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On Newhouse Domains of Two-Dimensional Diffeomorphisms Which are Close to a Diffeomorphism with a Structurally Unstable Heteroclinic Cycle

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It is known that a rapid development of the qualitative theory of multidimensional dynamical systems began in 1960s, which was stimulated, to a large extent, by the works of Anosov and Smale who laid the foundations for the hyperbolic theory. It was discovered, in the same years, that in contrast to two-dimensional vector fields, structurally unstable multidimensional fields can form domains in the space of dynamical systems. Smale [1] was the first to point out that fact. He constructed an example of three-dimensional diffeomorphism where structural instability was present on a wandering set, i.e., a structurally unstable one-dimensional manifold of a fixed saddle point unremovably touched the stable foliation of Anosov's torus. Somewhat later, the researchers discovered open domains of systems in which the instability was concentrated on nonwandering sets. Here we must, first of all, point out domains of everywhere dense structural instability connected with homoclinic tangencies (Newhouse domains [2, 3]) as well as systems with Lorenz attractors [4, 5]. However whereas only two invariants, kneading invariants, are required in a nonsymmetric case (one in a symmetric case) in order to describe Lorenz attractors [6], the situation is considerably more complicated in Newhouse domains [7, 8, 9], namely, infinitely many invariants (in particluar, the so-called Ω-moduli [10, 11]) are required. The materialization of the latter fact is that in Newhouse domains systems with a countable set of periodic motions of any order of generation are dense as well as systems with a countable set of homoclinic tangencies of any order. Another important characteristic property of systems in Newhouse domains is the property of coexistence of a countable set of periodic orbits of different topological types. As applied to two-dimensional diffeomorphisms, this property manifests itself as follows: in Newhouse domains connected with a homoclinic tangency of a fixed saddle point diffeomorphisms which, along with the saddle periodic orbits, have a countable set of stable (completely unstable) periodic orbits if the saddle value σ of the fixed point is smaller than unity (larger than unity) are everywhere dense. Here $\sigma = |\lambda\gamma|$, where λ and γ are eigenvalues of the mapping linearized at a fixed point.

In this work, we consider two-dimensional diffeomorphisms with structurally unstable heteroclinic cycle which contains fixed saddle points and heteroclinic orbits. We assume that exactly one of the latter orbits is structurally unstable and, along it, a stable and an unstable manifold have a quadratic tangency. When the saddle values of all fixed points of the cycle are simultaneously less than or larger than unity, the resluts do not differ, in principle, from the known results in the case of diffeomorphisms with a homoclinic tangency. However, if there are at least two fixed points in a cycle, one of which has a saddle value larger than unity, then a new phenomenon appears, namely, there are Newhouse domains in the vicinity of the diffeomorphism with such a cycle where the diffeomorphisms which have simultaneously a countable set of saddle orbits, a countable set of saddle orbits, and a countable set of completely unstable periodic orbits are dense. In their totality, these orbits are "unseparable" from one another since the closures of both the sets of stable and of the sets of completely unstable periodic orbits also contain saddle periodic orbits of nontrivial hyperbolic subsets.

Note that these statements are also valid for general one-parameter families of two-dimensional diffeomorphisms and three-dimensional flows. The last circumstance is especially important for problems of nonlinear dynamics since these new phenomena can be found in dynamical models with an alternating divergence (for instance, in Chua's circuits, see [12]).

Since homoclinic tangencies naturally appear upon small smooth perturbations of a diffeomorphism with a structurally unstable heteroclinic cycle, we shall first give a short review of some results connected with homoclinic bifurcations.

1. A SHORT REVIEW OF HOMOCLINIC BIFURCATIONS IN THE CASE OF TWO-DIMENSIONAL DIFFEOMORPHISMS

Let a C^r-smooth ($r \geq 2$) two-dimensional diffeomorphism g_0 have a structurally stable fixed saddle point O, whose stable W_0^s and unstable W_0^u manifolds have a quadratic tangency at the points of a certain homoclinic orbit Γ_0. Suppose that the point O has eigenvalues λ_0 and γ_0, where $|\lambda_0| < 1$, $|\gamma_0| > 1$. We assume that the saddle value $\sigma_0 \equiv |\lambda_0|\,|\gamma_0|$ of the point O is different from unity. Diffeomorphisms, close to g_0, which have a structurally unstable homoclinic orbit close to Γ_0 form, in the space of two-dimensional diffeomorphisms, a locally connected bifurcation surface H_0 of codimension 1. Let g_μ be a one-parameter family of C^r-smooth diffeomorphisms which is transversal to H_0 for $\mu = 0$.

Here is a brief review of some most important and well known properties of homoclinic bifurcations given with the use of the example of the family g_μ.

Note, first of all, the property of *nonisolatedness of homoclinic tangencies.* In the simplest version, it can be formulated as the following statement.

Let x_0 be a point of the orbit Γ_0. There exists a sequence μ_i of values of the parameter μ, such that the diffeomorphism g_{μ_i} has, at the point x_{μ_i}, a quadratic homoclinic tangency of manifolds of the fixed saddle point O_{μ_i}, where $\mu_i \to 0$, $x_{\mu_i} \to x_0$, $O_{\mu_i} \to O$ as $i \to \infty$.

This statement is obvious, and its geometric meaning can be seen from Fig. 1 in which it is demonstrated how the secondary (quadratic) homoclinic tangency arises.

Note, however, that the condition of genericity of the family g_μ does not yet guarantee that in its bifurcation set all values of the parameter are associated with only nondegenerable bifurcations, in particular, if there are homoclinic tangencies, then they are only quadratic. As is shown in [8, 13],

any generic family, which contains a system with homoclinic tangency, can be reduced, by an arbitrarily small smooth perturbation, again to a generic family in whose bifurcation interval there exist values of the parameter corresponding to an arbitrarily generate bifurcations.

It is shown in Fig. 2 how, for instance, cubic tangencies of manifolds of the point O_μ can arise. Thus the question concerning the type of new homoclinic tangencies arising at $\mu \neq 0$ requires a special attention since they are not automatically quadratic.

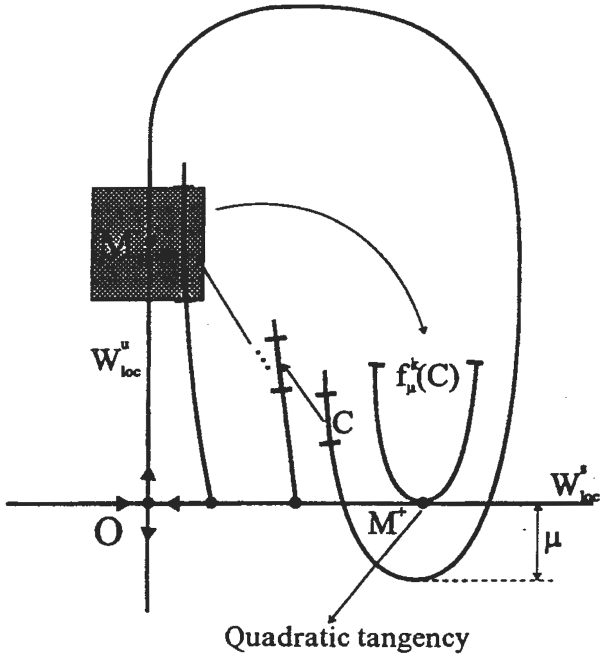


Fig. 1.

The property of nonisolatedness of homoclinic tangencies is also manifested in the fact that systems with homoclinic tangencies densely fill up whole domains (Newhouse domains) in the space of dynamical systems. Moreover, such domains can be found in generic one-parameter families containing a system with a homoclinic tangency, namely, the following result is proved in [3].

Newhouse theorem. *On the interval* $[-\mu_0, \mu_0]$*, for any* $\mu_0 > 0$ *there exists intervals where the values of the parameter* μ *for which the diffeomorphism* g_μ *has a quadratic tangency of invariant manifolds of a certain periodic saddle orbit are dense.*

This result is generalized in [14] to a multidimensional case for generic parametric families containing a system with a homoclinic tangency.

The *property of coexistence of periodic orbits of different topological types* is another important property which demonstrates homoclinic bifurcations. In the case of two-dimensional diffeomorphisms, which are close to a system with a homoclinic tangency, it is manifested in the fact that besides periodic saddle orbits belonging to nontrivial hyperbolic subsets [15], diffeomorphisms of this kind can also contain either stable or completely unstable periodic orbits according as the saddle value σ_0 of the point O is smaller or larger than unity respectively. For the first time, the statement concerning the coexostence of stable periodic orbits in the vicinity of a homoclinic tangency, which is often called *a theorem of the existence of a cascade of sinks (sources)*, was obtained in [15] and can be formulated as follows.

Assume that $\sigma < 1$ *(resp.* $\sigma > 1$*). Then, on the interval* $[-\mu_0, \mu_0]$ *for any* $\mu_0 > 0$ *there exists a sequence of nonintersecting intervals* $\delta_i = (\mu_i^*, \mu_i^{**})$ *contracting to* $\mu = 0$ *as* $i \to \infty$ *and such that for* $\mu \in \delta_i$ *the diffeomorphism* g_μ *has an asymptotically stable (completely unstable, resp.) one-circuit periodic orbit.*[1] *For* $\mu = \mu_i^*$ *the diffeomorphism* g_μ *has a one-circuit simplest structurally unstable*

[1]We can say that a periodic orbit, which lies entirely in a small fixed neighborhood of the contour $O \cup \Gamma_0$, is a k-circuit if its intersection with the small neighborhood of the point x_0 of the homoclinic tangency consists exactly

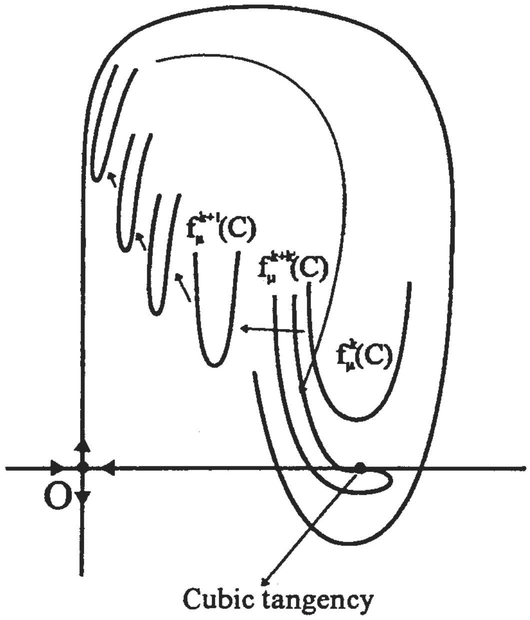


Fig. 2.

periodic orbit of the saddle-node type and, for $\mu = \mu_i^{**}$, *it has a one-circuit simplest structurally unstable periodic orbit with the multiplicator* -1.

A similar result for a generic one-parameter famliy of multidimensional diffeomorphisms, in the case, where the unstable manifold of the point O is one-dimensional, was established in [16] (see also [17], where the existence of a cascade of sinks was established for special families).

The existence of a cascade of sinks (sources) and the existence of Newhouse intervals associated with a homoclinic tangency make it possible to formulate the following result *(theorem on the coexistence of a countable set of sinks (sources))*.

Let $\sigma_0 < 1$ *(* $\sigma_0 > 1$, *resp.). Then, in Newhouse intervals, the values of the parameter* μ, *for which* g_μ *has a countable set of stable (completely unstable, resp.) periodic orbits, are dense.*

Note that the authors of [9, 13] found, for multidimensional systems which are close to a system with a homoclinic tangency, conditions for the existence as well as the absence, in a small neighborhood of a structurally unstable homoclinic orbit, of periodic orbits of some topological type.[2] In particular, for the case of two-dimensional diffeomorphisms with a homoclinic tangency it follows that if $\sigma_0 < 1$, then neither g_0 nor diffeomorphisms close to g_0 have completely unstable periodic orbits in the small neighborhood, $U(O \cup \Gamma_0)$ and if $\sigma_0 < 1$, they do not have stable orbits.

Thus, under general conditions, two-dimensional diffeomorphisms with a structurally unstable homoclinic orbit and diffeomorphisms close to them cannot contain simultaneously stable and complelely unstable orbits in its small neighborhood. As is shown in the present article, the

of k points.

[2]For instance, it was established in [9, 13] that stable periodic orbits could appear even in the case, where the dimension of the unstable manifold of the point O was 2 (in this case its unstable multiplicators must be complex conjugate) and that these orbits unremovably appeared only in two- or even three-parameter families. Correspondingly, the attainment of the boundary of stability can be followed by the appearance of orbits with two or even three, respectively, multiplicators modulo unity.

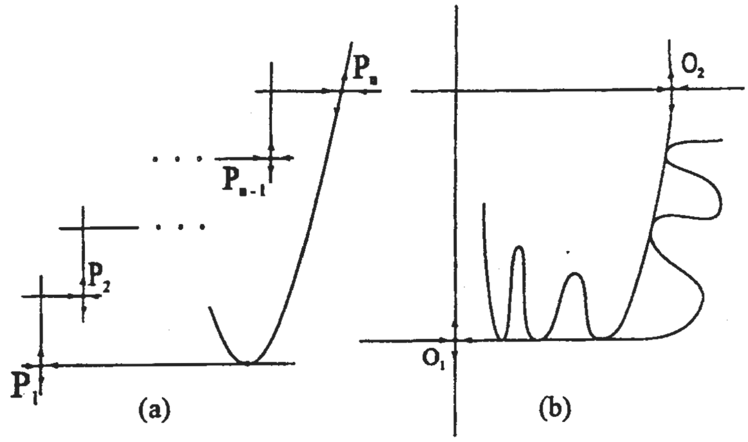


Fig. 3.

countable sets of these orbits can coexist for diffeomorphisms which are close to diffeomorphisms with a structurally unstable heteroclinic cycle.

2. STATEMENT OF THE PROBLEM AND FORMULATION OF THE MAIN RESULTS

In this article we shall study the bifurcations and the structure of the set of nonwandering orbits of two-dimensional diffeomorphisms which are close to a diffeomorphism with a structurally unstable heteroclinic cycle.

Let us recall how a heteroclinic cycle is interpreted. Let a dynamical system have structurally stable periodic saddle orbits $P_1, \ldots, P_n$ and heteroclinic orbits $\Gamma_{12}, \ldots, \Gamma_{n-1n}$ and Γ_{n1} such that $\Gamma_{ii+1} \subset W^u(L_i) \cap W^s(L_{i+1})$, $i = 1, \ldots, n-1$, $\Gamma_{n1} \subset W^u(L_n) \cap W^s(L_1)$. A *heteroclinic cycle* is a set of orbits $C = \{L_1, \ldots, L_n, \Gamma_{12}, \ldots, \Gamma_{n-1n}, \Gamma_{n1}\}$. The cycle is structurally stable if all indicated intersections of invariant manifolds along heteroclinic orbits are transversal and structurally unstable if at least one of the intersections is nontransversal.

Examples of two-dimensional diffeomorphisms with structurally unstable heteroclinic cycles are shown in Fig. 3. In the first case (Fig. 3a) the cycle contains several fixed saddle points (or periodic saddle orbits in the general case) and several heteroclinic orbits of which exactly one orbit (namely, Γ_{n1}) is structurally unstable. In the second case (Fig. 3b) a simple structurally unstable heteroclinic cycle is shown. Here O_1 and O_2 are fixed saddle points, the manifolds $W^u(O_1)$ and $W^s(O_2)$ intersect transversally at the points of the orbit Γ_{12}, and the manifolds $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency at the points of the orbit Γ_{21}.

The main attention will be paid to diffeomorphisms with the simplest structurally unstable cycles. Let f_0 be a diffeomorphism of this kind from the class C^r ($r \geq 3$) on the two-dimensional smooth manifold $\mathcal{M}^2$. We denote by λ_i, γ_i the eigenvalues of the point O_i such that $|\lambda_i| < 1$, $|\gamma_i| > 1$, $i = 1, 2$, and by σ_i the saddle value of the point O_i, i.e., $\sigma_i = |\lambda_i \gamma_i|$. We assume that $\sigma_i \neq 1$. Let U be a sufficiently small neighborhood of the heteroclinic cycle $C = O_1 \cup O_2 \cup \Gamma_{12} \cup \Gamma_{21}$.

The diffeomorphisms which are close to f_0 and have a structurally unstable heteroclinic orbit form a locally connected bifurcation surface H of codimension 1 in the space $\mathrm{Diff}^r(\mathcal{M}^2)$ of two-dimensional C^r-diffeomorphisms on $\mathcal{M}^2$. When investigating the bifurcations of systems with a structurally unstable heteroclinic cycle, it is natural to begin with the bifurcation in the one-parameter family f_μ of diffeomorphisms which includes f_0 and is transversal to H.

As is shown in the article, the solution of the problem concerning the coexistence in the neighborhood U of stable and completely unstable periodic orbits in systems which are close to f_0 essentially depends on the conditions satisfied by the saddle values σ_1 and σ_2. It seems to be natural (and is proved, see Statement 2 in Section 7) that if the saddle values σ_1 and σ_2 are both smaller than unity (both larger than unity, resp.), then neither f_0 nor diffeomorphisms which are close to it have completely unstable (stable, resp.) periodic orbits in U. Quite a different situation is observed in the case when the saddle values σ_1 and σ_2 are on different sides of unity. Here stable and completely unstable periodic orbits may coexist. Moreover, we establish the follwing general result (Theorem 4).

The fundamental theorem. *Let f_μ be a one-parameter family of two-dimensional diffeomorphisms from the class C^r ($r \geq 3$). We assume that for $\mu = 0$ the family f_μ is transversal to H, and $f_0 \in H$. We also assume that one of the saddle values σ_1 and σ_2 of f_0 is smaller than unity, and the other is larger than unity. Then, on any interval $[-\mu_0, \mu_0]$ of the values of the parameter μ there exists a countable set of intervals Δ_i^1 which accumulate to $\mu = 0$ as $i \to \infty$ and are such that*

(1) *on Δ_i^1 the values of the parameter μ, for which f_μ has a structurally unstable homoclinic to O_1 orbit, are dense and the values of of the parameter μ for which f_μ has a structurally unstable homoclinic to O_2 orbit are also dense;*

(2) *on Δ_i^1 the values of the parameter μ, for which f_μ has a structurally unstable heteroclinic cycle containing the points O_1, O_2 and the heteroclinic orbits $\Gamma_{12}(\mu)$, where $\Gamma_{12}(0) = \Gamma_{12}$. and $\tilde{\Gamma}_{21}(\mu) \subset W_\mu^u(O_2) \cap W_\mu^s(O_1)$ are dense (the orbit $\Gamma_{12}(\mu)$ is structurally stable and at the points of the orbit $\tilde{\Gamma}_{21}(\mu)$ the manifolds $W_\mu^u(O_2)$ and $W_\mu^s(O_1)$ have a quadratic tangency);*

(3) *on Δ_i^1 the values of the parameter μ, for which f_μ have simultaneoulsy a countable set of stable and a countable set of completely unstable periodic orbits, are dense.*

In addition to the Newhouse intervals indicated in the fundamental theorem (we shall call them interval of the first type), there can exist, in the family f_μ, Newhouse intervals of two more types which are characterized by the following main property which distinguish them from the intervals Δ_i^1: in the intervals of the second and the third type the values of the parameter μ, for which f_μ has a structurally unstable homoclinic orbit of only one fixed saddle point (the point O_1 or the point O_2 according as the type of the heteroclinic contour), are dense, and, in these intervals, there are no values of the parameter μ for which the diffeomorphism f_μ would have a homoclinic orbit of the other fixed saddle point. It should be also pointed out that the existence of Newhouse domains of the second and third types is possible not for any diffeomorphism with a structurally unstable heteroclinic cycle, namely, domains of this kind can only be in the vicinity of certain diffeomorphisms of the third class according to our classification. In Section 6 (by analogy with a structurally unstable homoclinic situation [15]) we divide diffeomorphisms with structurally unstable heteroclinic cycles into three classes according to the types of description of the set N_0 of orbits which lie entirely in the neighborhood U. Figure 4 shows four types of diffeomorphisms with the structurally unstable cycle in the case where λ_i and γ_i, $i = 1, 2$ are positive. For diffeomorphisms of the first class (such as shown in Fig. 4a), as is proved in [18], the set N_0 has a trivial structure: $N_0 = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$; for diffeomorphisms of the second class (Fig. 4b) the set N_0 admits a full

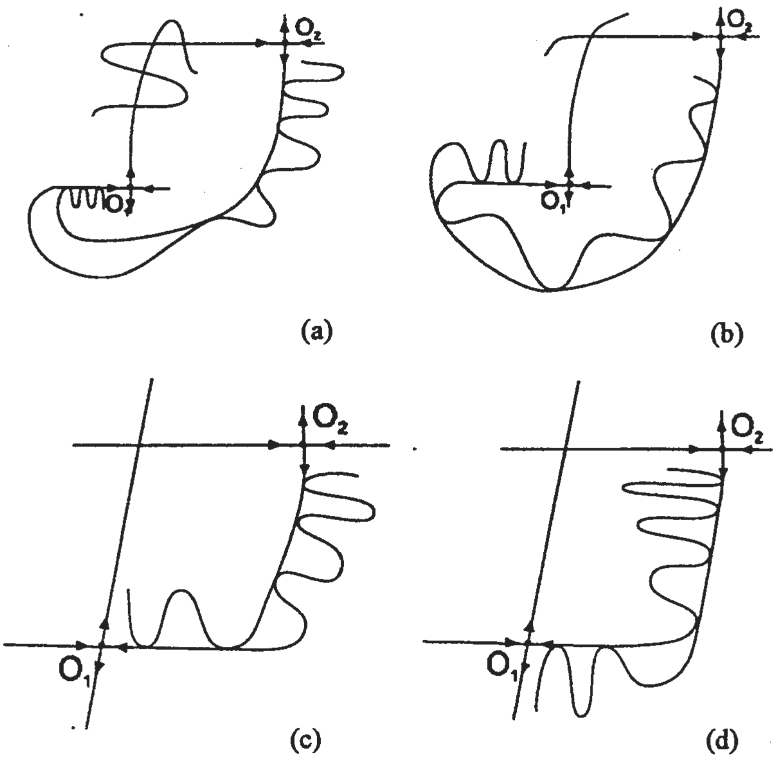


Fig. 4.

description in terms of symbolic dynamics (see Section 5); for diffeomorphisms of the third class (examples of these diffeomorphisms are given in Figs. 4c and 4d) the set N_0, in general, does no longer admit a full description, contains nontrivial hyperbolic subset (Theorem 3), and, on the bifurcation surface H_3 of these diffeomorphisms (see Sections 8 and 9) systems with structurally unstable periodic and homoclinic orbits are dense.

We introduce the quantities

$$\theta = -\frac{\ln|\lambda_2|}{\ln|\gamma_1|}$$

and

$$\alpha = \sigma_1^\theta \sigma_2.$$

Note that the invariant θ is a modulus of Ω-conjugacy (i.e., a continuous invariant of the topological conjugacy in a set of nonwandering orbits) of diffeomorphisms of the third class with a structurally unstable heteroclinic cycle (Theorem 6), and the quantity α characterizes the type of stability of one-circuit periodic orbits.

We assume that $\alpha < 1$ since the case $\alpha > 1$ reduces to the case under consideration for the diffeomorphism f^{-1}. Note that the condition $\alpha < 1$ is one of the sufficient conditions for the existence of stable periodic orbits for diffeomorphisms on H_3 (Theorem 12). In particular, for $\alpha < 1$ in Newhouse domains of all three types the values of the parameter μ corresponidng to the existence of a countable set of stable periodic orbits are dense.

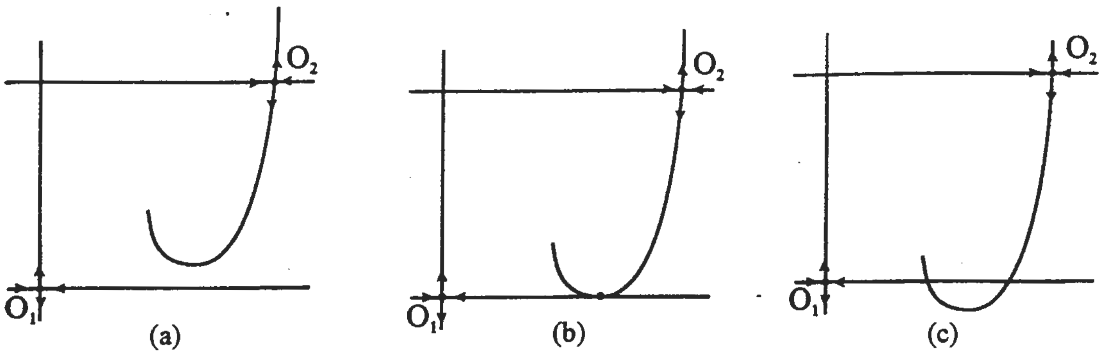


Fig. 5. $\mu > 0$ (a), $\mu = 0$ (b), $\mu < 0$ (c).

In order to give a general idea of Newhouse intervals of the second and third types and characterize the dynamical properties of the diffeomorphisms f_μ for the values of the parameter μ from these intervals, we shall consider, for definiteness, the family f_μ containing for $\mu = 0$ the diffeomorphism shown in Fig. 4c. We agree that the parameter μ belongs to the family in such a way that, for $\mu > 0$, f_μ does not have heteroclinic orbits which are close to Γ_{21}, and for $\mu < 0$ it has exactly two structurally stable heteroclinic orbits which are close to Γ_{21} (Fig. 5). We have the following statement for this generic family (see Theorems 15 and 16):

(1) *Irrespective of the quantity* α, *on the interval* $(-\mu_0, 0]$ *there exists, for any* $\mu_0 > 0$, *a countable set of intervals* Δ_i^1 *from the fundamental theorem.*

(2) *For* $\mu > 0$, *the diffeomorphism* f_μ *does not have in* U *any homoclinic orbits of the point* O_1 *or heteroclinic cycles which include the points* O_1 *and* O_2.

(3) *In the case* $\alpha < 1$, $\sigma_1 > 1$, $\sigma_2 < 1$, *on the interval* $(0, \mu_0]$ *there exists a countable set of intervals* Δ_i^2 *which accumulate to* $\mu_0 = 0$ *and are such that in* Δ_i^2 *the values of the parameter* μ, *for which* f_μ *has a structurally unstable homoclinic orbit of the point* O_2, *are dense, and the values of the parameter* μ, *for which* f_μ *has a countable set of stable peiodic orbits, are dense. In this case, the diffeomorphisms* f_μ *for* $\mu \geq 0$ *do not have in* U *any completely unstable periodic orbits.*

(4) *In the case* $\alpha < 1$, $\sigma_1 < 1$, $\sigma_2 > 1$, *on the interval* $(0, \mu_0]$ *there exists a countable set of intervals* Δ_i^3 *which accumulate to* $\mu_0 = 0$ *and are such that in* Δ_i^3 *the values of the parameter* μ, *for which* f_μ *has a structurally unstable homoclinic orbit of the point* O_2, *are dense and the values of the parameter* μ, *for which* f_μ *has simultaneously a countable set of stable and a countable set of completely unstable periodic orbits, are dense.*

Here is the plan of this work. In Section 3 we describe some general geometric and analytic properties of diffeomorphisms with the simplest structurally unstable heteroclinic cycle. In particular, we give here the definition of the generic one-parameter family f_μ containing a diffeomorphism with a structurally unstable heteroclinic cycle, describe the properties of local mappings defined in the neighborhoods of the fixed saddle points O_1 and O_2, and of global mappings defined in the neighborhoods of the heteroclinic orbits Γ_{12} and Γ_{21}, we also introduce special neighborhoods. In Sections 4 and 5 we study the question of the existence and the structure of nontrivial hyperbolic subsets of the diffeomorphisms f_μ. In Section 6 we divide diffeomorphisms with a structurally unstable heteroclinic cycle into three classes according as the structure of the set N_0 of orbits which lie entirely in the neightborhood of the contour. In Section 7 we prove the main result of the article, namely, the theorem on the existence of Newhouse intervals of the first type in the family f_μ which contains a diffeomorphism with the simplest structurally unstable heteroclinic cycle (Theorem 4) and generalize this theorem to the case of diffeomorphisms with an arbitrary structurally unstable heteroclinic cycle (Theorem 5). In Sections 8–10 we study certain dynamical properties of diffeomorphisms of the third class with a structurally unstable heteroclinic cycle (on the bifurcation surface H_3).

We prove in Section 8 that the diffeomorphisms on H_3 have moduli of Ω-conjugacy (in particluar, the invariant θ is the principal modulus of this kind (Theorem 6)) and that on H_3 the diffeomorphisms with a countable set of Ω-moduli are dense (Theorem 9). In Sections 9–10 we study the main bifurcations of periodic and homoclinic orbits of systems on H_3 in the framework of one-parameter families where θ is a parameter. On this basis, we establish in Section 10 the conditions for the existence and absence in U of stable and completely unstable periodic orbits of the difeomorphisms on H_3. Finally, in Section 11, we prove the existence of Newhouse intervals of the second and third types.

3. GEOMETRIC AND ANALYTIC PROPERTIES OF DIFFEOMORPHISMS WITH A STRUCTURALLY UNSTABLE HETEROCLINIC CYCLE

Let f_0 be a C^r-smooth ($r \geq 3$) diffeomorphism which is defined on the two-dimensional smooth manifold $\mathcal{M}^2$ and has the simplest structurally unstable heteroclinic cycle (Fig. 3b), i.e., f_0 has two structurally stable fixed saddle points O_1 and O_2 whose invariant mainfolds behave as follows: $W^u(O_1)$ transversally intersects $W^s(O_2)$ at the points of a certain heteroclinic orbit Γ_{12} and $W^u(O_2)$ has a quadratic tangency with $W^s(O_1)$ at the point of a certain heteroclinic orbit Γ_{21}. Let λ_i, γ_i be eigenvalues of the point O_i such that $|\lambda_i| < 1$, $|\gamma_i| > 1$, $i = 1, 2$. We denote by σ_i the saddle value of the point O_i, i.e., $\sigma_i = |\lambda_i \gamma_i|$. We assume that $\sigma_i \neq 1$.

We denote by $\mathrm{Diff}^r(\mathcal{M}^2)$ the space of C^r-smooth diffeomorphisms on $\mathcal{M}^2$ with a C^r-topology. Diffeomorphisms, which are close to f_0 and have a structurally unstable heteroclinic orbit which is close to Γ_{21}, form in $\mathrm{Diff}^r(\mathcal{M}^2)$ a locally connected bifurcation surface H of codimension 1.

When investigating the bifurcations of systems with a structurally unstable heteroclinic cycle, it is natural to begin with bifurcations in a one-parameter family of diffeomorphisms which includes f_0 and is transversal to H. We give the definition of these families and describe some of its properties.

3.1. Properties of transversal one-parameter families. Let g_μ be a one-parameter family of two-dimensional C^r-diffeomorphisms ($r \geq 2$) which is smooth with respect to the parameter μ. We assume that for sufficiently small μ the family g_μ has two C^r-smooth invariant curves $l_1(\mu)$ and $l_2(\mu)$ which smoothly depend on μ and are such that for $\mu = 0$ the curves $l_1(0)$ and $l_2(0)$ have tangency at a certain point x_0. It stands to reason that, first of all, we mean that $l_1(\mu)$ and $l_2(\mu)$ are, respectively, compact pieces of the invariant stable and unstable manifolds or of different periodic saddle orbits of the diffeomorphismg_μ, or of the same orbit. Then, for $\mu = 0$ we have, respectively, either a *heteroclinic* or *homoclinic* tangency at the point x_0.

Definition 1. We say that for $\mu = 0$ the family g_μ unfolds generically the tangency between the curves $l_1(\mu)$ and $l_2(\mu)$ if the following conditions are fulfilled.

1. The curves $l_1(0)$ and $l_2(0)$ have a quadratic tangency at the point x_0.
2. For $\mu \neq 0$ the curves $l_1(\mu)$ and $l_2(\mu)$ have no points of tangency in the vicinity of x_0.

Moreover, for a sufficiently small $\mu_0 > 0$ the interval $[-\mu_0, \mu_0]$ of values of μ is divided by the point $\mu = 0$ into two parts such that for $\mu < 0$ (or for $\mu > 0$) the curves $l_1(\mu)$ and $l_2(\mu)$ have no points of intersection close to x_0 and for $\mu > 0$ (or for $\mu < 0$ respectively) they have exactly two points $x_1(\mu)$ and $x_2(\mu)$ of transversal intersection, where $x_i(\mu) \to x_0$ as $\mu \to 0$.

3. The splitting function $\rho(\mu)$ of the curves $l_1(\mu)$ and $l_2(\mu)$ relative to the point x_0 is a smooth monotonic function of the parameter μ, and $\rho'(0) \neq 0$.

When $l_1(\mu)$ and $l_2(\mu)$ are, respectively, the pieces of a stable and an unstable manifold or of different periodic saddle orbits or of the same orbit, we say that for $\mu = 0$ the family g_μ *unfolds generically a heteroclinic or homoclinic tangency* respectively.

We can define the splitting function $\rho(\mu)$ for all sufficiently small μ, say, in the following way. Let V be a certain small fixed neighborhood of the point x_0. Then

(a) $\rho(0) = 0$ for $\mu = 0$;

(b) if $(l_1(\mu) \cap l_2(\mu)) \cap V = \varnothing$, then $\rho(\mu)$ is the distance between the curves $l_1(\mu) \cap V$ and $l_2(\mu) \cap V$;

(c) if $(l_1(\mu) \cap l_2(\mu)) \cap V = \{x_1(\mu), x_2(\mu)\}$, then $\rho(\mu)$ is defined as follows: let $\tilde{l}_1(\mu)$ and $\tilde{l}_2(\mu)$ be closed segments of the curves $l_1(\mu) \cap V$ and $l_2(\mu) \cap V$ with the endpoints $x_1(\mu)$ and $x_2(\mu)$. Then

$$\rho(\mu) = -\max_{x \in \tilde{l}_1(\mu)} d(x, \tilde{l}_2(\mu)) = -\max_{y \in \tilde{l}_2(\mu)} d(y, \tilde{l}_1(\mu)),$$

where $d(\cdot\,,\,\cdot)$ is the distance between the indicated point and the curve.

The important property of these generic families is their stability against smooth perturbations, namely, if for $\mu = 0$ the family $l_1(\mu)$ and $l_2(\mu)$, then the close family $\tilde{g}_\mu$ (C^r-close with respect to the coordinates and C^1-close with respect to the parameter) unfolds generically a tangency between the curves $\tilde{l}_1(\mu)$ and $\tilde{l}_2(\mu)$, which are close to $l_1(\mu)$ and $l_2(\mu)$, respectively, for the values of the parameter μ close to zero.

Definition 2. Let $l(\mu)$ be a smoothly dependent on μ one-parameter family of C^r-smooth curves. We say that the curves $l_j(\mu)$ *accumulate in a regular way to* $l(\mu)$ as $j \to \infty$ if $l_j(\mu)$ accumulates to $l(\mu)$ as $j \to \infty$ in the C^2-sense with respect to coordinates and in the C^1-sense with respect to the parameter.

The following statement is a simple consequence of the genericity of the family.

Statement 1. *Let $\{l_1^i(\mu)\}$ and $\{l_2^j(\mu)\}$ be two families of curves which accumulate in a regular way to the curves $l_1(\mu)$ and $l_2(\mu)$ respectively. Then there exist k_1 and k_2, such that if $i \geq k_1$ and $j \geq k_2$, then the family g_μ unfolds generically a tangency between any of the curves $\{l_1^i(\mu)\}$ and $\{l_2^j(\mu)\}$ for $\mu = \mu_{ij}$, where $\mu_{ij} \to 0$ as $i \to \infty$ and $j \to \infty$.*

3.2. Local and global mappings. Let U be a sufficiently small heighborhood of a heteroclinic cycle $C = O_1 \cup O_2 \cup \Gamma_{12} \cup \Gamma_{21}$. It is the union of two small disks U_1 and U_2 containing the points O_1 and O_2 and of a certain finite number of small neighborhoods of those points of the orbits Γ_{12} and Γ_{21}, which are outside of U_1 and U_2 (Fig. 6).

As is established in [10, 11], we can introduce on U_s, $s = 1, 2$, coordinates (x_s, y_s) such that the

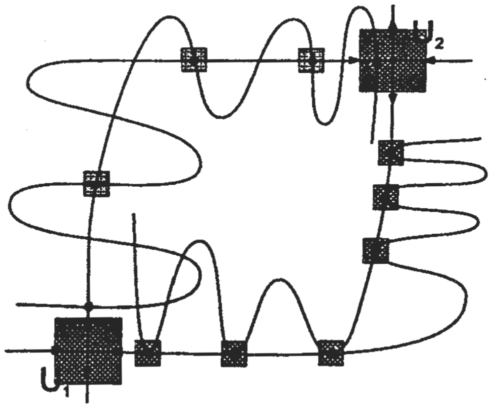


Fig. 6.

mapping $T_{0s}(\mu) \equiv f_{\mu|U_s}$ can be written as

$$\begin{aligned}\bar{x}_s &= \lambda_s(\mu)x_s + f_s(x_s, y_s, \mu)x_s y_s,\\ \bar{y}_s &= \gamma_s(\mu)y_s + g_s(x_s, y_s, \mu)x_s y_s,\end{aligned} \tag{3.1}$$

where $f_s(0, y_s, \mu) \equiv 0$, $g_s(x_s, 0, \mu) \equiv 0$, and, in addition, the functions on the right-hand sides of (3.1) are C^{r-1}-smooth with respect to the coordinates and C^1-smooth with respect to the parameter. In accordance with (3.1), the equations of the manifolds $W^s_{\text{loc}}(O_s(\mu))$ and $W^u_{\text{loc}}(O_s(\mu))$ in these coordinates are $y_s = 0$ and $x_s = 0$ respectively. The mappings $T_{01}(\mu)$ and $T_{02}(\mu)$ are callled *local mappings.*

For $\mu = 0$, in the diffeomorphismf_0 we choose in U_1 a pair of points $M_1^+(x_1^+, 0)$ and $M_1^-(0, y_1^-)$ belonging to the orbits Γ_{21} and Γ_{12} respectively. In U_2 we shall also consider a pair of points $M_2^+(x_2^+, 0)$ and $M_2^-(0, y_2^-)$ belonging to the orbits Γ_{12} and Γ_{21} respectively. Let $\Pi_s^+ \subset U_s$ and $\Pi_s^- \subset U_s$ be a sufficiently small rectangular neighborhoods of the points M_s^+ and M_s^-. We dentoe the coordinates on Π_s^+ and Π_s^- by (x_{0s}, y_{0s}) and (x_{1s}, y_{1s}) respectively.

For sufficiently large i and small μ the mapping $T_{0s}^i(\mu): \Pi_s^+ \to \Pi_s^-$ can be written in the form [10, 11]

$$\begin{aligned}\bar{x}_{1s} &= \lambda_s^i(\mu)x_{0s}(1 + (|\lambda_s|^i + |\gamma_s|^{-i})\xi_i^s(x_{0s}, \bar{y}_{1s}, \mu)),\\ y_{0s} &= \gamma_s^{-i}(\mu)\bar{y}_{1s}(1 + (|\lambda_s|^i + |\gamma_s|^{-i})\eta_i^s(x_{0s}, \bar{y}_{1s}, \mu)),\end{aligned} \tag{3.2}$$

where $(x_{0s}, y_{0s}) \in \Pi_s^+$, $(\bar{x}_{1s}, \bar{y}_{1s}) \in \Pi_s^-$, and the functions ξ_i^s and η_i^s are uniformly bounded with respect to i together with the derivatives with respect to the coordinates up to the order $(r-2)$ and with respect to the parameter. In addition, the derivatives of the order $(r-1)$ with respect to the coordinates of the functions on the right-hand sides of (3.2) tend to zero as $i \to \infty$. Thus, in these coordinates, the mapping T_{0s}^k for large k will be asymptotically close to a linear mapping.

We denote by N_μ the set of orbits of the diffeomorphism f_μ which lie entirely in U. Note that all orbits of the set N_μ, except for O_1 and O_2, must intersect the neighborhoods Π_s^+ and Π_s^- (otherwise these orbits will not be close to those of the cycle C). As we can easily see from

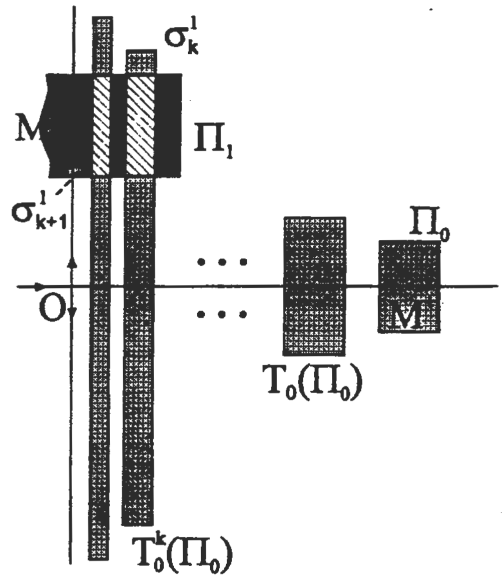


Fig. 7.

(3.2), the set of initial points on Π_s^+, whose orbits fall in Π_s^-, consists of a countable set of strips $\sigma_k^{0s} = \Pi_s^+ \cap T_{0s}^{-k}\Pi_s^-$, $k = \bar{k}_s, \bar{k}_s + 1, \ldots$, which accumulate to $W^s_{\mathrm{loc}}(O_s)$. The method of constructing these strips is obvious from Fig. 7. Correspondingly, the images of the strips σ_k^{0s} relative to the mappings T_{0s}^k are vertical strips $\sigma_k^{1s} \equiv T_{0s}^k(\sigma_k^{0s})$ on Π_s^- which accumulate to $W^u_{\mathrm{loc}}(O_s)$ (Fig. 8).

It is obvious that for $\mu = 0$ there exist natural numbers n_1 and n_2 such that $f_0^{n_1}(M_1^-) = M_2^+$, $f_0^{n_2}(M_2^-) = M_1^+$. Let us consider *global mappings*, namely, the mapping $T_{12} \equiv f_0^{n_1}\colon \Pi_1^- \to U_2$ with respect to orbits close to Γ_{12}. and the mapping $T_{21} \equiv f_0^{n_2}\colon \Pi_2^- \to U_1$ with respect to orbits close to Γ_{21}.

For $\mu = 0$ the mapping T_{12} can evidently be represented as

$$\begin{aligned} \bar{x}_{02} - x_2^+ &= a_{12}x_{11} + b_{12}(y_{11} - y_1^-) + \ldots, \\ \bar{y}_{02} &= c_{12}x_{11} + d_{12}(y_{11} - y_1^-) + \ldots, \end{aligned} \tag{3.3}$$

where the Jacobian $J_{12} \equiv a_{12}d_{12} - b_{12}c_{12}$ of the mapping T_{12} at the point M_1^- is nonzero since T_{12} is a diffeomorphism, and $d_{12} \neq 0$ since $W^u(O_1)$ intersects $W^s(O_2)$ at the point M_2^+ transversally.

For $\mu = 0$ the mapping T_{21} can be written as

$$\begin{aligned} \bar{x}_{01} - x_1^+ &= a_{21}x_{12} + b_{21}(y_{12} - y_2^-) + \ldots, \\ \bar{y}_{01} &= c_{21}x_{21} + d_{21}(y_{12} - y_2^-)^2 + \ldots, \end{aligned} \tag{3.4}$$

where $d_{21} \neq 0$ since the tangency $W^u(O_2)$ and $W^s(O_1)$ at the point M_1^+ is quadratic and the Jacobian $J_{21} \equiv b_{21}c_{21}$ of the mapping T_{21} at the point M_2^- is nonzero since T_{21} is a diffeomorphism.

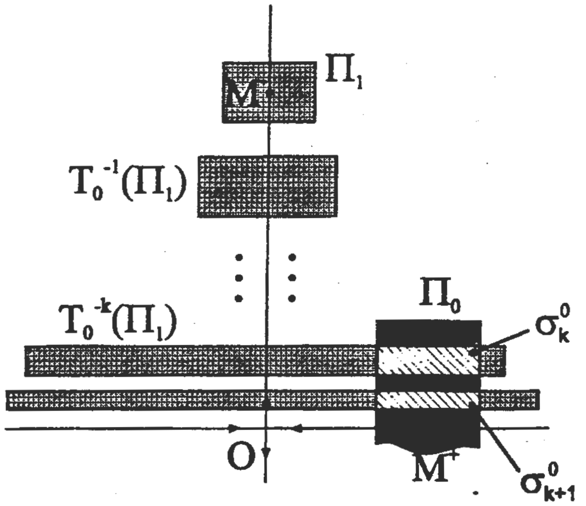


Fig. 8.

The mapping $T_{12}(\mu) \equiv f_\mu^{n_1}: \Pi_1^- \to U_2$ can be written as

$$\begin{aligned} \bar{x}_{02} - x_2^+(\mu) &= a_{12}x_{11} + b_{12}(y_{11} - y_1^-(\mu)) \\ &\quad + O[(|x_{11}| + |y_{11} - y_1^-(\mu)|)^2 + |\mu|(|x_{11}| + |y_{11} - y_1^-(\mu)|)], \\ \bar{y}_{02} &= c_{12}x_{11} + d_{12}(y_{11} - y_1^-(\mu)) \\ &\quad + O[(|x_{11}| + |y_{11} - y_1^-(\mu)|)^2 + |\mu|(|x_{11}| + |y_{11} - y_1^-(\mu)|)], \end{aligned} \tag{3.5}$$

where $x_2^+(0) = x_2^+$, $y_1^-(0) = y_1^-$, the points $(x_2^+(\mu), 0)$ and $(0, y_1^-(\mu))$ are, respectively, the points of intersection of the orbit $\Gamma_{12}(\mu)$ with the neighborhoods Π_2^+ and Π_1^-.

The mapping $T_{21}(\mu) \equiv f_\mu^{n_2}: \Pi_2^- \to U_1$ can be wirtten as

$$\begin{aligned} &\bar{x}_{01} - x_1^+(\mu) = a_{21}x_{12} + b_{21}(y_{12} - y_2^-) + O[(|x_{12}| + |y_{12} - y_2^-|)^2 + |\mu|(|x_{12}| + |y_{12} - y_2^-|)], \\ &\bar{y}_{01} = \mu + c_{21}x_{12} + d_{21}(y_{12} - y_2^-)^2 \\ &\qquad + O[x_{12}^2 + |\mu|(|x_{12}| + |y_{12} - y_2^-|) + |x_{12}||y_{12} - y_2^-|] + o[(y_{12} - y_2^-)^2], \end{aligned} \tag{3.6}$$

where $x_1^+(0) = x_1^+$.

Note that the parameter μ enters into the second equation of (3.6) additively in the principal order. This is a consequence of our requirement that the family f_μ should be transversal for $\mu = 0$ to the bifurcation surface H. Indeed, it follows from (3.6) that the equation of the piece $T_{21}(W^u_{\text{loc}}(O_2)) \cap \Pi_1^+$ of the unstable manifold of the point O_2 has the form (in (3.6) we must set x_{12} equal to 0)

$$y_{01} = \mu + \frac{d_{21}}{b_{21}^2}(x_{01} - x_1^+(\mu))^2 + \dots .$$

Thus, for $d_{21}\mu > 0$ the diffeomorphism f_μ does not have heteroclinic orbits, which would be close to Γ_{21}, and for $d_{21}\mu < 0$ it has exactly two structurally stable heteroclinic orbits, which are close to Γ_{21} and intersect the piece $W^s_{\text{loc}}(O_1) \cap \Pi_1^+$ of the stable manifold of the point O_1 at points with

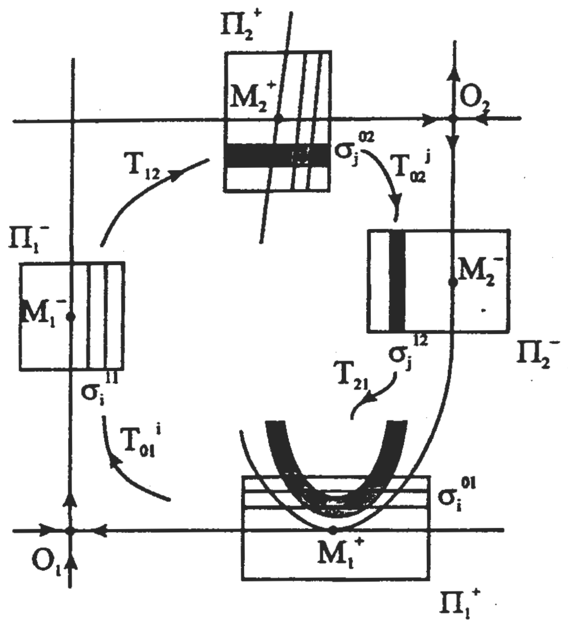


Fig. 9.

coordinates

$$x_{01}^{\alpha} = x_1^{+} + (-1)^{\alpha} b_{21}\sqrt{-\frac{\mu}{d_{21}}} + \ldots,$$

where $\alpha = 1, 2$, and $x_{01}^{\alpha} \to x_1^{+}$ as $\mu \to 0$. In this case, by virtue of (3.6), the splitting function relative to the structurally unstable heteroclinic point $M_1^{+}(x_1^{+}, 0)$ has the form

$$\rho(\mu) = \operatorname{sgn} d_{21} \times (\mu + o(\mu)).$$

3.3. A special neighborhood of a heteroclinic cycle. It is convenient to choose, as a neighborhood of the heteroclinic cycle C, a *special neghborhood* (by analogy with the special neighborhood of a structurally unstable homoclinic orbit [19, 20]), namely, we take sufficiently large integers $\bar{k}_1$ and $\bar{k}_2$ and consider only the orbits which (for all sufficiently small μ) get from Π_s^{+} into Π_s^{-} during no less than $\bar{k}_s$ iterations of the mapping f_μ. In particular, this means that Π_s^{+} and Π_s^{-} contain the strips σ_k^{0s} and σ_k^{1s} with the numbers $k \geq \bar{k}_s$ in their entirety, and do not contain strips with numbers smaller than $\bar{k}_s$. Note, in addition, that it suffices to choose the values of the parameter μ that belong to the "bifurcation interval"

$$|\mu| \leq C_1(|\gamma_1|^{-\bar{k}_1} + |\lambda_2|^{\bar{k}_2}) = \mu_0, \tag{3.7}$$

where C_1 is a positive constant such that, for instance, (for diffeomorphisms with a structurally unstable heteroclinic cycle shown in Fig. 4c), we have $T_{21}(\mu)(\Pi_2^{-}) \cap \Pi_1^{+} = \varnothing$ for $\mu > \mu_0$, and, for $\mu < -\mu_0$, the set $N(\mu)$ has a hyperbolic structure (Fig. 9b).

Then, without loss of generality, we can choose the neighborhoods Π_s^+ and Π_s^- such that

$$\begin{aligned}
\Pi_1^+ &= \{(x_{01}, y_{01}) \mid |x_{01} - x_1^+| \le \rho_{\bar{k}_1,\bar{k}_2}, |y_{01}| \le |\gamma_1|^{-\bar{k}_1}(y_1^- + \nu_{\bar{k}_1,\bar{k}_2})\},\\
\Pi_2^+ &= \{(x_{02}, y_{02}) \mid |x_{02} - x_2^+| \le \nu_{\bar{k}_1,\bar{k}_2}, |y_{02}| \le |\gamma_2|^{-\bar{k}_2}(y_2^- + \rho_{\bar{k}_1,\bar{k}_2})\},\\
\Pi_1^- &= \{(x_{11}, y_{11}) \mid |x_{11}| \le |\lambda_1|^{\bar{k}_1}(x_1^+ + \rho_{\bar{k}_1,\bar{k}_2}), |y_{11} - y_1^-| \le \nu_{\bar{k}_1,\bar{k}_2}\},\\
\Pi_2^- &= \{(x_{12}, y_{12}) \mid |x_{12}| \le |\lambda_2|^{\bar{k}_2}(x_2^+ + \nu_{\bar{k}_1,\bar{k}_2}), |y_{12} - y_2^-| \le \rho_{\bar{k}_1,\bar{k}_2}\},
\end{aligned} \tag{3.8}$$

where

$$\rho_{\bar{k}_1,\bar{k}_2} = C_2\sqrt{|\gamma_1|^{-\bar{k}_1} + |\lambda_2|^{\bar{k}_2}}, \qquad \nu_{\bar{k}_1,\bar{k}_2} = C_3(|\gamma_2|^{-\bar{k}_2} + |\lambda_1|^{\bar{k}_1}),$$

and C_2 and C_3 are positive constants independent of $\bar{k}_1$ and $\bar{k}_2$.

Let us prove this fact. We take, for definiteness, as initial neighborhoods Π_s^+ and Π_s^-, small square with centers at the points M_s^+ and M_s^- respectively, and with the side $2\varepsilon_0$ long. Since special neighborhoods must not contain points, which, during the number of iterations of the mapping f, smaller than $\bar{k}_s$, get from Π_s^+ into Π_s^-, we find from (3.2) that for sufficiently large $\bar{k}_1$ and $\bar{k}_2$ the neighborhoods Π_s^+ can be contracted in the direction of the coordinate y, and Π_s^- in the direction of the coordinate x so that

$$\begin{aligned}
|y_{01}| &\le |\gamma_1|^{-\bar{k}_1}(y_1^- + \varepsilon_0), \qquad & |x_{11}| &\le |\lambda_1|^{\bar{k}_1}(x_1^+ + \varepsilon_0),\\
|y_{02}| &\le |\gamma_2|^{-\bar{k}_2}(y_2^- + \varepsilon_0), \qquad & |x_{12}| &\le |\lambda_2|^{\bar{k}_2}(x_2^+ + \varepsilon_0).
\end{aligned} \tag{3.9}$$

Since $\bar{y}_{02} = c_{12}x_{11} + d_{12}(y_{11} - y_1^-) + \ldots$, by virtue of (3.5) and the validity of estimates (3.9) is required for the coordinates x_{11} and $\bar{y}_{02}$, the neighborhood Π_1^- can be narrowed so that the coordinate y_{11} will stisfy the inequality

$$|y_{11} - y_1^-| \le \max\left\{\frac{1}{|d_{12}|}(|\bar{y}_{02}| + |c_{12}|\,|x_{11}|\right\} \le C_4(|\gamma_2^{-\bar{k}_2}| + |\lambda_1^{\bar{k}_1}|). \tag{3.10}$$

By virtue of (3.6) $\bar{y}_{01} = \mu + c_{21}x_{12} + d_{21}(y_{12} - y_2^-)^2 + \ldots$. Since $d_{21} \ne 0$, it follows, by virtue of (3.9), that the neighborhood Π_2^- can, in turn, be narrowed so that the coordinate y_{12} will satisfy the inequaltiy

$$|y_{12} - y_2^-| \le C_5\sqrt{|\mu| + |\gamma_1|^{-\bar{k}_1} + |\lambda_2|^{\bar{k}_2}}, \tag{3.11}$$

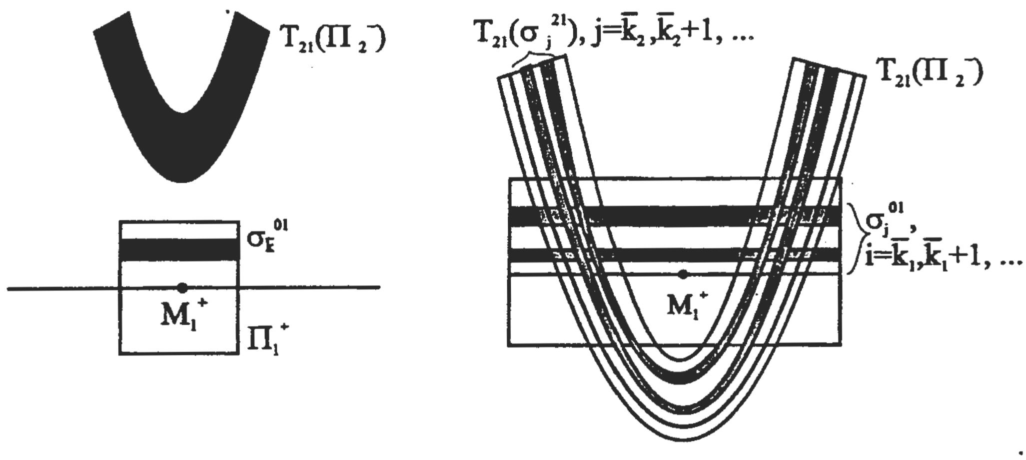
or, by virtue of (3.7), the inequality

$$|y_{12} - y_2^-| \le C_6\sqrt{|\gamma_1|^{-\bar{k}_1} + |\lambda_2|^{\bar{k}_2}}. \tag{3.12}$$

Now, by virtue of (3.6) and (3.9)–(3.12) the neighborhoods Π_1^+ and Π_2^+ can be narrowed so that the estimates

$$|x_{01} - x_1^+| \le C_7\sqrt{|\gamma_1|^{-\bar{k}_1} + |\lambda_2|^{\bar{k}_2}}, \qquad |x_{02} - x_2^+| \le C_8(|\gamma_2|^{-\bar{k}_2} + |\lambda_1|^{\bar{k}_1}). \tag{3.13}$$

will be satisfied for the cooridnates x_{01} and x_{02}. If we carry out the same operation for the obtained neighborhoods of the heteroclinic points once again, we get relations (3.8).



(a) (b)

Fig. 10. $\mu > \mu_0$ (a); $\mu < -\mu_0$ (b).

We call the neighborhoods defined by (3.8) *special neighborhoods of the points* M_1^+, M_2^+, M_1^- *and* M_2^- and will consider precisely these neighborhoods. We shall denote the corresponding special neighborhood of the heteroclinic cycle either simply by U or by $U(\bar{k}_1, \bar{k}_2)$, when we want to emphasize the dependence of its dimensions on the minimal numbers of strips. Similarly, we shall denote the set N_μ, when necessary, by $N_\mu(\bar{k}_1, \bar{k}_2)$.

4. CONDITIONS FOR INTERSECTION OF HORSESHOES AND STRIPS

Since Γ_{12} is the orbit of transversal intersection of the manifolds $W^u(O_1)$ and $W^s(O_2)$, the intersection of any strips $T_{12}\sigma_k^{11}$ with any strips σ_j^{02}, for sufficiently large k and j and sufficiently small μ, consists of one connection component (Fig. 10). The images $T_{21}(\sigma_j^{12})$ of the strips σ_j^{12} are shaped as horseshoes, which accumulate, as $j \to \infty$, to the "parabola" $T_{21}(W^u_{\text{loc}}(O_2)) \subset W^u(O_2) \cap \Pi_1^+$ (Fig. 10). It is clear that the orbits of the set N_μ must intersect the neighborhood Π_1^+ at the points of intersection of the horseshoes $T_{21}(\sigma_j^{12})$ and the strips σ_i^{01} for various $i \geq \bar{k}_1$ and $j \geq \bar{k}_2$. Consequently, the structure of the set N_μ essentially depends on the geometric properties of these intersections.

We say that the horseshoe $T_{21}(\sigma_j^{12})$ has a *regular intersection* with the strip σ_i^{01} if

(1) the set $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01}$ is nonempty and consists of two connection components;

(2) the mappings $T_{21}^{(1)}T_{02}^j$ and $T_{21}^{(2)}T_{02}^j$, which are defined on σ_j^{02} and have their range of values on $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01}$, are saddle mappings in the sense of [21] (roughly speaking, these mappings are expanding along the coordinate y_{02} and contracting along the coordinate x_{02} in Π_2^+).

Various kinds of intersections of the horseshoes $T_{21}(\sigma_j^{12})$ and the strips Π_1^+ are shown in Fig. 11. A horseshoe has a regular intersection with the strip σ_i^{01}, an irregular intersection with the strip σ_k^{01}, and an empty intersection with the strip σ_s^{01}.

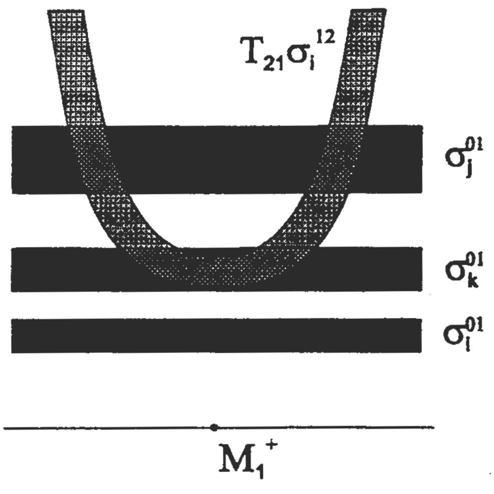


Fig. 11.

Theorem 1. *There exist a positive constant S_1 and sufficiently large integers $\bar{k_1}$ and $\bar{k_2}$, such that for $\mu \in [-\mu_0;\mu_0]$, for any $i \geq \bar{k_1}$, $j \geq \bar{k_2}$*

(1) *if the inequality*

$$d_{21}(\gamma_1^{-i}y_1^- - \mu - c_{21}\lambda_2^j x_2^+) < -S_{ij}(\bar{k}_1, \bar{k}_2), \tag{4.1}$$

where $S_{ij} = S_1(|\gamma_1|^{-i}+|\lambda_2|^j)(|\lambda_1|^{\bar{k}_1}+|\gamma_2|^{-\bar{k}_2}+|\gamma_1|^{-\bar{k}_1}+|\lambda_2|^{\bar{k}_2})$, is satisfied, then $T_{21}(\mu)(\sigma_j^{12})\cap\sigma_i^{01} = \varnothing$;

(2) *if the inequality*

$$d_{21}(\gamma_1^{-i}y_1^- - \mu - c_{21}\lambda_2^j x_2^+) > S_{ij}(\bar{k}_1, \bar{k}_2) \tag{4.2}$$

is satisfied, then the intersection of the horseshoe $T_{21}(\mu)(\sigma_j^{12})$ and the strip σ_i^{01} is regular;

(3) *the inequalities*

$$d_{21}(\gamma_1^{-i}y_1^- - \mu - c_{21}\lambda_2^j x_2^+) \geq -S_{ij}(\bar{k}_1, \bar{k}_2) \tag{4.3}$$

and

$$|d_{21}|\,|\gamma_1^{-i}y_1^- - \mu - c_{21}\lambda_2^j x_2^+| \leq S_{ij}(\bar{k}_1, \bar{k}_2) \tag{4.4}$$

are necessary for the horseshoe $T_{21}(\mu)(\sigma_j^{12})$ to have a nonempty and an irregular intersection, respectively, with the strip σ_i^{01}

Proof. Item (3) of the theorem is, obviously, a consequence of items (1) and (2).

By virtue of (3.2) and (3.8) the coordinates (x_{01}, y_{01}) of the points of the strip σ_i^{01} satisfy the inequaltites

$$|x_{01} - x_1^+| \leq \rho_{\bar{k}_1,\bar{k}_2}, \qquad |y_{01} - \gamma_1^{-i}y_1^-| \leq \gamma_1^{-i}(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\gamma_1|^{-i}), \tag{4.5}$$

and the coordinates (x_{12}, y_{12}) of the points of the strip σ_j^{12} satisfy the inequalities

$$|y_{02} - y_2^-| \leq \rho_{\bar{k}_1,\bar{k}_2}, \qquad |x_{12} - \lambda_2^j x_2^+| \leq \lambda_2^j(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\lambda_2|^j). \tag{4.6}$$

Let us consider, for definiteness, the case $\lambda_2 > 0$, $\gamma_1 > 0$, $c_{21} > 0$, $d_{21} > 0$ (in the other cases the proof is similar). Then, by virtue of (4.6) and (3.4) the horseshoe $T_{21}(\sigma_j^{12})$ on Π_1^+ is bounded by two "parabolas", the upper parabola

$$y_{01}^{(1)} = \mu + c_{21}\lambda_2^j(x_2^+ + |\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\lambda_2|^j) + \frac{d_{21}}{b_{21}^2}(x_{01} - x_1^+)^2 + \dots \tag{4.7}$$

and the lower parabola

$$y_{01}^{(2)} = \mu + c_{21}\lambda_2^j(x_2^+ - |\lambda_1|^{\bar{k}_1} - |\gamma_2|^{-\bar{k}_2} - |\lambda_2|^j) + \frac{d_{21}}{b_{21}^2}(x_{01} - x_1^+)^2 + \dots. \tag{4.8}$$

It is clear that the intersection of the strip σ_i^1 and the horseshoe $T_{21}(\sigma_j^{12})$ is empty if, for instance, the majorizing parabolas from (4.7) and (4.8) do not intersect the strip σ_i^0, i.e., if the inequality

$$\mu + c_{21}\lambda_2^j(x_2^+ - |\lambda_1|^{\bar{k}_1} - |\gamma_2|^{-\bar{k}_2} - |\lambda_2|^j) > \gamma_1^{-i}y_1^- + \gamma_1^{-i}(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\gamma_1|^{-i})$$

is satisfied. Thus (with due account of the sign of d_{21}), we have the inequality

$$\begin{aligned} &d_{21}(\gamma_1^{-i}y_1^- - \mu - c_{21}\lambda_2^j x_2^+) \\ &< -C_9[\lambda_2^j(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\lambda_2|^j) + |\gamma_1^{-i}|(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\gamma_1|^{-i})], \end{aligned} \tag{4.9}$$

which is similar to (4.1).

Let us now find the conditions for the regularity of the intersection of the strip σ_i^1 and the horseshoe $T_{21}(\sigma_j^{12})$. This intersection consists of two connection components if, for instance, the majorizing parabolas from (4.7) and (4.8) each intersects the strip σ_i^0 and this intersection consists of two connection components. For $d_{21} > 0$, this ensures the inequality

$$\mu + c_{21}\lambda_2^j(x_2^+ + |\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\lambda_2|^j) + C_{10}\lambda_2^{2j} < \gamma_1^{-i}y_1^- - \gamma_1^{-i}(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\gamma_1|^{-i}),$$

or

$$\begin{aligned} &d_{21}(\gamma_1^{-i}y_1^- - \mu - c_{21}\lambda_2^j x_2^+) \\ &> C_{11}[\lambda_2^j(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\lambda_2|^j) + |\gamma_1|^{-i}(|\lambda_1|^{\bar{k}_1} + |\gamma_2|^{-\bar{k}_2} + |\gamma_1|^{-i})]. \end{aligned} \tag{4.10}$$

This inequality is "similar" to (4.2). However, in order to prove the regularity of intersection we have to show that the mappings $T_{02}^jT_{21}$ defined on the inverse images of each of the connection components of the horseshoe $T_{21}(\sigma_j^{12})$, defined by the inequality (4.10), will be saddle mappings. We denote these components on Π_1^+ by Δ_{ij}^3 and Δ_{ij}^4 and denote the restrictions of the mapping T_{21} onto the components $T_{21}^{-1}(\Delta_{ij}^s)$, $s = 3, 4$, of the strip σ_j^{12} by $T_{21}^{(s)}$. We can rewrite the mapping $T_{21}^{(s)}$ as

$$\begin{aligned} \bar{x}_{01} - x_1^+ &= b_{21}(y_{12} - y_2^-) + a_{21}x_{12} + \dots, \\ (y_{12} - y_2^-) &= (-1)^s\sqrt{\frac{\mu + c_{21}x_{12} - \bar{y}_{01} + \dots}{d_{21}}}. \end{aligned} \tag{4.11}$$

Since by virtue of (4.5), (4.6) the relations

$$x_{12} = \lambda_2^j(x_{02} + (|\lambda_2|^j + |\gamma_2|^{-j})\xi_{j2}(x_{02}, y_{12})), \qquad y_{01} = \gamma_1^{-i}(y_{11} + (|\lambda_1|^i + |\gamma_1|^{-i})\eta_{i1}(x_{01}, y_{11})),$$

are satisfied for the coordinate x_{12} on the strip σ_{12}^j and for the coordinate y_{01} on the strip σ_{01}^i, respectively, it follows, by virtue of (4.11), that the mapping $T_{ij}^{(s)} \equiv T_{02}^j T_{21}^{(s)}: \sigma_j^{02} \to \Delta_{ij}^s$ can be written in the "cross-form"

$$\begin{aligned} \bar{x}_{01} - x_1^+ &= b_{21}(-1)^s\sqrt{\frac{\mu + c_{21}\lambda_2^j(x_{02} + \ldots) - \gamma_1^{-i}(\bar{y}_{11} + \ldots)}{d_{21}}} + a_{21}\lambda_2^j(x_{02} + \ldots), \\ (y_{12} - y_2^-) &= (-1)^s\sqrt{\frac{\mu + c_{21}\lambda_2^j(x_{02} + \ldots) - \gamma_1^{-i}(\bar{y}_{11} + \ldots)}{d_{21}}}. \end{aligned} \tag{4.12}$$

This is a saddle mapping (and contracting in cross-form coordinates) if, for instance, the inequalities

$$\left|\frac{\partial \bar{x}_{01}}{\partial x_{02}}\right| < \frac{1}{2}, \quad \left|\frac{\partial \bar{x}_{01}}{\partial \bar{y}_{11}}\right| < \frac{1}{2}, \quad \left|\frac{\partial y_{12}}{\partial x_{02}}\right| < \frac{1}{2}, \quad \left|\frac{\partial y_{12}}{\partial \bar{y}_{11}}\right| < \frac{1}{2} \tag{4.13}$$

are satisfied. It easily follows from (4.12) that these inequalities are satisfied if

$$\sqrt{\frac{\mu + c_{21}\lambda_2^j(x_{02} + \ldots) - \gamma_1^{-i}(\bar{y}_{11} + \ldots)}{d_{21}}} > 2(|\gamma_1|^{-i} + |\lambda_2|^j). \tag{4.14}$$

Note now that there obviously exists a positive constant S_1, independent of i and j, such that if inequality (4.1) is satisfied, then inequality (4.9) is also satisfied and if the inequality (4.2) is satisfied, then inequlatities (4.10) and (4.14) are simultaneously satisfied. This completes the proof of the theorem.

5. CODING NONWANDERING ORBITS AND NONTRIVIAL HYPERBOLIC SUBSETS

The convenient method of describing the structure of the set N_μ is the construction of the codes for its orbits. We denote the sets of strips σ_k^{01} on Π_1^+, σ_k^{11} on Π_1^-, σ_k^{02} on Π_2^+, and σ_k^{12} on Π_2^- by $\sigma_{01}, \sigma_{11}, \sigma_{02}$, and σ_{12} respectively. We assume that the orbit Λ belongs to N_μ and is not an asymptotic orbit to O_1 and O_2. Then it obviously intersects the neighborhoods Π_1^+, Π_1^-, Π_2^+, and Π_2^- only at the points belonging to the strips from the sets σ_{01}, σ_{11}, σ_{02}, and σ_{12} respectively. Let $(\ldots, M_{\alpha\beta}^{-s}, \ldots, M_{\alpha\beta}^s, \ldots)$, $\alpha = 0, 1$, $\beta = 1, 2$, be successive points of intersection of the orbit Λ and the strips from the sets $\sigma_{\alpha\beta}$. The relations

$$\begin{aligned} &M_{01}^s \in \sigma_{k_s(1)}^{01} \subset \Pi_1^+, \\ &M_{11}^s = T_{01}^{k_s(1)}(M_{01}^s) \in \sigma_{k_s(1)}^{11} \subset \Pi_1^-, \\ &M_{02}^s = T_{12}(M_{11}^s) \equiv f_\mu^{n_1}(M_{11}^s) \in \sigma_{k_s(2)}^{02} \subset \Pi_2^+, \\ &M_{12}^s = T_{02}^{k_s(2)}(M_{02}^s) \in \sigma_{k_s(2)}^{12} \subset \Pi_2^-, \\ &M_{01}^{s+1} = T_{21}(M_{21}^s) \equiv f_\mu^{n_2}(M_{12}^s) \in \sigma_{k_{s+1}(1)}^{01} \subset \Pi_1^+ \\ &\qquad s = 0, \pm 1, \ldots, \end{aligned} \tag{5.1}$$

must obviously be satisfied for these points. Recall that in these relations n_1 and n_2 are natural numbers such that $f_0^{n_1}(M_1^-) = M_2^+$, $f_0^{n_2}(M_2^-) = M_1^+$.

In accordance with (5.1) we define the *coding of the orbit* Λ as an infinite sequence of two symbols $\{1, 2\}$

$$(\ldots, 2, 2, \overbrace{1, 1, \ldots, 1}^{k_s(1)+n_1}, \overbrace{2, 2, \ldots, 2}^{k_s(2)+n_2}, 1, 1 \ldots), \tag{5.2}$$

in which the lengths of successive full segments consisting of symbols "1" and "2" are equal to $(k_s(1) + n_1)$ and $(k_s(2) + n_2)$ respectively. In other words, if a point of the orbit Λ falls in the neighborhood Π_1^+, then, in sequence (5.2) this point is associated with the symbol "1." In what follows, the symbol "1" will be associated with every successive point of the orbit Λ until a certain successive point falls in the neighborhood Π_2^+. We put this final point in correspondence with the symbol "2." We shall put the points of the next iterations into correspondence with the symbol "2" until a certain successive point of the orbit Λ again falls in the neighborhood Π_1^+. We put this final point into correspondence with the symbol "1", and so on.

Codings of form (5.2) can be generalized to orbits from N_μ asymptotic to O_1 and O_2. The orbit to O_1 will have the coding $(\ldots, 1, \ldots, 1, \ldots)$, the orbit to O_2 will have the coding $(\ldots, 2, \ldots, 2, \ldots)$, the orbit Γ_{12} will have the coding $(\ldots, 1, \ldots 1, 2, \ldots, 2, \ldots)$, and the orbit Γ_{21} the coding $(\ldots, 2, \ldots, 2, 1, \ldots, 1 \ldots)$. The orbit, which is α-limiting (ω-limiting, resp.), will be associated with a coding of form (5.2) which has an infinite sequence of symbols "1" appearing at the left (right, resp.) end. Similarly, the orbit from N which is α-limiting (ω-limiting) with respect to O_2 will be associated with a coding of form (5.2) which has an infinite sequence of symbols "2" appearing at the left (right) end.

If the heteroclinic orbit Γ_{21} were structurally stable, then there would exist a one-to-one correspondence between the set N_μ of orbits (all of which would be saddle orbits in this case) and the set of indicated codings. In our case, where Γ_{21} is a structurally unstable heteroclinic orbit, this is not the fact.

Note, first of all, that N_μ cannot contain orbits with codings in which, for a certain s, the numbers $j = k_s(2)$ and $i = k_{s+1}(1)$ satisfy inequality (4.1) since in this case, $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01} = \varnothing$ by virtue of Theorem 1. Second, even if we restrict ourselves to codings in which, for all $s = 0, \pm, 1, \ldots$ the numbers $k_s(2)$ and $k_{s+1}(1)$ satisfy inequality (4.2) with $j = k_s(2)$, $i = k_{s+1}(1)$, there will not be a one-to-one correspondence which is observed in a structurally stable case. To be more precise, here we have the following theorem.

Theorem 2. *For an arbitrary coding of form* (5.2), *in which* $\infty > k_s(1) \geq \bar{k}_1$, $\infty > k_s(2) \geq \bar{k}_2$, *and, for any* $s = 0, \pm 1, \ldots$, *the numbers* $k_s(2)$ *and* $k_{s+1}(1)$ *satisfy inequality* (4.2) *with* $j = k_s(2)$, $i = k_{s+1}(1)$, *there exists in* N_μ *a continuum of orbits of saddle type each of which has the given coding. The set of these orbits is in a one-to-one correspondence with a set of sequences, infinite in both directions, which are composed of two symbols.*

Proof. Inequality (4.2) guarantees that the intersection of the horseshoe $T_{21}\sigma_j^{12}$ and the strip σ_i^{01} is regular and consists of two connection components, which we denote by Δ_{ij}^3 and Δ_{ij}^4. It is clear that the orbits from N_μ which have the same coding of form (5.2) but have points of intersection with different connection components of this kind, must be distinguished, namely, the following sequence, infinite in both directions, will represent a coding which is more exact as compared to (5.2):

$$(\ldots,1,1,\overbrace{2,2,\ldots,2,\alpha_s}^{k_s(2)+n_2},\overbrace{1,1,\ldots,1}^{k_{s+1}(1)+n_1},\overbrace{2,2,\ldots,2,\alpha_{s+1}}^{k_{s+1}(2)+n_2},1,1,\ldots). \tag{5.3}$$

This sequence consists of four symbols $\{1,2,3,4\}$, where the symbol α_s is either "3" or "4" depending on whether the corresponding point of the orbit from N_μ belongs to the component Δ^3_{ij} or to the component Δ^4_{ij} of the strip σ^{01}_i (here $i=k_{s+1}(1)$, $j=k_s(2)$).

Let us show that each sequence (5.3), in which, for every s the numbers $i=k_{s+1}(1)$ and $j=k_s(2)$ satisfy inequality (4.2), is associated with exactly one orbit of the saddle type which has successive points of intersection with the neighborhoods Π^+_1, Π^-_1, Π^+_2, and Π^-_2 for which the following relations are satisfied:

$$\begin{aligned}
&M^s_{01}=T^{(\alpha_s-1)}_{21}(M^{s-1}_{12})\in\Delta^{\alpha_s-1}_{k_s(1)k_{s-1}(2)}\sigma^{01}_{k_s(1)}\subset\Pi^+_1,\\
&M^s_{11}=T^{k_s(1)}_{01}(M^s_{01})\in\sigma^{11}_{k_s(1)}\subset\Pi^-_1,\\
&M^s_{02}=T_{12}(M^s_{11})\in\sigma^{02}_{k_s(2)}\subset\Pi^+_2,\\
&M^s_{12}=T^{k_s(2)}_{02}(M^s_{02})\in\sigma^{12}_{k_s(2)}\subset\Pi^-_2,\\
&M^{s+1}_{01}=T^{(\alpha_s)}_{21}(M^s_{12})\in\Delta^{\alpha_s}_{k_{s+1}(1)k_s(2)}\in\sigma^{01}_{k_{s+1}(1)}\subset\Pi^+_1,\\
&\qquad s=0,\pm1,\ldots.
\end{aligned} \tag{5.4}$$

Let us consider the sequence of mappings (in accordance with (5.4))

$$\begin{aligned}
&(\bar x^s_{01},\bar y^s_{01})=T^{(\alpha_s-1)}_{21}(x^{s-1}_{12},y^{s-1}_{12}),\quad (\bar x^s_{11},\bar y^s_{11})=T^{k_s(1)}_{01}(x^s_{01},y^s_{01}),\\
&(\bar x^s_{02},\bar y^s_{02})=T_{12}(x^s_{11},y^s_{11}),\qquad\qquad (\bar x^s_{12},\bar y^s_{12})=T^{k_s(2)}_{02}(x^s_{02},y^s_{02}),\\
&(\bar x^{s+1}_{01},\bar y^{s+1}_{01})=T^{(\alpha_s)}_{21}(x^s_{12},y^s_{12}),\\
&\qquad s=0,\pm1,\ldots,
\end{aligned} \tag{5.5}$$

and show that system (5.5) has a unique fixed point.

We set $k_s(1)=k$, $k_s(2)=j$, $k_{s+1}(1)=i$ in order to simplify the calculations that will follow. Let us consider the mapping $T_{12}T^{k_s(1)}_{01}\equiv T_{12}T^k_{01}\colon\sigma^{01}_k\to\sigma^{02}_j$. Since (3.2) $y_{01}=\gamma_1^{-k}y_{11}(1+\ldots)$, $y_{02}=\gamma_2^{-j}y_{12}(1+\ldots)$ by virtue of (3.2), we can write the mapping $T_{12}T^k_{01}$ as

$$\begin{aligned}
&\bar x_{02}-x^+_2(\mu)=b_{12}(y_{11}-y^-_1(\mu))+a_{12}\lambda^k_1x_{01}+\ldots,\\
&\gamma_2^{-j}\bar y_{12}(1+\ldots)=c_{12}\lambda^k_1x_{01}+d_{12}(y_{11}-y^-_1(\mu))+\ldots.
\end{aligned} \tag{5.6}$$

Since $d_{12}\neq0$, we can express the coordinate $(y_{11}-y^-_1)$ from the second equation in (5.6), for all sufficiently small μ, in terms of x_{01} and $\bar y_{12}$, namely,

$$y_{11}-y^-_1(\mu)=\frac{\gamma_2^{-j}\bar y_{12}(1+\ldots)-c_{12}\lambda^k_1x_{01}(1+\ldots)}{d_{12}}.$$

Thus, we can rewrite the mapping $T_{12}T_{01}^k$ in the "cross-form"

$$\begin{aligned} \bar{x}_{02} - x_2^+(\mu) &= b_{12}\frac{\gamma_2^{-j}\bar{y}_{12}(1+\ldots) - (c_{12} - a_{12}d_{12})\lambda_1^k x_{01}(1+\ldots)}{d_{12}}, \\ y_{11} - y_1^-(\mu) &= \frac{\gamma_2^{-j}\bar{y}_{12}(1+\ldots) - c_{12}\lambda_1^k x_{01}(1+\ldots)}{d_{12}}. \end{aligned} \tag{5.7}$$

Let us consider the intervals

$$I_1 = \{x_{01} \mid |x_{01} - x_1^+| \le \rho_{\bar{k}_1,\bar{k}_2}\}, \qquad I_2 = \{x_{02} \mid |x_{02} - x_2^+| \le \nu_{\bar{k}_1,\bar{k}_2}\},$$

$$J_1 = \{y_{11} \mid |y_{11} - y_1^-| \le \nu_{\bar{k}_1,\bar{k}_2}\}, \qquad J_2 = \{y_{12} \mid |y_{12} - y_2^-| \le \rho_{\bar{k}_1,\bar{k}_2}\}.$$

We can see from (5.7) that the mapping $T_{12}T_{01}^k$ in the "cross-form" coordinates has the set $I_1 \times J_2$ as its domain of definition, and its range belongs to the set $I_2 \times J_1$. Note that mapping (5.7) is also contracting for sufficiently small i and j since, evidently, the estimates

$$\left|\frac{\partial \bar{x}_{02}}{\partial x_{01}}\right| + \left|\frac{\partial y_{11}}{\partial x_{01}}\right| < C_{12}|\lambda_1|^i < \frac{1}{2},$$

$$\left|\frac{\partial \bar{x}_{02}}{\partial \bar{y}_{12}}\right| + \left|\frac{\partial y_{11}}{\partial \bar{y}_{12}}\right| < C_{12}|\gamma_2|^{-j} < \frac{1}{2}$$

are valid.

When the conditions of Theorem 2 are satisfied, the mapping $T_{ij}^{(\alpha_s)}\colon \sigma_j^{12} \to \Delta_{ij}^{(\alpha_s)} \subset \sigma_i^{01}$ can also be rewritten in the "cross-form" (see the proof of Theorem 1 and, in particular, relation (4.12))

$$\begin{aligned} \bar{x}_{01} - x_1^+(\mu) &= b_{21}(-1)^{(\alpha_s)}\sqrt{\frac{\mu + c_{21}\lambda_2^j(x_{02}+\ldots) - \gamma_1^{-i}(\bar{y}_{11}+\ldots)}{d_{21}}} + a_{21}\lambda_2^j(x_{02}+\ldots), \\ (y_{12} - y_2^-) &= (-1)^{(\alpha_s)}\sqrt{\frac{\mu + c_{21}\lambda_2^j(x_{02}+\ldots) - \gamma_1^{-i}(\bar{y}_{11}+\ldots)}{d_{21}}}. \end{aligned} \tag{5.8}$$

In "cross-form" coordinates this mapping has the set $I_2 \times J_1$ as its domain of definition, and its range belongs to the set $I_1 \times J_2$. Note that mapping (5.8) is also contracting, which fact was established in the proof of Theorem 1.

Thus the sequence of points (5.4) and the sequence of mappings (5.5) corresponding to it are associatied with the sequence of saddle mappings infinite in both directions, which possess the following properties (in "cross-form" coordinates): (1) the range of each mapping belongs to the domain of definition of the successive mapping; (2) all mappings are contracting (with the contraction constant smaller than $\frac{1}{2}$). In this case, the *lemma on a fixed saddle point in a countable product of spaces* is applicable to this sequence [21]. According to this lemma, the sequence of mappings (5.5) has a unique fixed saddle point, which satisfies conditions (5.4). Consequently, when the conditions of Theorem 2 are satisfied, there exists in N a unique orbit, which has a given coding of form (5.3). We have proved the theorem.

6. CLASSES OF TWO-DIMENSIONAL DIFFEOMORPHISMS WITH THE SIMPLEST STRUCTURALLY UNSTABLE HETEROCLINIC CYCLE

In this section we show (by analogy with the homoclinic case [15]) that diffeomorphisms with structurally unstble heteroclinic cycles can be divided into three classes according to the types of description of the set of orbits N_0.

Note that we can always choose coordinates on U_1 and U_2 so that x_2^+ and y_1^- be positive. Then, for $\mu = 0$, the structure of the sets of solutions of inequalities (4.1)–(4.4) depends, first of all, on the signs of quantities $\lambda_2, \gamma_1, c_{21}$, and d_{21}.

The simplest structure of the set of solutions of inequalities (4.1)–(4.4) for $\mu = 0$ is observed in the case of diffeomorphisms corresponding to the following combination of signs: $\lambda_2 > 0$, $\gamma_1 > 0$, $c_{21} < 0$ (Figs. 4a and 4b).

We shall place the diffeomorphisms in which $d_{21} < 0$ (Fig. 4a) in the first class. It is easy to see that in this case, for any $i \geq \bar{k}_1$, $j \geq \bar{k}_2$, and $\mu \leq 0$, inequality (4.1) will always be satisfied, i.e., $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01} = \varnothing$ for all sufficiently large i and j and $\mu \leq 0$. Moreover, in this case the horseshoes $T_{21}(\sigma_j^{12})$ and strips σ_i^{01} will lie on Π_1^+ on different sides of $W^s_{\text{loc}}(O_1)$. As was shown in [18], here the srtucture of the set N_μ is trivial for $\mu \leq 0$, namely, $N_0 = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$, and $N_\mu = \{O_1, O_2, \Gamma_{12}\}$ for $\mu < 0$. As was pointed out in [18], diffeomorphisms of the first class with a structurally unstable heteroclinic cycle may lie on the boundary of the Morse-Smale system and systems with a complicated structure.

Diffeomorphisms with a structurally unstable heteroclinic cycle in which $\lambda_2 > 0, \gamma_1 > 0, c_{21} < 0$, $d_{21} > 0$ (Fig. 4b) are referred to the second class. Note that here, for $\mu \leq 0$ and for sufficiently large $\bar{k}_1$ and $\bar{k}_2$, inequality (4.2) will always be satisfied, i.e., for any $i \geq \bar{k}_1$ and $j \geq \bar{k}_2$ the horseshoes $T_{21}(\sigma_j^{12})$ and strips σ_i^{01} have regular intersections. Then, according to Theorem 2, for $\mu \leq 0$ all orbits of the set N_μ, except for Γ_{21} for $\mu = 0$, are saddle orbits. In this case, if $\mu < 0$, then the set N_μ has a hyperbolic structure and the orbits N_μ are in one-to-one correspondence with the orbits of the topological Bernoulli scheme consisting of four symbols $\{1, 2, 3, 4\}$. In this case, the attainment of the bifurcation surface H_2 for $\mu = 0$ is followed by a "merging" of two heteroclinic orbits with codings $(\ldots, 2, \ldots, 2, 3, 1, \ldots, 1, \ldots)$ and $(\ldots, 2, \ldots, 2, 4, 1, \ldots, 1, \ldots)$ into one (namely, the orbit Γ_{21}).

Diffeomorphisms with a structurally unstable heteroclinic cycle, which correspond to other combinations of signs of the quantities $\lambda_2, \gamma_1, c_{21}$ and d_{21}, are referred to the third class (these are, for instance, the diffeomorphisms shown in Figs. 4c and 4d). By virtue of Theorem 2, for all sufficiently small μ the set N_μ contains nontrivial hyperbolic subsets, which can be described as follows.

Let us consider the subsystem Ω_μ of the topological Bernoulli scheme of four symbols $\{1, 2, 3, 4\}$ which satisfies the following conditions:

(1) Ω_μ contains orbits $(\ldots, 1, \ldots, 1, \ldots)$ and $(\ldots, 2, \ldots, 2, \ldots)$;

(2) Ω does not contain orbits which would have segments of length exceeding unity and which would be composed of the symbols "3" and "4;"

(3) the symbol "1" cannot be followed by the symbol "3" or "4," and the symbol "2" cannot be followed by the symbol "1;" the symbol "3" or "4" is necessarily followed by the symbol "1;"

(4) the length of any full segment, which is composed of the symbol "1," is not smaller than $\bar{k}_1 + n_1$, and that composed of the symbol "2" is not smaller than $\bar{k}_2 + n_2 - 1$.

(5) Let $k_s(2) + n_2 - 1$ and $k_{s+1}(1) + n_1$ be the lengths of full segments which are composed of the symbols "2" and "1" and follow one another. Then, for any s, the numbers $j = k_s(2)$ and $i = k_{s+1}(1)$ satisfy inequality (4.2).

Similarly [15, 19], the following theorem can be deduced from Theorems 1 and 2.

Theorem 3. *Let f be a diffeomorphism of the third class. Then we can indicate in N_μ a subsytem $\tilde{N}_\mu$ such that, first, $f|_{\tilde{N}_\mu}$ is a conjugate of Ω_μ and, second, all orbits of the subsystem $\tilde{N}_\mu$ are of a saddle type.*

In turn, diffeomorphisms of the third class can be divided into types each of which will be associated with a definite combination of signs of the quantities $\lambda_1, \gamma_2, c_{21}$, and d_{21}. In the case where λ_1 or γ_2 are negative, the signs of the coefficients c_{21} and d_{21} may change depending on the choice of the heteroclinic points M_1^+ and M_2^-. It is easy to see from (3.3) and (3.4) that if we replace the points M_1^+ and M_2^- by $M_1^{+'} = T_{01}^{m_1} M_1^+$ and $M_2^{-'} = T_{02}^{-m_2} M_2^-$, then the signs of the new coefficients will be

$$\operatorname{sgn} c'_{21} = \operatorname{sgn} c_{21} \times \operatorname{sgn}(\lambda_2)^{m_2} \times \operatorname{sgn}(\gamma_1)^{m_1}, \qquad \operatorname{sgn} d'_{21} = \operatorname{sgn} d_{21} \times \operatorname{sgn}(\gamma_1)^{m_1}. \tag{6.1}$$

Then, without loss of generality, we can assume that $c_{21} > 0$ if λ_2 is negative and $d_{21} > 0$ if γ_1 is negative. Thus we have seven possible different combinations of signs of the quantities $\lambda_2, \gamma_1, c_{21}$, and d_{21}, indicated in

Table 1

	H_3^1	H_3^2	H_3^3	H_3^4	H_3^5	H_3^6	H_3^7
λ_2	+	+	+	+	−	−	−
γ_1	+	+	−	−	+	+	−
c_{21}	+	+	+	−	+	−	+
d_{21}	+	−	+	+	+	−	+

We denote by H_3 a locally connected bifurcation surface of codimension 1 in $\mathrm{Diff}^r(\mathcal{M}^2)$ corresponding to diffeomorphisms of the third class which have fixed saddle points close to O_1 and O_2 and a structurally stable heteroclinic orbit close to Γ_{12} and a structurally unstable heteroclinic orbit close to Γ_{21}. In order to show to which of the seven types the diffeomorphisms on H_3 belong, we shall denote the corresponding bifurcation surfaces by H_3^α, $\alpha = 1, \ldots, 7$.

7. THE EXISTENCE OF NEWHOUSE DOMAINS IN WHICH SYSTEMS HAVING COUNTABLE SETS OF STABLE AND COMPLETELY UNSTABLE PERIODIC ORBITS ARE DENSE

It is sufficiently obvious that by an arbitrarily small C^r-smooth perturbation of the diffeomorphism f_0 we can obtain a situation where the perturbed system f', which, generally speaking, no longer belongs to the bifurcation "film" H, will have structurally unstable homoclinic orbits either

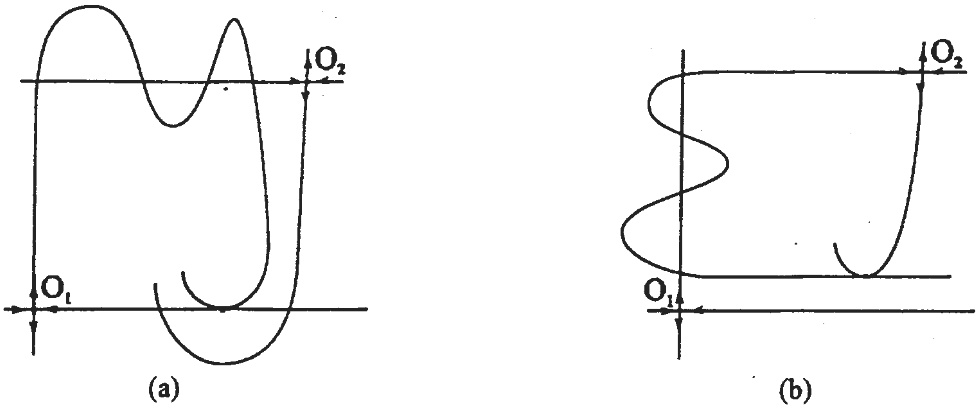


Fig. 12.

of the fixed point O_1 (Fig. 12a) or of the fixed point O_2 (Fig. 12b). Thus, by virtue of [3] we find that in any neighborhood of the diffeomorphism f_0 in $\mathrm{Diff}^r(\mathcal{M}^2)$ there exist Newhouse domains, which are connected with the homoclinic tangencies of both point O_1 and point O_2.

It seems to be natural that if the saddle values σ_1 and σ_2 of the points O_1 and O_2 are simultaneously smaller than unity (larger than unity), then, just as in homoclinic case, the indicated Newhouse domains will not have systems, which would possess completely unstable (stable) periodic orbits lying in the small neighborhood of the cycle. This is indeed the case since here the following statement is valid.

Statement 2. *Let f_0 be a diffeomorphism with the simplest structurally unstable heteroclinic cycle C and suppose that the saddle values σ_1 and σ_2 are simultaneously either smaller or larger than unity. Then there exists a neighborhood U of the cycle C such that neither f_0 nor the diffeomorphisms, which are sufficiently close to f_0, have in U either completely unstable if $\sigma_1 < 1$, $\sigma_2 < 1$, or completely stable if $\sigma_1 > 1$, $\sigma_2 > 1$, periodic orbits.*

Proof. Let, for definiteness, $\sigma_1 < 1$, $\sigma_2 < 1$. Suppose that $\tilde{f}$ is a diffeomorphism, which is sufficiently close to f_0, and let Λ be an n-circuit periodic orbit of the diffeomorphism $\tilde{f}$. Suppose that Λ intersects the neighborhood Π_1^+ at successively arranged points belonging to the strips $\sigma_{i_s}^{01}$, $s = 1, \ldots, n$, and intersects the neighborhood Π_2^+ at successively arranged points belonging to the strips $\sigma_{j_s}^{01}$, $s = 1, \ldots, n$. Then a point of the orbit Λ, say, the point M_{i_1}, belonging to the strip $\sigma_{i_1}^{01}$, is a fixed point of the following mapping performed in n circuits:

$$\tilde{T}_{i_1 j_1 \ldots i_n j_n} \equiv \tilde{T}_{21}\tilde{T}_{02}^{j_n} \ldots \tilde{T}_{21}\tilde{T}_{02}^{j_1}\tilde{T}_{12}\tilde{T}_{01}^{i_1}. \tag{7.1}$$

The Jacobian of this mapping, calculated at the point M_{i_1}, is equal to the product of the Jacobians of the factor-mappings from (7.1). Since f and $\tilde{f}$ are close, the Jacobians of the global mappings are also close (since they are mappings performed in a finite number of iterations), and the saddle values are close (in any event $\tilde{\sigma}_1 < 1$, $\tilde{\sigma}_2 < 1$). Consequently, the Jacobian of the mapping $\tilde{T}_{i_1 j_1 \ldots i_n j_n}$ is a quantity of the order

$$(J_{12}J_{21})^n \tilde{\sigma}_1^{i_1+\ldots+i_n} \tilde{\sigma}_2^{j_1+\ldots+j_n}, \tag{7.2}$$

i.e., smaller than unity by the hypothesis of the theorem. This means that a periodic orbit cannot be completely unstable.

The case $\sigma_1 > 1$, $\sigma_2 > 1$ can be considered by a complete analogy, i.e., it suffices to consider the diffeomorphism f^{-1} instead of f.

Thus, in the cases where $\sigma_1 < 1$, $\sigma_2 < 1$ or $\sigma_1 > 1$, $\sigma_2 > 1$ the existence of only "classical" Newhouse domains is possible in the vicinity of f_0 in which systems with a countable set of either only stable or completely unstable periodic orbits, respectively, are dense. Here, in the neighborhood U, stable and completely unstable orbits cannot coexist.

As we shall show below, a situation is completely different in the case where the saddle values σ_1 and σ_2 lie on different sides of unity. In this case, Newhouse domains connected with the homoclinic tangencies of the points O_1 and O_2 can "overlap" and, therefore, Newhouse domains may exist here in which diffeomorphisms having simultaneously a countable number of completely unstable orbits are dense.

We shall now prove the validity of this statement, but shall show that the Newhouse domains of the indicated type are observed when we consider parametric families, which are transversal to the bifurcation surface H of two-dimensional diffeomorphisms with a structurally unstable heteroclinic cycle.

The following fundamental theorem is valid.

Theorem 4. *Let f_μ be a one-parameter family of two-dimensional diffeomorphisms of the class C^r ($r \geq 3$) which is smooth with respect to the parameter μ. We assume that the family f_μ is transversal to the bifurcation surface H of diffeomorphisms with a structurally unstable heteroclinic cycle, and $f_0 \in H$. We also assume that in f_0 the saddle values of the points O_1 and O_2 are on different sides of unity. Then, on any interval $[-\mu_0, \mu_0]$ of the values of the parameter μ there exists a countable set of intervals Δ_i^1, which accumulate to $\mu = 0$ as $i \to \infty$, such that*

(1) on Δ_i^1 the values of μ, for which the family f_μ unfolds generically the homoclinic tangency of the point O_1, are also dense;

(2) on Δ_i^1 the values of μ, for which f_μ has a structurally unstable heteroclinic cycle containing the points O_1 and O_2 and the heteroclinic orbits $\Gamma_{12}(\mu)$, where $\Gamma_{12}(0) = \Gamma_{12}$, and $\tilde{\Gamma}_{21}(\mu) \subset W_\mu^u(O_2) \cap W_\mu^s(O_1)$, are dense. At the points of the orbit $\tilde{\Gamma}_{12}(\mu)$ the manifolds $W_\mu^u(O_2)$ and $W_\mu^s(O_1)$ have a quadratic tangency;

(3) on Δ_i^1 the values of μ, for which f_μ simultaneously has a countable number of stable and a countable number of completely unstable periodic orbits, are dense.

Proof. We shall show, first of all, that item 3 of the theorem follows from item 1. We shall assume, for definiteness, that $\sigma_1 > 1$, $\sigma_2 < 1$. Let $\mu = \mu_i \in \Delta_i^1$. Then, by virtue of statement 1 of the theorem, there exists, arbitrarily close to μ_i, a $\mu = \mu_i^1$ such that $f_{\mu_i^1}$ has a structurally unstable homoclinic orbit $\tilde{\Gamma}_1$ of the point O_1. Since $\sigma_1 > 1$, it follows, according to the theorem on a cascade of sinks (sources), that in any heighborhood of the point μ_i^1 there exists an interval $\delta_1 \in \Delta_i^1$ of values of μ such that for $\mu \in \delta_1$ the family f_μ has a completely unstable periodic orbit. Furthermore, on the interval δ_1, again by virtue of Statement 1 of the theorem, there exists a $\mu = \mu_i^2$ such that $f_{\mu_i^2}$ has a structurally unstable homoclinic orbit $\tilde{\Gamma}_2$, this time of the point O_2. Since $\sigma_2 < 1$, it follows, again according to the theorem on a cascade of sinks (sources), that there exists an interval $\delta_2 \subset \delta_1$ such that for $\mu \in \delta_2$ the family f_μ has a stable periodic orbit. Thus we find that for $\mu \in \delta_2$ the diffeomorphism f_μ simultaneously has a stable periodic orbit and a completely unstable periodic orbit which lie in U (these orbits lie entirely in certain small neighborhoods of the homoclinic orbits $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$). Similarly, on the interval δ_2 we find a subinterval δ_4 such that for $\mu \in \delta_4$ the diffeomorphism f_μ has two stable orbits and two completely unstable periodic orbits lying in U.

Thus we obtain a countable set of nested intervals

$$\delta_2 \supset \delta_4 \supset \ldots \supset \delta_{2n} \supset \ldots$$

such that for $\mu \in \delta_{2n}$ the diffeomorphism f_μ simultaneously has n stable and n completely unstable periodic orbits. This completes the proof of item 2 of the theorem.

For definiteness, we shall begin the proof of item 1 of the theorem with the case of a cycle in which $\gamma_1 > 0$, $\lambda_2 > 0$, $c_{21} > 0$, $d_{21} > 0$ (Fig. 4c), and shall assume that $\sigma_1 > 1$, $\sigma_2 < 1$. Then, for $\mu = 0$, the diffeomorphism f_μ has in U a structurally unstable heteroclinic orbit (nàmely, Γ_{21}), for $\mu < 0$ it has two structurally stable heteroclinic orbits, which are close to Γ_{21}, and for $\mu > 0$ it does not have in U any heteroclinic orbits passing through the points O_2 and O_1. Note, that in the case under consideration, for $\mu \geq 0$ the diffeomorphism f_μ cannot have orbits which would be homoclinic to O_1 and would lie in U since all curves from the set $W^u(O_1) \cap \Pi_1^+$ lie above the "parabola" $T_{21}(\mu)(W^u_{\text{loc}}(O_2)) \cap \Pi_2^- \subset W^u(O_2)$, and, consequently, (for $\mu \geq 0$), above $W^s_{\text{loc}}(O_1)$. However, for $\mu < 0$ the family f_μ already can have orbits homoclinic to O_1. Moreover, we have the following lemma.

Lemma 1. *There exists a sequence $\{\mu_i\}$ of values of the parameter μ such that $\mu_i \to 0$ as $i \to \infty$ and the diffeomorphism f_{μ_i} has a structurally unstable one-circuit orbit Γ_{1i} homoclinic to O_1. In this case, the tangency of the manifolds $W^s(O_1)$ and $W^s(O_2)$ along Γ_{1i} is quadratic and, for $\mu = \mu_i$, the family f_μ is transversal to the bifurcation surface H_{1i} of systems with a structurally unstable homoclinic orbit, which is close to Γ_{1i}.*

Proof. We denote by $l^1_u(\mu)$ the segment $T_{12}(W^u_{\text{loc}}(O_1)) \cap \Pi_2^+$ of the unstable manifold of the point O_1. For small μ the segment $l^1_u(\mu)$ intersects the manifold $W^s_{\text{loc}}(O_2)$ transversally. It is easy to see from (3.2) that the curves $l^{1i}_u \equiv T^i_{02}(l^1_u \cap \sigma^{02}_i)$ on Π_2^- accumulate regularly (see Definition 2) to the segment $l^2_u \equiv W^u_{\text{loc}}(O_2) \cap \Pi_2^-$, which is defined by the equation $x_{12} = 0$ on Π_2^-. In this case, by virtue of (3.2) and (3.5), the equation of the curve l^{1i}_u is $x_{12} = \lambda_2(\mu)^i x_2^+(\mu)(1 + \ldots)$ and, by virtue of (3.6), for $\mu = \mu_i \equiv -c_{21}\lambda_2^i x_2^+(1 + \ldots)$ the diffeomorphism f_μ has in U a one-circuit structurally unstable homoclinic orbit Γ_{1i} of the point O_1. For large i, the tangency of the stable and the unstable manifold of the point O_1 at the points of the orbit Γ_{1i} is quadratic, and, by virtue of Statement 1, the family f_μ is transversal to the bifurcation surface H_i of diffeomorphisms with a structurally unstable homoclinic curve which is close to Γ_{1i}.

Lemma 1 and the Newhouse theorem [3] yield the following lemma.

Lemma 2. *There exists a sequence $\{\tilde{\Delta}^1_i\}$ of ranges of values of the parameter μ which accumulate to $\mu = 0$, the sequence being such that on the interval $\tilde{\Delta}^1_i$ the values of the parameter μ, for which the family of diffeomorphisms f_μ unfolds generically the homoclinic tangency of the point O_1, are dense.*

Lemma 3. *Let $\mu = \mu_i^*$ be a value of the parameter μ from the interval $\tilde{\Delta}^1_i$ such that $f_{\mu_i^*}$ has* *a structurally unstable homoclinic orbit of the point O_1. Then there accumulates to the point μ_i^* a countable set of values μ_{ij}^* of the parameter μ such that for $\mu = \mu_{ij}^*$ the family of diffeomorphisms f_μ unfolds generically a homoclinic tangency of the point O_2.*

Proof. Suppose that for $\mu = \mu_i^* \in \Delta_i^1$ the diffeomorphism f_μ has a structurally unstable homoclinic orbit of the point O_1 and let $M_{1*}^- \in W_{\mathrm{loc}}^u(O_1) \cap \Pi_1^-$ and $M_{1*}^+ \in W_{\mathrm{loc}}^s(O_1) \cap \Pi_1^+$ be a pair of homoclinic points of this orbit. Then $f_{\mu_i^*}^{q_i}(M_{1*}^-) = M_{2*}^+$ for a certain natural q_i. Note that for $\mu = \mu_i^*$, since $\mu_i^* < 0$, the piece $l_u \equiv T_{21}(W_{\mathrm{loc}}^u(O_2))$ of the unstable manifold of the point O_2 intersects the piece $W_{\mathrm{loc}}^s(O_1)$ of the stable manifold of the point O_1 transversally at two points (which are close to M_1^+). We denote these points by M_{11} and M_{12}. We take a piece l_{u1} of the curve l_u, that contains one of these points, say, M_{11}. As $k \to \infty$, the curves $l_{u1}^k \equiv T_{01}^k(l_{u1}) \cap \Pi_1^-$ accumulate regularly to the piece $W_{\mathrm{loc}}^u(O_1) \cap \Pi_1^-$ of the unstable manifold of the point O_1 and, consequently, to the piece $f_\mu^{q_i}(W_{\mathrm{loc}}^u(O_1) \cap \Pi_1^-)$ (for a fixed i). For $\mu = \mu_i^*$ the latter has a quadratic tangency with $W_{\mathrm{loc}}^s(O_1) \cap \Pi_1^+$ at a certain point M_i^*, which is close to M_1^+. Furthermore, for all sufficiently small μ, the piece $l_{s2} \equiv T_{12}^{-1}(W_{\mathrm{loc}}^s(O_2)) \cap \Pi_1^-$ of the stable manifold of the point O_2 transversally intersects the piece $W_{\mathrm{loc}}^u(O_1) \cap \Pi_1^-$ of the unstable manifold of the point O_1 (at a point which is close to M_1^-). Consequently, the curves $l_{s2}^j \equiv T_{01}^{-j}(l_{s2}) \cap \Pi_1^+$ accumulate regularly to the piece $W_{\mathrm{loc}}^s(O_1) \cap \Pi_1^+$ of the stable manifold of the point O_1. Thus we have two families $\{l_{u1}^k\}$ and $\{l_{s2}^j\}$ of curves, C^r-smooth and smoothly dependent on the parameter which accumulate regularly to the curves $f_\mu^{q_i}(W_{\mathrm{loc}}^u(O_1) \cap \Pi_1^-)$ and $W_{\mathrm{loc}}^s(O_1) \cap \Pi_1^+$ respectively. For $\mu = \mu_i^*$ the latter have a quadratic tangency. Moreover, as follows from Lemma 2, for $\mu = \mu_i^*$ the family f_μ unfolds generically the tangency of the curves $f_\mu^{q_i}(W_{\mathrm{loc}}^u(O_1) \cap \Pi_1^-)$ and $W_{\mathrm{loc}}^s(O_1) \cap \Pi_1^+$. Then Lemma 3 follows immediately from Statement 1.

The Newhouse theorem and Lemma 3 give the following result.

Lemma 4. *To the value $\mu = \mu_i^*$ there accumulate a countable set of intervals $\tilde{\Delta}_{ij}^2$ of values of the parameter μ such that on $\tilde{\Delta}_{ij}^2$ the values of μ, for which the family f_μ unfolds generically the homoclinic tangency of the point O_2, are dense.*

It is clear that for large j the intervals $\tilde{\Delta}_{ij}^2$ lie within $\tilde{\Delta}_i^1$. Then, by virtue of Lemma 2, in the intervals $\tilde{\Delta}_{ij}^2$ the values of the parameter μ, for which f_μ has a structurally unstable homoclinic orbit of the point O_1, are dense and the values of the parameter μ, for which f_μ has a structurally unstable homoclinic orbit of the point O_2, are also dense.

Now the proof of item 2 of the theorem is sufficiently obvious. Indeed, first, for all sufficiently small μ (including the values of μ from the interval $\tilde{\Delta}_{ij}^2$) there is in U a heteroclinic orbit $\Gamma_{12}(\mu)$, $\Gamma_{12}(\mu) \to \Gamma_{12}$ as $\mu \to 0$, along which the mainfolds $W^u(O_1)$ and $W^s(O_2)$ intersect transversally. Second, as follows from the proofs of Lemmas 1–4, on the interval $\tilde{\Delta}_{ij}^2$ the values of μ, for which the family f_μ unfolds generically the heteroclinic tangency of the manifolds $W^u(O_2)$ and $W^s(O_1)$ are dense. The moments of these tangencies correspond to the exstence in the diffeomorphism f_μ of a structurally unstable heteroclinic cycle containing the points O_1 and O_2. This completes the proof of Theorem 4 for the case under consideration (i.e., the case $f_0 \in H_3^1$).

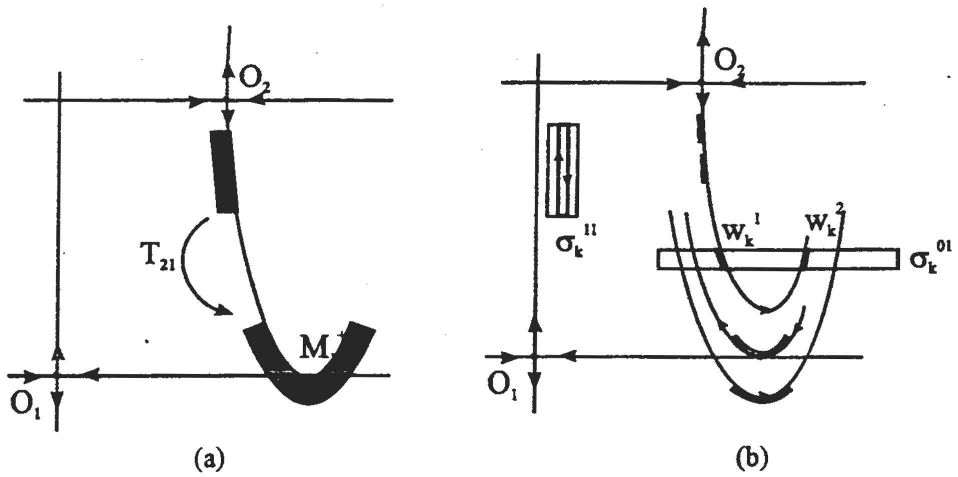


Fig. 13.

In the case of diffeomorphisms with a structurally unstable heteroclinic cycle of a different type, the proof is completely similar, except for the case where f_0 is a diffeomorphism of the second class ($f_0 \in H_2$) (Fig. 4b), (i.e., the case $\lambda_2 > 0$, $\gamma_1 > 0$, $c_{21} < 0$, and $d_{21} > 0$). The matter is that the existence of Newhouse intervals Δ_i^1 was proved precisely on the subinterval of the interval $[-\mu_0, \mu_0]$, where f_μ has two structurally stable heteroclinic orbits, which are close to Γ_{21}. However, as was established above (see Section 5), for diffeomorphisms of the family f_μ, where $f_0 \in H_2$, the set N_μ has a hyperbolic sturcture for the values of the parameter μ from this subinterval. Naturally, this family can have Newhouse intervals only for positive μ, i.e., in the class of diffeomorphisms without heteroclinic orbits, which are close to Γ_{21}. Nevertheless, the fundamental theorem is also valid in this case and the proof follows immediately from Lemma 5.

Lemma 5. *Let f_μ be a one-parameter family, which is transversal for $\mu = 0$ to the bifurcation surface H_2. Then, on the interval $(0, \mu_0]$ there exists a countable set of values of the parameter μ: $\mu = \tilde{\mu}_k$ such that $\tilde{\mu}_k \to 0$ as $k \to \infty$ and, for $\mu = \tilde{\mu}_k$, the family f_μ unfolds generically the second heteroclinic tangency of the manifolds $W^s(O_1)$ and $W^u(O_2)$ which corresponds to a third-class diffeomorphism with the simples structurally unstable heteroclinic cycle.*

Figure 13 illustrates the method for proving this lemma. First we take the diffeomorphism f_μ, which is close to f_0 (Fig. 13a) and such that the piece $T_{21}(W^u_{\mathrm{loc}}(O_2)) \cap \Pi_1^+$ of the unstable manifold of the point O_2 lies above the piece $W^s_{\mathrm{loc}}(O_1) \cap \Pi_1^+$ of the stable manifold of the point O_1 and intersects the strip σ_k^{01} along two components W_k^1 and W_k^2 (Fig. 13b). It is obvious that the value of μ can be chosen such ($\mu \sim -c_{21}\lambda_2^j x_2^+$) that the "parabola" $T_{21}T_{02}^j T_{12} T_{01}^k (W_k^2)$ (for a certain $j \geq \bar{k}_2$) touches the interval $W^s_{\mathrm{loc}}(O_1) \cap \Pi_1^+$. It is easy to make sure that this tangency is associated with a third-class structurally unstable cycle (corresponds to the diffeomorphism from H_3^1).

7.1. Newhouse domains in the vicinity of diffeomorphisms with a structurally unstable heteroclininc cycle of the general type. The fundamental theorem (to be more precise, its second part) is also valid in the case of one-parameter families which are transversal to the bifurcation surface of diffeomorphisms with a structurally unstable heteroclinic cycle of the genral type (Fig. 3a).

Namely, let the diffeomorphismf_0 have structurally stable saddle periodic orbits $P_1, \dots, P_n$ such that $\tilde{\Gamma}_{ii+1} \subset W^u(P_i) \cap W^s(P_{i+1})$, $\tilde{\Gamma}_{n1} \subset W^u(P_n) \cap W^s(P_1)$, $i = 1, \dots, n-1$. We assume that all the indicated intersections are transversal and only one intersection, say, the intersecion of the manifolds $W^u(P_n)$ and $W^s(P_1)$, is nontransversal and, moreover, $W^u(P_n)$ and $W^s(P_1)$ have a quadratic tangency at the points of the heteroclinic orbit $\tilde{\Gamma}_{n1}$.

Let us consider a one-parameter family f_μ of C^r-smooth ($r \geq 3$) diffeomorphisms, smoothly dependent on μ, which is transversal to the bifurcation surface of diffeomorphisms, with a structurally unstable heteroclinic cycle, which are close to f_0.

Theorem 5. *Let f_μ be a one-parameter family of C^r-smooth ($r \geq 3$) diffeomorphisms, which, for $\mu = 0$ is transversal to the bifurcation surface of diffeomorphisms with a structurally unstable heteroclinic cycle which are close to f_0. We assume that at least two periodic orbits from the set $\{P_1, \dots, P_n\}$ have saddle values (the moduluds of the product of multiplicators), one of which is larger and the other is smaller than unity. Then, on any interval $[-\mu_0, \mu_0]$, where $\mu_0 > 0$, there exists a countable set of Newhouse subintervals Δ_i^1 such that in Δ_i^1 the values of the parameter μ, for which f_μ simultaneously has a countable set of stable orbits and a countable set of completely unstable orbits, are dense.*

Proof. Let q be a degree of the diffeomorphism of f_μ such that the points of periodic orbits $P_1, \dots, P_n$ are fixed for $F_\mu \equiv f_\mu^q$. We choose exactly one points O_i, $i = 1, \dots, n$, from each cycle P_i and consider for $\mu = 0$, for the diffeomorphism F_0, a heteroclinic cycle, which includes the fixed points $O_1, \dots, O_n$, and the heteroclinic orbits $\Gamma_{ii+1} \subset \tilde{\Gamma}_{ii+1}$, $\Gamma_{n1} \subset \tilde{\Gamma}_{n1}$, where the orbits Γ_{ii+1} of the diffeomorphismF_0 consist of the corresponding points of the orbits $\tilde{\Gamma}_{ii+1}$ of the diffeomorphismf_0, taken in q iterations. For the diffeomorphism F_0 we have a cycle such that the intersection of the manifolds $W^u(O_i)$ and $W^s(O_i + 1)$, $i = 1, \dots, n-1$, along the trajectory Γ_{ii+1} is transversal and $W^u(O_n)$ has a quadratic tangency with $W^s(O_1)$ along the orbit Γ_{n1}. It is clear that for $\mu = 0$ the family F_μ unfolds generically the heteroclinic tangency.

By assumption, the saddle values of at least two points from $O_1, \dots, O_n$ lie on different sides of unity. We shall first consider the case where O_1 and O_n are these points. Since the intersections of the manifolds $W^u(O_i)$ and $W^s(O_i + 1)$, $i = 1, \dots, n-1$, are transversal, it follows, by the C^r-λ-lemma, that there exists in U a heteroclinic orbit Γ_{1n} along which the manifolds $W^u(O_1)$ and $W^s(O_n)$ intersect transversally. Let us consider the heteroclinic cycle $C = \{O_1, O_2, \Gamma_{1n}, \Gamma_{n1}\}$. Obviously, it is the simplest structurally unstable heteroclinic cycle, and, for the family F_μ containing the diffeomorphism F_0 with such a cycle Theorem 5 immedialtely follows from the fundamental theorem.

Let us now consider the case where the saddle values of the points O_1 and O_j, with $j \in \{2, \dots, n-1\}$, lie on different sides of unity. First, we take some heteroclinic orbit $\Gamma_{1j} \subset U$ along which the manifolds $W^u(O_1)$ and $W^s(O_j)$ intersect transversally. Second, we shall consider the structurally unstable heteroclinic point $M_1^+ \in W^s_{\text{loc}}(O_1) \cap W^u(O_n)$ and its neighborhood Π_1^+. We denote by l_0^u a connected piece of the set $W^u(O_n) \cap \Pi_1^+$, which contains the point M_1^+. It follows from the C^r-λ-lemma that in Π_1^+ there lies a countable set of curves w_k from the set $W^u(O_j) \cap \Pi_1^+$, which for sufficiently small μ accumulate regularly to l_μ^u as $k \to \infty$. Then there exists a countable set of values of the parameter μ, $\mu = \mu_k$, for which the curves $w_k(\mu_k)$ have a quadratic tangency with $W^s_{loc}(O_1) \cap \Pi_1^+$. Correspondingly, for $\mu = \mu_k$ the diffeomorphism F_μ has a structurally unstable heteroclinic orbit Γ_{jk} at the points of which the manifolds $W^u(O_j)$ and $W^s(O_1)$ have a quadratic tangency. By virtue of Statement 1, for $\mu = \mu_k$, the family F_μ unfolds generically a heteroclinic tangency which corresponds to the existence in F_{μ_k} of the simplest structurally unstable cycle $C_\mu = \{O_1, O_j, \Gamma_{1j}, \Gamma_{jk}\}$, for which the saddle values of the points O_1 and O_j lie on the different sides of unity. This completes the proof of the theorem.

8. MODULI OF Ω-CONJUGACY OF THE THIRD CLASS DIFFEOMORPHISMS WITH A STRUCTURALLY UNSTABLE HETEROCLINIC CYCLE

In this section and in next two sections we shall study the structure of the set of nonwandering orbits and their bifurcations, properly of third-class diffeomorphisms with the structurally unstable heteroclinic cycle (i.e., in the class of systems on the bifurcation surface H_3). The aim that we pursue is to find the conditions for the existence of stable and completely unstable orbits of these systems. We shall show that orbits of this kind appear as a result of the simplest saddle-node bifurcations the control parameter of which is the quantity $\theta = -\ln|\lambda_2|/\ln|\gamma_1|$. We shall also show that when θ varies continuously, homoclinic bifurcations will also "continuously" occur in diffeomorphisms on H_3. Such a considerable dependence of the structure of the set of periodic and homoclinic orbits precisely on θ is not accidental. This is a consequnce of the fact that θ is a modulus of Ω-conjugacy of diffeomorphisms on H_3 (i.e., a continuous invariant of topological conjugacy on a set of nonwandering orbits). This section is devoted to the proof of the last statement and of some other results concerning the moduli of Ω-conjugacy of diffeomorphisms with structurally unstable heteroclinic cycles.

Recall the definition of the modulus.

Definition 3 [11, 7]. We say that the system f has a *modulus* if, in the space of dynamical systems, f lies in a certain Banach manifold $\mathcal{M}$, on which the continuous, locally nonconstant functional h is defined which possesses the following property: if $f_1, f_2 \in \mathcal{M}$ and f_1 and f_2 are equivalent, then $h(f_1) = h(f_2)$. The system f has m *modulus* if f lies in a certain Banach manifold on which there exist m independent moduli. Finally, we say that f has a *countable set of moduli* if f has m moduli for any preassigned m.

It follows directly from [22] that the invariant

$$\theta = -\frac{\ln|\lambda_2|}{\ln|\gamma_1|}$$

is a modulus of topological conjugacy of diffeomorphisms with a structurally unstable heteroclinic cycle, including first-class and second-class diffeomorphisms. However, if we restrict the consideration to the conditions of Ω-conjugacy, then, on the corresponding bifurcation surfaces, the latter will be Ω-structurally stable. At the same time, third-class diffeomorphisms will possess moduli of Ω-conjugacy.

Let U and U' be some neighborhoods of the heteroclinic cycles C and C' of the diffeomorphisms f and f'. Suppose that $\Omega(f)$ and $\Omega(f')$ are sets of nonwandering orbits lying entirely in U and U' respectively. Note that $\Omega(f)$ does not always coincide with N_0 (namely, $\Omega(f) \subseteq N_0$), but all the same, the nontrivial set $\tilde{N}_0$ (from Theorem 3) is contained in $\Omega(f)$.

Definition 4. We say that f and f' are *locally Ω-conjugate* if there exist neighborhoods U and U' of the heteroclinic cycles C and C' and a homeomorphism $h:\Omega(f) \to \Omega(f')$ such that $h(O_s) = O'_s$, $s = 1,2$, $h(\Gamma_{12}) = \Gamma'_{12}$, $h(\Gamma_{21}) = \Gamma'_{21}$, and the diagram

$$\begin{array}{ccc} \Omega(f) & \xrightarrow{f} & \Omega(f) \\ \downarrow h & & \downarrow h \\ \Omega(f') & \xrightarrow{f'} & \Omega(f') \end{array}$$

is commutative.

Suppose that the heteroclinic points M_s^+, M_s^- and $M_s'^+$, $M_s'^-$, $s = 1,2$, are chosen such that

$$h(M_s^+) = M_s'^+, \qquad h(M_s^-) = M_s'^-. \tag{8.1}$$

We say that these points are *conjugate*. For the diffeomorphism f we shall consider a special neighborhood $V \equiv V(\bar{k}_1, \bar{k}_2) \subseteq U$. By virtue of the commutativity of the diagram, the continuity of h, and condition (8.1), for sufficiently large $\bar{k}_1$ and $\bar{k}_2$ we find that there exists a special neighborhood $V' \equiv V'(\bar{k}_1, \bar{k}_2) \subseteq U'$ such that $h(\Omega(f|_V)) \subset V'$ and the homeomorphism $h:\Omega(f|_V) \to \Omega(f'|_{V'})$ preserves codings of form (5.2). Thus we find that codings of form (5.2) of the corresponding orbits from $\Omega(f)$ and $\Omega(f')$ must coincide. In particular, since $\tilde{N}_0 \subset \Omega(f)$, $\Omega(f')$ must contain a set of orbits codings of which coincide with those of the orbits from $\tilde{N}_0$. Using these obvious properties of Ω-conjugate diffeomorphisms, we can now prove the following result.

Theorem 6. *Let $f, f' \in H_3$ and let f and f' be locally Ω-conjugate in certain neighborhoods U and U' of the heteroclinic cycles C and C'. Then $\theta = \theta'$.*

Proof. We assume that f and f' are Ω-conjugate in certain neighborhoods U and U' of heteroclinic cycles, but $\theta > \theta'$. Let M_s^+, M_s^- and $M_s'^+$, $M_s'^-$, $s = 1,2$, be pairs of conjugate heteroclinic points.

We assume, for definiteness, that the diffeomorphisms f and f' are of the same type as those shown in Fig. 4c, i.e., $\lambda_1 > 0$, $\gamma_2 > 0$, $c_{21} > 0$, $d_{21} > 0$. For f we shall consider the set of pairs (i,j) of natural numbers, which satisfy ineqality (4.2), i.e., i and j such that the horseshoe $T_{21}(\sigma_j^{12})$ regularly intersects the strip σ_i^{01}. If we take the logarithms of inequality (4.2), we get

$$i < j\theta - \tau - S_2(\bar{k}_1, \bar{k}_2), \tag{8.2}$$

where $S_2(\bar{k}_1, \bar{k}_2) \to 0$ as $\bar{k}_1, \bar{k}_2 \to \infty$, and

$$\tau = \frac{1}{\ln|\gamma_1|} \ln \frac{|c_{21}x_2^+|}{|y_1^-|}. \tag{8.3}$$

For f' we shall consider the set of pairs (i,j) of natural numbers satisfying inequality (4.3), i.e., the inequality

$$i \le j\theta' - \tau' + S_2'(\bar{k}_1, \bar{k}_2). \tag{8.4}$$

Note that, in any event, inequality (8.4) is satisfied by all numbers i and j for which $T'_{21}(\sigma_j^{12}) \cap \sigma_i^{01} \neq \varnothing$.

Finally, let us consider the set of pairs (i, j) of natural numbers, which satisfy inequality (8.2) but do not satisfy inequality (8.4). For these i and j the inequality

$$j\theta' - \tau' + \ldots < i < j\theta - \tau + \ldots \tag{8.5}$$

is satisfied, where the dots denote the terms which tend to zero as $\bar{k}_1, \bar{k}_2 \to \infty$. For sufficiently large $\bar{k}_1$ and $\bar{k}_2$, inequality (8.5) has a countable set of integer-valued solutions since $\theta > \theta'$ by assumption. Let $i = i^*$, $j = j^*$ be one of those solutions. Then, since $i = i^*$, $j = j^*$ satisfy inequality (8.2), by virtue of Theorem 2 the diffeomorphism f has nonwandering orbits (of the saddle type that have codings of form (5.2) in which $k_s(2) = j^*$, $k_{s+1}(1) = i^*$ for a certain s. In particular, such a coding exists for a one-circuit periodic orbits which has one point of intersection with the strips $\sigma^{01}_{i^*}$ and $\sigma^{02}_{j^*}$. In this case, its point of intersection with the strip $\sigma^{01}_{i^*}$ is fixed for the mapping $T_{21}T_{02}^{j^*}T_{12}T_{01}^{i^*}\colon \sigma^{01}_{i^*} \to \sigma^{01}_{i^*}$, which, in this case, is similar to the familiar map of Smale's horseshoe. On the other hand, the numbers $i = i^*$, $j = j^*$ do not satisy inequality (8.4), and therefore $T'_{21}(\sigma^{12}_{j^*}) \cap \sigma^{01}_{i^*} = \varnothing$ for f'. Thus, f' cannot have orbits with a coding, which would have adjoining symbols $k_s(2) = j^*, k_{s+1}(1) = i^*$. Consequently, f and f' cannot be Ω-conjugate. We have got a contradiction with the previous assumption that $\theta > \theta'$. The case $\theta < \theta'$ can be considered by analogy. It suffices to change the places of f and f' here. We have thus proved the theorem.

Thus, by virtue of Definition 3 the functional θ is the modulus of Ω-conjugacy of the third-class diffeomorphisms. By analogy with systems with a structurally unstable homoclinic orbit [10, 11, 8], we can show that θ is not a unique Ω-modulus (see, e.g., [23]). Moreover, below we prove the existence of a countable set of Ω-moduli for the third-class diffeomorphisms.

First of all, we have the following theorem.

Theorem 7. *In H_3, the set B such that any diffeomorphism from B has a structurally stable saddle periodic orbit with a structurally unstable homoclinic orbit is dense.*

Proof. Let us consider, for definiteness, the case of diffeomorphisms on H_3^1. We shall consider a one-parameter family f_θ of diffeomorphisms on H_3^1 and show that for any $\theta = \theta_0$ on the interval $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, for any $\varepsilon > 0$, the values of θ^* such that the diffeomorphism f_{θ^*} has a structurally unstable homoclinic orbit of the point O_2 are dense.

We fix $\theta = \theta_0$, and consider, for the diffeomorphism f_{θ_0}, the set of pairs (i, j) for which inequality (4.2) is satisfied. In the case where $\lambda_1 > 0$, $\gamma_2 > 0$, $c_{21} > 0$, $d_{21} > 0$ it assumes the form

$$i < j\theta_0 - \tau + S_2(\bar{k}_1, \bar{k}_2). \tag{8.6}$$

Now we set $\theta = \theta_0 - \varepsilon$ and consider, for the diffeomorphism $f_{\theta_0 - \varepsilon}$, the set of pairs (i, j) for which inequality (4.1) is satisfied, i.e.,

$$i > j(\theta_0 - \tau - \varepsilon) - S_2(\bar{k}_1, \bar{k}_2). \tag{8.7}$$

Let us now consider the set of pairs (i, j) for which inequalities (8.6) and (8.7) are simultaneously satisfied. The set of these pairs is obviously countable for any $\varepsilon > 0$. Let (i^*, j^*) be one of these pairs. For the diffeomorphism f_{θ_0} the strip $\sigma^{01}_{i^*}$ and the horseshoe $T_{21}(\sigma^{21}_{j^*})$ intersect regularly. Then the curve

$$W^s_{i^*} \equiv T_{01}^{-i^*}(T_{12}^{-1} W^s_{\text{loc}}(O_2)) \subset W^s(O_2)$$

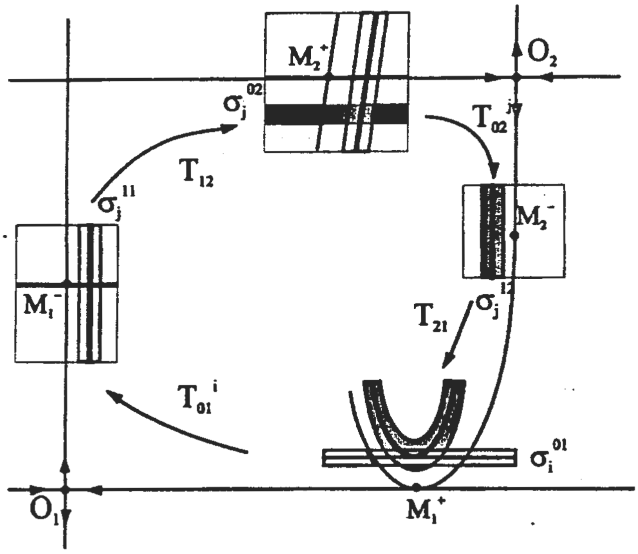


Fig. 14.

lying in the strip $\sigma_{i^*}^{01}$ and the curves

$$W_{i^*j^*}^u \equiv T_{21}T_{02}^{j^*}T_{12}T_{01}^{i^*}T_{21}(W_{\rm loc}^u(O_2)) \subset W^u(O_2)$$

lying in the horseshoe $T_{21}(\sigma_{j^*}^{21})$ intersect. The points of intersection of these curves are homoclinic points of the saddle O_2. For $\theta = \theta_0 - \varepsilon$ the horseshoe $T_{21}(\sigma_{j^*}^{21})$ and the strip $\sigma_{i^*}^{01}$ do not intersect, and therefore there are no corresponding homoclinic points. Consequently, there exists $\theta^* \in (\theta_0 - \varepsilon, \theta_0)$ for which f_{θ^*} has a structurally unstable homoclinic orbit of the point O_2 (Fig. 14). It also follows from (3.2) and (3.4) that if the curves $W_{i^*}^s$ and $W_{i^*j^*}^u$ are tangent, then this tangency is quadratic.

We have thus found that on the bifurcation surface H_3^1 systems with a structurally unstable homoclinic orbit of the saddle O_2 are dense. Note, however, that diffeomorphisms of this kind do not have in U any homoclinic orbits of the saddle O_1. This obviously follows from the fact that for $\lambda_1 > 0$, $\gamma_2 > 0$, $c_{21} > 0$, $d_{21} > 0$ all curves from the set $W^u(O_1) \cap \Pi_1^+$ lie above the curve $T_1 W_{\rm loc}^u(O_2) \cap \Pi_1^+$.

In the case of diffeomorphisms from H_3^2 (i.e., when $\lambda_1 > 0$, $\gamma_2 > 0$, $c_{21} > 0$, $d_{21} < 0$, see Fig. 4d) the density of the values of the parameter θ for which f_θ has a structurally unstable homoclinic orbit of the point O_1 can be proved by analogy. It is also obvious that the diffeomorphisms in H_3^2 do not have in U any homoclinic orbits of the saddle O_2.

Theorem 7 will also be vaild in other cases of third-class diffeomorphisms, but the problem concerning the existence of homoclinic orbits precisely of the saddles O_1 and O_2 will be solved differently in different cases, and here we have the following theorem.

Theorem 8. *The following statements are valid:*

(1) *In $H_3^1 \cup H_3^3$, systems with a structurally unstable homoclinic orbit of the saddle O_2 are dense and there are no systems with a homoclinic orbit of the saddle O_1.*

(2) *In $H_3^2 \cup H_3^6$, systems with a structurally unstable homoclinic orbit of the saddle O_1 are dense*

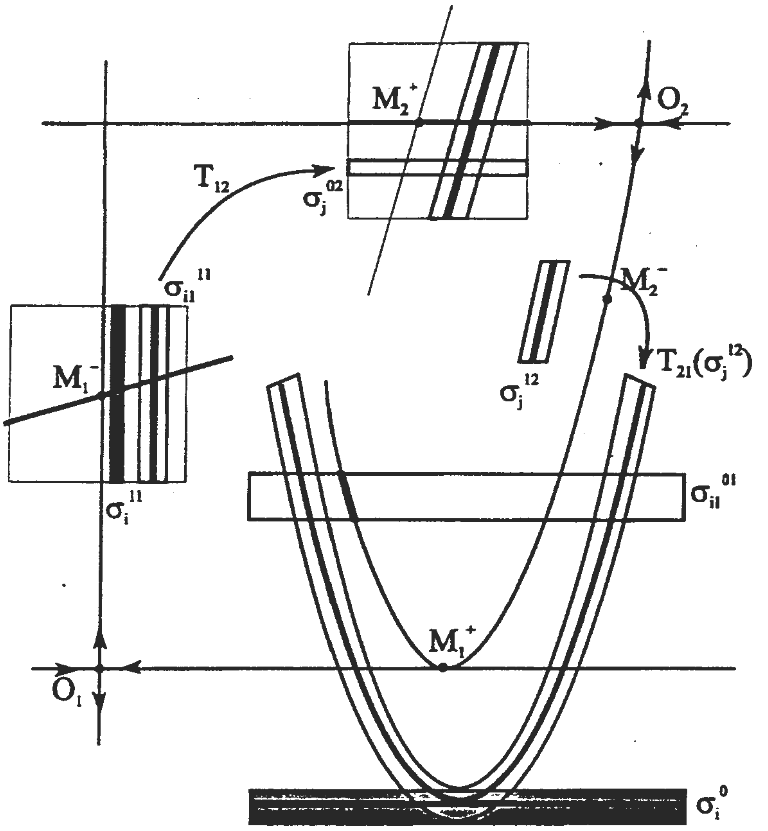


Fig. 15.

and there are no systems with a homoclinic orbit of the saddle O_2.

(3) *In $H_3^4 \cup H_3^5 \cup H_3^7$, systems with a structurally unstable homoclinic orbit of the saddle O_1 and systems with a strcturally unstable homoclinic orbit of the saddle O_2 are dense.*

Figures 15 and 16 illustrate the main geometrical idea of item 3 of this theorem for the case of systems on H_3^4 (i.e., for the case where $\lambda_2 > 0$, $\gamma_1 < 0$, $c_{21} < 0$, $d_{21} > 0$; see Table 1). Figure 15 shows the moment of a nonregular intersection of the strip σ_i^{01} (where i is odd in this case) and the horseshoe $T_{21}(\sigma_j^{12})$ (the oddness of j does not matter) when there is a homoclinic tangency of the piece

$$T_{21}T_{02}^{j}T_{12}T_{01}^{i_1}[T_{21}(W_{\rm loc}^{u}(O_2)) \cap \sigma_{i_1}^{01}]$$

of the unstable manifold of the point O_2 and the piece

$$T_{01}^{-i}[T_{12}^{-1}(W_{\rm loc}^{s}(O_2)) \cap \sigma_i^{11}]$$

of the stable manifold of the point O_2. Similarly, Fig. 16 shows the moment of the homoclinic tangency of the piece

$$T_{21}T_{02}^{j}[T_{12}(W_{\rm loc}^{u}(O_1)) \cap \sigma_j^{02}]$$

of the unstable manifold of the point O_1 and the piece

$$T_{01}^{-i}T_{12}^{-1}T_{02}^{-j_2}[T_{21}^{-1}(W_{\rm loc}^{s}(O_1)) \cap \sigma_{j_2}^{0}]$$

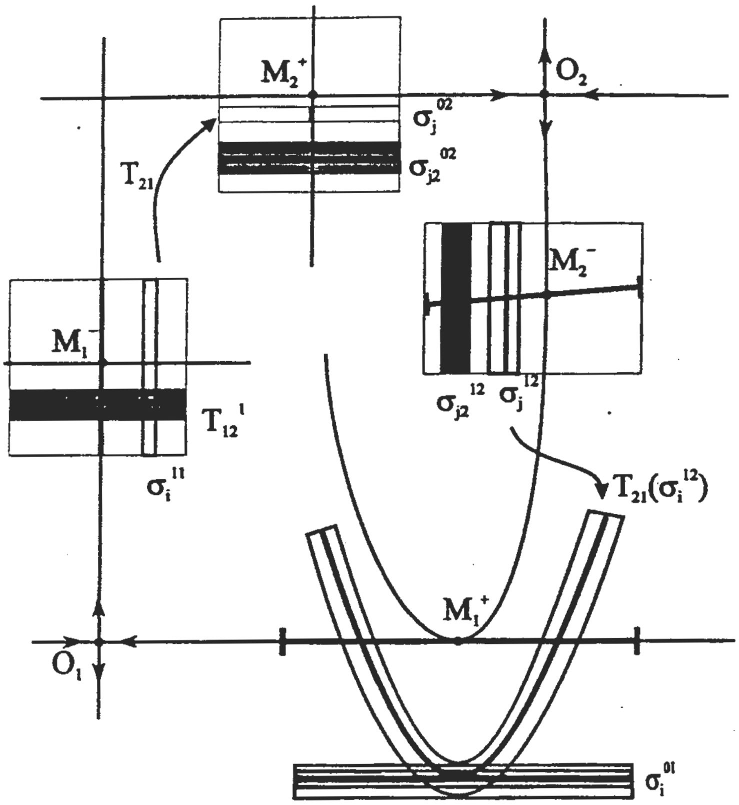


Fig. 16.

of the stable manifold of the point O_1. Note that the numbers i_1 and j_2 are arbitrary, in principle, only i_1 must be even, and the numbers i and j are connected by the condition of nonregularity of the intersection of the corresponding strips and horseshoes. By virtue of Theorem 1, i and j satisfy inequaltiy (4.4) for $\mu = 0$, i.e., the inequality

$$|i - j\theta_0 + \tau| \leq S_2(\bar{k}_1, \bar{k}_2)$$

in which the number i is odd. This is connected with the geometry of the arrangement of the strips and horseshoes in the case of the systems on H_3^4.

Using now Theorems 7 and 8, we shall prove the following result.

Theorem 9. *In H_3, systems, which have a countable set of moduli of Ω-conjugacy, are dense.*

Proof. We shall use the results of [8] in which it is shown that any system with a structurally unstable homoclinic point can be arbitrarily slightly permuted (by a permutation of the class C^{r-1}) so that a system will result with a countable set of structurally stable saddle periodic orbits each of which has a structurally unstable homoclinic orbit. Note that we can choose these pemutations such that they will be localized in a small neighborhood of a structurally unstable homoclinic orbit, and, consequently, will not bring diffeomorphisms with a structurally unstable heteroclinic cycle of the third class out of the "film" H_3. Thus we obtain from Theorem 7 that in H_3 the set B^* such that any diffeomorphism from B^* has a countable set of structurally stable saddle periodic orbits, each of which has a structurally unstable homoclinic orbit, is dense.

Let us consider the diffeomorphism $f^* \in B^*$. Suppose that it has structurally stable saddle periodic orbits $O_1^*, \dots, O_k^*, \dots$ with a structurally unstable homoclinic orbits Γ_k^* respectively. Obviously, f^* lies at the intersection of the countable set of smooth Banach manifolds $\mathcal{M}_n$ such that any diffeomorphism from $\mathcal{M}_n$ has n periodic orbits $\tilde{O}_1^*, \dots, \tilde{O}_n^*$ close to $O_1^*, \dots, O_n^*$ with structurally unstable homoclinic orbits $\tilde{\Gamma}_1^*, \dots, \tilde{\Gamma}_n^*$. Let ν_k, ρ_k ($|\nu_k| < 1$, $|\rho_k| > 1$) be the multiplicators of the orbit O_k^*. It is shown in [10, 11] that the quantities

$$\theta_k = -\frac{\ln|\nu_k|}{\ln|\rho_k|}$$

are moduli of Ω-conjugacy of the systems on $\mathcal{M}_n$. Obviously, the functionals θ_k, $k = 1, 2, \dots, n$, defined on the smooth Banach manifold $\mathcal{M}_n$, are independent since, first, for different k they can be expressed in terms of multiplicators of different periodic orbits and, second, their values change independently of one another when we pass from one system on $\mathcal{M}_n$ to another. The latter statement is connected, for instance, with the fact that the diffeomorphism f belonging to $\mathcal{M}_n$ also lies in the smooth Banach manifold $\mathcal{M}_{ni} \subset \mathcal{M}_n$ of codimension $(n-1)$, which contains diffeomorphisms where the values θ_k, $k = 1, 2, \dots, i-1, i+1, n$, are fixed and θ_i is not locally constant (i.e., is a modulus for systems on $\mathcal{M}_{ni}$). Then, according to Definition 3, the diffeomorphism f^* has a countable set of Ω-moduli. This completes the proof of the theorem.

9. STRUCTURALLY UNSTABLE PERIODIC ORBITS OF THE THIRD-CLASS DIFFEOMORPHISMS

From the viewpoint of the bifurcation theory the important property of Ω-moduli is that they can be regarded as controlling parameters in the investigation of nonwandering orbits, in particular, periodic and homoclinic orbits. In this section, we shall mainly consider bifurcations of periodic orbits.

Theorem 10. *In H_3 systems with structurally unstable periodic orbits, are dense.*

Proof. We say that the periodic orbit L which lies entirely in U, is the *k-circuit* orbit if the intersection $L \cap \Pi_1^+$ (and, hence, the intersection L with Π_1^- or Π_2^+, or Π_2^-) consists of exactly k points. Such an orbit has exactly one intersection point with each of the neighborhoods Π_s^+ and Π_s^-, $s = 1, 2$. Let $M_{01} \in \Pi_1^+$, $M_{11} \in \Pi_1^-$, $M_{02} \in \Pi_2^+$, $M_{12} \in \Pi_2^-$ be successive points of this kind. Then, for certain $i \geq \bar{k}_1$, $j \geq \bar{k}_2$ we have

$$M_{11} = T_{01}^i(M_{01}), \quad M_{02} = T_{12}(M_{11}), \quad M_{12} = T_{02}^j(M_{02}), \quad M_{01} = T_{21}(M_{12}).$$

Correspondingly,

$$M_{01}(x_{01}, y_{01}) \in \sigma_i^{01}, \qquad M_{11}(x_{11}, y_{11}) \in \sigma_i^{11},$$
$$M_{02}(x_{02}, y_{02}) \in \sigma_j^{02}, \qquad M_{12}(x_{12}, y_{12}) \in \sigma_j^{12}.$$

The point M_{01} is, obviously, a fixed point of the mapping T_{ij} in one circuit along the cycle, $T_{ij} \equiv T_{21} T_{02}^j T_{12} T_{01}^i \colon \sigma_i^{01} \to \sigma_i^{01}$.

By virtue of (3.2)–(3.4), the mapping T_{ij} can be written as:

$$\begin{aligned}
&\bar{x}_{02} - x_2^+ = a_{12}\lambda_1^i x_{01} + b_{12}(y_{11} - y_1^-) + \ldots,\\
&\gamma_2^{-j}\bar{y}_{12}(1+\ldots) = c_{12}\lambda_1^i x_{01} + d_{12}(y_{11} - y_1^-) + \ldots,\\
&\bar{x}_{01} - x_1^+ = a_{21}\lambda_2^j x_{02} + b_{21}(y_{12} - y_2^-) + \ldots,\\
&\gamma_1^{-i}\bar{y}_{11}(1+\ldots) = c_{21}\lambda_2^j x_{02} + d_{21}(y_{12} - y_2^-)^2 + \ldots .
\end{aligned} \tag{9.1}$$

The coordinates of its fixed points satisfy the system of equations

$$\begin{aligned}
&x_{02} - x_2^+ = a_{12}\lambda_1^i x_{01} + b_{12}(y_{11} - y_1^-) + \ldots,\\
&\gamma_2^{-j}y_{12}(1+\ldots) = c_{12}\lambda_1^i x_{01} + d_{12}(y_{11} - y_1^-) + \ldots,\\
&x_{01} - x_1^+ = a_{21}\lambda_2^j x_{02} + b_{21}(y_{12} - y_2^-) + \ldots,\\
&\gamma_1^{-i}y_{11}(1+\ldots) = c_{21}\lambda_2^j x_{02} + d_{21}(y_{12} - y_2^-)^2 + \ldots .
\end{aligned} \tag{9.2}$$

Since $d_{12} \neq 0$, $d_{21} \neq 0$, it is easy to see that if system (9.2) has a solution, then the estimates

$$|\eta_1| < L_1(|\lambda_1^i| + |\gamma_2^{-j}|), \qquad |\eta_2| < L_2\sqrt{|\lambda_2^j| + |\gamma_1^{-i}|} \tag{9.3}$$

hold for the coordinates $\eta_1 = y_{11} - y_1^-$ and $\eta_2 = y_{12} - y_2^-$ of this solution for sufficiently large i and j.

For large i and j the first and third equations in (9.2) are resolvable for x_{01} and x_{02}. When we substitute them into the second and fourth equations of system (9.2), we get the following system for η_1 and η_2:

$$\begin{aligned}
&(d_{12}\eta_1 + \ldots) - (\gamma_2^{-j}(\eta_2 + \ldots) - \lambda_1^i(b_{21}c_{12}\eta_2 + \ldots)) - (\gamma_2^{-j}(y_2^- + \ldots) - \lambda_1^i(c_{12}x_1^+ + \ldots)) = 0,\\
&(d_{21}\eta_2^2 + \ldots) - (\gamma_1^{-i}(\eta_1 + \ldots) - \lambda_2^j(b_{12}c_{21}\eta_1 + \ldots)) - (\gamma_1^{-i}y_1^-(1+\ldots) - c_{21}\lambda_2^j x_2^+(1+\ldots)) = 0,
\end{aligned} \tag{9.4}$$

where the dots denote the terms, which have, together with the first derivatives, the order $o(|\lambda_1^i| + |\gamma_2^{-j}| + \sqrt{|\lambda_2^j| + |\gamma_1^{-i}|})$, and, in addition, their second derivatives tend to zero as $i, j \to \infty$.

Since $d_{12} \neq 0$, the first equation of system (9.4), for large i and j, is resolvable for η_1. When we substitute this solution into the second equation of system (9.4), we get the following equation for η_2:

$$d_{21}\eta_2^2 + o(\eta_2^2) - \frac{1}{d_{12}}[\gamma_1^{-i}(\eta_2 + \ldots) - b_{12}c_{21}\lambda_2^j(\eta_2 + \ldots)][\gamma_2^{-j}(1+\ldots) - b_{21}c_{12}\lambda_1^i(1+\ldots)]$$

$$-(\gamma_1^{-i}y_1^-(1+\ldots) - c_{21}\lambda_2^j x_2^+(1+\ldots)) = 0. \tag{9.5}$$

Obviously, for sufficiently large i and j, (9.5) does not have roots of multiplicity exceeding two. In addition, there exists a postive constant L_3, which is independent of i and j, such that if the inequality

$$\frac{\gamma_1^{-i}y_1^- - c_{21}\lambda_2^j x_2^+}{d_{21}} > L_3(|\gamma_1|^{-i} + |\lambda_2|^j)(|\lambda_1|^i + |\gamma_2|^{-j}) \tag{9.6}$$

is satisfied, then Eq. (9.5) has exactly two roots of the form

$$\eta_2^{1,2} = \pm\sqrt{\frac{\gamma_1^{-i}y_1^- - c_{21}\lambda_2^j x_2^+}{d_{21}}}(1+\ldots).$$

On the other hand, if the inequality

$$\frac{\gamma_1^{-i}y_1^- - c_{21}\lambda_2^j x_2^+}{d_{21}} < -L_3(|\gamma_1|^{-i} + |\lambda_2|^j)(|\lambda_1|^i + |\gamma_2|^{-j}) \tag{9.7}$$

is satisfied, then Eq. (9.5) has no roots.

Let us now consider, for definiteness, the case $\gamma_1 > 0$, $\lambda_2 > 0$, $c_{21} > 0$, $d_{21} > 0$ (in other cases the proof is similar). After taking logarithms, inequality (9.6) assumes the form

$$i < j\theta - \tau - L_4(\lambda_2^j + \gamma_1^{-i}). \tag{9.8}$$

Correspondingly, if we take logarithms of inequality (9.7), we get

$$i > j\theta - \tau + L_4(\lambda_2^j + \gamma_1^{-i}). \tag{9.9}$$

Thus, if the numbers i and j of the strips satisfy inequality (9.9), then the mapping T_{ij} does not have fixed points, and if i and j satisfy inequality (9.8), then T_{ij} has exactly two fixed points.

Let us consider now the one-parameter family f_θ of diffeomorphisms on H_3^1. We fix $\theta = \theta_0$ and consider, for the diffeomorphismf_{θ_0}, the set of pairs (i, j) for which the inequaltity

$$i < j\theta_0 - \tau - L_4(\lambda_2^j + \gamma_1^{-i}) \tag{9.10}$$

is satisfied. The set of these pairs is countable. We set $\theta = \theta_0 - \delta$ and consider the set of pairs (i, j) for which the inequality

$$i > j(\theta_0 - \delta) - \tau + L_4(\lambda_2^j + \gamma_1^{-i}) \tag{9.11}$$

is satisfied. The set of these pairs is also countable.

Let us now consider the set of pairs (i, j) for which inequalities (9.10) and (9.11) are simultaneously satisfied. Obviously, the set of these pairs is countable for any $\delta > 0$. Let us consider one of these pairs, say, (i^*, j^*). Then we find that the diffeomorphismf_{θ_0} has two one-circuit periodic orbits which successively intersect the strips $\sigma_{j^*}^{02}$ and $\sigma_{i^*}^{01}$ and $f_{\theta_0-\delta}$ does not have any periodic orbits of this type. Since the solutions of system (9.4) continuously depend on the parameters and all of them lie in a bounded domain, we find, by virtue of (9.3), that there exists $\theta^* \in (\theta_0 - \delta, \theta_0)$ such that the diffeomorphism f_{θ^*} has a structurally unstable one-circuit periodic orbit. We have proved the theorem.

It follows from our discussion that this structurally unstable periodic orbit has at least one multiplicator equal to +1 and is a double multiplicator. If the second multiplicator is not equal to unity in absolute value, then this periodic orbit is of a saddle-node type with the first Lyapunov value not equal to zero. If such an orbit is subjected to a bifurcation, either stable or completely unstable periodic orbit may be generated according as the absolute value of the second multiplicator is smaller or larger than unity. We shall consider questions concerning the existence of stable and completely unstable periodic orbits in third-class diffeomorphisms in the next sections.

10. STABLE AND COMPLETELY UNSTABLE PERIODIC ORBITS OF DIFFEOMORPHISMS ON H_3

Note that the product of the multiplicators ν_1 and ν_2 of the fixed point M_{01} of the mapping T_{ij} is equal to the Jacobian J of this mapping calculated at the fixed point. Since $T_{ij} \equiv T_{21}T_{02}^{j}T_{12}T_{01}^{i}$, we have

$$J(T_{ij}) \equiv J(T_{21}(M_{12}))J(T_{02}^{j}(M_{02}))J(T_{12}(M_{11}))J(T_{01}^{i}(M_{01})). \tag{10.1}$$

From (9.1) and (9.2) we find that

$$\nu_1\nu_2 \equiv J(T_{ij}(M_{01}))$$

$$= J_{12}J_{21}\sigma_1^{i}\sigma_2^{j}(1 + O(|\lambda_1|^{i} + |\gamma_1|^{-i} + |\lambda_2|^{j} + |\gamma_2|^{-j})), \tag{10.2}$$

where $J_{12} \equiv (a_{12}d_{12} - b_{12}c_{12})$ is the Jacobian of the mapping T_{12} calculated for $\mu = 0$ at the point M_1^-, and $J_{21} \equiv -b_{21}c_{21}$ is the Jacobian of the mapping T_{21} calculated for $\mu = 0$ at the point M_2^-. Since T_{12} and T_{21} are diffeomorphisms, it follows that $J_{12} \neq 0$, $J_{21} \neq 0$.

Thus we find from (10.2) that for large i and j, by virtue of (10.2) the Jcaobian of the mapping T_{ij} is a quantity of the order $\sigma_1^{i}\sigma_2^{j}$.

10.1. A Case, where saddle values lie on the same side of unity. As was established in Section 7 (Statement 2), if both saddle values σ_1 and σ_2 are either smaller or larger than unity, then neither f nor diffeomorphisms sufficiently close to f have, in a sufficiently small neighborhood U of the cycle, either completely unstable or stable periodic orbits respectively. On the other hand, we have the following result.

Theorem 11. *In the case $\sigma_1 < 1$, $\sigma_2 < 1$ ($\sigma_1 > 1$, $\sigma_2 > 1$ resp.) in H_3 the systems with a countable set of stable periodic orbits (with a countable set of completely unstable periodic orbits, resp.) are dense.*

Proof. Let us consider the case $\sigma_1 < 1$, $\sigma_2 < 1$. Obviously, the case $\sigma_1 > 1$, $\sigma_2 > 1$ can be reduced to it by the substitution of f^{-1} for f. Let f_θ be a one-parameter family of diffeomorphisms in H_3. By virtue of Theorem 10, for this family the values of the parameter θ are dense if $\theta = \theta_{ij}^*$ for them f_θ has a one-circuit two-fold periodic orbits for which one multiplicator ν_1 is equal to "+1" and the other, ν_2, by virtue of (10.2) is a quantity of order $\sigma_1^{i}\sigma_2^{j}$. According to the hypothesis of the theorem, $\nu_2 \ll 1$ for sufficiently large i and j. When the parameter θ changes appropriately (for instance, $\gamma_1 > 0$, $\lambda_2 > 0$, $c_{21} > 0$, $d_{21} > 0$ when it decreases), then the saddle-node periodic orbit decomposes into two. One of them is of saddle type and the other is asymptotically stable for the values of the parameter θ from a certain interval $\delta_{ij} = (\theta_{ij}^*, \theta_{ij}^{**})$. Since the points θ_{ij}^* are dense, the values of θ, for which the diffeomorphisms f_θ already have a countable set of stable periodic orbits, are also dense.

This fact can easily be proved by the method of nested intervals. Indeed, we fix $\theta = \theta_0$ and consider the interval $\delta_0 = (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$. We have shown that for any $\varepsilon > 0$ on the interval δ_0 there exists a subinterval δ_1 such that for $\theta \in \delta_1$ the mapping $T_{i_1j_1}$, for sufficiently large i_1 and j_1, has an asymptotically stable fixed point. On the interval δ_1 we again find a subinterval δ_2 such that for $\theta \in \delta_2$ already the mapping $T_{i_2j_2}$ has an asymptotically stable fixed point for sufficiently large i_2 and j_2, and, consequently, the diffeomorphism f_θ has two stable one-circuit periodic orbits.

Proceeding in this way, we get an infinite sequence of nested intervals

$$\delta_1 \supset \delta_2 \supset \ldots \supset \delta_n \supset \ldots$$

such that for $\theta \in \delta_n$ the diffeomorphism f_θ has n asymptotically stable one-circuit periodic orbits. All these intervals have a common point, say, the point θ^*. Then f_{θ^*} has a countable set of stable one-circuit orbits. These values of the parameter θ are dense by virtue of the arbitrariness of the choice of the initial θ_0 and ε.

10.2. Stable and completely unstable periodic orbits (the case when saddle values lie on different sides of unity). We introduce the quantity $\alpha = \sigma_1^\theta \sigma_2$. In the case where either $\sigma_1 < 1 < \sigma_2$ or $\sigma_1 > 1 > \sigma_2$, we divide the bifurcation surface H_3 into two parts and denote by H_s (H_u, resp.) the part, which consists of diffeomorphisms with $\alpha < 1$ ($\alpha > 1$, resp.).

Theorem 12. *In H_s (in H_u, resp.) sustems with a countable set of stable (completely unstable, resp.) periodic orbits are dense.*

Proof. By virtue of Theorem 10 in H_3 structurally unstable one-circuit periodic orbits are dense. We shall consider one of these orbits. By virtue of (10.1), its Poincaré mapping T_{ij} has a Jacobian

$$J(T_{ij}) = J(T_{12})J(T_{21})\sigma_1^i\sigma_2^j(1+\ldots). \tag{10.3}$$

Note now that the numbers i and j in this relation are not arbitrary. Since by hypothesis we consider a structurally unstable one-circuit orbit, the numbers of strips i and j, by virtue of (9.6) and (9.7), for $\mu = 0$, must satisfy the inequality

$$|i - j\theta - \tau| < L_3(|\lambda_2|^j + |\gamma_1|^{-i}). \tag{10.4}$$

Thus we find that the numbers i are j related as

$$i = j\theta + \tau + O(|\lambda_2|^j + |\gamma_1|^{-i}). \tag{10.5}$$

It follows from (10.3) and (10.5) that the Jacobian J of the mapping T_{ij}, calculated at a saddle-node point, is a quantity of order

$$J = J(T_{12})J(T_{21})\sigma_1^{j\theta+\tau}\sigma_2^j(1+\ldots) \sim \sigma_1^{j\theta}\sigma_2^j = \alpha^j. \tag{10.6}$$

Therefore (for sufficiently large i and j) $J < 1$ if $\alpha < 1$, and $J > 1$ if $\alpha > 1$. Following now the scheme of the proof of Theorem 11, we get the required statement.

Let us consider now the problem of coexistence of stable and completely unstable periodic orbits of diffeomorphisms on H_3. As follows from Statement 2 and Theorem 11, the coexistence of stable and completely unstable periodic orbits is possible, in the general case when σ_1 and σ_2 lie on different sides of unity.

Let $H_s^1 = H_s \cap (H_3^1 \cup H_3^2 \cup H_3^3 \cup H_3^6)$ (see Table 1). We denote by H_{ss} the subset H_s^1, which includes the systems on H_3^1 and H_3^3 for which $\sigma_1 > 1$ and $\sigma_2 < 1$, and also systems on H_3^2 and H_3^6 for which $\sigma_1 < 1$ and $\sigma_2 > 1$. Correspondingly, let $H_u^1 = H_u \cap (H_3^1 \cup H_3^2 \cup H_3^3 \cup H_3^6)$ and denote by H_{uu} the subset H_u^1, which includes systems on H_3^1 and H_3^3 for which $\sigma_1 < 1$ and $\sigma_2 > 1$ and also systems on H_3^2 and H_3^6 for which $\sigma_1 > 1$ and $\sigma_2 < 1$.

Theorem 13. (1) *Systems on H_{ss} do not have any completely unstable periodic orbits, and on H_{uu} they do not have any stable periodic orbits.*

(2) *On $(H_s^1 \backslash H_{ss}) \cup (H_u^1 \backslash H_{uu})$ systems, which simultaneously have a countable set of stable and a countable set of unstable periodic orbits, are dense.*

Proof. It should be pointed out at once that the case $\alpha < 1$ reduces to $\alpha > 1$ when we pass from the mapping f to the mapping f^{-1}. It is easy to see that upon this transform we again get a diffeomorphism of the third class with a structurally unstable cycle, but O_1 is replaced by O_2 and O_2 by O_1. Respectively, σ_1 is replaced by σ_2^{-1}, σ_2 by σ_1^{-1}, d_{21} by $-\frac{d_{21}}{c_{21}b_{21}^2}$, c_{21} by $\frac{1}{c_{21}}$. In addition,

$$\theta(f^{-1}) = (\theta(f))^{-1}, \qquad \alpha(f^{-1}) = (\alpha(f)^{1/\theta(f)})^{-1}.$$

Therefore, it suffices to prove the theorem for diffeomorphisms on H_s. For definiteness, we shall again consider the case of systems on H_3^1, i.e., the case $\lambda_1 > 0$, $\gamma_2 > 0$, $c_{21} > 0$, $d_{21} > 0$. In the other cases the proof is completely similar.

First suppose that $\sigma_1 > 1$ and $\sigma_2 < 1$, i.e., $f \in H_{ss}$. We shall show that f does not have in U completely unstable periodic orbits.

Let Λ be an s-circuit periodic orbit of the diffeomorphism$f$. Let Λ intersect the neighborhoods Π_1^+ and Π_2^+ at successively arranged points belonging to the strips $\sigma_{i_n}^{01}$ and $\sigma_{j_n}^{02}$, $n = 1, \dots, s$, respectively. The point of the orbit Λ belonging to the strip $\sigma_{i_1}^{01}$ is, obviously, a fixed point of the following mapping in s circuits:

$$T_{i_1 j_1 \dots i_s j_s} \equiv T_{21} T_{02}^{j_s} \dots T_{21} T_{02}^{j_1} T_{12} T_{01}^{i_1}. \tag{10.7}$$

Note now that the numbers $i_n \geq \bar{k}_1$ and $j_n \geq \bar{k}_2$ in (10.7), are not, in general, arbitrary. In any case, for the mapping $T_{i_1 j_1 \dots i_s j_s}$ to have a fixed point, it is necessary that the coditions

$$\begin{aligned} &T_{21}(\sigma_{j_n}^{12}) \cap \sigma_{i_{n+1}}^{01} \neq \varnothing, \qquad n = 1, \dots, s-1, \\ &T_{21}(\sigma_{j_s}^{12}) \cap \sigma_{i_1}^{01} \neq \varnothing \end{aligned} \tag{10.8}$$

be satisfied.

By virtue of Theorem 1, since $\gamma_1 > 0$, $\lambda_2 > 0$, $c_{21} > 0$, $d_{21} > 0$, and $\mu = 0$, the following inequalities must be satisfied (cf. inequalities (9.8)–(9.9)):

$$\begin{aligned} &i_{n+1} \leq j_n \theta + \tau + \dots, \qquad n = 1, \dots, s-1, \\ &i_1 \leq j_s \theta + \tau + \dots . \end{aligned} \tag{10.9}$$

The Jacobian I of the mapping $T_{i_1 j_1 \dots i_s j_s}$ is equal to the product of the Jacobian of the factor-mappings in (10.7) and, consequently, there exists a quantity of order

$$I \sim \sigma_1^{i_1 + \dots + i_s} \sigma_2^{j_1 + \dots + j_s}. \tag{10.10}$$

Since $\sigma_1 > 1$ and $\sigma_2 < 1$, inequalities (10.9) yield

$$I \leq \sigma_1^{(j_1 + \dots + j_s)\theta + s|\tau| + \dots} \sigma_2^{j_1 + \dots + j_s} \leq \sigma_1^{s|\tau| + \dots} \alpha^{(j_1 + \dots + j_s)}. \tag{10.11}$$

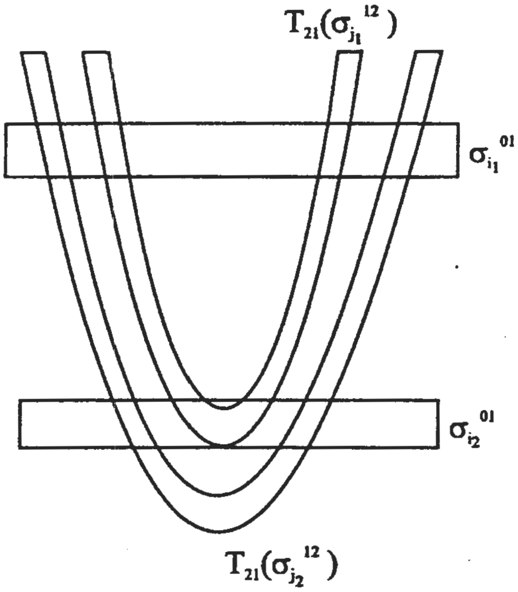


Fig. 17.

Since $\alpha < 1$, it follows that $I < 1$ for large $j_1, \dots, j_s$. This completes the proof of the first part of the theorem.

Let us prove the second part of the theorem. Let now $\sigma_1 < 1$ and $\sigma_2 > 1$, i.e., $f \in H_s^1 \backslash H_{ss}$.

Since $\alpha < 1$, it follows that on $H_s^1 \backslash H_{ss}$ the systems with a countable number of stable one-circuit periodic orbits are dense (Theorem 12). Thus it remains to prove that completely unstable periodic orbits can also exist here.

In order to show that this is so, we shall consider the mapping $T_{i_1 j_1 i_2 j_2}$ in two circuits along the heteroclinic cycle C. We assume that the geometry of the corresponding intersections of the strips and horseshoes is the following (Fig. 17). The horseshoe $T_{21}(\sigma_{j_1}^{12})$ intersects the strip $\sigma_{i_1}^{01}$ regularly, and the strip $\sigma_{i_2}^{01}$ irregularly; the horseshoe $T_{21}(\sigma_{j_2}^{12})$ intersects the strips $\sigma_{i_1}^{01}$ and $\sigma_{i_2}^{01}$ regularly. Note that by analogy with the proof of Theorem 10, we can show that by small variations of the value of the parameter θ (the variations may be the smaller, the larger the values of i_1, i_2, j_1, and j_2) we can obtain the situation where the mapping $T_{i_1 j_1 i_2 j_2}$ will have a structurally unstable fixed point, one of whose multiplicators is equal to $+1$. The Jacobian of the mapping $T_{i_1 j_1 i_2 j_2}$ at this point is a quantity of order

$$J \sim \sigma_1^{i_1+i_2} \sigma_2^{j_1+j_2} = \sigma_1^{i_1} \sigma_2^{j_2} \alpha^{j_1}. \tag{10.12}$$

Note that here the numbers i_2 and j_1 are related as $i_2 = j_1\theta - \tau + \dots$ since the intersection of the horseshoe $T_{21}(\sigma_{j_1}^{12})$ and the strip $\sigma_{i_2}^{01}$ is irregular and the numbers i_1 and j_2 can be chosen, in principle, arbitrarily. To be more precise, the inequalities

$$i_1 < j_1\theta - \tau + \dots, \qquad i_1 < j_2\theta - \tau + \dots, \qquad i_2 < j_2\theta - \tau + \dots \tag{10.13}$$

must be satisfied for them. Recall that the first of inequalities (10.13) guarantees that the horseshoe $T_{21}(\sigma_{j_1}^{12})$ regularly intersects the strip $\sigma_{i_1}^{01}$, and the second and third inequalities guarantee that the horseshoe $T_{21}(\sigma_{j_2}^{12})$ regularly intersects the strips $\sigma_{i_1}^{01}$ and $\sigma_{i_2}^{01}$. We fix i_1, and take j_2 so large that the Jacobian of the mapping $T_{i_1 j_1 i_2 j_2}$ is much larger than unity. By virtue of (10.12) and (10.13), we can do this since, by hypothesis, $\sigma_2 > 1$.

Theorem 14. *In the case where the saddle values lie on different sides of unity, on $H_3^4 \cup H_3^5 \cup H_3^7$ the systems, which simultaneously have a countable set of stable and a countable set of completely unstable periodic orbits are, dense.*

Indeed, let us consider a one-parameter family f_θ of diffeomorphisms on H_3^l, where $l \in \{4, 5, 7\}$. By virtue of Theorem 8, the values of θ (we denote them by θ^*), for which f_θ has a structurally unstable homoclinic orbit of the saddle O_1, are dense and the values of θ (we denote them by θ^{**}), for which f_θ has a structurally unstable homoclinic orbit of the saddle O_2, are also dense. It follows from [15] that to each of the values θ^* and θ^{**} there accumulates a countable set of intervals of the parameter θ, for which f_θ has a stable and, respectively, completely unstable periodic orbit. Now the statement can be proved by means of a standard procedure of the method of nested intervals.

11. NEWHOUSE INTERVALS OF THE SECOND AND THIRD TYPES

It was established above (see Section 7) that in the one-parameter family f_μ, which is transversal to the bifurcation surface of diffeomorphisms with the simplest structurally unstable heteroclinic cycle, there exist, in any neighborhood of the point $\mu = 0$, Newhouse intervals Δ_i^1 of the first type, where the values of the parameter μ are dense, for which there is

(a) a homoclinic tangency of the point O_1;

(b) a homoclinic tangency of the point O_2;

(c) a structurally unstable heteroclinic cycle containing the points O_1 and O_2;

(d) simultaneously a countable set of stable, completely unstable, and saddle periodic orbits (if the saddle values σ_1 and σ_2 lie on different sides of unity).

We can also describe some properties concerning the arrangement of the intervals Δ_i^1 on the μ-axis.

For instance, if f_0 is a first-class diffeomorphism, then the intervals Δ_i^1 exist only for $d_{21}\mu < 0$ (i.e., in a class of systems with two heteroclinic orbits, which are close to Γ_{21}), and for $d_{21}\mu > 0$ the structure of the set $N(\mu)$ is trivial, namely, $N(\mu) = \{O_1; O_2; \Gamma_{12}\}$.

If f_0 is a second-class diffeomorphism, then, as follows from the proofs of Theorem 4 and Theorem 2, the intervals Δ_i^1 exist only for $d_{21}\mu > 0$ (i.e., in the class of systems without heteroclinic orbits, which are close to Γ_{12}), and for $d_{21}\mu < 0$ the set $N(\mu)$ has a hyperbolic structure.

For the third-class diffeomorphisms the situation is more complicated. It is easy to realize, for instance, from Theorems 8 and 14 that in the family f_μ, where $f_0 \in H_3^4 \cup H_3^5 \cup H_3^7$, the intervals Δ_i^1 exist both for positive and for negative μ.

As was proved in Theorem 4, in the case of family f_μ, where $f_0 \in H_3^1 \cup H_3^2 \cup H_3^3 \cup H_3^6$, the intervals Δ_i^1 exist for $d_{21}\mu < 0$ (i.e., in the class of systems with two heteroclinic orbits, which are close to Γ_{21}). Here we are certainly interested in the question concerning the existence and structure of Newhouse intervals on the half-interval of values of μ, where $d_{21}\mu > 0$.

Note, first of all, that

for $d_{21}\mu > 0$ the diffeomorphisms of the family f_μ, where $f_0 \in H_3^1 \cup H_3^2 \cup H_3^3 \cup H_3^6$, do not have in U homoclinic orbits of the point O_1 in the case $d_{21} > 0$, or the point O_2 in the case $d_{21} < 0$, *neither they have heteroclinic cycles containing the points O_1 and O_2.*

This statement is an obvious consequence of the geometry of the mutual positions of the invariant manifolds of the points O_1 and O_2:

in the case $d_{21} > 0$ all curves of the set $W^u(O_1) \cap \Pi_1^+$ for $\mu > 0$ lie above the piece $T_{21} W^u_{\rm loc}(O_2) \cap \Pi_1^+$ of the unstable manifold of the point O_2 which, in turn, lies above $W^s_{\rm loc}(O_1) \cap \Pi_1^+$;

in the case $d_{21} < 0$, for $\mu < 0$, the piece $T_{21} W^u_{\rm loc}(O_2) \cap \Pi_1^+$ of the unstable manifold of the point O_2 lies below $W^s_{\rm loc}(O_1) \cap \Pi_1^+$, and all curves of the set $W^s(O_2) \cap \Pi_1^+$ lie above the latter.

We shall show that for $d_{21}\mu > 0$ the indicated family f_μ includes Newhouse intervals of the second and third types. However, as distinct from Newhouse intervals of the first type, property (c) is no longer fulfilled for them, and properties (a) and (b) formulated at the beginning of this section cannot be fulfilled simultaneously. In addition, property (d) will not be fulfilled for Newhouse intervals of the second type.

Let us consider, as before, the one-parameter family f_μ, which is transversal to H_3, but we shall introduce one more condition of the general position, namely, $\alpha \neq 1$. We shall assume that $\alpha < 1$, since the case $\alpha > 1$ reduces to $\alpha < 1$ upon the substitution of f^{-1} for f. Thus we shall consider the problem of the existence and structure of Newhouse intervals for $d_{21}\mu > 0$ in the one-parameter family f_μ, where $f_0 \in H_s$ (see Section 10).

We shall begin with the case $f_0 \in H_{ss}$. Here we have the following theorem.

Theorem 15. *Let f_μ be a one-parameter family of diffeomorphisms which is transversal to H_{ss} for $\mu = 0$. On the interval $d_{21}\mu > 0$ there are no completely unstable periodic orbits in N_μ and, in addition, there accumulates to $\mu = 0$ a countable set of intervals Δ_i^2 such that*

(1) on Δ_i^2 the values of the parameter μ, for which f_μ has a structurally unstable homoclinic orbit of the point O_2 (of the point O_1, resp.) in the case $d_{21} > 0$ (in the case $d_{21} < 0$), are dense,

(2) on Δ_i^2 the values of the parameter μ, for which f_μ has a countable set of stable and saddle periodic orbits, are dense.

Proof. Let us consider, for definiteness, a one-parameter family f_μ such that $f_0 \in H_3^1 \cap H_{ss}$, i.e., the main parameters of the diffeomorphism f_0 satisfy the relations $\lambda_2 > 0$, $\gamma_1 > 0$, $c_{21} > 0$, $d_{21} > 0$, $\alpha < 1$, and $\sigma_1 > 1$, $\sigma_2 < 1$. As was established above (Theorem 13), there are no systems on the bufurcation surface $H_3^1 \cap H_{ss}$ which would have completely unstable periodic orbits in U. The main analytic condition for this is the inequality (see relation (10.9))

$$i \leq j\theta + \tau + \ldots, \tag{11.1}$$

which is necessary for the intersection of the strip σ_{01}^i with the horseshoe $T_{21}\sigma_{12}^j$ to be nonempty for $\mu = 0$. For $d_{21}\mu > 0$ inequality (11.1) remains necessary. To be more precise, in this case the set of solutions of inequalities (4.2) and (4.3) belongs to the set of solutions of inequality (11.1). Since the condition $\alpha < 1$ is fulfilled for the systems on $H_3^1 \cap H_{ss}$, it will be fulfilled for all systems, which are close to them. Thus, by analogy with Theorem 13, we can prove that

for $d_{21}\mu > 0$ the diffeomorphisms f_μ, where $f_0 \in H_3^1 \cap H_{ss}$, do not have in U any completely unstable periodic orbits.

Thus, if the family f_μ includes, for $d_{21}\mu > 0$, Newhouse intervals, they can only be "classical" Newhouse intervals, i.e., intervals in which the values of the parameter μ, corresponding to the existence in f_μ of a countable set of stable periodic orbits, are dense (if $f_0 \in H_{uu}$, then completely unstable orbits, resp.). It remains to prove the existence of intervals of this kind, and this is a simple corollary of the following lemma.

Lemma 6. *There exists a countable set of values μ_k^* of the parameter μ such that $d_{21}\mu_k^* > 0$, $\mu_k^* \to 0$ as $k \to \infty$, and, for $\mu = \mu_k^*$, the diffeomorphism f_μ has a structurally unstable one-circuit homoclinic orbit of the point O_2.*

Proof. The piece $T_{21}(W^u_{\rm loc}(O_2) \cap \Pi_2^-)$ of the unstable manifold is, by virtue of (3.6), a "parabola" $l_u(\mu)$

$$y_{01} = \mu + d_{21}\Big(\frac{x_{01} - x_1^+(\mu)}{b_{21}}\Big)^2 + \ldots,$$

which, for $\mu = 0$, touches the segment $y_{01} = 0$ on Π_1^+ and for $\mu > 0$ lies above it (at a distance of order μ). To the segment $y_{01} = 0$ on Π_1^+ there always regularly accumulates a countable set of segments of the stable manifold of the point O_2, say, segments, such that

$$l_k^s \equiv T_{01}^{-k} T_{12}^{-1}(W^s_{\rm loc}(O_2) \cap \Pi_2^+),$$

which, by virtue of (3.2) and (3.5), have an equation

$$y_{01} = \gamma_1^{-k}(\mu)\Big(y_1^-(\mu) - \frac{c_{12}}{d_{12}}\lambda_1^k(\mu)x_{01} + \ldots\Big).$$

For $\mu \leq 0$ the parabola l_u transversally cuts each of the curves l_k^s with a sufficiently large number k at two points. The intersection points are associated with a one-circuit homoclinic orbit of the fixed point O_2. For $\mu \geq 0$, the parabola l_u can already touch one of the curves l_k^s, and this corresponds to the appearance of a structurally unstable homoclinic orbit of the point O_2. The moment of tangency is associated with the value $\mu_k^* = \gamma_1^{-k} y_1^-(1 + \ldots)$ of the parameter μ. It is obvious that this tangency is quadratic and the family f_μ, for $\mu = \mu_k^*$, is transversal to the corresponding bifurcation surface h_k of diffeomorphisms with a one-circuit structurally unstable homoclinic orbit of the point O_2. We have proved the lemma.

Theorem 15 follows from this lemma and from the Newhouse theorem [3].

Let us now consider the case of the family f_μ, where $f_0 \in H_s^1 \backslash H_{ss}$. Here we have the following theorem.

Theorem 16. *Let f_μ be a one-parameter family of diffeomorphisms which is transversal to $H_s^1 \backslash H_{ss}$ for $\mu = 0$. Then, to the value $\mu = 0$ on the interval $d_{21}\mu > 0$ there accumulates a countable set of intervals Δ_i^3 such that*

(1) on Δ_i^3, the values of the parameter μ, for which f_μ has a structurally unstable homoclinic orbit of the point O_2 in the case $d_{21} > 0$, and of the point O_1 in the case $d_{21} < 0$, are dense,

(2) on Δ_i^3, the values of the parameter μ, for which f_μ simultaneoulsy has a countable set of stable, completely unstable, and saddle periodic orbits, are dense.

Proof. Let us again consider, for definiteness, a one-parameter family f_μ such that $f_0 \in H_3^1$, i.e., the main parameters of the diffeomorphism f_0 satisfy the relations $\lambda_2 > 0$, $\gamma_1 > 0$, $c_{21} > 0$, $d_{21} > 0$, $\alpha < 1$, and $\sigma_1 < 1$, $\sigma_2 > 1$.

Let us, first of all, consider in detail the diffeomorphism f_0 (for $\mu = 0$).

Lemma 7. *The diffeomorphism f_0 has in U a countable set of structurally stable saddle one-circuit periodic orbits for which the product of multiplicators is less than unity.*

Proof. As follows from Theorems 1 and 3, f_0 has a countable set of structurally stable saddle one-circuit periodic orbits. The point of intersection of this orbit with Π_1^+ is a fixed point of the mapping $T_{ij} \equiv T_{21}(0)T_{02}^j T_{12}(0)T_{01}^i$, where the natural numbers i and j satisfy the inequalities

$$i < j\theta + \tau - S(\bar{k}_1, \bar{k}_2), \qquad i \geq \bar{k}_1, \quad j \geq \bar{k}_2. \tag{11.2}$$

The product of multiplicators σ_{ij} (a saddle value) of this periodic orbit is a quantity of order

$$\sigma_{ij} \sim \sigma_1^i \sigma_2^j.$$

Let us consider the first inequality in (11.2) and set $i = j\theta + \tau - s$ in it, where s is a positive number such that $j\theta + \tau - s$ is an integer and $i \geq \bar{k}_1$. In this case we have

$$\sigma_{ij} \sim \sigma_1^{j\theta+\tau-s}\sigma_2^j \sim \alpha^j \sigma_1^{-s}. \tag{11.3}$$

Since $\alpha < 1$ and $\sigma_1 < 1$, it follows that for a fixed j there exist a finite number of values of s such that

$$\alpha^j \sigma_1^{-s} < 1.$$

It follows that the numbers s must satisfy the inequalities

$$s \geq j\theta + \tau - \bar{k}_1, \qquad s < j\frac{|\ln \alpha|}{|\ln \sigma_1|}. \tag{11.4}$$

The number of solutions of inequaliy (11.4) (provided that $j\theta + \tau - s$ is an integer), for every fixed j, is finite but tends to infinity as $j \to \infty$, and this proves the lemma.

Let us fix now a sufficiently large $j \geq \bar{k}_2$ and consider a one-circuit periodic motion of P_i^j which intersects the strip σ_j^{21} and the strip σ_i^{01} with the number $i = i^* - s_0$ such that the horseshoe $T_{21}(\sigma_j^{21})$ intersects the strips $\sigma_i^{01}, \sigma_{i+1}^{01}, \ldots, \sigma_{i^*-1}^{01}$ obviously regularly and the strip $\sigma_{i^*}^{01}$, where $i^* = j\theta + \tau \ldots$, possibly irregularly (Fig. 18). By virtue of Lemma 7, we can choose the integer s_0 such that inequalities (11.4) will be satisfied for $s = s_0$. In this case, the product of the multiplicatiors of the periodic orbit P_i^j will be smaller than unity. Thus, the fixed point p_{ij} of the mapping $T_{21}T_{02}^j T_{12}T_{01}^i$ corresponding to P_i^j has a saddle value smaller than unity.

Let us show that in the generic one-parameter family f_μ there will accumulate to $\mu = 0$ positive (since $d_{21} > 0$) values of μ for which the diffeomorphism f_μ will have a structurally unstable heteroclinic cycle, which includes the saddles O_2 and P_i^j.

Note that we can deduce from Theorem 3 that for sufficiently small μ the invariant manifolds of the saddles O_2 and P_i^j possess the following properties:

(1) $W^u(P_i^j)$ has a point of transversal intersection with $W^s(O_2)$,

(2) $W^s(P_i^j)$ has a point of transversal intersection with $W^u(O_1)$,

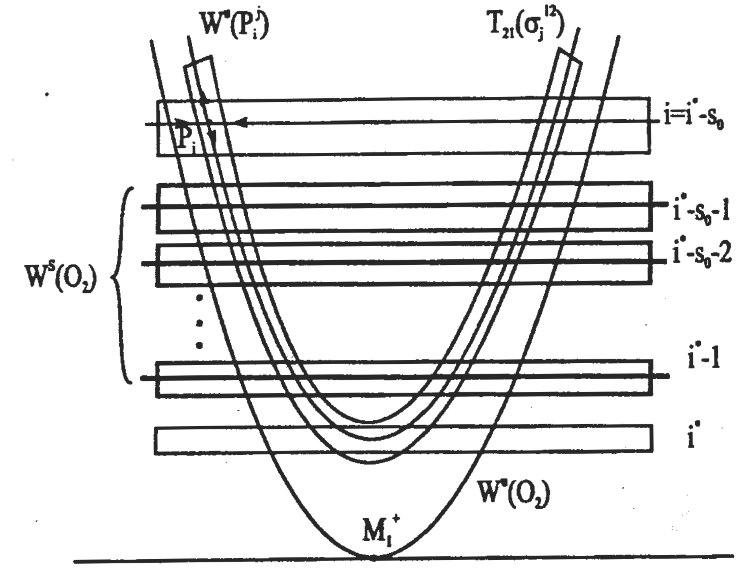


Fig. 18.

Indeed, in the first case, one of these heteroclinic orbits is associated with the coding

$$(\ldots\overbrace{2,2,\ldots,2,3}^{j+n_2},\overbrace{1,1,\ldots,1}^{i+n_1},\overbrace{2,2,\ldots,2,3}^{j+n_2},\overbrace{1,1,\ldots,1}^{i+n_1},\overbrace{2,2,\ldots}^{\infty})$$

of form (5.3) which is associated with a heteroclinic orbit passing through P_i^j and O_2. This orbit is structurally stable since the numbers i and j satisfy inequality (4.2).

Similarly, in the second case, one of the heteroclinic orbits lying at the intersection of the manifolds $W^s(P_i^j)$ and $W^u(O_1)$ is associated with the coding

$$(\overbrace{\ldots,1,1}^{\infty},\overbrace{2,2,\ldots,2,3}^{j+n_2},\overbrace{1,1,\ldots,1}^{i+n_1},\overbrace{2,2,\ldots,2,3}^{j+n_2},\ldots)$$

of form (5.3) which is also associated with a structurally stable heteroclinic orbits.

It follows from (1) that to the piece $T_{21}(W^u_{\text{loc}}(O_2)) \cap \Pi_1^+$ of the unstable manifold of the point O_2 there regularly accumulates, for all sufficiently small μ, a countable set of compact pieces of the unstable manifold of the orbit P_i^j.

In turn, it follows from (2) that to the segment $W^s_{\text{loc}}(O_1) \cap \Pi_1^+$ of the stable manifold of the point O_1 there regularly accumulates, for sufficiently small μ, a countable set of compact pieces of the stable manifold of the orbit P_i^j.

Then, by virtue of Statement 1 there will accumulate to $\mu = 0$ a countable set of values of the parameter μ: $\mu = \mu_k$ (μ_k are positive) such that for $\mu = \mu_k$ the family f_μ unfolds generically the heteroclinic tangency of the manifolds $W^s(P_i^j)$ and $W^u(O_2)$. Since $W^u(P_i^j)$ transversally intersects $W^s(O_2)$ (by virtue of Statement 1), the diffeomorphism f_μ has, for $\mu = \mu_k$, a structurally unstable heteroclinic cycle containing saddle periodic orbits P_i^j and O_2, saddle values of which lie on different sides of unity.

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