8c).

In this way, Shilnikov obtained the following

Theorem 5 ([10]). *Let V be a small neighborhood of $O \cup \Gamma_1 \cup \dots \cup \Gamma_k$. For all sufficiently small $\mu > 0$ the set of all orbits that lie entirely in V is in a one-to-one correspondence with the set of all bi-infinite sequences of symbols $1, \dots, k$.*

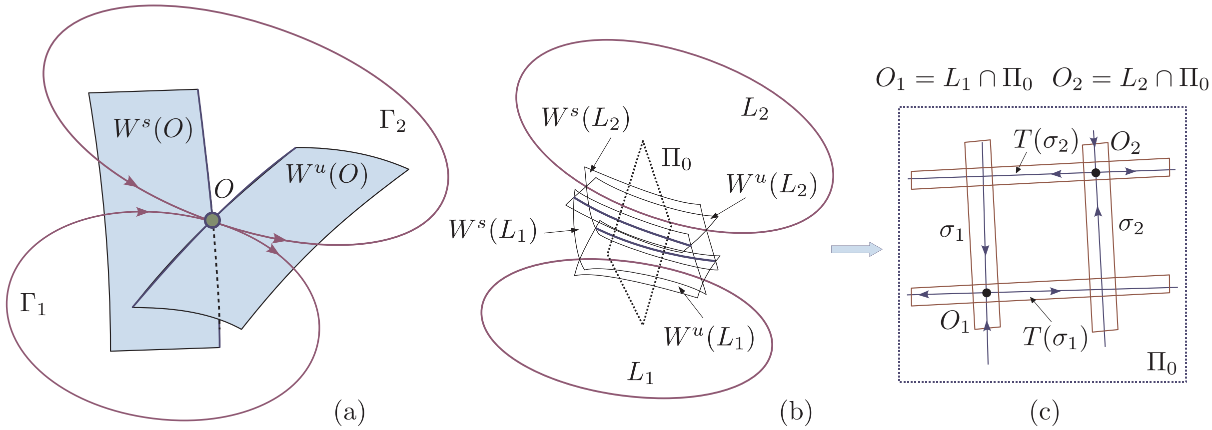

Fig. 8. An illustration to Theorem 5 for $k = 2$: (a) $\mu = 0$, the homoclinic loops Γ_1 and Γ_2 are tangent to each other at O; (b) $\mu > 0$, saddle limit cycles L_1 and L_2 are born such that $W^u(L_1)$ intersects with $W^s(L_2)$ and $W^u(L_2)$ intersects with $W^s(L_1)$; (c) a Smale horseshoe for the Poincaré map on a cross-section Π_0.

Further generalizations of this result were obtained in a series of papers with V. Afraimovich [13, 14]. Along with the existence of a saddle-saddle equilibrium O and homoclinic loops to it, one also assumes the existence of a number of saddle periodic orbits $L_1, \ldots, L_p$ such that $\dim W^s(L_i) = m+1$, $\dim W^u(L_i) = n+1$ for all $i = 1, \ldots, p$. Denote $L_0 = O$, and assume that

- for certain pairs $i, j = 0, \ldots, p$ there is a number $q_{ij} \geqslant 1$ of orbits Γ_{sij}, $s = 1, \ldots, q_{ij}$, of transverse intersection of the manifolds $W^u(L_i)$ and $W^s(L_j)$ such that if $i = 0$ or $j = 0$, then none of these orbits lies in $W^{uu}(O) \cup W^{ss}(O)$.

Consider an oriented graph G whose vertices we identify with $L_0, L_1, \ldots, L_p$, and pairs of the vertices L_i, L_j are connected by q_{ij} edges, which are identified with Γ_{sij}.

Theorem 6 ([14][4]). *Let V be a small neighborhood of the union of all the orbits L_i and Γ_{sij} under consideration. Let the saddle-saddle equilibrium O disappear at $\mu > 0$. Then for all sufficiently small $\mu > 0$ the set N of all orbits that lie entirely in V is in a one-to-one correspondence with the set of all bi-infinite paths in the graph G. Moreover, the system restricted to the set N is topologically equivalent to a suspension flow over the topological Markov chain defined by the graph G.*

Returning to Theorem 5, we note that at $\mu \leqslant 0$, when the saddle-saddle de-couples into two saddles (or more, in the case of a degenerate saddle-saddle), there are only finitely many orbits, which entirely lie in V: the saddles, and heteroclinic orbits connecting them. Thus, the theorem describes the so-called Ω-explosion, a sudden transition from a simple behavior (at $\mu < 0$) to chaotic dynamics (at $\mu > 0$) in the of system. This was (and remains) an amazing result, first of its kind. While the principal fact of the existence of chaotic dynamics was firmly established in the end of 60s, the routes from simple dynamics to chaos were unknown at the time (the problem is not completely resolved even today and many outstanding issues remain). The possibility of a simple transition to chaos described by Theorem 5, along with a complete description of the dynamics before and after the transition, came in as an absolute surprise. When L. P. Shilnikov told about this result to E. A. Leontovich-Andronova, she responded by quoting Salieri from A. S. Pushkin's "Little Tragedies":

— You, Mozart, are a god, not knowing that yourself.

[4]The theory built in [14] includes also the analysis of various cases of the dynamics described by the topological Markov chains that emerge at this bifurcation, and the study of a simultaneous disappearance of several saddle-saddle equilibria.

2. HOMOCLINIC SADDLE-FOCUS AND SPIRAL CHAOS

After the study of a homoclinic loop to a saddle with a negative saddle value σ [2], the next logical step was to consider the case $\sigma > 0$. In this case, unlike Theorem 2, one cannot rely on the general contraction mapping principle, i.e. a more detailed analysis of the behavior of orbits near the saddle equilibrium was necessary. The simplest case corresponds to a three-dimensional system, which has a homoclinic loop to an equilibrium with one positive characteristic exponent and two characteristic exponents with negative real parts. The case where the characteristic exponents are real does not differ much from the two-dimensional one (see the end of Section 1.3). Thus, Shilnikov focused on the case of complex characteristic exponents [3].

By placing such equilibrium at the origin, the system is written in the form

$$\begin{aligned} \dot{x} &= -\lambda x - \omega y + F_1(x, y, z), \\ \dot{y} &= \omega x - \lambda y + F_2(x, y, z), \\ \dot{z} &= \gamma z + F_3(x, y, z), \end{aligned}$$

where smooth functions F_i, along with their first derivatives, vanish at the origin. The characteristic exponents here are $\gamma > 0$, and $\lambda_{1,2} = -\lambda \pm i\omega$ (where $\lambda > 0$, $\omega > 0$). The unstable manifold is a curve, tangent to the z-axis. The stable manifold W^s is a two-dimensional surface, tangent to the plane $z = 0$. If we restrict the system to the stable manifold only, the equilibrium O will be a stable focus, i.e. the orbits in W^s spiral onto O as $t \to +\infty$. Therefore, in the full system, O is called a *saddle-focus*. Assume that

- the system has a homoclinic orbit Γ to the saddle-focus O, see Fig. 9.

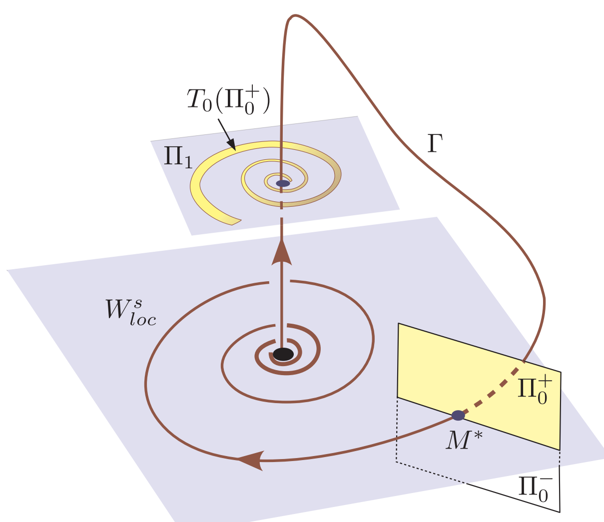

Fig. 9. Homoclinic saddle-focus in a three-dimensional phase space.

Theorem 7 ([3][5]). *Let the saddle value $\sigma = \gamma - \lambda$ be positive. Then, in any, arbitrarily small neighborhood of the homoclinic loop to the saddle-focus there exist infinitely many saddle periodic orbits.*

In [6] Shilnikov also proved that the same result holds true for the four-dimensional case (Fig. 10) where the equilibrium O has all characteristic exponents complex ($\lambda_{1,2} = -\lambda \pm i\omega$, $\gamma_{1,2} = \gamma \pm i\alpha$, $\lambda > 0$, $\gamma > 0$, $\gamma - \lambda \neq 0$).

[5] While the paper came out in the journal of short communications, it, however, contained a full proof of the result. In particular, the analysis of the behavior of the three-dimensional nonlinear system near the saddle-focus was done using asymptotic expansions due to Lyapunov [39].

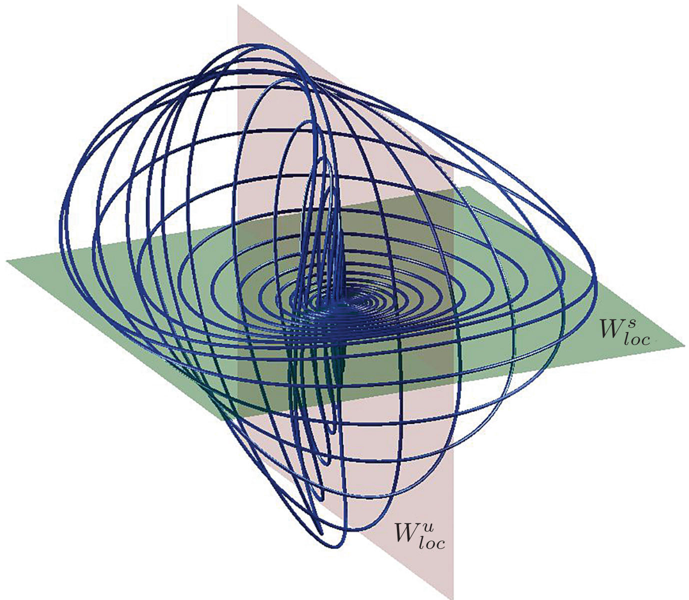

Fig. 10. A homoclinic loop to a saddle-focus (with all complex characteristic exponents) in a four-dimensional case.

Note a difference between Theorem 7 and the results of the previous Section (Theorems 2 and 3): in Theorem 7 the periodic orbits do not emerge from the separatrix loop, they exist simultaneously with it. Thus, Theorem 7 is an example of a "criterion of chaos", when having a small set of orbits (here — an equilibrium O and a homoclinic Γ) one can make a conclusion on the complex behavior in a neighborhood of this set[6]. This approach became a de-facto standard in the modern bifurcation theory of systems with chaotic dynamics.

Shilnikov understood very well the significance of his discovery. At that time the theory of dynamical systems lacked the language to properly describe such phenomena; necessary methods were not developed either. In general, dynamics of non-planar systems was somewhat of a mistery[7]. Shilnikov realized that the problem of a description of chaotic dynamics must be closely related to the problem of the structure of a neighborhood of the Poincaré's homoclinic curve (see Section "Homoclinic Chaos" in [85]). He solved this problem and its generalizations in [7, 8]. After that he returned to the study of the homoclinic loop of a saddle-focus and obtained [11] a quite peculiar description of chaotic dynamics near the loop in the general multi-dimensional case (Theorem 8).

In order to explain the results of [11], let us consider the three-dimensional case more closely. First, we reduce the analysis of the orbit behavior near the homoclinic loop Γ to that of a Poincaré map T on a small cross-section Π_0 transversal to Γ. This map is the composition of the local map T_0 and the global map T_1. The map T_0 takes points from Π_0 to the second cross-section Π_1 transversal to W^u_{loc}, while T_1 takes them back from Π_1 to Π_0. The stable manifold breaks Π_0 into the top and bottom components, $\Pi_0^{\pm}$. The orbits that start at Π_0^- do not get to Π_1 and leave a neighborhood of the homoclinic loop; the orbits that start in Π_0^+ will reach Π_1, and then follow

[6] An example of a criterion of chaos derived from the Shilnikov theorem is due to A. Arneodo, P. H. Coullet, E. A. Spiegel and C. Tresser [67, 68] who found that the normal forms for the bifurcations of equilibria wih three zero eigenvalues have, generically, a saddle-focus loop at certain parameter values. A free of computer assistance proof for this fact was given in [73]. This provides a convenient way for an analytic proof of the existence of Shilnikov chaos in systems of differential equations: one just needs to find the triply degenerate equilibrium at certain prameter values, then the existence of a region of parameter values that correspond to chaos is guranteed. This criterion was e.g. used in [58] for the analytic proof of the existence of infinite-dimensional (space-time) version of Shilnikov chaos in equations of Ginzburg – Landau type.

[7] It was very characteristic how Leontovich-Andronova later described the moment when Shilnikov told her about his findings: "At once I wanted to say: Nonsence! Then I thought: this is a [three-dimensional] space, so told him: Maybe you are right."

the homoclinic loop until they will return to Π_0. If the orbit returns to Π_0^-, then it will leave the neighborhood of Γ; otherwise it will make another round following Γ and return to Π_0 again, and so forth. The image of Π_0^+ on Π_1 has the shape of a spiral with infinitely many curls accumulating to W^u_{loc} (see Fig. 9; cf. Fig. 4). The map from Π_1 to Π_0 is a regular diffeomorphism (because it takes a bounded time for an orbit to complete the excursion), therefore the image of Π_0^+ by the Poincaré map $T = T_1 \circ T_0$ preserves the spiraling shape too. It intersects W^s_{loc} infinitely many times around the point $M^* = \Gamma \cap \Pi_0$, as shown in Fig. 11.

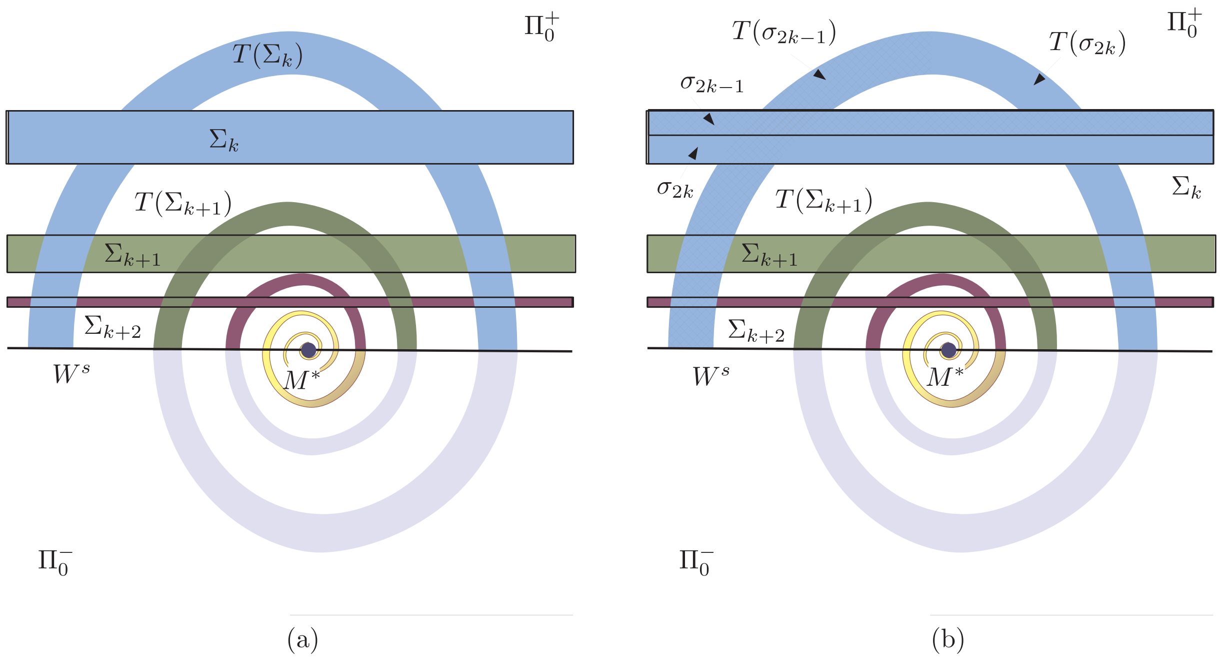

Fig. 11. (a) The Poincare return map near a homoclinic loop to a saddle-focus with $\sigma > 0$ generates infinitely many Smale horseshoes. (b) The refined partition of the cross-section (following the original construction of [11]).

Let us select in the upper section Π_0^+ a countable number of strips Σ_k such that the image $T\Sigma_k$ is a connected component of $T\Pi_0^+ \cap \Pi_0^+$ (one half of one curl of the spiraling "snake" $T\Pi_0^+$, see Fig. 11). Observe that if an orbit remains in a small neighborhood of Γ for all times, then all the points of its intersection with Π_0 belong to the strips Σ_k.

The homoclinic orbit Γ tends to O in a spiral. Therefore, we may take a small piece of the plane $x = 0$ as our cross-section Π_0. If so, let us use (y, z) as a local coordinates on Π_0. One can estimate that Σ_k stays at $z \sim z_k = e^{-2\pi k \frac{\gamma}{\omega}}$, while the top of the curl $T\Sigma_k$ lies at $z \sim (z_k)^\nu = e^{-2\pi k \frac{\lambda}{\omega}}$, where $\nu = \lambda/\gamma$. As $\nu < 1$, we have $z_k \ll (z_k)^\nu$. Therefore, for each k large enough the intersection $T\Sigma_k \cap \Sigma_k$ is non-empty and consists of two connected components (Fig. 11). It is geometrically evident that there is a fixed point of the return map T within each of the components, which gives us Theorem 7 (recall that fixed points of the Poincaré map correspond to periodic orbits of the system).

The computation of the mutual position of the strips and their images is easy when the system is exactly linear near O (i.e. the nonlinearities F_i vanish identically at small (x, y, z)). Then the solution $(x(t), y(t), z(t))$ that starts from a point $(0, y, z) \in \Pi_0^+$ at $t = 0$ and ends up at a point $(x_1, y_1, z_1 = d \ll 1) \in \Pi_1$ when $t = \tau$ satisfies $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \exp\left[\tau \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}\right] \begin{pmatrix} 0 \\ y \end{pmatrix} =$ $\exp^{-\lambda\tau} \begin{pmatrix} \cos\omega\tau & -\sin\omega\tau \\ \sin\omega\tau & \cos\omega\tau \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix}$, $z = e^{-\gamma\tau} d$. We can now evaluate the dwelling time $\tau = -\frac{1}{\gamma} \ln \frac{z}{d}$

between Π_0^+ and Π_1, and obtain the following formula for the map $T_0 : (y, z) \mapsto (x_1, y_1)$:

$$(x_1, y_1) = y\left(\frac{z}{d}\right)^\nu \left(\sin\frac{\omega}{\gamma}\ln\frac{z}{d},\ \cos\frac{\omega}{\gamma}\ln\frac{z}{d}\right).$$

It is seen that the image of every line $y = \text{const}$ by T_0 is a logarithmic spiral on Π_1. We pass from Σ_k to Σ_{k+1} as these spirals rotate to 2π in the (x_1, y_1)-plane, which gives us an estimate for the position $z \sim z_k$ of Σ_k in Π_0^+ and the distance $\sim (z_k)^\nu$ of the half-curl $T_0\Sigma_k$ from the origin in Π_1. As the map T_1 adds only a bounded distortion factor to this picture, the distance of the image of this half-curl in Π_0^+ from M^* is of the same order.

It is seen from these formulas that if we ignore a contribution of the y-coordinate, then the action of the map T on the z-variable can be modeled by the equation (cf. [69, 70])

$$\bar{z} = \mu + Az^\nu \cos\left(\frac{\omega}{\gamma}\ln z + \theta\right),$$

where A and θ are constants, and the bifurcation parameter μ controls the splitting of the primary homoclinic loop Γ (the loop exists at $\mu = 0$). The graph of this one-dimensional map is shown in Fig. 12. From it one can clearly see the difference between the case $\nu > 1$ (simple dynamics), and the case under consideration, $\nu < 1$. In the later case, infinitely many fixed points accumulate to $z = 0$ at $\mu = 0$. Variations of μ move the graph of the map up or down, so that the fixed points undergo saddle-node and period-doubling bifurcations; multi-round homoclinic loops are created too [72]. A two-parameter analysis of these bifurcations (in the (μ, ν)-plane) see in [32, 59].

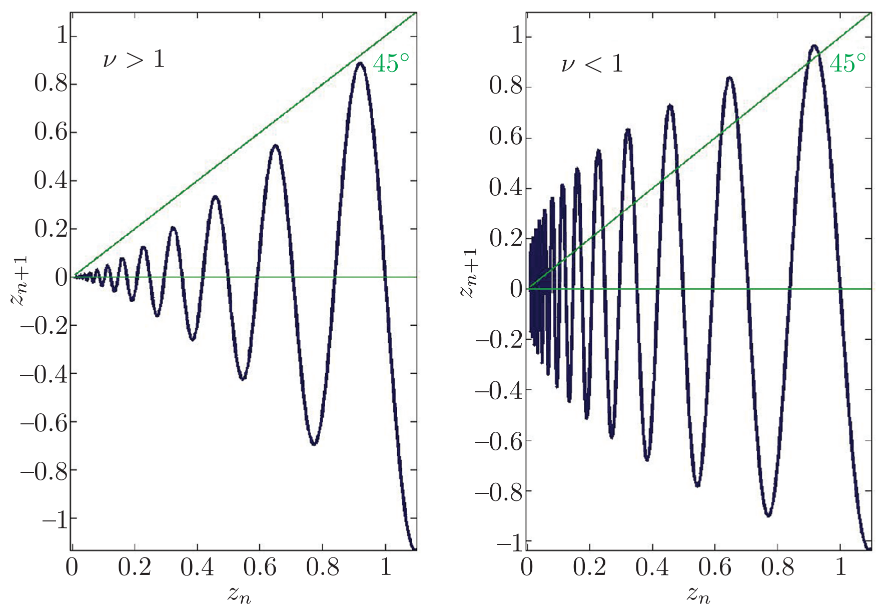

Fig. 12. One-dimensional return maps for $\nu > 1$ and $\nu < 1$.

Assessing the geometry of the *two-dimensional* map near the Shilnikov saddle-focus (see Fig. 11), a modern reader may recognize the topological Smale horseshoe in the picture of Σ_k and its image. In fact, Shilnikov proved in [11] the hyperbolicity of the map T in restriction to the intersection $T\Sigma_k \cap \Sigma_k$, so it is a Smale horseshoe indeed for each k large enough. However, the set of all orbits that lie in a small neighborhood of the homoclinic loop is larger and much more complicated than the union of the hyperbolic sets that correspond to these horseshoes. The reason is the orbits may jump between different strips Σ_k. For a jump from Σ_i to Σ_j one needs that $(z_i)^\nu \gtrsim z_j$, or $i\nu \lesssim j$. This shows us that the structure of the set of orbits that lie in a small neighborhood of the homoclinic loop depends on the value of the saddle index ν. It is convenient to split each of our strips Σ_k to two, a strip σ_{2k-1} that corresponds to the left part of the k-th curl, and a strip σ_{2k} that corresponds to the right part (see Fig. 11b). Then the intersection $T\sigma_i \cap \sigma_j$ is either empty (if $i\nu \gg j$) or consists of one connected component (if $i\nu \lesssim j$). Every orbit from a small neighborhood of Γ acquires a natural coding, a sequence of integers k_s such that the consecutive points of intersection of the orbit with the cross-section Π_0 belong to the strips σ_{k_s}. The structure of the set of possible codings, and the correspondence between the orbits and their codings is described by the following result.

Theorem 8 ([11]). *Choose two numbers $\nu' < \nu$ and $\nu'' > \nu$. There exists a small neighborhood V of the homoclinic loop Γ and a small cross-section Π_0 to the loop such that for every orbit from V its intersections with Π_0 belong to a disjoint union of strips σ_k, which accumulate to $W^s_{loc} \cap \Pi_0$ as $k \to +\infty$. The corresponding coding sequence $\{k_s\}$ satisfies*

$$k_{s+1} > k_s \nu' \tag{2.1}$$

for all s. There exist $\bar{k} \geqslant 0$ such that given any sequence $\{k_s\}$, which satisfies

$$k_{s+1} > k_s \nu'', \qquad k_s \geqslant \bar{k} \tag{2.2}$$

for all s, there exists a unique orbit in V with this coding.

Consider the set $\Omega = \Omega(\nu'', \bar{k})$ of the orbits whose codings satisfy (2.2). The uniqueness (the bijectivity) of the correspondence between the coding and the orbit in $\Omega(\nu'', \bar{k})$ is important, as it implies that a periodic coding satisfying (2.2) corresponds to a periodic orbit, while a recurrent coding corresponds to a recurrent orbit, and a coding that is asymptotic to a periodic one corresponds to a Poincaré homoclinic (an orbit which is homoclinic to periodic), etc. In fact, Shilnikov proved the uniqueness by showing that the Poincaré map T is uniformly hyperbolic when restricted to the intersection of $\Omega(\nu'', \bar{k})$ with Π_0. The set $\Omega(\nu'', \bar{k})$ itself (as an invariant set of the system of differential equations) is not uniformly-hyperbolic (since its closure contains the equilibrium O), however it is the limit, as $k \to +\infty$, of the increasing sequence of closed uniformly-hyperbolic sets $\Omega_k(\nu'', \bar{k})$ formed by the orbits whose codings satisfy (2.2) and are bounded by k (i.e. $k_s \leqslant k$ for all s). Shilnikov remarked that the Poincaré map T is conjugate to a finite topological Markov chain when restricted to any of the sets $\Omega_k(\nu'', \bar{k})$, and that the sets $\Omega_k(\nu'', \bar{k})$ are structurally-stable, i.e. for each fixed k the set $\Omega_k(\nu'', \bar{k})$ persists for all systems, which are sufficiently close (in C^1) to the given one, even when the homoclinic loop splits. The complete set $\Omega(\nu'', \bar{k})$ does not entirely survive the breakdown of the loop (the horseshoes that are too close to the loop get disappeared).

The result of Theorem 8 was proven in [11] for systems of arbitrary dimension. In the case where the dimension is higher than 3, it requires an imposition of additional non-degeneracy assumptions on the homoclinic loop. We list assumptions of [11] as follows.

- The nearest to the imaginary axis characteristic exponent of the equilibrium state O is complex.

It means that if we denote the characteristic exponents of O as $\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_m$ where $\mathrm{Re}\gamma_j > 0$ and $\mathrm{Re}\lambda_i < 0$ $(i = 1, \ldots, m; j = 1, \ldots, n)$, then either

$$\gamma_{1,2} = \gamma \pm i\beta, \qquad \gamma < \min_{j \geqslant 3} \ \mathrm{Re}\ \gamma_j, \qquad \text{and} \qquad \gamma < \min |\mathrm{Re}\ \lambda_i|,$$

or

$$\lambda_{1,2} = -\lambda \pm i\omega, \qquad \lambda < \min_{i \geqslant 3} |\mathrm{Re}\ \lambda_i|, \qquad \text{and} \qquad \lambda < \min\ \mathrm{Re}\ \gamma_j.$$

We call $\gamma_{1,2}$ in the first case and $\lambda_{1,2}$ in the second case the leading complex exponents. Define the saddle index ν as

$$\nu = \frac{\gamma}{\min_i |\mathrm{Re}\ \lambda_i|} \qquad \text{in the first case,}$$

and

$$\nu = \frac{\lambda}{\min_j\ \mathrm{Re}\ \gamma_j} \qquad \text{in the second case.}$$

By construction, $\nu < 1$.

- There exists a homoclinic loop Γ to O.
- The intersection of W^s and W^u along Γ has the least possible degeneracy, i.e.
$$\dim(T_M W^u \cap T_M W^s) = 1$$
for any point $M \in \Gamma$.
- $\Gamma \notin W^{uu}$ in the case of the leading complex exponent with positive real part, and $\Gamma \notin W^{ss}$ in the case of the leading complex exponent with negative real part.
- A certain coefficient Δ is non-zero[8)].

Note that the last three non-degeneracy conditions are automatically fulfilled for systems of dimension $m + n = 3$. We are not aware of any works where the violation of these conditions was studied systematically. Codimension-2 bifurcations, which correspond to the Andronov – Hopf bifurcation of a saddle-focus with a homoclinic loop (the so-called Shilnikov – Hopf bifurcation), crossing the boundary between the saddle and saddle-focus, and the vanishing of the saddle value, were studied by Shilnikov's student L. A. Belyakov [75–77].

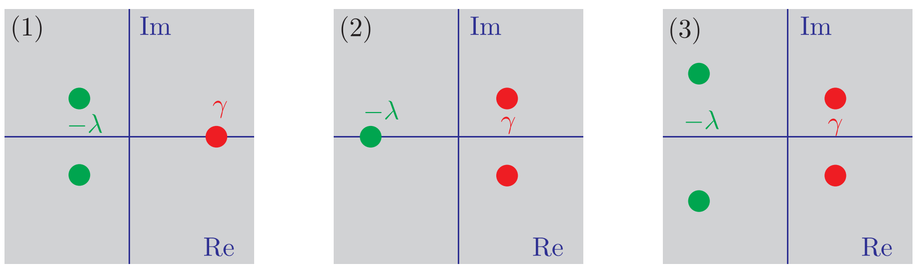

Fig. 13. Examples of different configurations of characteristic exponents of saddle-foci.

The chaotic dynamics described by Theorem 8 possesses several important properties. First, the hyperbolic set Ω does not, typically, admit a symbolic description with finitely many symbols. Secondly, and more importantly, the hyperbolic set Ω does not necessarily contain all orbits from the neighborhood V of the homoclinic loop. To see this, note that Ω depends sensitively on the saddle index ν [9)] In particular, for any two, arbitrarily close values ν_1 and ν_2 of the saddle index ν, if $\nu_1 > \nu_2$, then one can always find a pair of sufficiently large integers k_s, k_{s+1}, for which both inequalities (2.2) and (2.1) will be satisfied at $\nu = \nu_2$ and will be violated at $\nu = \nu_1$. Thus, orbits with certain codings (e.g. periodic ones, and homoclinic orbits to periodic ones) will disappear in V at an arbitrarily small increase of ν. This means bifurcations of non-hyperbolic periodic orbits and homoclinic tangencies must occur in V.

This observation was made rigorous in two papers by I. M. Ovsyannikov and L. P. Shilnikov [26, 27] (similar ideas were earlier developed in the works by N. K. Gavrilov, S. V. Gonchenko and L. P. Shilnikov [12, 24], see Section "Homoclinic Tangency" in [85]). Let H^1_{sf} be the set of C^r-smooth systems ($r \geqslant 4$) with a homoclinic loop to a saddle-focus with one-dimensional unstable manifold, and H^2_{sf} be the set of systems with a homoclinic loop to a saddle-focus with two-dimensional unstable manifold such that the two characteristic exponents with positive real parts are complex.

8) This is an analogue of the separatrix value A; see comments after Theorem 3. In geometrical terms, $\Delta \neq 0$ is the condition of a transverse intersection, at the points of Γ, of the extended stable manifold and the unstable manifold (in the case where the leading complex exponent has positive real part), or a transverse intersection of the extended unstable manifold and the stable manifold if the leading complex exponent has negative real part; see [38] for definitions. Unlike Theorem 3, these non-degeneracy conditions do not provide a reduction to an invariant manifold of a lower dimension, so the proof of Theorem 8 in full generality required a full multidimensional and nonlinear computation.

9) It was e.g. proven in [37] that ν is a modulus (a continuous invariant) of topological equivalence of systems with a homoclinic loop to a saddle-focus; curiously, this result holds in the case $\nu \geqslant 1$ too.

Theorem 9 ([26, 27]).

1. *Systems with homoclinic tangencies are* C^r*-dense in* H^1_{sf}.
2. *Systems with non-hyperbolic periodic orbits are dense in* H^1_{sf} *and* H^2_{sf}.

The importance of the homoclinic tangencies is that their existence implies the complexity of dynamics, full details of which are (in a certain precise sense, see Section "Homoclinic Tangency" in [85]) beyond a human comprehension [29]. So, item 1 of Theorem 9 implies, in fact, that a complete description of the set of all orbits that lie in a small neighborhood of a saddle-focus loop is impossible to give (the set Ω of Theorem 8 provides a quite good approximation of this set, but an essential refinement of this approximation can never be achieved). The importance of non-hyperbolic periodic orbits (item 2 of Theorem 9) is that their bifurcations can produce stable periodic regimes (periodic attractors). Theorem 9 has to be true for a larger class of systems with a saddle-focus loop. However, Ovsyannikov and Shilnikov focused on these particular cases as they were specifically interested in finding criteria for the existence of stable periodic orbits alongside chaos[10)].

These criteria are given in [27] in terms of the so-called *second saddle value* σ_2. For systems in H^1_{sf} the characteristic exponents at O are $\gamma > 0, \lambda_1, \ldots, \lambda_m$ where

$$\lambda_{1,2} = -\lambda \pm i\omega, \qquad \max_{i\geqslant 3} \mathrm{Re}\ \lambda_i < -\lambda < 0, \qquad \gamma - \lambda > 0.$$

For systems in H^2_{sf} the characteristic exponents are $\gamma_{1,2} = \gamma \pm i\beta(\gamma > 0), \lambda_1, \ldots, \lambda_m$ where

$$\lambda_1 = -\lambda, \qquad \max_{i\geqslant 2} \mathrm{Re}\ \lambda_i < -\lambda < 0, \qquad \gamma - \lambda < 0,$$

or

$$\lambda_{1,2} = -\lambda \pm i\omega, \qquad \max_{i\geqslant 3} \mathrm{Re}\ \lambda_i < -\lambda < 0, \qquad \gamma - \lambda \neq 0$$

(see Fig. 13). The second saddle value is defined as

$$\begin{aligned} \sigma_2 &= \gamma - 2\lambda && \text{in the first case,} \\ \sigma_2 &= 2\gamma - \lambda && \text{in the second case,} \\ \sigma_2 &= 2\gamma - 2\lambda && \text{in the third case.} \end{aligned}$$

This sign of this value determines whether the linearized system at O expands ($\sigma_2 > 0$) or contracts ($\sigma_2 < 0$) volumes in the so-called leading subspace (spanned by the eigen-directions that correspond to the characteristic exponents nearest to the imaginary axis from both sides, see [38]). The existence of stable periodic orbits requires volume contraction (see more discussion in [55]), so the following result is natural.

Theorem 10 ([26, 27]).

1. *If* $\sigma_2 < 0$*, then systems with infinitely many coexisting stable periodic orbits are* C^r*-dense in* H^1_{sf} *and* H^2_{sf}.
2. *If* $\sigma_2 > 0$*, then no system from* H^1_{sf} *or* H^2_{sf}*, nor any close system, can have a stable periodic orbit in a sufficiently small neighborhood of the loop.*

[10)] Increasing the dimension of the unstable manifold of the equilibrium O would typically prohibit the stable periodic orbits in a small neighborhood of the homoclinic loop [55].

This theorem means that the chaotic dynamics produced by the saddle-focus loop in the case $\sigma_2 < 0$ belongs to the most common type of chaos known and frequently observed today: an extremely complex and structurally unstable mixture of hyperbolic sets and co-existing stable periodic orbits of long periods. To categorize and disclose unclear, "fuzzy" structures of numerically or experimentally observed strange attractors of this type, including spiral attractors due to the Shilnikov saddle-focus, Afraimovich and Shilnikov [21] introduced the notion of a "quasi-attractor." We will discuss it in detail in Sections "Homoclinic Tangency", "Saddle-Node Bifurcation" and "Lorenz Attractor" in [85]. Here, we notice that the result of item 2 of Theorem 10 poses a question about the possibility of the existence of true strange attractors (i.e. those which contain *no* stable periodic orbits) which include a saddle-focus loop and the chaotic set associated with it. Indeed, the results of [26, 27] motivated the work of D. Turaev and L. P. Shilnikov [31] who presented an example of such attractor and describe its dynamical and bifurcation features (see Section "Lorenz Attractor" in [85]).

The behavior near the saddle-focus loop in the conservative case with $\sigma_2 = 0$ was studied by V. Biragov and L. P. Shilnikov in [28]; codimension-2 bifurcations at the change of sign of σ_2 were considered by V. S. Gonchenko and L. P. Shilnikov in [35]. See also Section "Hamiltonian Chaos" in [85] for homoclinics of saddle-foci in Hamiltonian systems.

2.1. Routes to Spiral Chaos

The discovery of spiral chaos near a saddle-focus loop was a genuinely paradigmatic shift for the bifurcation theory. Its importance for dynamical systems research became clear practically from the very beginning. However the value of Shilnikov discovery for applied and cross-disciplinary sciences could not be evident then in 1965[11)]. Only starting from mid 70s–80s, when researchers became interested in computer studies of chaotic behavior in nonlinear models, it became clear that the Shilnikov saddle-focus is a pivotal element of chaotic dynamics in a broad range of real-world applications. In particular, Afraimovich, Bykov and Shilnikov themselves found the saddle-focus loops in the Lorenz model [19]. In general, the number of various models from hydrodynamics, optics, chemical kinetics, biology etc, which demonstrated the numerically or experimentally strange attractors with the characteristic spiral structure suggesting the occurrence of a saddle-focus homoclinic loop, was overwhelming.

By Theorems 7, 8, the occurrence of the Shilnikov saddle-focus loop implies chaos. Why the converse is also so often true: how can chaos imply a homoclinic loop of a saddle-focus? This question preoccupied Shilnikov in the middle of 80s. He found [23] that whenever a system, depending on some bifurcation parameter, evolves from a stable ("laminar") to a chaotic ("turbulent") regime, then the transition is naturally accompanied with the formation of a saddle-focus equilibrium in the phase space. Furthermore, regardless of a particular route to chaos, it is also natural for the stable and unstable manifolds of this saddle-focus to come sufficiently close to each other, which makes the creation of a homoclinic loop feasible.

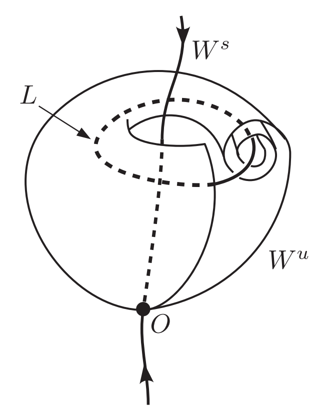

Fig. 14. A funnel-type configuration of $W^s(O)$.

This idea is hardly mathematically formalizable, it is more of an empirical statement that makes it even more valuable. The construction links the beginning of the transition to chaos (through an Andronov – Hopf bifurcation) with the end of it (the formation of a spiral attractor) in a simple and model-independent way. Indeed, this scenario has turned out to be typical for a variety of systems and models of very diverse origins.

The main scenario proposed in [23] was as follows. Consider a smooth three-dimensional system

$$\dot{x} = X(x, R) \tag{2.3}$$

[11)]For instance, in the paper [5] devoted to a problem of the birth of periodic regimes in piece-wise linear systems of automatic control the result of [3] was mentioned only in passing, as only stable periodic orbits were thought to be meaningful for applications.

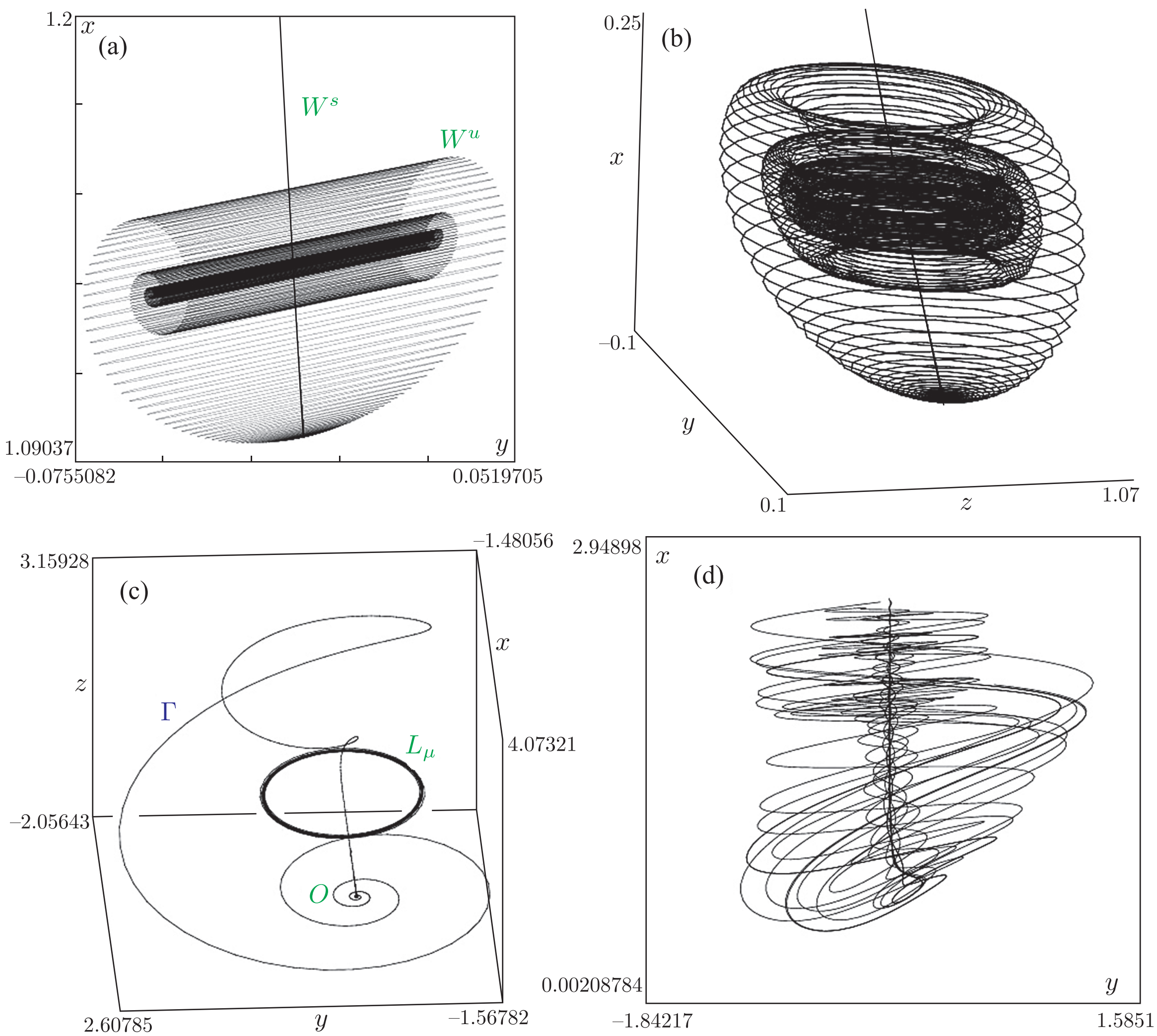

Fig. 15. Evolution towards the spiral attractor in the Lorenz-84 model of atmospheric global circulation [53].

that depends on a certain parameter R (this choice of notation for the parameter had a hydrodynamic motivation, e.g. one may think of R as being somehow related to the Reynolds number). Let the increase of R make the system evolve from a stable regime to chaotic dynamics. That is, at some $R < R_1$ the system has a stable equilibrium state O, which at $R = R_1$ loses the stability through a supercritical Andronov – Hopf bifurcation, and a stable periodic orbit L is born. The point O becomes a saddle-focus at $R > R_1$, and at small positive values of $R - R_1$ the boundary of its two-dimensional unstable manifold W^u_O is the stable periodic orbit L. With further increase of R, the new stable periodic regime L also loses stability, say through a period doubling bifurcation, or through a bifurcation of the birth of a quasiperiodic regime (a two-dimensional stable torus)[12]. In any case, before the periodic orbit loses stability its multipliers must become complex at some $R = R_2 > R_1$ (the multipliers of L are real positive at R close to R_1, so they must become complex before one of them becomes equal to -1). At $R > R_2$ the manifold W^u_O starts winding

[12] The transition to spiral chaos via a period-doubling occurs, for example, in the Rössler system [79]

$$\dot{x} = -(y + z), \qquad \dot{y} = x + ay, \qquad \dot{z} = b + (x - c)z.$$

The transition involving the breakdown of an invariant torus is a feature of the Anischenko – Astakhov electronic generator [80] described by the equations

$$\dot{x} = ax + y - xz, \qquad \dot{y} = -x + y, \qquad \dot{z} = -bz + xH(x),$$

where H is the Heaviside function.

onto L and forms a funnel-type configuration (Fig. 14). After the funnel is formed, the creation of a homoclinic loop of the saddle-focus O as R grows further becomes quite natural: the throat of the funnel may become smaller, or it may change its position, so that W^u_O and W^s_O gets closer and closer to each other until the primary homoclinic loop is formed at some $R = R_3$. If the complex characteristic exponents of the saddle-focus O are the nearest to the imaginary axis (closer than the real negative one), which is automatically fulfilled at $R = R_1$, so we may assume it continues to hold at $R = R_3$ too, then the occurrence of the homoclinic loop to O directly implies chaotic orbit behavior in a neighborhood of the loop (Theorem 8). In the case where the throat of the funnel can be cut through by a cross-section so that all the orbits intersecting the cross-section come inside the funnel, the unstable manifold W^u_O (more precisely, its part from O up to the cross-section, plus the cross-section itself) will bound a forward-invariant region; at $R = R_3$ the attractor, which resides within this region will contain the homoclinic loop and the chaotic set around the loop. All orbits in this chaotic set will spiral around the saddle-focus, forming the characteristic shape of the "spiral attractor", as one depicted in Fig. 15. As R is increased more and the homoclinic loop breaks down, the large portion of the chaotic set shall nevertheless persist, with new, multi-round homoclinic loops possibly emerging. This scenario of the transition to chaos can occur in n-dimensional systems with any $n \geqslant 3$ too, e.g. just by adding $(n-3)$ contracting directions.

The main observation here is that the Andronov–Hopf bifurcation of the primary stable equilibrium O transforms it to a saddle-focus, and instead of following details of the further evolution of the stable regimes (the periodic orbit L, the periodic orbit born from L after, for example, the first period-doubling, etc.) it may be more useful for the understanding of the transition to chaos to follow the evolution of the shape of the unstable manifold of the saddle-focus as parameters of the system vary. Studying dynamical and bifurcation features of various attractors that can occur and [co]exist in the Shilnikov funnel would be a very compelling research direction. Shilnikov proposed a model for the Poincaré map in the funnel [23]. Based on the analysis of this map a birth of an invariant torus in the funnel was studied in [81, 82]. In [83] there was shown that a certain type of the funnel is consistent with the existence of a hyperbolic Plykin attractor. The wild attractor proposed in [84] can also be attributed to the Shilnikov funnel (in dimension $n \geqslant 5$).

The above described scenario has proved to present the simplest (hence, the most general) route to chaos in dissipative systems. It employs only a minimal number of objects required for chaos formation: the equilibrium O, its unstable manifold, and the periodic orbit L. Other, more complicated scenarios can also be based on the same idea. For example, Shilnikov also noticed in [23] that the Andronov–Hopf bifurcation at $R = R_1$ can be different from one described above. Namely, we assumed that this bifurcation is soft (supercritical), i.e. the stability of the equilibrium O is transferred to the stable periodic orbit L bifurcating from O at $R \geqslant R_1$. The alternative is a subcritical Andronov–Hopf bifurcation at $R = R_1$: the periodic orbit L of the saddle type, existing at $R < R_1$, collapses onto O at $R = R_1$ thus making it a (weak) saddle-focus. The unstable manifold W^u_O at $R = R_1$ is the limit of the unstable manifold of L. Thus, already at $R = R_1$, the manifold W^u_O may have a non-trivial shape, e.g. it may form the necessary funnel, so a large forward-invariant region associated with the funnel is formed at $R = R_1$. If a chaotic set Λ (not necessarily an attractor) had already existed at $R < R_1$ inside this region, then one should observe a sudden transition from the stable stationary regime O to a well-developed spiral chaos at $R = R_1$. A similar observation of a sudden transition from a stable equilibrium state to a large invariant torus was made in [81, 82]. The chaotic set Λ can be produced by several ways. For example, at some R smaller than R_1 a saddle-node periodic orbit emerges and, as R grows, decouples into saddle and stable periodic orbits, L and L_+. In the three-dimensional case, the stable manifold of L is two-dimensional, and L divides it into two halves; one of the halves tends to O and the other half, W^{u+}_L converges to L_+. As R grows, the orbit L_+ loses stability in some way and, eventually, homoclinic intersections of W^{u+}_L and W^s_L form. A homoclinic trajectory to a saddle periodic orbit is accompanied with a nontrivial hyperbolic set Λ' [7]. If L preserves its homoclinics as it merges with O, then the weak saddle-focus O will possess a homoclinic loop at $R = R_1$. Chaotic dynamics associated with this so-called Shilnikov–Hopf bifurcation was studied in [75, 78], for example. If the saddle orbit L loses its homoclinics near $R = R_1$, a portion Λ of the hyperbolic set Λ' may still survive until $R = R_1$.

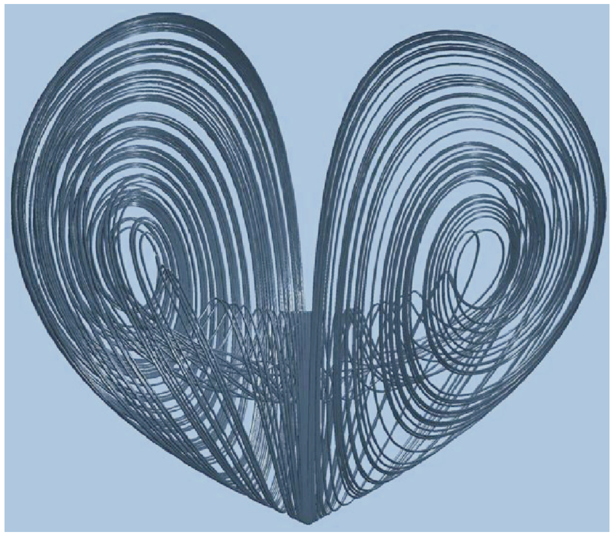

Fig. 16. A wild Lorenz-like attractor with a saddle-focus.

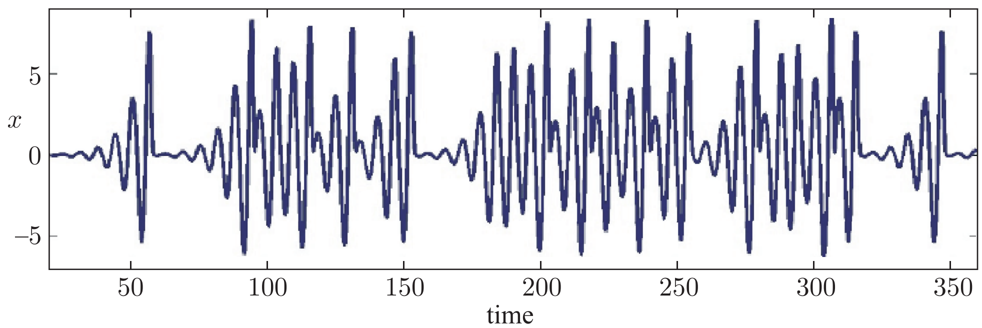

Fig. 17. Bursts in the time series generated by a spiral attractor.

In a system with symmetry, the stable equilibrium O can typically loose the stability through a pitchfork bifurcation. Then, instead of a stable periodic orbit L, there will be a new pair of stable equilibria O_1 and O_2, whereas the equilibrium O becomes a saddle with one-dimensional unstable manifold (of two symmetric separatrices that tend to $O_{1,2}$). After the equilibria $O_{1,2}$ acquire complex characteristic exponents, the unstable separatrices of O will start spiral around $O_{1,2}$. A further increase of a parameter may lead then to the formation of a symmetric pair of homoclinic loops and, next, to the onset of chaotic dynamics, like in the Lorenz model [17] or in systems with "double-scroll" attractors [66, 68, 74]. In dimension $n \geqslant 4$ a symmetric wild Lorenz-like attractor (see Fig. 16) may emerge in this way as well [31].

In another scenario of [23] Shilnikov discussed the formation of a strange attractor due to creation of a Poincaré homoclinic orbit. In later works [36, 60] this route to chaos was shown to be typical for three-dimensional diffeomorphisms.

Returning to the simplest scenario of the transition to chaos, we note that a typical spiral attractor formed inside the funnel in a three-dimensional dissipative system is a quasi-attractor in the sense of [21], because bifurcations of a homoclinic loop of the Shilnikov saddle-focus lead to the birth of stable periodic orbits along with the hyperbolic sets according to Theorem 10. Practically, these stable periodic orbits are hardly distinguishable within chaotic attractors, because they have long periods and thin attraction basins, which can be fuzzed out by noise inevitable in any real system. They may, however, influence the statistics of the repetition of typical patterns in such seemingly chaotic behavior. The signature pattern of the spiral chaos is shown in Fig. 17: one can observe that the quiescence periods corresponding to the phase point passing close by the saddle-focus are alternated with bursts of oscillations, which amplitude rapidly increases from zero.

The occurrence of such characteristic patterns has allowed for easy identifications of the Shilnikov homoclinic saddle-focus underlying the chaotic dynamics in a large variety of numerical simulations as well as experimental studies, including but not limited to nonlinear optics, electronic circuits, life sciences, and fluid dynamics, to name a few.

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