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Breakdown of Symmetry in Reversible Systems

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Abstract—We review results on local bifurcations of codimension 1 in reversible systems (flows and diffeomorphisms) which lead to the birth of attractor-repeller pairs from symmetric equilibria (for flows) or periodic points (for diffeomorphisms).

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1. INTRODUCTION

Reversible systems serve as mathematical models for various processes in physics and, in particular, in mechanics [3, 17, 24, 32, 33, 40]. Usually, the reversibility results from certain special properties of the process under consideration. For example, if a differential equation contains only derivatives of even orders (like it often happens for Euler–Lagrange equations and their generalizations), then the corresponding system of first order ODEs will be reversible with respect to some involution of the phase space. Recall the definition of a reversible system on a smooth manifold M (see [9, 34]). Suppose an involution acts on M, i.e. a diffeomorphism $R : M \to M$ is defined such that $R^2 = id_M$. A vector field v on M is called *reversible* with respect to R if $DR \circ v(x) = -v(Rx)$ for all $x \in M$. This identity implies the main property of the reversible system: if $x(t)$ is a solution of the system, then $x_1(t) = Rx(-t)$ is also a solution. A special role in the study of a reversible system is played by the set of the fixed points of the involution, $Fix(R)$. This set is assumed to be a smooth submanifold of M. Everywhere in this paper we assume the vector fields and diffeomorphisms to have smoothness sufficiently high for our purposes.

If an orbit of a vector field is invariant with respect to the involution, this orbit is called *symmetric*. Such orbit must intersect $Fix(R)$. It is easy to see that any orbit may intersect $Fix(R)$ not more than in two points, and the existence of two intersections implies that the orbit is periodic (the symmetric periodic orbit intersects $Fix(R)$ exactly at each half-period) [9, 41]. An equilibrium p is symmetric if $p \in Fix(R)$. Non-periodic symmetric orbits (for example, homoclinic to symmetric equilibria) intersect $Fix(R)$ only at one point. Asymmetric orbits appear in pairs: $\gamma, R(\gamma)$. It is known that the behavior of a reversible system near symmetric orbits reminds the behavior of a Hamiltonian system. The behavior near asymmetric orbits can be arbitrary, like in any dissipative system.

In the case of discrete time one considers reversible diffeomorphisms. Again, assume an involution R acts smoothly on a smooth manifold M. A diffeomorphism $T\colon M \to M$ is called reversible if the following identity holds: $T^{-1} = R \circ T \circ R$. Note that a reversible diffeomorphism is a superposition

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of two involutions: $T = (R \circ T^{-1}) \circ R$. This representation is useful in constructing reversible diffeomorphisms with various properties.

Like in the case of a flow, one introduces the notion of symmetric and asymmetric orbits. However, the properties of symmetric orbits here are somewhat different. By definition, the map T is reversible if and only if $T \circ R$ is an involution, moreover the map T is reversible with respect to the involution $T \circ R$. An orbit of the reversible diffeomorphism T is symmetric if and only if it intersects $Fix(R) \cup Fix(T \circ R)$. A symmetric orbit is periodic (with period larger than 1) if it intersects the set $Fix(R) \cup Fix(T \circ R)$ exactly at two points. The symmetric periodic orbit has odd period if and only if it intersects both $Fix(R)$ and $Fix(T \circ R)$. Any symmetric orbit of even period $2p$ contains a point in $Fix(R) \cap T^p(Fix(R))$ or in $Fix(T \circ R) \cap T^p(Fix(T \circ R))$, and a symmetric orbit of period $2p+1$ has a point in $Fix(R) \cap T^p(Fix(T \circ R))$ (see [24] for more detail).

In applied problems described by reversible systems the following phenomenon is often observed: in some region of parameter values the system exhibits a stable symmetric regime, while upon crossing some critical parameter value a pair of asymmetric regimes appears, one asymptotically stable and one asymptotically unstable (these regimes are symmetric to each other). Our aim in this work is to list main local bifurcations of reversible systems which can lead to the birth of such pairs (we call them *symmetry breaking bifurcations*). Since the results on bifurcations of reversible systems are scattered over many publications (see e.g. [19, 25, 39]), we have decided to write a short survey, in the hope it can be useful both for the experts and for those who would like to get acquainted with the existing results and to start their own research in this field. We restrict ourselves with the simplest bifurcations (i.e. bifurcations of codimension 1) and by systems of low dimension (2, 3 and 4). We substantially extended an earlier version [30] of this survey by adding an overview of symmetry-breaking bifurcations in non-orientable reversible diffeomorphisms and in other types of reversible diffeomorphisms which cannot serve as Poincare maps of reversible flows. We show that various types of symmetry-breaking bifurcations are described by one of the normal forms: system (2.4) under condition (2.5), system (3.4) under condition (3.5), system (4.9) under condition (4.10), system (4.13) under condition (4.14), and system (5.2) under condition (5.3). The phase portraits for the two-dimensional normal forms (2.4) and (3.4) are given in Figs. 1 and 2; while we do not give here a complete analysis of the three-dimensional normal forms (4.9), (4.13), (5.2), it should be doable, as these systems can be made piece-wise linear by a coordinate transformation.

2. TWO-DIMENSIONAL REVERSIBLE FLOWS

Let us start with the bifurcations of symmetric equilibrium states in two-dimensional systems. Since the problem is local, we can always assume that the involution acts linearly in some properly chosen coordinates (x, y) in a neighborhood of the equilibrium state (such coordinates always exist by the Bochner–Montgomery theorem [4]). The main case here corresponds to the involution $R(x, y) = (x, -y)$. Indeed, there are only two cases when a linear involution can not be brought to this form by a linear coordinate transformation: $R(x, y) = (x, y)$ and $R(x, y) = (-x, -y)$. In the first case, the only reversible system is $\dot{x} = 0$, $\dot{y} = 0$, and its dynamics is trivial. In the second case, the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is reversible if and only if both functions P and Q are even: $P(-x, -y) = P(x, y)$, $Q(-x, -y) = Q(x, y)$. A symmetric equilibrium is always a fixed point of the involution. In the given case there is only one such point (at zero), therefore for the existence of a symmetric equilibrium two independent equality-type conditions have to be imposed on the system ($P(0, 0) = 0$ and $Q(0, 0) = 0$ in our coordinates). Thus, if the involution is given by $R(x, y) = (-x, -y)$, the existence of a symmetric equilibrium is an event of codimension 2, and we do not consider such bifurcations here.

So, we further assume $R(x, y) = (x, -y)$. Then the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is reversible if and only if $P(x, -y) = -P(x, y)$, $Q(x, -y) = Q(x, y)$. We assume the vector field is smooth enough. Then the reversibility condition is written as $P(x, y) = yf(x, y^2)$, $Q(x, y) = g(x, y^2)$, where f and g are arbitrary sufficiently smooth functions (and $g(0, 0) = 0$). Thus, we write a reversible system in a neighborhood of a symmetric equilibrium as follows:

$$\dot{x} = yf(x, y^2), \quad \dot{y} = g(x, y^2). \tag{2.1}$$

The linearization matrix A for the reversible vector field (2.1) at the symmetric equilibrium at zero has the form

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \tag{2.2}$$

where $\alpha = f(0,0)$ and $\beta = g'_x(0,0)$, and the eigenvalues of A are equal to $\lambda_{1,2} = \pm\sqrt{\alpha\beta}$. In general $\alpha\beta \neq 0$, and the equilibrium is either a saddle (when $\alpha\beta > 0$), or a center (if $\alpha\beta < 0$). In the latter case the eigenvalues are purely imaginary, which implies that the orbits from a small neighborhood of the equilibrium state must rotate around it, hence they must intersect the line $y = 0$ more than once. Since $y = 0$ is the line of the fixed points of the involution R, all these orbits are closed, i.e. the equilibrium is surrounded by a continuous family of periodic orbits and is a center indeed (in particular, it is Lyapunov stable).

The degenerate case corresponds to $\alpha\beta = 0$, then $\lambda_{1,2} = 0$ is a multiple eigenvalue. If $\alpha \neq 0$, then the function f does not vanish near the origin, so both the system itself and any reversible system close to it can have equilibrium states only on the line $y = 0$, i.e. only symmetric equilibrium states. Therefore, asymptotically stable equilibria cannot appear in this case. In fact, the system here is integrable in a neighborhood of zero: the integral curves are given by the equation

$$\frac{y^2}{2} = p(x, C),$$

where C is an arbitrary constant, and the smooth function p is defined as the solution of the Cauchy problem

$$\frac{dp}{dx} = \frac{g(x, 2p)}{f(x, 2p)}, \quad p(0, C) = C$$

(recall that $f \neq 0$). In other words, dynamics is always conservative in this case. In particular, if we have a non-degenerate parabolic equilibrium (i.e. $g'_x(0,0) = 0$ and $f(0,0) \neq 0$, $g''_{xx}(0,0) \neq 0$), then for any general one-parameter unfolding in the class of reversible systems the equilibrium either disappears or (when the parameter changes in the opposite direction) splits into two symmetric equilibria, a center and a saddle, with a symmetric homoclinic loop that surrounds the center, completely analogously to the bifurcations of a parabolic equilibrium in a Hamiltonian system.

The case $\alpha = 0$ is essentially non-Hamiltonian. As we consider only cases of codimension 1, we will further assume $\beta \neq 0$ in the matrix A. In this case $g'_x \neq 0$, so we can choose coordinates such that the identity $g(x, y^2) \equiv x$ holds. Then, an arbitrary reversible system close to (2.1) takes the form

$$\dot{x} = y(\varepsilon + ax + by^2 + O(x^2 + y^4)), \quad \dot{y} = x, \tag{2.3}$$

where ε is a small parameter. By scaling $y \to Y\delta$, $x \to X\delta^2$, $t \to \tau/\delta$ (for a small δ), the system is brought to the form

$$\dot{X} = Y(E + aX + bY^2 + O(\delta)), \quad \dot{Y} = X,$$

where the parameter $E = \varepsilon/\delta^2$ can run arbitrary finite values. It is easy to check that for a general choice of a and b (precisely, for $b > 0$ or $b < 0, a \neq 0, a^2 \neq 8|b|$, see below) the small terms $O(\delta)$ do not influence the bifurcations of the phase portraits as E varies. Thus, if we drop these terms and return to the non-rescaled variables, we will obtain the versal family (cf. [33]):

$$\dot{x} = y(\varepsilon + ax + by^2), \quad \dot{y} = x. \tag{2.4}$$

Bifurcations of the phase portrait are shown in Fig. 1. The symmetric equilibrium at zero does not disappear as ε varies, but it changes its stability type: the zero equilibrium is a center at $\varepsilon < 0$ and a saddle at $\varepsilon > 0$. No other symmetric equilibria appear here, however a pair of asymmetric equilibria is born from zero in the region $b\varepsilon < 0$ (these equilibria lie on the axis $x = 0$). At $b > 0$ both asymmetric equilibria are saddle, their separatrices bound a cell filled by periodic orbits. As we see, dynamics is conservative in the case $b > 0$ (eventhough the divergence of the vector field

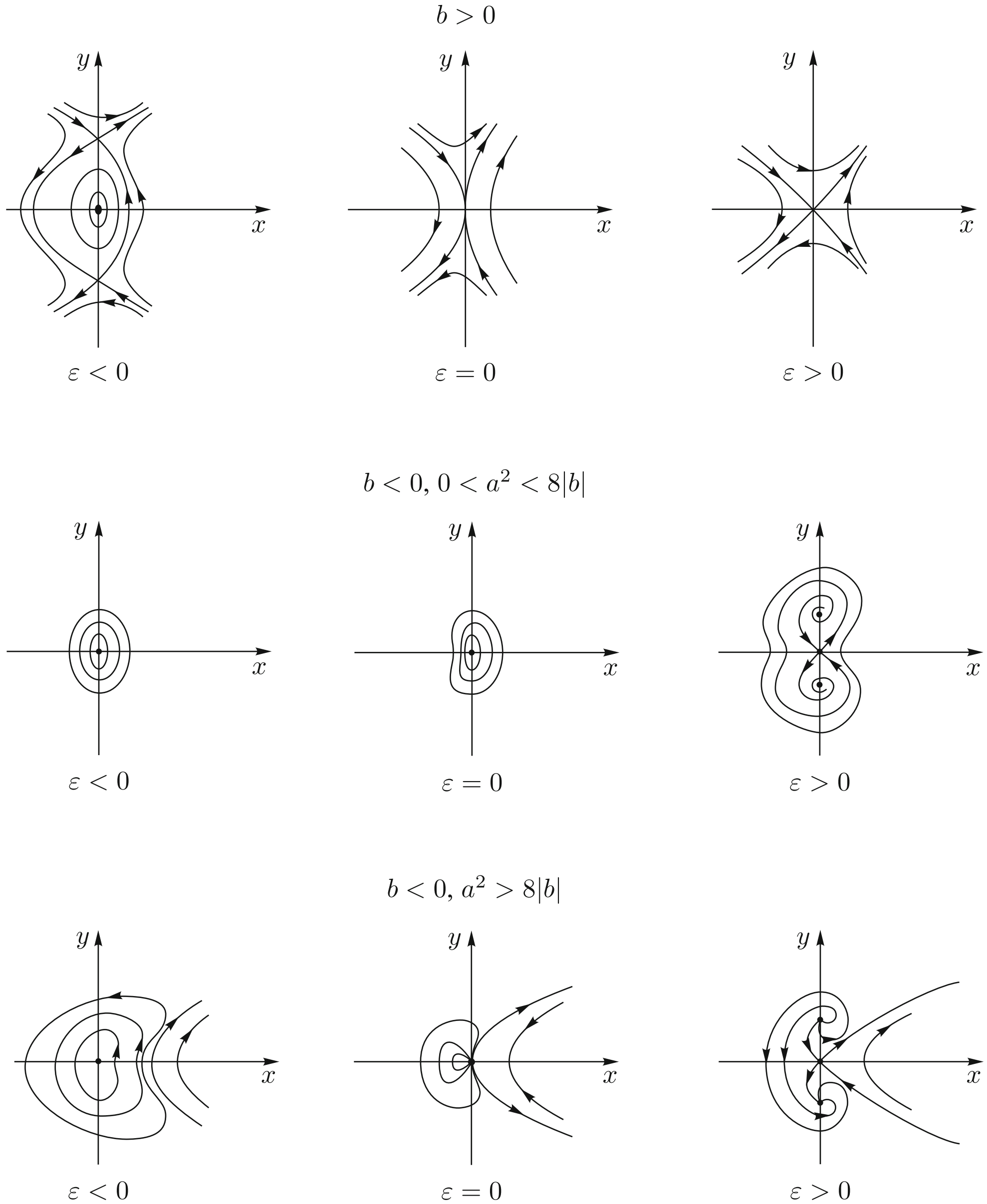

Fig. 1. Phase portraits of the normal forms for symmetry breaking in two-dimensional flows and orientation-preserving maps.

at the asymmetric saddles is non-zero at $a \neq 0$ and the integrating factor is, therefore, singular at these points). Really non-conservative dynamics emerges when the condition

$$b < 0 \tag{2.5}$$

is fulfilled. Here, if $a \neq 0$, one of the asymmetric equilibria that are born at $\varepsilon > 0$ is asymptotically stable and the other is asymptotically unstable. Note that the topological type of the equilibria that exist at $\varepsilon > 0$ is the same for all $a \neq 0$, however the phase portraits at $a^2 > 8|b|$ and at $0 < a^2 < 8|b|$ are different (in particular, the asymmetric equilibria are nodes at $a^2 > 8|b|$, and they become foci at $0 < a^2 < 8|b|$). The symmetric equilibrium also has different types at $\varepsilon = 0$: it is a center at $0 < a^2 < 8|b|$, while at $a^2 > 8|b|$ the zero equilibrium is unstable and has elliptic and hyperbolic sectors.

Bifurcations in one-parameter families of reversible vector fields on the plane were studied in [25], bifurcations in two-parameter families were studied in [39] (see also the recent book [19]); both papers used different versal families.

3. TWO-DIMENSIONAL REVERSIBLE DIFFEOMORPHISMS

Now let us consider analogues of the above described bifurcations for the case of two-dimensional reversible maps in a neighborhood of a symmetric fixed point. Such maps appear, in particular, as Poincaré maps in a neighborhood of a symmetric periodic orbit of a three-dimensional reversible vector field. One can always choose a cross-section to a symmetric periodic orbit γ in such a way that this cross-section will be R-invariant and will contain all points of the set $Fix(R)$ from a small neighborhood of the point of intersection of the cross-section with γ. Note that the Poincare map is orientation-preserving in this case (if the flow is defined on an orientable manifold). We, however, will also consider the case of reversible diffeomorphisms which do not preserve the orientation (such diffeomorphisms appear naturally as quotient maps in reversible systems with additional symmetries [8, 27]).

In general, if we have a symmetric periodic orbit of a reversible map T, then this orbit contains a fixed point of one of the involutions, R or $R' = T \circ R$. If q is the period of the orbit, then this point is a symmetric fixed point of the map T^q (this map is reversible with respect to both R and R', as the map T is). Therefore, the problem of studying symmetric periodic orbits of reversible maps is reduced to the study of symmetric fixed points.

A symmetric fixed point of a map T belongs to the intersection of $Fix(R)$ and $Fix(T \circ R)$. Since T is reversible with respect to both the involutions R and $T \circ R$, we can always choose $R' = T^k \circ R$ (for any integer k we want) to be a "primary" involution R. We will restrict this freedom by further assuming $dim(Fix(R)) \geqslant dim(Fix(T \circ R))$. In the two-dimensional case, if $dim(Fix(R)) = 2$, then $R = id$. In this case the reversible map T is an involution, i.e. $T^2 = id$, so its dynamics is trivial. If $dim(Fix(R)) = dim(Fix(T \circ R)) = 0$, then the existence of an intersection of $dim(Fix(R))$ and $dim(Fix(T \circ R))$ (the existence of a symmetric fixed point) is a phenomenon of codimension 2, and we do not consider such bifurcations here. Therefore, like in the case of reversible vector fields on a plane, we further assume $dim(Fix(R)) = 1$. In this case, the involution acts as $R\colon (x, y) \to (x, -y)$ in appropriate coordinates (x, y) near the fixed point $(0, 0)$.

The linearization matrix A for the reversible map T at the symmetric fixed point satisfies the relation $RAR = A^{-1}$. Therefore, if λ is an eigenvalue, then λ^{-1} is an eigenvalue too. Thus, generically, a symmetric fixed point of a reversible orientation-preserving two-dimensional map T is either an elliptic point ($\lambda_{1,2} = e^{\pm i\alpha}$, $\alpha \neq 0, \pi$), or an orientable saddle ($\lambda_{1,2} = \{\lambda, \lambda^{-1}\}$, $0 < \lambda < 1$), or a non-orientable saddle ($\lambda_{1,2} = \{\lambda, \lambda^{-1}\}$, $-1 < \lambda < 0$). For symmetric fixed points of non-orientable two-dimensional reversible maps the only possible set of multipliers is $\{\lambda_1 = 1, \lambda_2 = -1\}$.

Dynamics of a reversible map near an elliptic point reminds the dynamics near an equilibrium of the center type: a generic elliptic point is surrounded by a family of closed invariant KAM-curves [34], which occupy a set of almost full measure in a small neighborhood of the fixed point. On each KAM-curve the map is conjugate to an irrational rotation; as the distance to the fixed point increases, the rotation number changes monotonically, i.e. the rotation number is different for different KAM-curves. Note that the KAM-curves do not completely fill the neighborhood of the fixed point: rational rotation numbers correspond to resonant zones between the KAM-curves, and dynamics in these zones is, typically, chaotic. Note also that as the argument α of the complex eigenvalues $\lambda_{1,2}$ varies (when the map depends on a parameter), the elliptic point undergoes continuous bifurcations: when $\alpha/2\pi$ crosses a rational value p/q points of period q are born from the fixed point (and the corresponding resonant zone is formed). Eventhough all this reminds very much what happens near an elliptic fixed point of an area-preserving map, an essential difference exists: the resonant zones of a generic reversible map near an elliptic fixed point contain asymmetric asymptotically stable and asymptotically unstable periodic orbits [16].

The transition from an elliptic point to a saddle happens at parameter values which corresponds to a double eigenvalue $\lambda_{1,2} = 1$ or $\lambda_{1,2} = -1$. Generically (for one-parameter families of reversible diffeomorphisms), the double eigenvalue correspond to a single eigenvector, i.e. the matrix A is similar to 2-dim Jordan block. Let ξ be this eigenvector. By the reversibility, $A\xi = \lambda\xi$ implies $A(R\xi) = \lambda^{-1}R\xi$, and, since $\lambda = \lambda^{-1}$, it follows that either $R\xi = \xi$ or $R\xi = -\xi$. In the first case (like in the case of reversible vector fields with $\alpha \neq 0$ in (2.2)), bifurcations are analogous to the bifurcations of symplectic diffeomorphisms of a plane near a parabolic point. Thus, if $\lambda = 1$ and

$R\xi = \xi$, the map (and every smooth reversible map close to it) can be brought near the symmetric fixed point to the following form [28]:

$$\bar{x} = x + 2y + h_n(x+y) + \ldots, \qquad \bar{y} = y + h_n(x+y) + \ldots, \tag{3.1}$$

where h_n is a polynomial of order n, and the dots stand for terms of order $n+1$ and higher. Thus, up to terms of an arbitrarily high order, the map in a neighborhood of zero is approximated by an area-preserving map. In a generic one-parameter family, the function $h_n(u)$ is $h(u) = \varepsilon + au^2 + \ldots$, where $a \neq 0$. An analogous conservative normal form can be written in the case $\lambda = -1$:

$$\bar{x} = -x - 2y - h_n(x+y) + \ldots, \qquad \bar{y} = -y - h_n(x+y) + \ldots, \tag{3.2}$$

where h_n is an odd polynomial of order n. In a generic one-parameter family the function $h_n(u)$ takes the form $h(u) = \varepsilon u + au^3 + \ldots$, $a \neq 0$.

Note that eventhough we obtain conservative approximations up to an arbitrarily high order, the dynamics is not completely conservative. Near symmetric elliptic orbits which are born at these bifurcations the chaotic dynamics in the resonant zones is, generically, non-conservative (as we have already mentioned). The non-conservative behavior is also typical for the stochastic layer near the split separatrices (see below).

Consider now the case $R\xi = -\xi$. Note that if $\lambda = -1$, then $R\xi = -\xi$ implies $AR\xi = \xi$, where A is the linearization matrix of T at the fixed point. Therefore, in the coordinates where the involution $R' = T \circ R$ is linear, we have $R'\xi = \xi$, so the map is again brought to the conservative normal form (3.2) (recall that T is reversible with respect to R' as well). Thus, we are left with the case $R\xi = -\xi$ and $\lambda = 1$. It is well known that a map near a fixed point whose all multipliers are equal to 1 can be approximated, up to the terms of arbitrarily high order, by the time-1 shift along the orbits of a certain system of differential equations [38]. We call such system a flow normal form. Let $v(u)$ be a vector field (on the plane of variables $u = (x, y)$) such that the shift by the orbits defined by this vector field approximate the map T near a fixed point up to the terms of degree n. Then the shift by the orbits of the system defined by the vector field $w(u) = -Rv(Ru)$ approximates the map $RT^{-1}R$ up to the terms of order n too. If T is reversible, then $T = RT^{-1}R$; therefore the time-1 shift by the orbits defined by the vector field w approximates the time-1 shift by the orbits defined by the vector field v (and, hence, the map T) up to the terms of order n, hence the same is true for the vector field $(v + w)/2$. The latter vector field is reversible by construction. Therefore, we find that the reversible map near a symmetric fixed point with all multipliers equal to 1 has a reversible flow normal form up to any order [27].

A fixed point of the map corresponds to an equilibrium of the flow normal form, and the linearization matrix at the equilibrium is a logarithm of the linearization matrix of the original map at the fixed point (see more detail in [35]). So, the flow normal form for the reversible map with a pair of multipliers equal to 1 is given by (2.1). Moreover, the case $R\xi = -\xi$ corresponds to $\alpha = 0$ in (2.2), hence the normal form is given by system (2.3).

Thus, Fig. 1 shows us the bifurcations of the map under consideration. Of course, the orbits of the map do not coincide with the phase curves of the flow normal form, but they follow these curves sufficiently closely. In particular, the pair of asymptotically stable and asymptotically unstable equilibria in the case $b < 0$ corresponds to a pair of asymptotically stable and asymptotically unstable fixed points, i.e. we have here the symmetry-breakdown bifurcation. Saddle equilibria in the flow normal form correspond to saddle fixed points of the original map, symmetric centers correspond to elliptic fixed points, symmetric periodic orbits of the flow normal form correspond to KAM-curves (for irrational rotation numbers which satisfy Diophantine conditions).

An important difference with the case of vector fields is that the stable and unstable separatrices for a generic reversible map do not coincide (like they do in the flow normal forms in the cases $b > 0$ and $b < 0$, $a^2 > 8|b|$). Instead, the separatrices acquire transverse intersections which leads to a chaotic dynamics in their small neighborhood (the so-called chaotic layer). Moreover, in generic one-parameter families of such diffeomorphisms secondary tangencies of the stable and unstable separatrices appear. As a consequence, asymptotically stable, asymptotically unstable, and elliptic periodic orbits coexist in the stochastic layer (this is the so-called phenomenon of "mixed dynamics" for which the conservative and non-conservative behaviors are, in a sense, inseparable) [8, 26]. Thus, even in the case $b > 0$, when the dynamics of the flow normal form is conservative, the original map can exhibit non-conservative features.

The next case of the symmetry breaking bifurcation corresponds to the pair of multipliers $\lambda_1 = 1$, $\lambda_2 = -1$ (note that this bifurcation was reported in [8] for a system of three coupled oscillators [32]). In this case the reversible diffeomorphism T is orientation-reversing (since $\det(A) = \lambda_1\lambda_2 = -1$, where A is the linearization of T). Since we take the convention $\dim(Fix(T \circ R)) \leqslant \dim(Fix(R)) = 1$, it follows that $\dim(Fix(AR)) \leqslant 1$, and since $\det(AR) = \det(A)\det(R) = 1$, this implies $\dim(Fix(AR)) = 0$. Since AR is an involution, we have $AR = -id$, hence $A = -R$. Now, choose the coordinate axes such that x is the projection to the eigenvector that corresponds to the eigenvalue $\lambda_1 = 1$ and y is the projection to the eigenvector that corresponds to the eigenvalue $\lambda_2 = -1$. Then, the involution acts as $R : (x, y) \mapsto (-x, y)$, and the map T takes the form

$$\bar{x} = x + O(x^2 + y^2), \qquad \bar{y} = -y + O(x^2 + y^2).$$

Note that since $Fix(T \circ R)$ is a single point here, we can always perturb the map such that the line $Fix(R)$ would not pass though this point, i.e. we can make a perturbation within the class of reversible maps such that the symmetric fixed point would disappear. In order to analyze what happens at this bifurcation, we write a flow normal form. Namely, the linear part of the map AT is identity, which implies that T can be approximated up to terms of an arbitrarily high order by the product of the operator A to the time-1 shift of a reversible flow (the flow normal form). Moreover, the coordinates (x, y) can be chosen such (see [27]) that the vector field will be equivariant with respect to the multiplication by A (i.e. with respect to the transformation $y \to -y$ in our case). The equivariance implies, in particular, that the time-2 shift by the flow approximates the map T^2. The same conclusions are true for any reversible map close to T. The general form of the R-reversible and A-equivariant flow is given by

$$\dot{x} = f(x^2, y^2), \qquad \dot{y} = xyg(x^2, y^2). \tag{3.3}$$

An equilibrium of this system may correspond to a fixed point of the original map T only if it lies on the symmetry axis $y = 0$. A pair of equilibria which are symmetric to each other by the change $y \to -y$ correspond to a period-2 point of T. Thus, the existence of a symmetric fixed point is possible only when $f(0, 0) = 0$, i.e. this is a phenomenon of codimension 1. Generically, $g(0, 0) \neq 0$ at this bifurcation, so by scaling time we can make $g(x, y) \equiv 1$. We introduce a small parameter $\mu = f(0, 0)$ and study what happens as μ varies across zero. By retaining the leading order terms only, we obtain the flow normal form

$$\dot{x} = \mu + ax^2 + by^2, \qquad \dot{y} = xy. \tag{3.4}$$

We assume the coefficients a and b are non-zero. This gives 4 cases of the combinations of the signs of a and b. All four cases are easy to study, the corresponding phase portraits are shown in Fig. 2 (we show only the part $y \geqslant 0$ as the part $y \leqslant 0$ is symmetric).

The phase portrait for the original map T is obtained from that of flow normal form (3.4) by flipping around the axis $y = 0$, as well as by splitting the separatrices of the saddles (at $a < 0$, $\mu > 0$) and replacing the resonant closed curves (at $b < 0$, $\mu > 0$) by chaotic zones. Thus, the asymmetric equilibria (stable, unstable, saddle) on the axis $y = 0$ correspond to asymmetric fixed points of the same type in the map T. As we see, the attractor-repeller pairs of fixed points are born at this bifurcation if the condition

$$a > 0 \tag{3.5}$$

is satisfied. The R-symmetric equilibria on the axis $x = 0$ correspond to symmetric points of period 2 for the map T. The periodic orbits that surround centers correspond to KAM-curves of T (each curve consists of two closed components, one at $y > 0$, the other at $y < 0$, each taken into the other by T), provided the rotation number is Diophantine. Resonant zones and stochastic layers near the separatrices produce regions of mixed dynamics (like in the previous case at $b > 0, \varepsilon < 0$).

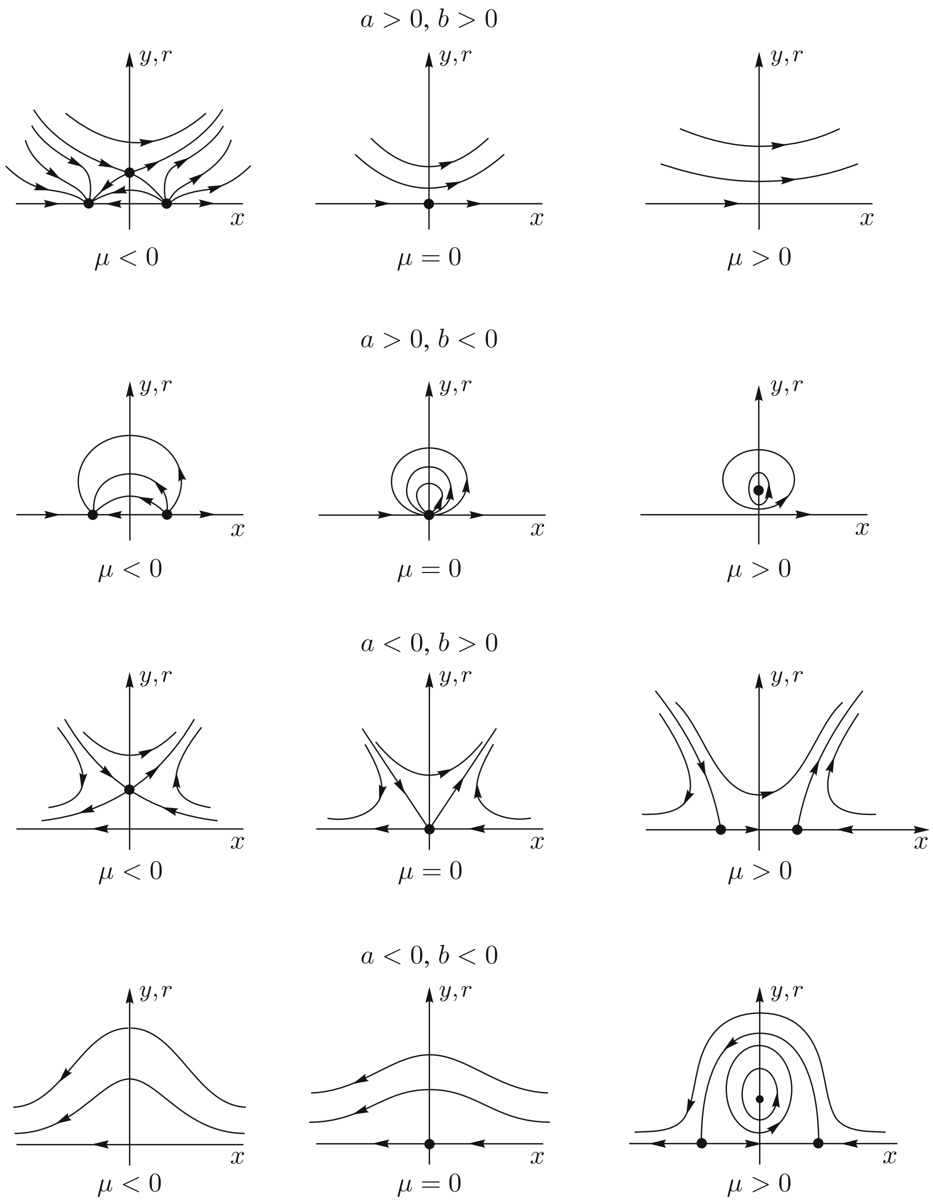

Fig. 2. Phase portraits of normal forms for symmetry breaking in orientation-reversing two-dimensional maps and three-dimensional flows.

4. REVERSIBLE 3- AND 4-DIMENSIONAL VECTOR FIELDS

Consider a three-dimensional reversible vector field v which has a symmetric equilibrium state O. Let O be at the origin of coordinates. By the Bochner–Montgomery theorem, coordinates (x, y, z) can be chosen such that the action of the involution R near the origin is linear. The linearization operator A of the vector field v at the point O satisfies $RA = -AR$. The spectrum of such operator is $(0, \lambda, -\lambda)$ (notice that $RA = -AR$ implies, as the dimension is odd, $\det(A) = -\det(A) = 0$).

Further we consider two different cases, depending on the dimension of the set of fixed points of the involution, the subspace $Fix(R)$. These are the cases $\dim(Fix(R)) = 1$ and $\dim(Fix(R)) = 2$ (as before, we do not consider the case $\dim(Fix(R)) = 0$ since the existence of an equilibrium state at the locally unique fixed point of the involution is a phenomenon of codimension-3 in the three-dimensional space; the case $\dim(Fix(R)) = 3$ is also not interesting, since the only vector field reversible with respect to the identity map is $v = 0$).

We start with the case $\dim(Fix(R)) = 1$ (an example of a reversible system of this type is given e.g. by a model of class-B lasers with injected signal [33]). We may assume that the involution acts as $R\colon (x,y,z)\to(-x,-y,z)$. Then the general form of a reversible vector field is

$$\dot x = f(x,y,z), \quad \dot y = g(x,y,z), \quad \dot z = xh_1(x,y,z) + yh_2(x,y,z), \tag{4.1}$$

where the functions $f, g, h_{1,2}$ are even with respect to (x,y) (i.e. they are R-invariant). The line $Fix(R)$ is the z-axis. The system has an equilibrium on this axis when $f(0,0,z)=0$ and $g(0,0,z)=0$. These are two equations for one variable z; hence the existence of a symmetric equilibrium in system (4.1) is a phenomenon of codimension 1. Therefore, we must consider bifurcations in one-parameter families of systems of type (4.1). Since we do not consider bifurcations of codimension 2 and higher, we are allowed to impose any number of additional inequality type conditions on the system at the moment of existence of the symmetric equilibrium. In particular, assuming that at the moment of bifurcation the symmetric equilibrium state is at zero, we will require

$$\frac{\partial(f,g)}{\partial z}(0,0,0)\neq 0. \tag{4.2}$$

Since f and g are even with respect to (x,y), the linearization matrix A of system (4.1) at zero is given by

$$\begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha \\ \beta_1 & \beta & 0 \end{pmatrix},$$

where $\alpha_1 = f_z(0,0,0)$, $\alpha = g_z(0,0,0)$, $\beta_1 = h_1(0,0,0)$ and $\beta = h_2(0,0,0)$. By (4.2), we have $\alpha_1^2 + \alpha^2 \neq 0$, and we can always choose the coordinates (x,y) such that $\alpha_1 = 0$ and $\alpha\neq 0$. We also assume $\beta\neq 0$. Then, by an additional change of the coordinate y, we make $\beta_1 = 0$. In this way, the system takes the form

$$\dot x = \mu + \dots, \quad \dot y = \alpha z + \dots, \quad \dot z = \beta y + \dots, \tag{4.3}$$

where μ is a small parameter, the dots stand for the second and higher order terms, even with respect to (x,y) in the equations for $\dot x$ and $\dot y$, and odd with respect to (x,y) in the equation for $\dot z$; the coefficients α and β are nonzero.

A symmetric equilibrium for such system exists at $\mu = 0$; the eigenvalues of the linearization matrix A are equal $0, \pm\lambda$ where $\lambda = \sqrt{\alpha\beta}$. If $\alpha\beta > 0$ (the quasi-hyperbolic case) the eigenvalues $(\lambda, -\lambda)$ are real and non-zero. In this case, in a neighborhood of zero there exists a smooth center manifold: a smooth invariant curve which, at $\mu = 0$, is tangent at zero to the x-axis (the eigendirection that corresponds to the zero eigenvalue). Transverse to the center manifold the behavior is saddle: we have a contraction in the direction which corresponds to the eigenvalue $-\lambda$ and expansion in the direction which corresponds to the eigenvalue λ; at all small μ all the orbits with the initial conditions outside the center manifold leave the neighborhood of zero as $t\to+\infty$ or as $t\to-\infty$. Thus, it remains to study the behavior on the center manifold. It is well known (see [19]) that the local center manifold of a reversible vector field near a symmetric equilibrium is invariant with respect to the involution R. In our case it means that the center manifold is a smooth invariant curve of the form $\{y = p(x,\mu), z = q(x,\mu)\}$, where p is odd and q is even (as functions of x). The system restricted on such curve is given by $\dot x = \psi(x,\mu)$ with the function ψ even with respect to x. In the generic case we have

$$\dot x = \mu + ax^2 + o(x^2),$$

where $a\neq 0$. Thus, system (4.3) has a saddle-saddle [35] equilibrium at $\mu = 0$. At $a\mu < 0$ this equilibrium disintegrates into two asymmetric saddles connected by a symmetric heteroclinic orbit, while at $a\mu > 0$ the equilibrium disappears. In other words, the bifurcation of a saddle-saddle in reversible three-dimensional vector fields in the case $\dim(Fix(R)) = 1$ does not differ from that in the case of general vector fields.

Note that the center-unstable manifold of the saddle-saddle at $\mu = 0$ is two-dimensional, therefore its unstable part (composed by the orbits which go out of the saddle-saddle at $t = -\infty$)

may intersect the line of the fixed points of R transversely at some point different from zero. The corresponding orbit is symmetric, hence it makes a homoclinic loop to the saddle-saddle, i.e. it belongs to the intersection of the unstable part of the center-unstable manifold with the stable part of the center-stable manifold. In general we may assume that this intersection is transverse. When the saddle-saddle disappears, a symmetric saddle periodic orbit is born from the homoclinic loop. If there are more than one symmetric homoclinic loops at the bifurcation moment, then the disappearance of the saddle-saddle leads to the birth of a non-trivial symmetric hyperbolic set. If the saddle-saddle has an asymmetric homoclinic loop, then there must exist another asymmetric homoclinic loop, symmetric to the first one. After the saddle-saddle disappears, a symmetric non-trivial hyperbolic set is born from this pair of loops too (see [36]).

Let us now consider the quasi-elliptic case. It corresponds to $\alpha\beta < 0$ in (4.3). Then, the linearization at the symmetric equilibrium at $\mu = 0$ has one zero and a pair of pure imaginary eigenvalues $\pm i\omega$ (where $\omega = \sqrt{-\alpha\beta}$). In this case the one-dimensional center manifold does not, typically, exist, so a complete reduction to a case of lower dimension cannot be conducted. Nevertheless, a sufficiently detailed analysis can be performed with the help of the normal form method: the non-resonant terms can be killed by normalizing transformations which do not alter the form of the involution R: $(x, y, z) \to (-x, -y, z)$. Thus, up to the terms of arbitrarily high order, system (4.3) is brought to the following normal form (cf. [19]):

$$\begin{cases} \dot{x} = \mu + f(x^2, y^2 + z^2), \\ \dot{y} = xyg(x^2, y^2 + z^2) - \Omega(x^2, y^2 + z^2)z, \\ \dot{z} = xzg(x^2, y^2 + z^2) + \Omega(x^2, y^2 + z^2)y, \end{cases} \tag{4.4}$$

where $f(0,0) = 0$, $\Omega(0,0) = \omega \neq 0$, and we will assume $g(0,0) \neq 0$. In the polar coordinates $y + iz = re^{i\phi}$ the system is written as

$$\dot{x} = \mu + p(x^2, r^2), \quad \dot{r} = xrg(x^2, r^2), \quad \dot{\phi} = \Omega(x^2, r^2) \neq 0,$$

and we see that the evolution of the variables (x, r) in the normal form is independent of ϕ, while the evolution of the ϕ-variable is trivial. The system for variables (x, r) is reversible with respect involutions $(x, r) \to (-x, r)$. To the main order (after an additional rescaling) this system transforms into

$$\dot{x} = \mu + ax^2 + br^2, \quad \dot{r} = xr, \tag{4.5}$$

which coincides with normal form (3.4), so the phase portraits can be found in Fig. 2. Note also that system (4.5) (augmented by an equation $\dot{\phi} = \Omega$ for the phase ϕ) is a partial case of normal forms for bifurcations of equilibria with eigenvalues $(0, \pm i\omega)$ which appear in families of general and divergence-free vector fields, see in [6, 11, 12, 18, 23].

The phase portraits of (4.4) are obtained from the phase portraits of system (4.5) by the rotation around the x-axis. The asymmetric equilibria of system (4.5) at $r = 0$ correspond to asymmetric equilibria in system (4.4), and their stability type remains the same (in fact, saddles become saddle-foci: saddles which are stable on the x-axis acquire a two-dimensional unstable manifold, while saddles which are unstable on the x-axis acquire a two-dimensional stable manifold). Symmetric equilibria (saddles and centers) on the axis $(x = 0, r \neq 0)$ correspond to symmetric periodic orbits of system (4.4). Symmetric periodic orbits of system (4.5) (the case $b < 0, \mu > 0$) correspond to symmetric two-dimensional invariant tori of (4.4). At small perturbations that preserve the reversibility (in particular, at the transition from the truncated normal form to the full original system), symmetric equilibria and periodic orbits of system (4.5) persist. Symmetric two-dimensional invariant tori can be destroyed, however a set of almost full measure filled by two-dimensional KAM-tori remains (see [34]). The heteroclinic connection (the segment of the x-axis) between the two asymmetric saddle-foci in the case $a < 0, \mu > 0$ does not persist in the full system (which, unlike normal form (4.4), does not need to possess a rotational symmetry), but it may exist for a discrete sequence of values of the parameter μ: this connecting orbit is symmetric and appears when the one-dimensional unstable separatrix of the saddle-focus intersects the one-dimensional line $Fix(R)$, so it is a codimension-1 phenomenon here. Note also that in the case $a < 0$, $b < 0$,

$\mu > 0$ the two-dimensional separatrices of the saddle-foci coincide in the normal form (4.4). This is a degeneracy of infinite codimension (due to the rotational symmetry), therefore in the original system we should expect the splitting of the two-dimensional separatrix, and transverse intersections should emerge. Moreover, the one-dimensional separatrices may lie onto the two-dimensional ones, i.e. similar to how it happens in the case of bifurcations of an equilibrium with the eigenvalues of the linearization $(0, \pm i\omega)$ in vector fields of general type [18, 37] homoclinic loops of saddle-foci may appear, and Shilnikov chaos emerges.

Like in the case of a saddle-saddle, it would be interesting in this case to consider global bifurcations at the disappearance of the symmetric quasi-elliptic equilibrium, which are related to possible homoclinic orbits. In the non-reversible situation such bifurcations were considered in [1, 13, 22].

Consider now the case $\dim(Fix(R)) = 2$. Without loss of generality we assume that the involution acts as $R\colon (x, y, z) \to (x, y, -z)$, and the general form of the smooth reversible vector field is

$$\dot{x} = zf(x, y, z^2), \quad \dot{y} = zg(x, y, z^2), \quad \dot{z} = h(x, y, z^2). \tag{4.6}$$

Before we start studying bifurcations let us make several general remarks on the behavior of systems of type (4.6). First, any orbit (unless it is an equilibrium) which intersects the plane $z = 0$ filled with fixed points of the involution must intersect this plane transversely. Thus, for any symmetric periodic orbit, i.e. for such orbit which intersects $z = 0$ twice, all close orbits will also intersect $z = 0$ twice, hence all of them are symmetric periodic orbits. Therefore, symmetric periodic orbits of system (4.6) always fill open regions. Another observation is that symmetric equilibria here are defined by a single relation $h(x, y, 0) = 0$, i.e. such equilibria (in general) fill curves on the plane $z = 0$. The zero eigenvalue corresponds to the direction tangent to such line. In the directions transverse to the line of the equilibria the typical behavior is either saddle (at $h'_x f + h'_y g > 0$) or of a center type (at $h'_x f + h'_y g < 0$). All the orbits near the line of centers rotate around this line, hence they intersect the plane $z = 0$, i.e. all of them are symmetric periodic. The line of saddle equilibria is the intersection its two-dimensional stable and unstable manifolds. If some orbit from the unstable manifold intersects the plane $z = 0$, then, by the reversibility, this orbit also belongs to the stable manifold and forms a homoclinic loop. Moreover, since the intersection of this orbit with $z = 0$ is always transverse, all close orbits in the unstable manifold must also intersect this plane, i.e. the symmetric homoclinic loops always form one-parameter families. Obviously, the surfaces formed by the symmetric homoclinic loops bound the regions filled by symmetric periodic orbits.

As we see, in order to study systems of form (4.6), it is necessary to find the lines of symmetric equilibria, to determine regions filled by the symmetric periodic orbits, and to understand how the symmetric orbits coexist in the phase space with asymmetric orbits. The latter do not intersect the plane $z = 0$, hence they lie entirely on one side of this plane. By the reversibility, the behavior in the region $z < 0$ is completely recovered by the behavior in the region $z > 0$. In order to study the orbits that lie at $z > 0$, we may make the change of variables and time: $z^2 = u$ and $z\,dt = d\tau$. The system takes the form

$$\dot{x} = f(x, y, u), \quad \dot{y} = g(x, y, u), \quad \dot{u} = 2h(x, y, u).$$

In principle, an arbitrary and very complicated behavior is possible here. However, as we will see, in the case of symmetry-breaking bifurcation of codimension 1 this system is linear in the main order, and its non-wandering set consists of a single equilibrium. This allows one to conduct a sufficiently complete study of this system.

Let us now discuss bifurcations of symmetric equilibria in system (4.6). Since these equilibria fill the curves, the type of the equilibria can change along the curve. In other words, bifurcations can happen here even when the system does not depend on parameters (cf. [10]). Nevertheless, we are primarily interested in the symmetry-breakdown bifurcation, and it turns out to be a phenomenon of codimension 1 in the given class of systems. Indeed, let $(x_0, y_0, 0)$ be an equilibrium state (i.e. $h(x_0, y_0, 0) = 0$). If one of the functions f, g does not vanish at the point $(x_0, y_0, 0)$, then the system is integrable in a neighborhood of this point. Indeed, let $f(x_0, y_0, 0) \neq 0$, for example. Then we can write the system (4.6) near $(x_0, y_0, 0)$ in the form

$$\frac{dy}{dx} = \frac{g(x, y, u)}{f(x, y, u)}, \quad \frac{du}{dx} = 2\frac{h(x, y, u)}{f(x, y, u)},$$

where $u = z^2$. Let $u = p(x, C_1, C_2)$, $y = q(x, C_1, C_2)$ be the solution of the Cauchy problem $y(x_0) = C_1$, $u(x_0) = C_2$ for this equation. Near $y = y_0$, $u = 0$, $x = x_0$ values C_1, C_2 are smooth functions of $(x, y, u = z^2)$, and they are, obviously, integrals of system (4.6). In particular, we can take C_1 as the new variable y. Then system will take the form

$$\dot{x} = zf(x, y, z^2), \quad \dot{y} = 0, \quad z^2 = p(x, y, C).$$

Here, on each invariant plane $y = const$, we have an integrable system, i.e. the problem is reduced to the study of bifurcations in families of two-dimensional conservative systems.

Thus, in order to have a symmetry breaking bifurcation, we must require $f(x_0, y_0, 0) = 0$ and $g(x_0, y_0, 0) = 0$. Along with the condition $h(x_0, y_0, 0) = 0$, this makes three conditions on two variables x_0 and y_0, i.e. we indeed deal here with the degeneracy of codimension 1. Without loss of generality we set $x_0 = y_0 = 0$. Note that the linearization matrix of system (4.6) at the symmetric equilibrium has the following form when the conditions $f = g = 0$ are fulfilled:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{pmatrix}.$$

This matrix has two independent eigenvectors, $\mathbf{e} = (\beta, -\alpha, 0)$ and $\mathbf{f} = (0, 0, 1)$, which correspond to the zero eigenvalue, moreover

$$R\mathbf{e} = \mathbf{e}, \quad R\mathbf{f} = -\mathbf{f} \tag{4.7}$$

(if $\alpha = \beta = 0$, take $\mathbf{e} = (1, 0, 0)$). As we see, the existence of a pair of eigenvectors that correspond to the zero eigenvalue and satisfy condition (4.7) is a necessary condition for the symmetry breaking bifurcation.

Impose additional inequality type condition on the system near zero:

$$(h'_x)^2 + (h'_y)^2 \neq 0.$$

We can always choose the coordinates (x, y) such that $h'_y \neq 0$. Then we can choose the variable y such that $y \equiv h(x, y, z^2)$. Under this condition, we require $g'_x \neq 0$, which allows for choosing the variable x such that $x \equiv g(x, y, z^2)$. System (4.6) takes the form

$$\dot{x} = z(\varepsilon + ax + by + cz^2 + O(x^2 + y^2 + z^4)), \quad \dot{y} = zx, \quad \dot{z} = y, \tag{4.8}$$

where ε is a small parameter. We require $c \neq 0$. Then, by scaling the variables z and y, we can make $c = -1$. Thus, in the main order, the bifurcation under consideration is described by the following normal form:

$$\dot{x} = z(\varepsilon + ax + by - z^2), \quad \dot{y} = zx, \quad \dot{z} = y. \tag{4.9}$$

At $\varepsilon > 0$ this system has a pair of asymmetric equilibria outside the plane $z = 0$:

$$O_\pm = (0, 0, \pm\sqrt{\varepsilon}).$$

The linearization matrix at O_+ equals

$$\begin{pmatrix} a\sqrt{\varepsilon} & b\sqrt{\varepsilon} & -2\varepsilon \\ \sqrt{\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is given by

$$\lambda^3 - a\sqrt{\varepsilon}\lambda^2 - b\varepsilon\lambda + 2\varepsilon\sqrt{\varepsilon}.$$

We obtain that the equilibrium O_+ is asymptotically stable at

$$a < 0, \quad b < 0, \quad ab > 2. \tag{4.10}$$

The point O_- is asymptotically unstable in the same domain of the parameters a, b. For the values of a and b from the interior of the complement to domain (4.10) the equilibrium states $O_\pm$ are saddles (O_+ has a two-dimensional unstable manifold, and O_- has a one-dimensional one).

Note that the dynamics of normal form (4.9) (and, hence, the dynamics of the original system in a sufficiently small neighborhood of the equilibrium state under consideration) is sufficiently simple. Indeed, let $z > 0$. After the change $u = z^2$, $z\,dt = d\tau$ system (4.9) becomes linear:

$$\dot{x} = \varepsilon + ax + by - u, \quad \dot{y} = x, \quad \dot{u} = 2y.$$

Thus, for a, b from the region

$$\{ab \neq 2\} \cup \{a \geqslant 0\} \cup \{b \geqslant 0\}, \tag{4.11}$$

those orbits of this system which do not tend to infinity must tend to the equilibrium state $(x = 0, y = 0, u = \varepsilon)$. For system (4.9) this means that its orbits either tend to infinity (for the original system this means leaving a certain fixed neighborhood of zero), or tend to an equilibrium state, or intersect the plane $z = 0$. By the reversibility, the same is true for the orbits from the region $z < 0$. Since an orbit which intersects $z = 0$ twice must be periodic (recall that $z = 0$ is the set of fixed points of the involution), we obtain that if a, b belong to region (4.11), then every bounded orbit either is symmetric periodic or tends to an equilibrium state. It follows that the phase space is decomposed into regions filled either by symmetric periodic orbits, or by the orbits which connect the repeller O_- with the attractor O_+ (at $\varepsilon > 0$ and a, b from region (4.10)), or by unbounded orbits; the boundaries of these regions are formed by lines of symmetric equilibria and by the separatrices of the saddles (symmetric and, if exist, asymmetric).

In conclusion we consider an example of a codimension-1 bifurcation that causes the breakdown of symmetry in four-dimensional reversible vector fields (following [19, 42]). Here we assume that $\dim(Fix(R)) = 2$ and the linearization matrix A of the vector field at a symmetric equilibrium has a double zero eigenvalue (with a Jordan box), and a pair of pure imaginary eigenvalues. We consider the case where there exists a basis $\xi_0, \xi_1, \xi_2, \xi_3$ such that

$$\begin{gathered} A\xi_0 = 0, \quad A\xi_1 = \xi_0, \quad A\xi_2 = -\omega\xi_3, \quad A\xi_3 = \omega\xi_2, \\ R\xi_0 = -\xi_0, \quad R\xi_1 = \xi_1, \quad R\xi_2 = \xi_2, \quad R\xi_3 = -\xi_3 \end{gathered}$$

(one can check that such basis exists for an open set of reversible systems for which the matrix A has a double zero eigenvalue). In the coordinates $z\xi_0 + y\xi_1 + u\xi_2 + v\xi_3$, the involution acts as $R(z, y, u, v) = (-z, y, u, -v)$. By killing resonant terms and dropping terms of high orders, we obtain the following shortened normal form [19]:

$$\begin{cases} \dot{z} = y, \\ \dot{y} = z(\mu + ay + bz^2 + c(u^2 + v^2)), \\ \dot{u} = -v\omega + zu, \\ \dot{v} = \omega u + zv. \end{cases} \tag{4.12}$$

By introducing the cylindrical coordinates $(z, y, x\cos\theta, x\sin\theta)$, we obtain the following three-dimensional autonomous system for the variables (x, y, z):

$$\begin{cases} \dot{x} = xz, \\ \dot{z} = y, \\ \dot{y} = z(\varepsilon + ay + bz^2 + cx^2), \end{cases} \tag{4.13}$$

(the equation for θ is just $\dot{\theta} = \omega$). This system has an invariant plane $x = 0$, and at $b\varepsilon < 0$ a pair of asymmetric equilibria with $z \neq 0$ is born in this plane. These equilibria form an attractor-repeller pair at

$$a > 0, \quad b < 0. \tag{4.14}$$

Like system (4.9), the normal form (4.13) may have regions filled by symmetric periodic orbits. By taking into account the rotation in θ, we find that these periodic orbits correspond to symmetric two-dimensional invariant tori of system (4.12). In the original system we, thus, obtain a region where a set of almost full measure is filled by two-dimensional KAM-tori.

5. REVERSIBLE 3-DIMENSIONAL DIFFEOMORPHISMS

A symmetric periodic point of a reversible diffeomorphism is a fixed point of the Poincare map T; this map is reversible with respect to a certain involution R, and the fixed point O is also a fixed point of R. By the definition of reversibility, $(T \circ R)^2 = id$, i.e. $R' = T \circ R$ is also an involution, and the fixed point O of the map T and the involution R is also a fixed point of R', i.e. $O \in Fix(R) \cap Fix(R')$ (the converse is also true). As before, we assume $\dim(Fix(R)) \geqslant \dim(Fix(R'))$. Note that in the case where the reversible diffeomorphism is obtained as a Poincare map of a reversible flow we always have $\dim(Fix(R)) = \dim(Fix(R'))$, we however will also consider cases where $\dim(Fix(R)) > \dim(Fix(R'))$ (e.g. the case of an orientation-reversing map T).

If $\dim(Fix(R)) < 2$ and $\dim(Fix(R')) = 0$, then the existence of a symmetric fixed point of a three-dimensional map is a phenomenon of codimension at least 2, and we do not consider such degenerate cases here. Therefore, we will assume either $\dim(Fix(R)) = \dim(Fix(R')) = 1$, or $2 = \dim(Fix(R)) \geqslant \dim(Fix(R'))$ (if $\dim(Fix(R)) = 3$, then $R = id$ by the Bochner–Montgomery theorem, so the map T must be an involution itself in this case, and its dynamics is trivial).

We start with the case where the sets $Fix(R)$ and $Fix(T \circ R)$ are both one-dimensional. As we mentioned, this is the case when a four-dimensional vector field, reversible with respect to an involution such that $\dim(Fix(R)) = 1$ has a symmetric periodic orbit γ (an example of such situation is given by the models of a rattleback, or Celtic stone [3, 17]). Then, by choosing a cross-section to γ such that it contains $Fix(R)$ and is invariant with respect to R, we obtain the case under consideration; the intersection of γ with the cross-section is a symmetric fixed point of the Poincaré map.

Since $Fix(R)$ and $Fix(T \circ R)$ are one-dimensional, the existence of their intersection (the symmetric fixed point O) in the three-dimensional space is a phenomenon of codimension 1 (in a generic one-parameter family of reversible diffeomorphisms the smooth curves $Fix(R)$ and $Fix(T \circ R)$ depend smoothly on the parameter). As before, we will assume that the involution R acts linearly in a small neighborhood of O. Let A be the linearization matrix of the map T at O. Note that $\dim(Fix(R)) = 1$ implies $\det(R) = 1$, and $\dim(Fix(T \circ R)) = 1$ implies $\det(AR) = 1$, hence $\det(A) = 1$, i.e. the diffeomorphism T is orientation-preserving. By the reversibility, if λ is an eigenvalue of A (i.e. λ is a multiplier of O), then λ^{-1} must also be an eigenvalue. For a three-dimensional orientation-preserving map, this implies that one of the multipliers must be equal to 1. Since the curves $Fix(R)$ and $Fix(T \circ R)$ are not tangent at the point O in the case of codimension 1, we assume that the eigenvector $\mathbf{e}$ that corresponds to the unit multiplier does not lie in $Fix(R)$. It follows that $R\mathbf{e} = -\mathbf{e}$ (since $AR\mathbf{e} = R\mathbf{e}$ by the reversibility, the vector $\mathbf{f} = R\mathbf{e} + \mathbf{e}$ satisfies both $AR\mathbf{f} = \mathbf{f}$ and $R\mathbf{f} = \mathbf{f}$, i.e. it belongs to the intersection of $Fix(R)$ with the tangent to $Fix(T \circ R)$, hence it is zero).

The other two multipliers form a pair λ and λ^{-1}. If λ is real and differs from ± 1, the center manifold is one-dimensional, and O is a saddle-saddle [23, 35]. Like in the case of a saddle-saddle equilibrium (see the previous Section), if there are no additional degeneracies, then we have here practically the same bifurcation scenario as in the case of general (i.e. not reversible) diffeomorphisms. Namely, the map on the center manifold is topologically conjugate to the time-1 shift by the orbits of the flow

$$\dot{x} = \mu + x^2 + \ldots,$$

reversible with respect to the involution $x \to -x$; at $\mu < 0$ we have two symmetric fixed points, these points collide at $\mu = 0$ and disappear at $\mu > 0$. Note that similar to the case of a saddle-saddle equilibrium, if the stable and unstable manifolds of the saddle-saddle fixed point have a transverse intersection along some curve, then non-trivial partially hyperbolic sets may be born as the saddle-saddle disappear (see [2, 20]).

The second case, $\lambda = e^{\pm i\omega}$, is more interesting. If ω/π is irrational (or rational, $\omega/(2\pi) = m/n$, but the denominator n is large enough), then the diffeomorphism near the fixed point admits an approximation by a flow normal form up to terms of a sufficiently high order (up to terms of order $n - 1$) (cf. [27, 38]). The linear part of the flow normal form is

$$\dot{x} = 0, \qquad \dot{y} = \omega z, \qquad \dot{z} = -\omega y,$$

and the involution R acts as

$$x \to -x, \quad y \to -y, \quad z \to z.$$

Nonlinear terms must be resonant and satisfy the reversibility condition. In this way we obtain the normal form given by (4.4). As we explained, the three-dimensional phase portraits for this normal form are obtained from those shown in Fig. 2 by the rotation around the x-axis. The equilibria of system (4.4) correspond to the fixed points of the original map; in particular, the attractor-repeller pair of equilibria corresponds to the attractor-repeller pair of fixed points in the original map. Symmetric two-dimensional invariant tori of normal form (4.4) (the closed curves in Fig. 2) correspond to symmetric two-dimensional KAM-tori of the original map (at Diophantine sets of rotation numbers; the resonant zones, in analogy to the case of two-dimensional maps, should correspond to a chaotic dynamics of mixed type, i. e. to coexistence of conservative and non-conservative dynamical regimes). The symmetric saddle periodic orbit of system (4.4) (in Fig. 2, this is the saddle equilibrium on the line $(x = 0, r \neq 0)$) corresponds to a normally-hyperbolic symmetric closed invariant curve of the original map (normally-hyperbolic manifolds persist at small perturbations [14], therefore this curve persists when the flow normal form is replaced by the original map). The symmetric elliptic periodic orbit of system (4.4) (in Fig. 2, this is the center equilibrium on the line $(x = 0, r \neq 0)$) corresponds to a symmetric closed invariant curve of the original map for a discrete set of values of the parameter μ, which correspond to Diophantine rotation numbers; at other parameter values this curve may disintegrate into resonances (we do not study a detailed picture of this effect here).

Now, consider the case $\dim(Fix(R)) = 2$. Let O be a symmetric fixed point of the map T, and let A be the linearization of T at O. By the reversibility, if λ is an eigenvalue of A, then λ^{-1} must also be an eigenvalue, which means that A has at least one multiplier equal to 1 or -1. Let the two other multipliers be different from ± 1. Then the spectrum of A is either $(\lambda, \lambda^{-1}, 1)$ (in the case of orientation-preserving T) or $(\lambda, \lambda^{-1}, -1)$ (in the case of orientation-reversing T). The image of the plane $Fix(R)$ by the map T is a two-dimensional smooth surface. It follows from our assumption that ± 1 is a simple eigenvalue, that $T(Fix(R))$ must intersect $Fix(R)$ transversely. Indeed, if a vector $\mathbf{e}$ is tangent to both $Fix(R)$ and $T(Fix(R))$, then $R\mathbf{e} = \mathbf{e}$ and $RA^{-1}\mathbf{e} = A^{-1}\mathbf{e}$. Since, by the reversibility, $A^{-1} = RAR$, we obtain $A^2\mathbf{e} = \mathbf{e}$. Therefore, would $T(Fix(R))$ be tangent to $Fix(R)$, then the plane $Fix(R)$ would be invariant with respect to the action of A, with $A^2 = id$ on $Fix(R)$, which implies that A would have at least two (hence all) eigenvalues equal to ± 1 in this case.

The line of transverse intersection of $T(Fix(R))$ and $Fix(R)$ is a smooth curve ℓ, to which O belongs. By construction, this curve is the set of all points M in a neighborhood of O such that $RM = M$ and $RTM = TM$. Since $T^{-1} = R \circ T \circ R$, it follows that $RT^2M = T^2M = M$ for every $M \in \ell$. In other words, the curve ℓ is invariant with respect to T, and $T^2 = id$ on ℓ. We have therefore two possibilities: either the curve ℓ is filled by symmetric fixed points of T, then the tangent vector to ℓ is the eigenvector of the linearization matrix A that corresponds to the eigenvalue $+1$, or $\ell \backslash O$ is filled by symmetric orbits of period two, then the tangent vector to ℓ is the eigenvector that corresponds to the eigenvalue -1 (i.e. this happens in the case of orientation-reversing T).

If the multipliers (λ, λ^{-1}) are real, then the fixed point O is called quasi-hyperbolic. All the points of ℓ near O are also quasi-hyperbolic in this case. Thus, the line ℓ is a normally-hyperbolic invariant manifold, and it has two-dimensional stable and unstable manifolds. If (λ, λ^{-1}) lie on the unit circle, then O is a quasi-elliptic point. The behavior near ℓ in this case is described by the KAM-theory [34]. Here, the set of almost full measure is filled by closed invariant KAM-curves (if T is orientation-reversing, then each each KAM-curve consists of two connected components, one mapped to the other by the map T). In analogy to the case of elliptic fixed points of two-dimensional map, in the resonant zones near the quasi-elliptic line ℓ we expect a chaotic behavior and the violation of conservativity (a mixed dynamics).

When the Poincaré map T is orientation-preserving, the line ℓ is filled by the fixed points. When moving along this curve, a transition is possible between the points of quasi-hyperbolic and quasi-elliptic types (see e.g. [9]). This corresponds to $\lambda = \pm 1$ (the third multiplier corresponds to the eigen-direction tangent to the line of the fixed points and also equals to 1). We note that the transition of the multiplier across ± 1 happens without a change in the parameters of the system, i.e. we have here a "bifurcation of codimension 0". Such bifurcation with $\lambda = 1$ was studied in [31] for the case where the matrix A has a full Jordan box (i.e. A has no eigenvectors other than the tangent to ℓ). It was shown in [31] that near the transition point the symmetric quasi-hyperbolic points have symmetric homoclinic orbits. In fact, the behavior here is pretty much analogous to the behavior of the Poincaré map of a Hamiltonian system near a parabolic periodic orbit, i.e. the dynamics is essentially conservative (still, in the chaotic layer near the homoclinics one should again expect a mixed dynamics). Similar results can be obtained when the transition from quasi-elliptic to quasi-hyperbolic points happens as the multiplier λ becomes double and equal to -1, and a Jordan box corresponds to it. Crossing a resonance ($\lambda = e^{\pm 2\pi i \frac{p}{q}}$) when moving along the line of quasi-elliptic points is, in the absence of additional degeneracies (i.e. in the codimension 0 situation), also similar to the Hamiltonian case.

Thus, one should expect the symmetry-breaking bifurcation in cases of codimension 1 at least. We will not study here degenerate (codimension 1) bifurcations of resonant quasi-elliptic points. This leaves us with the case where all the eigenvalues of the linearization matrix A are equal to ± 1. By [27], the map T near the fixed point admits an arbitrarily good approximation by the time-1 shift of an R-reversible flow (the flow normal form) multiplied by S, the semi-simple part of the matrix A (the diagonal part of A in the Jordan basis). Moreover, the flow normal form is symmetric with respect to the action of S.

As we discussed in the previous Section, bifurcations of a symmetric equilibrium of an R-reversible flow with $dim(Fix(R)) = 2$ can lead to a non-conservative dynamics only if the linearization matrix has at least two linearly independent eigenvectors which correspond to the zero eigenvalue, and these eigenvectors must satisfy (4.7). Thus, the only possibility for the flow normal form to produce a symmetry-breaking bifurcation is the case where the matrix A has two eigenvectors that satisfy (4.7). Moreover, when A has an eigenvalue (-1), the linear operator S is non-trivial, so the symmetry with respect to S further restricts the dynamics of the flow normal form. In particular, the flow normal form is reversible with respect to the involution SR. This implies one more necessary condition for a possible non-conservative behavior in the flow normal form: the matrix A must have two eigenvectors which satisfy conditions (4.7) with R replaced by SR.

The vector $\mathbf{f}$ in (4.7) such that $R\mathbf{f} = -\mathbf{f}$ is unique (since $\dim(Fix(R)) = 2$). As we just explained, we must require $A\mathbf{f} = \pm\mathbf{f}$. In fact, one may check that if $A\mathbf{f} = -\mathbf{f}$, then the flow normal form produces only conservative dynamics in cases of codimension 1 (the symmetry with respect to S prevents of the non-conservative behavior here, unless additional degeneracy appears). Therefore, we further assume $A\mathbf{f} = +\mathbf{f}$. We will always choose coordinates such that the vector $\mathbf{f}$ will coincide with the z-axis, so the involution R acts as $R(x, y, z) = (x, y, -z)$. It is easy to see, that now there remain only three cases of the matrices A which satisfy $(AR)^2 = id$ and conditions (4.7) (with the involutions R and SR), so that after an appropriate choice of the coordinates (x, y) we can bring A to one of the following forms:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}. \tag{5.1}$$

In the first case, the operator S is identity, so the flow normal form (4.6) has no additional symmetries, hence it can be brought to the form (4.8) at $\beta \neq 0$. The non-degeneracy condition $\beta \neq 0$ corresponds to the case of codimension 1. Under this condition the surfaces $Fix(R)$ (i.e. $z = 0$) and $T(Fix(R))$ intersect transversely. Thus, this bifurcation corresponds to a symmetric fixed point that lies on the curve $\ell = Fix(R) \cap T(Fix(R))$ and separates the quasi-elliptic and quasi-hyperbolic segments of ℓ; this point corresponds specifically to a triple eigenvalue 1 with the additional degeneracy (that makes it a bifurcation of codimension 1), an incomplete Jordan box. Now we can use the analysis of system (4.8) which was conducted in the previous Section. In particular, we find that under condition (4.10) the given bifurcation leads (in the original map) to the birth of a pair of asymptotically stable and asymptotically unstable fixed points $O_\pm$ from the curve ℓ of symmetric fixed points. The regions in the phase space of the flow normal form which are filled by the orbits connecting the repeller O_- with the attractor O_+ correspond to the regions of the same behavior in the original map. The regions filled by symmetric periodic orbits corresponds, in the original map, to the regions where a set of large measure is filled by symmetric KAM-curves.

In the second case we have $AR = -id$, so $\dim(Fix(T\circ R)) = \dim(Fix(AR)) = 0$, i.e. $Fix(T\circ R)$ consists of a single point. This point is a symmetric fixed point of T only if it belongs to the plane $Fix(R)$, so the existence of a symmetric fixed point is an event of codimension 1 for the three-dimensional maps. Note that since $\dim(Fix(T\circ R)) \neq \dim(Fix(R)) = 2$, the diffeomorphism under consideration is not a Poincaré map of a 4-dimensional reversible flow, eventhough the map T is orientation-preserving. The flow normal form here is reversible with respect to the involution $R: z \to -z$ and symmetric with respect to $S: (x,y) \to -(x,y)$. The general form of such system is given by

$$\begin{aligned}\dot{x} &= z(xc_{11}(x^2,xy,y^2,z^2) + yc_{12}(x^2,xy,y^2,z^2)),\\ \dot{y} &= z(xc_{21}(x^2,xy,y^2,z^2) + yc_{22}(x^2,xy,y^2,z^2)),\\ \dot{z} &= h(x^2,xy,y^2)\end{aligned}$$

(cf. (4.6)). By retaining terms up to the second order only, we obtain the following normal form (for a proper choice of the coordinates (x,y)):

$$\left\{\begin{aligned}&\frac{d}{dt}\begin{pmatrix}x\\ y\end{pmatrix} = zC\begin{pmatrix}x\\ y\end{pmatrix},\\ &\frac{d}{dt}z = \varepsilon + ax^2 + by^2 - z^2,\end{aligned}\right. \tag{5.2}$$

where C is a matrix with constant coefficients, and ε is a small parameter that governs the bifurcation (in particular, the equilibrium at zero that corresponds to the symmetric fixed point of T disappears at $\varepsilon \neq 0$). Non-zero R-symmetric equilibria at $z = 0$ correspond to symmetric points of period 2 in the original map T (recall that the time-1 shift by the flow of (5.2) approximates the composition $S\circ T$). These equilibria are given by the equation

$$\varepsilon + ax^2 + by^2 = 0.$$

At $ab > 0$ this line is a closed curve born from zero at $a\varepsilon < 0$, while at $a\varepsilon > 0, b\varepsilon > 0$ the R-symmetric equilibria (the symmetric points of period 2 in the map T) disappear. In the case $ab < 0$ the set of symmetric period-2 points is a pair of lines intersecting at the symmetric fixed point $(x,y) = 0$ at $\varepsilon = 0$; at $\varepsilon \neq 0$ this set becomes a pair of hyperbolas. The change in the structure of the set of symmetric points of period 2 is accompanied by the birth, at $\varepsilon > 0$, of a pair of asymmetric fixed points — they correspond to the pair of asymmetric equilibria of system (5.2) on the z-axis. These points are an attractor-repeller pair if the spectrum of the matrix C lies strictly to the left of the imaginary axis, i.e. at

$$det(C) > 0, \qquad tr(C) < 0. \tag{5.3}$$

In the last case, $A = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, the map T is orientation-reversing. The flow normal form must be reversible with respect to $R: z \to -z$, i.e. it must have form (4.6). Moreover, it must be symmetric with respect to $S: x \to -x$, which gives us the following system:

$$\dot{x} = zxf(x^2,y,z^2), \quad \dot{y} = zg(x^2,y,z^2), \quad \dot{z} = h(x^2,y,z^2).$$

This system has an equilibrium $(0,y_0,0)$ with a triple zero eigenvalue if $h(0,y_0,0) = 0$ and $g(0,y_0,0) = 0$. It is two conditions on one variable (y_0), i.e. we have here a bifurcation of codimension 1. We always may assume $y_0 = 0$. In order not to increase the codimension of the bifurcation, we impose the non-degeneracy conditions $f(0,0,0) \neq 0$ and $h'_y(0,0,0) = \beta \neq 0$. Then, by scaling time, we can make $f \equiv 1$, and we can also choose the y-variable such that $h \equiv y$. After that, the system to the main order coincides with normal form (4.13). Therefore, the bifurcation under consideration (double multiplier 1 for a symmetric fixed point of an orientation-reversing reversible map) leads to a breakup of symmetry under condition (4.14).

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