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Universal Dynamics in a Neighborhood of a Generic Elliptic Periodic Point

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Abstract—We show that a generic area-preserving two-dimensional map with an elliptic periodic point is C^ω-universal, i.e., its renormalized iterates are dense in the set of all real-analytic symplectic maps of a two-dimensional disk. The results naturally extend onto Hamiltonian and volume-preserving flows.

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Generically, periodic orbits of a smooth area-preserving two-dimensional map are either hyperbolic (saddle) or elliptic depending on the values of their multipliers. The multipliers of an elliptic periodic point belong to the unit circle, $\lambda_{1,2} = e^{\pm i\omega}$, and are not real. According to KAM-theory, when the map is C^r with $r = 5, \dots, \infty, \omega$ the generic elliptic periodic point is surrounded by closed smooth invariant curves filled by quasiperiodic orbits. The KAM-curves cover the most part of a small neighborhood of the elliptic point, however between any two KAM curves there are resonant zones with at least two "subharmonic" periodic orbits inside each one. Generically, one of this orbits is saddle and its stable and unstable manifolds intersect transversely along a homoclinic orbit [1]. In other words, while for a majority of initial conditions the behavior near a generic elliptic point is quasiperiodic, there are also thin zones of chaotic behavior which accumulate to such point.

In this paper we show that the chaotic dynamics in the resonant zones is ultimately rich and *complex beyond comprehension*. In order to give a rigorous meaning to this statement we use the notion of C^r*-universality* introduced in [2]. The definition is based on the following construction.

Let $f : \mathcal{M} \to \mathcal{M}$ be an area-preserving C^r-map of a two-dimensional symplectic manifold $\mathcal{M}$ and $\mathcal{D}_1 \subset \mathbb{R}^2$ be the unit disk. Consider an arbitrary C^r-embedding $\psi : V \to \mathcal{M}$ of an open neighborhood V of $\mathcal{D}_1$ into $\mathcal{M}$ and assume that the ratio of the area of image to the area of preimage is constant, i.e., $area_{\mathcal{M}}(\psi(B))/area(B)$ is the same for every disc $B \subset V$. If $f^n(\psi(\mathcal{D}_1)) \subset \psi(V)$ for some $n \in \mathbb{N}$, we can define a *renormalized iteration* of f by

$$f_{n,\psi} = \psi^{-1} \circ f^n \circ \psi|_{\mathcal{D}_1}.$$

Obviously the map $f_{n,\psi} : \mathcal{D}_1 \to V$ is area-preserving and of class C^r. The set of all possible renormalized iteration is called the *dynamical conjugacy class* of f.

Note that since the transformations ψ can contract areas (with the requirement that the contraction rate should be constant), the discs $\psi(\mathcal{D}_1)$ can be arbitrarily small (and they can be situated anywhere in $\mathcal{M}$). Thus, the dynamical conjugacy class of f contains information about the behavior of arbitrarily long iterations of f on arbitrarily fine spatial scales.

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We say that the map f is C^r*-universal* if its dynamical conjugacy class is *dense* in the C^r-topology among *all* symplectic maps $\mathcal{D}_1 \to \mathbb{R}^2$.

In this case the dynamics of f is extremely rich as its iterations approximate arbitrarily well (in appropriately chosen coordinates) every two-dimensional symplectic map. This property is a manifestation of the utter degree of dynamical complexity.

The main result of the paper is the following

Theorem 1. *C^r-generic maps with an elliptic periodic point are C^r-universal.*

In other words, in the class of area-preserving maps which have elliptic periodic points, the universal maps form a residual subset, i.e., they belong to an intersection of at most countably many open and dense subsets, in the topology of C^r-uniform convergence on compacta. In the case of real-analytic maps, i.e., when $r = \omega$, this can be clarified as follows: given any C^ω-map f with an elliptic periodic point O we will take any, arbitrarily large compact neighborhood $K \subseteq M$ of the point O and take an open complex neighborhood Q of K such that f is analytic in some open neighborhood of $cl(Q)$. Take a sufficiently small neighborhood $\mathcal{O}$ of f in the space of real-analytic maps with the topology of uniform convergence in $cl(Q)$. We will show that a generic map from $\mathcal{O}$ is C^ω-universal.

Note that it is enough to prove that the universal maps form a dense subset — the genericity will follow automatically. Indeed, let $g_1, g_2, \ldots$ be a countable dense subset of the set of all symplectic maps $\mathcal{D}_1 \to \mathbb{R}^2$. Let U_m be the set of all area-preserving maps f of $\mathcal{M}$ for which the dynamical conjugacy class intersects with an open 2^{-m}-neighborhood of each of the maps $g_1, \ldots, g_m$, i.e., there exist numbers $n_1, \ldots, n_m$ and coordinate transformations $\psi_1, \ldots, \psi_m$ such that the corresponding renormalized iterations $f_{n_1,\psi_1}, \ldots, f_{n_m,\psi_m}$ satisfy $\|f_{n_1,\psi_1} - g_1\| < 2^{-m}, \ldots, \|f_{n_m,\psi_m} - g_m\| < 2^{-m}$. The sets U_m are open by definition. Moreover the intersection $\bigcap_{m=1}^{\infty} U_m$ coincides with the set $\mathcal{U}$ of all universal maps. Thus $\mathcal{U}$ is an intersection of a countable sequence of open sets, i.e., we have proved that when $\mathcal{U}$ is dense, it is also residual.

The proof of Theorem 1 is heavily based on the results of [3], where it was shown that *universal area-preserving maps exist in any neighborhood of any area-preserving map with a homoclinic tangency*. Therefore it remains to show that given any area-preserving map with an elliptic point, *a homoclinic tangency can be created by an arbitrarily C^r-small ($r = 1, \ldots, \infty, \omega$) perturbation* within the class of area-preserving maps.

In the non-analytic case this claim, as we will see, easily follows from the integrability of the normal form near the elliptic point and similar statements can be found in [4]. The case of real-analytic maps (i.e., $r = \omega$) is more involved.

Let f be an area-preserving map and O be a period-m point of f. We assume that O is elliptic and not strongly resonant, i.e., the eigenvalues of the derivative of f^m at O are $e^{\pm i\nu}$ with $0 < \nu < \pi$ and $\nu \notin \{2\pi/3, \pi/2\}$. We can always imbed f into a one-parameter family of area-preserving maps for which ν will change monotonically as the parameter varies. So we may assume from the very beginning that $\nu = \pi p/q$ where p, q are mutually prime integers and $q \geqslant 3$. A standard fact from normal form theory is that one can choose canonical polar coordinates near O in such a way that the map $T := f^m$ will take the form

$$T = R_{\pi p/q} \circ \mathcal{F} + o(r^q), \tag{1}$$

where $R_{\pi p/q}$ stands for the rotation to the angle $\pi p/q$, and $\mathcal{F}$ is the time-1 map of the flow defined by the Hamiltonian

$$H(r, \varphi) = -\mu r + \sum_{j=1}^{q-1} \frac{\Omega_j(\mu)}{j+1} r^{j+1} + A(\mu) r^q \cos(2q\varphi) \tag{2}$$

with some coefficients $\Omega_j(\mu)$; the small parameter μ is given by $\mu = \pi p/q - \nu$. In order to be reducible to the form (1), the map T should be at least C^{2q-1} locally. It is possible that our original map f is not sufficiently smooth to ensure that property. In this case we note that a finitely smooth family can be locally approximated by a C^∞-smooth one without loosing the area-preserving property. Therefore we may always assume that it is at least C^∞ in a neighborhood of each point of the orbit of the periodic point O without loosing in generality.

We can also assume $\Omega_1(0) \neq 0$ and $A(0) \neq 0$. These inequalities can be achieved by an arbitrarily small perturbation of the map f (a precise description of the perturbation construction can be found *e.g.* in [3, 5]). Consider values of μ such that $\mu\Omega_1(0) > 0$ and make the linear coordinate transformation $r \mapsto \rho$:

$$r = r_0 + 2qr_0^{q-1}\rho/\Omega_1,$$

where $r_0 \approx \mu/\Omega_1(0)$ solves the equation $\mu = \sum_{j=1}^{q-1} \Omega_j(\mu) r_0^j$. Then the vector field defined by (2) takes the form

$$\dot{\rho} = 2qr_0^{q-1}A(0)\sin(2q\varphi) + o(r_0^{q-1}), \qquad \dot{\varphi} = 2qr_0^{q-1}\rho + o(r_0^{q-1}),$$

i.e., in these coordinates the map f^m takes the form

$$T = R_{\pi p/q} \circ \mathcal{T}_\mu + o(\mu^{q-1}), \tag{3}$$

where $\mathcal{T}_\mu$ is the shift for the time $2qr_0^{q-1} \sim \mu^{q-1}$ by the flow of

$$\dot{\rho} = \sin(2q\varphi), \qquad \dot{\varphi} = \rho. \tag{4}$$

This system is invariant with respect to the rotation $R_{\pi p/q}$ and has $2q$ saddle equilibria $P_k^0 : (\rho = 0, \varphi = \pi k/q)$, $k = 0, \ldots, 2q-1$. All P_k^0 belong to one orbit of $R_{\pi p/q}$, hence they form one saddle periodic orbit of the map $R_{\pi p/q} \circ \mathcal{T}_\mu$. The multipliers of this orbit are of order $\sim \mu^{\frac{q-1}{2}}$, so an $o(\mu^{q-1})$-perturbation will not destroy the saddle, that is the map T for all small μ has a period-$(2q)$ saddle orbit $\mathcal{P}_\mu$ composed of points $P_k(\mu)$ such that $P_k(\mu) \to P_k^0$ as $\mu \to 0$, and the multipliers of this orbit are of order $\sim \mu^{\frac{q-1}{2}}$. The stable and unstable invariant manifolds $W_k^s(\mu)$ and $W_k^u(\mu)$ of the points $P_k(\mu)$ depend continuously on μ and tend, as $\mu \to 0$, to the corresponding stable and unstable manifolds, l_k^s and l_k^u, of the equilibria P_k^0 of system (4). System (4) has a Hamiltonian: $H_0 = \frac{1}{2}\rho^2 + \frac{1}{2q}(\cos(2q\varphi) - 1)$, and the union of the curves l_k^s and l_k^u forms the zero level of H_0. Namely, l_k^u is given by the equation

$$\rho = \xi_k^0(\varphi) := \operatorname{sign}(\varphi - \pi k/q)\sqrt{\frac{1}{q}(\cos(2q\varphi) - 1)}, \quad |\varphi - \pi k/q| < \pi/q,$$

and l_k^s is given by $\rho = -\xi_k^0(\varphi)$. Thus, given any $\delta > 0$, a curve $L_k^u(\mu)$ of the form $\rho = \xi_k^u(\varphi, \mu)$, where ξ_k^u is defined on the segment $|\varphi - \pi k/q| < \pi/q - \delta$, represents a piece of the unstable manifold $W_k^u(\mu)$ passing through $P_k(\mu)$, and $\xi_k^u(\varphi, \mu) \to \xi_k^0(\varphi)$ on this segment as $\mu \to 0$. A passing through $P_k(\mu)$ piece $L_k^s(\mu)$ of the stable manifold $W_k^s(\mu)$ is a curve $\rho = \xi_k^s(\varphi, \mu)$ where ξ_k^s is defined at $|\varphi - \pi k/q| < \pi/q - \delta$ and tends to $-\xi_k^0$ on this segment.

As the curve l_k^u coincides with l_{k+1}^s at $\pi k/q < \varphi < \pi(k+1)/q$, it follows that the curve $L_k^u(\mu)$ is close to $L_{k+1}^s(\mu)$ at small μ, and the preservation of the area implies the existence of intersections of $L_k^u(\mu)$ and $L_{k+1}^s(\mu)$ for all small μ. These are homoclinic intersections, i.e., the orbits of the intersection points tend to $\mathcal{P}$ both at forward and at backward iterations of the map T. Note that if we drop the $o(r^q)$-terms in (1), the curves $L_k^u(\mu)$ and $L_{k+1}^s(\mu)$ will coincide for small μ (as they will correspond to the same level lines of Hamiltonian (2)). In other words, we will have orbits of a homoclinic tangency (of infinite order) in this case. Since making $o(r^q)$-terms locally vanish is a C^q-small perturbation, and we can always choose arbitrarily large q, this gives us homoclinic tangencies by arbitrary small smooth perturbations of f. This complete the proof of Theorem 1 for smooth maps.

The above argument with dropping the $o(r^q)$-terms does not work in the analytic category. Therefore in this case we have to choose another line of reasoning. Namely, we will show that there is a tending to zero sequence of values of μ for which a homoclinic tangency between the stable and unstable manifolds of $\mathcal{P}_\mu$ exists. Indeed, if this is not the case, then all the intersections between $W^u_k(\mu)$ and $W^s_{k+1}(\mu)$ are transverse for all sufficiently small μ, hence every such intersection depends smoothly on μ. Pick one of the intersection points of $L^u_k(\mu)$ and $L^s_{k+1}(\mu)$. Every point in the orbit Γ_μ of this point corresponds to a transverse intersection of the stable and unstable manifolds of $\mathcal{P}_\mu$, so each of the points of Γ_μ depends smoothly on μ. Choose any $\varphi_0 \in (\pi k/q, \pi(k+1)/q)$. Denote as Q^0_μ that point in Γ_μ which lies on $L^u_k(\mu)$ between the point $M_1 = (\varphi = \varphi_0, \rho = \xi^u_k(\varphi, \mu))$ and the point $M_2 = T^q M_1$ (the line $L^u_k(\mu)$ is invariant with respect to T^q). Let Q^+_μ and Q^-_μ be the two nearest to Q^0_μ points of (transverse) intersection of $L^u_k(\mu)$ and $L^s_{k+1}(\mu)$, one to one side from Q^0_μ and the other to the other side, i.e., Q^-_μ belongs to the piece of $L^u_k(\mu)$ between $P_k(\mu)$ and Q^0_μ while Q^+_μ belongs to the piece of $L^s_{k+1}(\mu)$ between Q^0_μ and $P_{k+1}(\mu)$. By construction, all forward iterations of the points $Q^\pm_\mu$ by the map T^q lie in $L^s_{k+1}(\mu)$ between Q^+_μ and $P_{k+1}(\mu)$, while all their iterations by T^{-q} lie in $L^u_k(\mu)$ between $P_k(\mu)$ and Q^-_μ. Denote the segment of $L^u_k(\mu)$ between Q^-_μ and Q^+_μ as $w^u(\mu)$ and the segment of $L^s_{k+1}(\mu)$ between Q^-_μ and Q^+_μ as $w^s(\mu)$. As we just explained, neither of the iterations of the points $Q^\pm_\mu$ lie in $w^s \cup w^u$. Note that both the segments w^s and w^u have length of order $\sim \mu^{q-1}$, as the points $Q^\pm_\mu$ are no farther from Q^0_μ than the points $T^{-q}Q^0_\mu$ and $T^{+q}Q^0_\mu$ and the distance between the latter points is $O(\mu^{q-1})$, see (3).

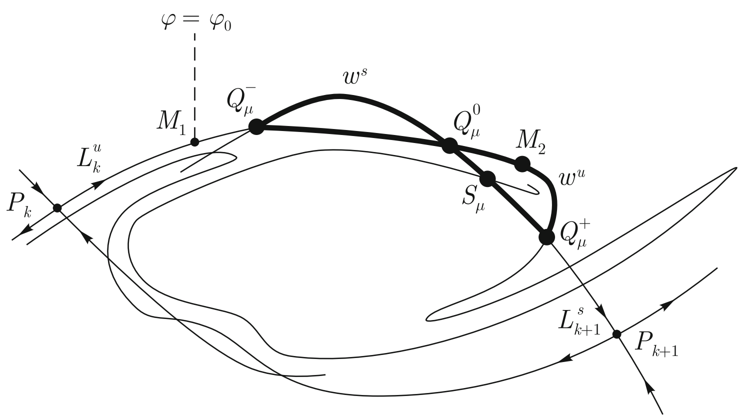

Figure. Illustration to the proof of Theorem 1.

Choose some sufficiently small μ_0. There is a positive integer j such that $T^j(w^u(\mu_0)) \cap w^s(\mu_0) \neq \emptyset$ (by Poincaré recurrence theorem, as the union $w^u(\mu_0) \cup w^s(\mu_0)$ bounds a set of positive measure, some forward iterate of this set should intersect it, giving rise to the desired intersections, for details see e.g. [6]). Fix any such j. Let S_μ be a point of intersection of $T^j(w^u(\mu))$ with $w^s(\mu)$. By construction, S_μ cannot be an end point of either of the curves w^s and $T^j(w^u)$ (otherwise, one of the iterations of a point Q^+_μ or Q^-_μ would get into $w^s \cup w^u$, which is not the case, as we mentioned). Recall also that the intersection of the curves w^s and $T^j(w^u)$ at S_μ is transverse (this point corresponds to a homoclinic orbit, and since we assume that there are no homoclinic tangencies for all small μ). Thus, the intersection cannot disappear as μ changes.

Indeed, both the curves w^s and w^u remain in a bounded region of the phase plane and, as we mentioned, their length remains bounded, hence so does the length of $T^j(w^u)$ (recall that j is fixed and independent of μ). Therefore, by transversality, the intersection point S_μ will persist for all μ for which the curves w^s and $T^j(w^u)$ depend on μ continuously. We do have the continuous dependence for those values of μ for which the point Q^0_μ lies in the interior of the segment (M_1, M_2). At the moment the point Q^0_μ arrives at the boundary of the segment $[M_1, M_2]$ and leaves it, the image of Q^0_μ by either T^q or T^{-q} enters the segment $[M_1, M_2]$ and becomes the new point Q^0_μ. This means that both the segments w^u and w^s are replaced by either $T^q(w^u)$ and $T^q(w^s)$ or $T^{-q}(w^u)$ and $T^{-q}(w^s)$. Accordingly, the point S_μ of the intersection of $T^j w^u$ with w^s changes to $T^q S_\mu$ or

$T^{-q}S_\mu$ (and the corresponding homoclinic orbit remains the same). Thus, at each μ between 0 and μ_0 we have a point of intersection of $T^j(w^u)$ and w^s, with the same $j > 0$.

However, it follows from (3) that since the map $\mathcal{T}_\mu$ is $O(\mu^{q-1})$-close to identity, the first j iterations of the map T are $O(\mu^{q-1})$-close to the iterations of $R_{\pi p/q}$. Thus, if μ is small enough, the curve $T^j(w^u)$ will be close to $R_{\pi jp/q}w^u$. Hence, in order to have an intersection of $T^j(w^u)$ with w^s, the number of iterations j must be a multiple of $2q$, and the arc $T^j(w^u)$ will be a part of the curve $L_k^u(\mu)$. This is a contradiction: as $j > 0$, the arc $T^j(w^u)$ must lie in $L_k^u(\mu)$ between Q_μ^+ and P_{k+1}, i.e., it cannot intersect w^s.

Thus, in a generic family where the multipliers $e^{\pm i\nu}$ of the elliptic point change monotonically, every parameter value is a limit of a sequence of parameter values which correspond to homoclinic tangencies to some saddle periodic orbit. This completes the proof of our theorem. □

Note also that given any area-preserving map with an elliptic periodic point and any finite number of orbits of homoclinic tangency we can imbed it into a family of area-preserving maps in which the multipliers of the elliptic point will change monotonically, and neither of the tangencies will split [3, 5]. Thus, we can perturb our map in such a way that new tangencies will be born one by one, without destroying the homoclinic tangencies obtained on previous steps. This proves that arbitrarily C^r-close ($r = 1, \ldots, \infty, \omega$) to any map with an elliptic point there exists a map with infinitely many orbits of a homoclinic tangency which accumulate to the elliptic point. By [3] these tangencies can be of arbitrarily high orders. By [7], an arbitrarily small C^ω perturbation of a homoclinic tangency (within the class of area-preserving maps) can create a wild hyperbolic set [8], so by combining our construction with the results of [7] we find that a C^ω-generic elliptic point is accumulated by wild hyperbolic sets. As a corollary, one also obtains that *the two following open domains in the space of two-dimensional area-preserving C^ω maps have equal closure: the set of maps which have an elliptic periodic point and the set of maps which have a wild hyperbolic set (the Newhouse domain).*

Two main sources of area-preserving maps are Hamiltonian and volume-preserving flows. A Hamiltonian flow on a 4-dimensional symplectic manifold is volume-preserving on every energy level, and the Poincaré map near a periodic orbit of a volume-preserving flow on a 3-dimensional manifold is a two-dimensional area-preserving map. An analogue of Theorem 1 for the flows reads as follows.

Theorem 2. *For a C^r-generic volume-preserving flow on a 3-dimensional manifold the Poincaré map near every elliptic periodic orbit is C^r-universal. For a generic Hamiltonian system on a 4-dimensional symplectic manifold there exists a residual set of energy values such that on every corresponding energy level the Poincaré map near every elliptic periodic orbit is C^r-universal.*

As the number of periodic orbits for a generic volume-preserving flow (as well as for a generic Hamiltonian flow on every energy level) is at most countable (see e.g. [5]), it is enough to show the generic universality of the Poincaré map near a single elliptic periodic orbit. In fact, in the non-analytic case ($r = 1, \ldots, \infty$), Theorem 2 follows directly from Theorem 1, as it is obvious that any C^r-small perturbation of the Poincaré map which is localized in a disc of a small radius can be obtained via a C^r-small perturbation of the flow, and if the perturbed map remains area-preserving, then the corresponding perturbation of the flow will keep the flow volume-preserving (or Hamiltonian if we work in the Hamiltonian setting). However, this is not true in the real-analytic case: an analytic map which is defined locally (in a neighborhood of the intersection of the elliptic periodic orbit under consideration with some local cross-section) is not necessarily generated by any flow which is analytic on the whole of the phase manifold. To overcome the problem, one has to repeat for the flow case the arguments we made for 2-dimensional maps. In fact, to carry on the reasoning made in the proof of Theorem 1 and in [3] one just needs to be able to imbed any real-analytic volume-preserving or Hamiltonian flow X_0 into an analytic finite-parameter family X_ε of volume-preserving or, respectively, Hamiltonian flows such that given any fixed number of orbits of homoclinic tangency and periodic orbits (saddle and elliptic) of the original flow X_0, every homoclinic tangency can be split in a generic way (and without splitting the rest of the tangencies) as parameters ε vary, and the multipliers of the periodic orbits can as well be changed without splitting any of the tangencies. Such construction is done (as a part of a general scheme) in [5]. After that, the arguments of [3] are applied with no modifications. □

It is commonly believed that a generic 2-degrees-of-freedom Hamiltonian system which, at least for one energy value, is not Anosov must have an elliptic periodic orbit. The proof is known only for the C^1-case [9], and no approach is known up to date for the really interesting case of a higher smoothness. However, elliptic periodic orbits do appear easily in Hamiltonian systems. Indeed, they always exist near generic elliptic equilibria (the so-called Lyapunov families), i.e., near a minimum of the Hamilton function. Far from integrability, elliptic periodic orbits appear via bifurcations of homoclinic loops (see e.g. [10, 11] for the bifurcation of a homoclinic loop to a saddle-focus in Hamiltonian and volume-preserving systems), via homoclinic tangencies [3, 7, 9, 12, 13], in slow-fast systems [14], in billiard-like potentials for high energies [15, 16], etc. In essence, Theorem 2 shows that incomprehensibly complex behavior on very long time scales should be typical for basically all 2-degrees-of-freedom Hamiltonian systems that appear in natural applications.

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