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# On Three Types of Dynamics and the Notion of Attractor

**S. V. Gonchenko**[a] **and D. V. Turaev**[a,b]

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**Abstract**—We propose a theoretical framework for explaining the numerically discovered phenomenon of the attractor–repeller merger. We identify regimes observed in dynamical systems with attractors as defined in a paper by Ruelle and show that these attractors can be of three different types. The first two types correspond to the well-known types of chaotic behavior, conservative and dissipative, while the attractors of the third type, reversible cores, provide a new type of chaos, the so-called mixed dynamics, characterized by the inseparability of dissipative and conservative regimes. We prove that every elliptic orbit of a generic non-conservative time-reversible system is a reversible core. We also prove that a generic reversible system with an elliptic orbit is universal; i.e., it displays dynamics of maximum possible richness and complexity.

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## 1. INTRODUCTION

**1.1. Attractor–repeller merger.** When we speak of dynamical chaos, we usually mean one of two quite different types of dynamics. In Hamiltonian systems, we have conservative chaos, which is something like a “chaotic sea” with elliptic islands inside. Chaos in dissipative systems is quite different and is associated with strange attractors. Our goal in this paper is to attract attention to another type of chaos, the third one, which was called “mixed dynamics” in [9, 20]. This type of behavior is characterized by *inseparability* of attractors, repellers, and conservative elements in the phase space [23].

In order to have both attractors and repellers, the system must contract the phase volume somewhere and somewhere expand it. For example, any diffeomorphism of a compact phase space will have an attractor and a repeller, unless the whole phase space is a chain transitive set (for the definition of chain transitivity, see, e.g., [2] and Section 2). Attractors and repellers may be separated from each other, as in Morse–Smale systems. However, it was recently established for many examples [12, 13, 27, 28, 36] that when parameters of a system are varied, the numerically obtained attractor and repeller may *collide* and start to occupy approximately the same part of the phase space. Moreover, a further change of parameters does not seem to break the attractor–repeller merger. Examples of such behavior in models of a Celtic stone and a “rubber ellipsoid” that roll on a horizontal plane are shown in Fig. 1.

A theorem by Conley [7] establishes the existence of a Lyapunov function, non-increasing along the orbits of the system, which attains its maxima on repellers and minima on attractors. The attractor–repeller collision means that this function must have very degenerate critical points, so the intuition based on the Conley theorem might suggest that this phenomenon should be very exotic. However, in reality, the attractor–repeller merger appears quite robustly and persists for significant regions of parameter values in the models where it was detected [12, 13]. As we will

[a] Lobachevsky State University of Nizhni Novgorod, pr. Gagarina 23, Nizhni Novgorod, 603950 Russia.
[b] Department of Mathematics, Imperial College London, London SW7 2AZ, UK.
E-mail addresses: sergey.gonchenko@mail.ru (S.V. Gonchenko), dturaev@imperial.ac.uk (D.V. Turaev).

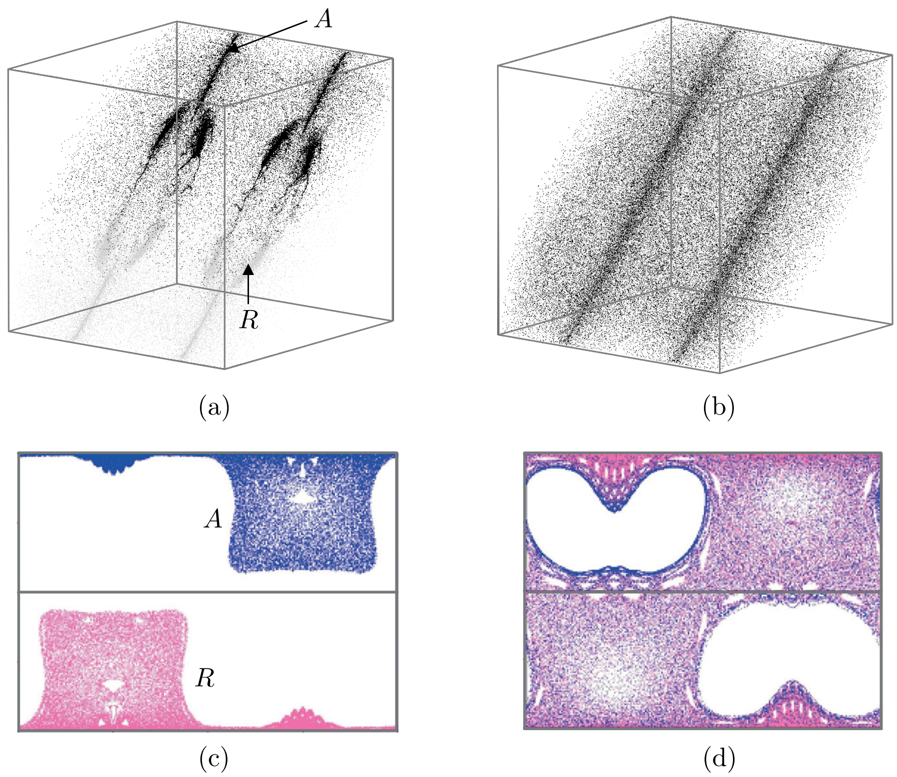



**Fig. 1.** Examples of attractor–repeller merger for the Poincaré map: (a, b) in a model of the Celtic stone (the attractor is shown in black, and the repeller, in grey) [12] and (c, d) in a model of the Chaplygin ball (rubber body) [27]. Here, the numerically obtained attractor $A$ and repeller $R$ are depicted for different values of the energy of the system.

argue below, it is a generic phenomenon for the non-conservative time-reversible systems and, more generally, for systems belonging to the so-called absolute Newhouse domain [41, 42].

A systematic study of this phenomenon requires a precise definition of what one means by attractor or repeller. In this paper we discuss two closely related definitions, going back to Ruelle's work [34], which we think are most relevant in describing numerically obtained portraits of attractors/repellers. Accordingly, we propose two paradigmatic models (see Subsections 1.2 and 1.3 below) for the attractor–repeller collision, the second less restrictive than the first one, and also discuss basic dynamical phenomena associated with them in the context of time-reversible systems. The models are different but not mutually exclusive. In fact, the dynamics underlying the attractor–repeller collision is so extremely rich that both our models appear to be applicable at the same time.

**1.2. Reversible core.** The first model employs the notion of attractor as defined in the works of Ruelle [34] and Hurley [26]. Both authors attribute important ideas to Conley [7], so we use the term Conley–Ruelle–Hurley (CRH) attractor, which is a stable (with respect to permanently acting perturbations) and chain transitive closed invariant set. A CRH repeller is a CRH attractor in the reverse time (see precise definitions in Section 2). We also show that only three types of CRH attractors are possible for a homeomorphism $f$ of a compact connected separable metric space $\mathcal{M}$:

- *conservative*, when the whole phase space $\mathcal{M}$ is a chain transitive set and, hence, is a unique CRH attractor and repeller (this includes, e.g., the case of volume-preserving maps on compact manifolds);
- *dissipative*, when an $\varepsilon$-orbit from outside converges to a neighborhood of the CRH attractor;

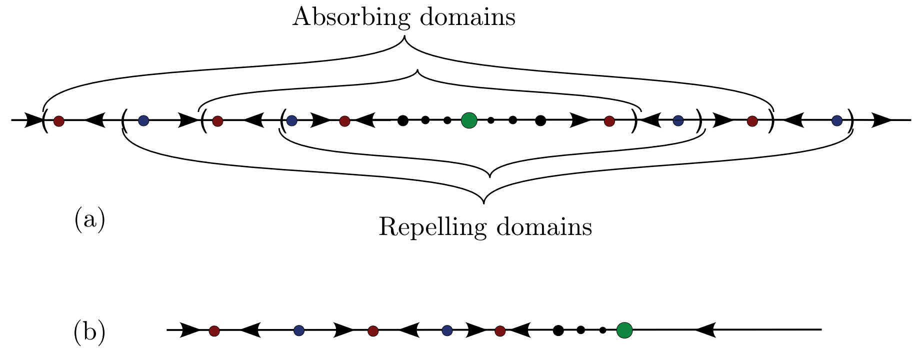



**Fig. 2.** Simple examples of a CRH attractor that is (a) a reversible core and (b) not a reversible core, though it is a limit of CRH repellers.

- *mixed*, when a CRH attractor is, at the same time, a CRH repeller[1] (it is then called a *reversible core*); see Subsection 2.3 and Theorem 2.

The reversible core should be distinguished from a dissipative attractor, as the reversible core does not attract any orbit. The dynamics here is not exactly conservative either, since the reversible core, as we show in Section 2 (Theorem 1), is always a limit of a sequence of attractors (and a limit of a sequence of repellers as well). Therefore, we can associate this type of stable sets with the third, mixed, type of dynamical behavior.

We stress that there are no further possibilities in this scheme (no "fourth" type of chaos). The CRH attractors and repellers cannot have non-trivial intersections: a CRH attractor either does not intersect any CRH repeller or coincides with one of them. In the latter case such a CRH attractor is a reversible core.

It is easy to construct trivial examples of a reversible core (see Fig. 2a).[2] However, we also provide a non-trivial example in Section 3 (see Theorem 3) and, in fact, show that the reversible cores are present *generically* in non-conservative time-reversible systems and, hence, are relevant beyond abstract schemes of topological dynamics. Numerical simulations with several models of mechanics provide direct evidence for the possible existence of a reversible core in these models: the attractor and repeller in Figs. 1a and 1c are separated, while those in Figs. 1b and 1d appear to really coincide!

In theory, one could perform a quite straightforward procedure for the numerical detection of a reversible core: if a numerically obtained attractor does not separate from a numerically obtained repeller with the increase of the accuracy of computation, then this is a reversible core. As we mentioned, a reversible core is always a limit of an infinite sequence of attractors (see Theorem 1), so this observation gives a numerical criterion for the coexistence of infinitely many attractors in the phase space. Other known criteria for this phenomenon are based on completely different ideas [9, 17, 19, 21, 23, 30, 31, 41, 42].

**1.3. Full attractor.** In reality, however, the computations are rarely repeated many times with ever increasing accuracy. It is difficult, and may be unnecessary, to distinguish whether the attractor and repeller coincide exactly or are just very close to each other: in both cases one has a right to speak about a mix of attracting and repelling dynamics.

This leads to a more relaxed idea of the intersection of attractors and repellers, based on a different notion of attractor. We define the *full attractor* of a map $f$ as the closure of the union of

[1] So it retains both forward and backward orbits of $f$ in its small neighborhood.

[2] It is also easy to construct examples where a CRH attractor is a limit of CRH repellers but is not a reversible core (see Fig. 2b). Non-trivial generic examples of such a situation can be found in [6].

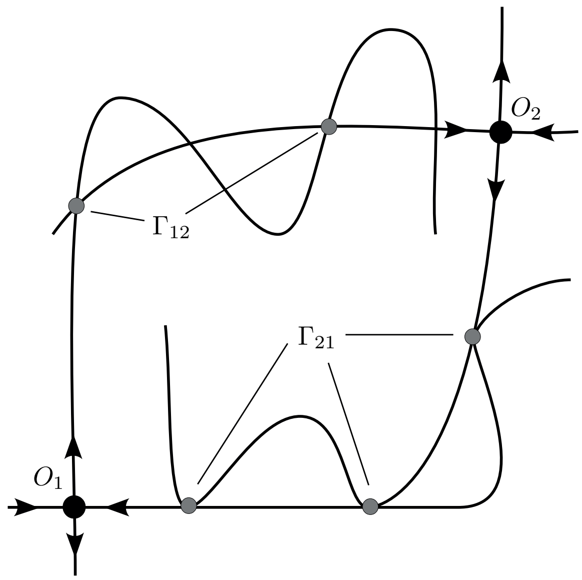



**Fig. 3.** A non-transverse heteroclinic cycle for a two-dimensional diffeomorphism with two saddle fixed points, $O_1$ and $O_2$, and two heteroclinic orbits, $\Gamma_{12}$ and $\Gamma_{21}$, such that $W^{\mathrm{u}}(O_1)$ and $W^{\mathrm{s}}(O_2)$ intersect transversely at the points of $\Gamma_{12}$ and $W^{\mathrm{u}}(O_2)$ and $W^{\mathrm{s}}(O_1)$ have a quadratic tangency at the points of $\Gamma_{21}$. When $(|J(O_1)|-1)(|J(O_2)|-1)<0$, bifurcations of this cycle lead the system into the absolute Newhouse domain and create a non-destructible intersection of the full attractor and full repeller.

all its CRH attractors, and the *full Ruelle attractor*[3] as the prolongation of this set. Recall that the prolongation of a set $A$ is the set of all points attainable from $A$ by $\varepsilon$-orbits for arbitrarily small $\varepsilon$ (see [2]). The precise definition of the full attractor is given in Section 2.

While closely related to the behavior of the system subject to a small bounded noise, i.e., to the pictures of the dynamics obtained from numerical or other experiments, the attractors thus defined are purely dynamical objects. Namely, these are closed invariant sets of the map $f$ and they are preserved by the topological conjugacy: if maps $f$ and $g$ are conjugate by a homeomorphism $h$, then the full attractor of $f$ is taken by $h$ to the full attractor of $g$, and the full Ruelle attractor of $f$ is taken to the full Ruelle attractor of $g$. The full repeller and the full Ruelle repeller of $f$ are defined as the full and, respectively, full Ruelle attractors of $f^{-1}$.

If the entire phase space is chain transitive, then the attractor and repeller are equal to the whole of the phase space, as we already mentioned; we consider such dynamics conservative from the topological point of view. When the full Ruelle attractor and the full Ruelle repeller of $f$ do not intersect, we say that the global dynamics of $f$ is dissipative. In the last remaining case, where the phase space is not chain transitive but the full Ruelle attractor and repeller have a non-empty intersection, we say that the global dynamics of $f$ is mixed. As any reversible core is simultaneously a CRH attractor and a CRH repeller, it belongs to both the full attractor and the full repeller, so the existence of reversible cores implies the mixed dynamics of $f$. However, there are more general possibilities for the mixed dynamics, as discussed below.

**1.4. Heteroclinic cycles and the intersection of attractor and repeller.** In retrospect, the phenomenon of a non-removable intersection of the full attractor and repeller was discovered in [23], where we proved that the closure of the set of asymptotically stable periodic orbits (sinks) and the closure of the set of repelling periodic orbits (sources) may have a persistently non-empty

[3] Ruelle did not call this set an attractor, but he introduced it for the study of the behavior of epsilon-orbits. This is the set $A^*$ in [34, Corollary 5.6].

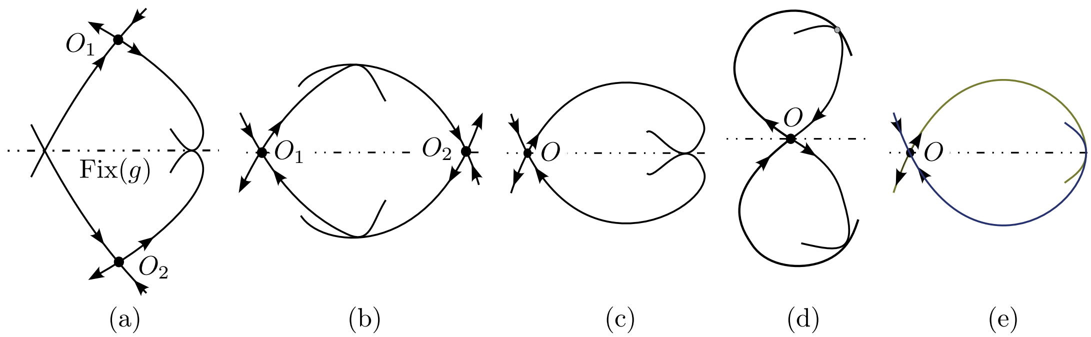



**Fig. 4.** Examples of codimension 1 bifurcations of homoclinic and heteroclinic tangencies in two-dimensional reversible maps. The first two examples show maps with non-transverse heteroclinic cycles: (a) with $O_1 = g(O_2)$ and $J(O_1) = J(O_2)^{-1} < 1$ and (b) with $J(O_1) = J(O_2) = 1$. The last three examples show maps with homoclinic tangencies, with the point $O$ symmetric in all three cases: (c) quadratic tangency, with a symmetric homoclinic orbit; (d) example of a reversible map with a symmetric pair of quadratic homoclinic tangencies to $O$; (e) cubic tangency, with a symmetric homoclinic orbit.

intersection. Namely, for a class of heteroclinic cycles shown in Fig. 3, in any continuous family of smooth two-dimensional maps for which the heteroclinic tangency splits, there exist open regions (Newhouse regions) in the parameter space where a residual set of parameter values corresponds to a non-empty intersection of the closure of sinks and the closure of sources.

The main property of the heteroclinic cycles that produce mixed dynamics is that they contain two saddles such that the map is area-contracting near one of the saddles and area-expanding near the other (the maps having cycles with this property are dense in the absolute Newhouse domain of [41, 42]).

Heteroclinic cycles with this property appear naturally in reversible diffeomorphisms (a diffeomorphism $f$ is reversible if it is conjugate to its own inverse $f^{-1}$ by means of a certain involution $g$). An example of a heteroclinic cycle of the type described above for a reversible diffeomorphism is shown in Fig. 4a. Note a symmetry in this picture: the involution $g\colon x \to x$, $y \to -y$ takes the saddle fixed point $O_1$ to the saddle fixed point $O_2$, the invariant manifold $W^{\mathrm{u}}(O_1)$ to $W^{\mathrm{s}}(O_2)$, and $W^{\mathrm{u}}(O_2)$ to $W^{\mathrm{s}}(O_1)$. In general, the Jacobian $J$ of the map at $O_1$ can be arbitrary; for instance, let $|J(O_1)| < 1$. Since $f^{-1}$ near $O_2$ is smoothly conjugate to $f$ near $O_1$, it follows that $|J(O_2)| = |J^{-1}(O_1)| > 1$. It was shown in [30] that if the map $f$ is embedded into a family of reversible maps for which the non-transverse heteroclinic cycle splits, then for generic values of parameters from the corresponding Newhouse regions there exist infinitely many periodic sinks, sources, saddles, and *elliptic periodic orbits*. Moreover, the closure of the set of periodic orbits of each of these types contains the points $O_1$ and $O_2$; i.e., *the attractors, repellers, saddles, and elliptic orbits are inseparable from each other.*

Similar results for reversible maps with another type of heteroclinic cycle (see Fig. 4b) were obtained in [9]. Unlike the previous case, the saddle points $O_1$ and $O_2$ belong to the line of the fixed points of the involution $g$; hence $J(O_1) = J(O_2) = 1$. The involution $g$ takes $W^{\mathrm{u}}(O_1)$ to $W^{\mathrm{s}}(O_1)$ and $W^{\mathrm{u}}(O_2)$ to $W^{\mathrm{s}}(O_2)$, and the map $f$ has a symmetric pair of non-transverse heteroclinic orbits. In contrast to the case of Fig. 4a, the map $f$ near the fixed points $O_1$ and $O_2$ is conservative. However, the conservativity is violated in this situation near the heteroclinic tangencies: the maps along heteroclinic tangencies have, in general, a non-constant Jacobian.[4] As shown in [9], bifurcations of such a heteroclinic cycle also lead to reversible mixed dynamics (i.e., an unbreakable intersection of the closures of the sets of attractors, repellers, saddles, and elliptic orbits). The same phenomenon takes place at the bifurcations of a symmetric pair of homoclinic tangencies as in Fig. 4d (here

[4]Corresponding explicit conditions of general position are given in [9].

$g(O) = O$ and $g(W^{\mathrm{u}}(O)) = W^{\mathrm{s}}(O)$; see [8]). We also plan to prove the existence of absolute Newhouse intervals in one-parameter families of reversible maps which unfold symmetric quadratic and cubic homoclinic tangencies, as in Figs. 4c and 4e, respectively.

**1.5. Richness of reversible mixed dynamics.** In this paper we investigate further the mixed dynamics in reversible maps. We show (Section 3) that every elliptic periodic orbit of a generic two-dimensional reversible diffeomorphism is a reversible core. We also show (Section 4) that every elliptic periodic orbit of a generic $n$-dimensional ($n \geq 2$) reversible diffeomorphism is *a limit of a sequence of uniformly hyperbolic attractors and repellers of all topological types possible in an $n$-dimensional ball*.

We recall basic definitions from the theory of non-conservative reversible systems in Section 3. Such systems are known to appear in various applications. In particular, non-conservative time-reversible dynamical systems are natural models for mechanical systems with nonholonomic constraints (for instance, the examples of dynamics shown in Fig. 1 are obtained for reversible systems of this type). The existence of symmetric elliptic periodic orbits is a characteristic property of reversible maps for which the dimension of the set $\mathrm{Fix}(g)$ of the fixed points of the involution $g$ is at least half the dimension of the phase space. In particular, they emerge in various homoclinic bifurcations [8–10, 30]. According to the reversible mixed dynamics conjecture of [9], the mixed dynamics emerging at the most typical homoclinic bifurcations of reversible systems must always include a large number of elliptic periodic orbits.

It is known [35] that the dynamics of a reversible map near a symmetric elliptic periodic orbit is, to a large extent, conservative: a significant portion of the phase space in a neighborhood of such an orbit is filled with KAM tori. However, the dynamics in the resonant zones between the KAM tori does not, in general, need to be conservative [17]. The results of Section 4 show that the dynamics near a typical elliptic orbit is *universal* in the sense of [24, 25, 40–42]; i.e., the iterations of the Poincaré map in the resonant zones near the elliptic orbit approximate, with arbitrarily good precision, every dynamics possible in the given dimension of the phase space. Thus, on the one hand, non-conservative reversible systems are of immediate importance for applications. On the other hand, their dynamics is an example of ultimate richness and complexity: any generic non-conservative reversible system with an elliptic point exhibits *all robust dynamical phenomena possible*.

## 2. ATTRACTORS, REPELLERS, AND A REVERSIBLE CORE

In several papers [9, 13–17, 20, 22] a new, third, type of chaotic dynamics was identified, the so-called mixed dynamics characterized by the attractor–repeller merger. Below we propose a scheme which could formalize this idea.

**2.1. Definitions of the attractor.** We start with recalling definitions and simple facts from topological dynamics. Consider a homeomorphism $f$ of a compact separable metric space $\mathcal{M}$. A sequence of points $x_1, \dots, x_N$ is called an *$\varepsilon$-orbit* of the map $f$ if $\mathrm{dist}(f(x_j), x_{j+1}) < \varepsilon$ for all $j = 1, \dots, N-1$. We will say that the $\varepsilon$-orbit $x_1, \dots, x_N$ *connects* the point $x_1$ to $x_N$ and that $x_N$ is *attainable* from $x_1$ by an $\varepsilon$-orbit of length $N$.

A closed invariant set $\Lambda$ is called *chain transitive* if for every $\varepsilon > 0$ and every two points $x \in \Lambda$ and $y \in \Lambda$ there exists an $\varepsilon$-orbit that lies in $\Lambda$ and connects $x$ and $y$. If two chain transitive sets $\Lambda_1$ and $\Lambda_2$ have a non-empty intersection, their union $\Lambda_1 \cup \Lambda_2$ is also a chain transitive set.

An open set $U \subset \mathcal{M}$ is an *absorbing domain* if $f(\mathrm{cl}(U)) \subset U$. An important fact is that

- the set $D_{K,\varepsilon}(x)$ of all points attainable from $x$ by $\varepsilon$-orbits of length $N \geq K$ is *always an absorbing domain.*

Indeed, this set is obviously open, so we just need to show that for any point $z$ in the closure of $D_{K,\varepsilon}(x)$ there is an $\varepsilon$-orbit of length at least $K$ that connects $x$ and $f(z)$. By definition, $z$ can be approximated arbitrarily well by an end point $x_N$ of some $\varepsilon$-orbit $x_1 = x, \ldots, x_N$, where $N \geq K$. Let $\mathrm{dist}(x_N, z) < \delta$ where $\delta$ is such that the images of any two $\delta$-close points under $f$ lie at a distance smaller than $\varepsilon$ from each other. Then $\mathrm{dist}(f(x_N), f(z)) < \varepsilon$, i.e., $x_1, \ldots, x_N, f(z)$ is an $\varepsilon$-orbit that connects $x_1 = x$ to $f(z)$; i.e., $f(z) \in D_{K,\varepsilon}(x)$.

The only absorbing domain that a *volume-preserving* map of a compact manifold $\mathcal{M}$ can have is $\mathcal{M}$ itself. Therefore, for any volume-preserving map, $D_{K,\varepsilon}(x) = \mathcal{M}$ for every $x$; i.e., every two points of $\mathcal{M}$ are connected by an $\varepsilon$-orbit for any $\varepsilon > 0$. Thus, $\mathcal{M}$ is a chain transitive set in this case. There are other, non-volume-preserving, examples. For instance, for any homeomorphism which is topologically conjugate to a volume-preserving map (like, e.g., any Anosov map on a torus [1, 3]), the whole phase space is chain transitive (if the phase space is compact); another example is given by the map $\varphi \mapsto \varphi + \sin^2(\varphi/2)$ of a circle, which has a single semi-stable fixed point.

We will no longer consider the case where the whole phase space is chain transitive. Thus, we assume that some points of $\mathcal{M}$ are not chain recurrent (a point $x$ is *chain recurrent* if its $\varepsilon$-orbits return to it infinitely many times, i.e., if $x \in D_{K,\varepsilon}(x)$ for all $\varepsilon > 0$ and $K > 0$; see also [2]).

Define $D(x) = \bigcap_{K,\varepsilon} D_{K,\varepsilon}(x)$. This is a closed invariant set, and it is an intersection of a family of nested absorbing domains. Such sets are *stable*, meaning that given any $\delta > 0$ there exists a sufficiently small $\varepsilon > 0$ such that no $\varepsilon$-orbit starting in this set can leave its $\delta$-neighborhood. If a stable set $A$ is chain transitive, it has no proper stable subsets. If not, it contains a point $y$ which is not chain recurrent, and the set $D(y)$ is a proper and stable subset of $A$. By (transfinite) induction, one can show that *every stable set has a chain transitive stable subset* (cf. [26, 34]).

Following Ruelle and Hurley, we will call a *chain transitive and stable invariant set* an attractor of the map $f$ or a *Conley–Ruelle–Hurley attractor* (CRH attractor).

We identify observable dynamical regimes with trajectories which stay in a neighborhood of CRH attractors. The logic is as follows: Whenever a certain dynamical process is observed, there is never a guarantee that the dynamical system which generates it is known precisely (for example, when we compute orbits of a given map numerically, the resulting sequence of points is, in fact, an orbit of a slightly different map, due to rounding off). One therefore may claim that the observed regimes are $\varepsilon$-orbits with a sufficiently small $\varepsilon$.

As a simplified model, one can consider the $\varepsilon$-orbits $\{x_j\}$ of the map $f$ as realizations of a random process such that the deviations $x_{j+1}$ from $f(x_j)$ are *independent random variables* $\xi_j$ with probability densities supported in the ball $\|\xi_j\| \leq \varepsilon$ and bounded away from zero and continuous in this ball. Then, for a fixed $\varepsilon > 0$ it is natural to define the *$\varepsilon$-attractor* of a point $x_0$ as the set $A_\varepsilon(x_0)$ of all points which are $\omega$-limit for the $\varepsilon$-orbits of $x_0$ with positive probability. The result does not depend on the choice of the probability density for $\xi_j$; it is a union of finitely many closed sets such that no forward $\varepsilon$-orbit starting in any of these sets can leave it, so the interior of this set is an absorbing domain, and in each of these absorbing domains every two points are connected by an $\varepsilon$-orbit. It immediately follows that if some point $x^*$ belongs to the intersection of some $\varepsilon_j$-attractors for a sequence $\varepsilon_j \to 0$ as $j \to +\infty$, then $x^*$ belongs to the intersection of a sequence of nested absorbing domains, and every two points in this intersection are connected by $\varepsilon_j$-orbits for all $j$; i.e., $x^*$ belongs to a stable and chain transitive set, a CRH attractor. Thus, every point which is not in a CRH attractor stays outside the $\varepsilon$-attractors for all sufficiently small $\varepsilon$.

Another way to express the same idea is to note that the intersection $A_0(x) = \bigcap_{\rho>0} \bigcup_{0<\varepsilon\leq\rho} A_\varepsilon(x)$ consists only of CRH attractors (moreover, each of these CRH attractors is attainable from $x$ by $\varepsilon$-orbits for every $\varepsilon$; we will further simply say that it is attainable from $x$). We note that the sets $\bigcup_{0<\varepsilon\leq\rho} A_\varepsilon(x)$ are not closed, so it is difficult to investigate the structure of their intersection $A_0(x)$. However, one can show that $A_0(x)$ is non-empty. Moreover, its closure $\overline{A_0(x)}$ contains *all* CRH attractors attainable from $x$, so it is the closure of the union of all CRH attractors attainable from $x$.

Thus, the following definition makes sense:

- an *attractor of a point* $x$ is any CRH attractor attainable from $x$.

If the number of such attractors is finite, then their union is the *full attractor* of $x$. In the case of an infinite number of such attractors, there is less certainty in the definition of the full attractor. One candidate would be *the closure of the union of all attractors of* $x$, i.e., the above defined set $\overline{A_0(x)}$. It is a closed invariant set, but it may not be stable. It is easy to show that the minimal closed stable set that contains $\overline{A_0(x)}$ is the *prolongation* of $\overline{A_0(x)}$, i.e., the set of all points which are attainable from $\overline{A_0(x)}$ for every arbitrarily small $\varepsilon$. We call it the *full Ruelle attractor* of $x$. Similarly, one introduces the following definition:

- the *full attractor of the map* $f$ is *the closure of the union of all its CRH attractors*, and the *full Ruelle attractor of* $f$ is *the prolongation of the full attractor*.

In the same way one produces definitions of repellers as attractors for the inverse map $f^{-1}$.

**2.2. Absolute Newhouse domain.** With these definitions, our goal is to investigate how attractors and repellers can intersect. The possibility of intersection of the full attractor and the full repeller was discovered in [23]. The main idea of that paper is that a generalization of the classical Newhouse phenomenon to maps which are not area-contracting leads to a persistent intersection of the full attractor and full repeller (as we defined them here).

It was shown by Newhouse in [31, 33] that a two-dimensional diffeomorphism can have a *wild hyperbolic set*, i.e., a zero-dimensional compact transitive hyperbolic set $\Lambda$ whose stable and unstable sets $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$ have a tangency which is not removable by any $C^2$-small perturbation. By the definition, maps with wild hyperbolic sets form a $C^2$-open region in the space of two-dimensional diffeomorphisms; we call this region the *Newhouse domain*. If we take a map from the Newhouse domain, then every map from its small neighborhood $\mathcal{D}$ in the space of $C^2$ diffeomorphisms will have a hyperbolic set $\Lambda$ such that $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$ are tangent. Since saddle periodic orbits are dense in $\Lambda$ and their stable and unstable manifolds are dense in $W^{\mathrm{s}}(\Lambda)$ and $W^{\mathrm{u}}(\Lambda)$, respectively, the maps which have a homoclinic tangency of the stable and unstable manifolds of some periodic orbit in $\Lambda$ are $C^r$-dense in $\mathcal{D}$ (for every $r \geq 2$).

As was first shown by Gavrilov and Shilnikov in [11], bifurcations of a homoclinic tangency to a saddle periodic orbit of a two-dimensional map lead to the birth of *stable periodic orbits if the Jacobian $J$ of the first return map at the saddle periodic point is smaller than* 1 *in absolute value*. Using this fact (which he established independently), Newhouse proved in [31] that maps with *infinitely many periodic attractors* (*stable periodic orbits*) are *dense* (*and form a residual subset*) in the part of the Newhouse domain of the space of $C^r$ diffeomorphisms that corresponds to maps which are *area-contracting* in a neighborhood of the wild hyperbolic set. Moreover, for a $C^r$-generic map of this class *the closure of the set of stable periodic orbits contains a wild hyperbolic set* $\Lambda$ *and all points homoclinic to* $\Lambda$.

In [23] we considered heteroclinic cycles with two saddle periodic orbits, $O_1$ and $O_2$, such that $|J(O_1)| < 1$ and $|J(O_2)| > 1$ (where $J(O)$ is the product of the multipliers of $O$, i.e., the Jacobian of the first return map at $O$). The cycle contains orbits of the heteroclinic intersection of $W^{\mathrm{u}}(O_1)$ with $W^{\mathrm{s}}(O_2)$ and of $W^{\mathrm{u}}(O_2)$ with $W^{\mathrm{s}}(O_1)$, and we assume that one of these orbits corresponds to a tangency of the corresponding stable and unstable manifolds. We showed in [23] that a generic splitting of this tangency leads to the creation of a wild hyperbolic set $\Lambda$ which contains both orbits $O_1$ and $O_2$, i.e., $\Lambda$ contains a pair of saddle periodic orbits such that the map is area-contracting near one of the orbits and area-expanding near the other. In [42] such wild hyperbolic sets were called *ultimately wild*; the part of the Newhouse domain which contains maps with ultimately wild sets is called the *absolute Newhouse domain* [41, 42]. Using this terminology, we can formulate the main result of [23] as follows:

- *A generic two-dimensional map in the absolute Newhouse domain has infinitely many coexisting periodic attractors and repellers, and the closure of the set of these attractors and the closure of the set of these repellers both contain the ultimately wild hyperbolic set* $\Lambda$ *and all points homoclinic to it.*[5]

For such maps the full attractor and full repeller both contain the wild set $\Lambda$ and its homoclinic points, so they have a non-empty intersection. In spite of the intersection, the full Ruelle attractor and the full Ruelle repeller can differ significantly, as the full Ruelle attractor also contains the whole unstable manifold of $\Lambda$, while the full Ruelle repeller contains the whole stable manifold of $\Lambda$.[6]

We stress that this particular instance of the intersection of the attractor and repeller is, probably, the most basic model for the mixed dynamics in two-dimensional diffeomorphisms. Indeed, if we see something which looks like a chaotic attractor, then it is natural to expect that it contains a hyperbolic set and areas are contracted near this set. Similarly, it is natural to expect that a chaotic repeller contains a hyperbolic set near which areas are expanded by the map. Collision of such sets would involve creation of heteroclinic connections between the saddle periodic orbits belonging to these two sets. This means the existence of heteroclinic tangencies at a sequence of parameter values during the attractor–repeller collision process. Thus, the above described heteroclinic cycles appear, so the maps undergoing the attractor–repeller merger can be considered with a good degree of certainty as belonging to the absolute Newhouse domain.

Note that the generic ultimately wild hyperbolic sets serve as the limits of objects much more complicated than just stable and unstable periodic orbits. It was shown in [20, 22] that they, generically, are the limits of sequences of stable and unstable closed invariant curves (i.e., quasiperiodic attractors and repellers). In [42] it was shown that a generic ultimately wild hyperbolic set is a limit of a sequence of hyperbolic attractors and hyperbolic repellers of all topological types possible for a diffeomorphism of a two-dimensional disc. In fact, it was shown in [42] that the dynamics of a $C^\infty$-generic two-dimensional diffeomorphism in an arbitrarily small neighborhood of the closure of the set of points homoclinic to an ultimately wild set is *universal*; i.e., iterations of such a map provide arbitrarily good $C^r$ approximations to *all* orientation-preserving $C^r$ diffeomorphisms of a two-dimensional disc into $\mathbb{R}^2$, for every $r$. In this sense, the mixed dynamics near the intersection of the full attractor and full repeller of a generic diffeomorphism belonging to the absolute Newhouse domain is of maximal possible richness.

**2.3. Reversible core.** Next, we report a different, previously unnoticed, mechanism for the coexistence of infinitely many attractors and repellers and their intersection. It is based on the observation that there are, in fact, two types of CRH attractors: those which actually attract something, and those which attract nothing.

Namely,

- we call a CRH attractor $A$ a *dissipative attractor* if there is a point $x \notin A$ such that for any $\varepsilon > 0$ there is an $\varepsilon$ orbit that connects $x$ to a point in $A$. Otherwise we will call the CRH attractor $A$ a *reversible core.*

It is obvious that every absorbing domain (different from the whole phase space $\mathcal{M}$) must contain at least one dissipative attractor. However, reversible cores may also exist (and they can

[5] For three-dimensional diffeomorphisms, analogous results were established in [18, 19, 21]; the existence of an absolute Newhouse domain in any dimension was shown in [39].

[6] Note that in the case of *area-preserving* maps on compact surfaces, the closures of stable and unstable manifolds of $\Lambda$ coincide [37, 38], and this is in complete agreement with the fact that the attractor coincides with the repeller in this case.

exist $C^\infty$-generically, as we show in the next section). As explained next, the existence of a reversible core immediately implies mixed dynamics.

By definition, given any $\delta > 0$ there exists $\varepsilon > 0$ such that no $\varepsilon$-orbit that starts at a distance greater than $\delta$ from the reversible core $C$ can end at a point of $C$. Note that the set $U_\varepsilon(C)$ of all points $x$ such that some $\varepsilon$-orbit of $x$ ends in $C$ is open, and it is easy to see that it satisfies the inclusion $f^{-1}(\mathrm{cl}(U)) \in U$; i.e., it is an absorbing domain for the inverse map $f^{-1}$. Thus, any neighborhood of $C$ contains an absorbing domain for the map $f^{-1}$; hence the reversible core is a CRH attractor for the inverse map as well, i.e., it is also a CRH repeller. Thus, the reversible core is an intersection of a sequence of embedded absorbing domains for the map $f$ and absorbing domains for the map $f^{-1}$ (see Fig. 2a). Any absorbing domain that encloses $C$ must contain at least one dissipative attractor, and any absorbing domain for the inverse map must contain at least one dissipative repeller (i.e., a dissipative attractor for the map $f^{-1}$). These attractors and repellers are different from $C$ (because they are dissipative) and stay at a finite distance from $C$ (because they are compact sets). Thus, we can take smaller absorbing domains around $C$ and obtain one more dissipative attractor even closer to $C$, and a dissipative repeller as well. This procedure can be repeated infinitely many times, which gives the following result.

**Theorem 1.** *Every reversible core contains a limit of an infinite sequence of dissipative attractors and a limit of an infinite sequence of dissipative repellers.*

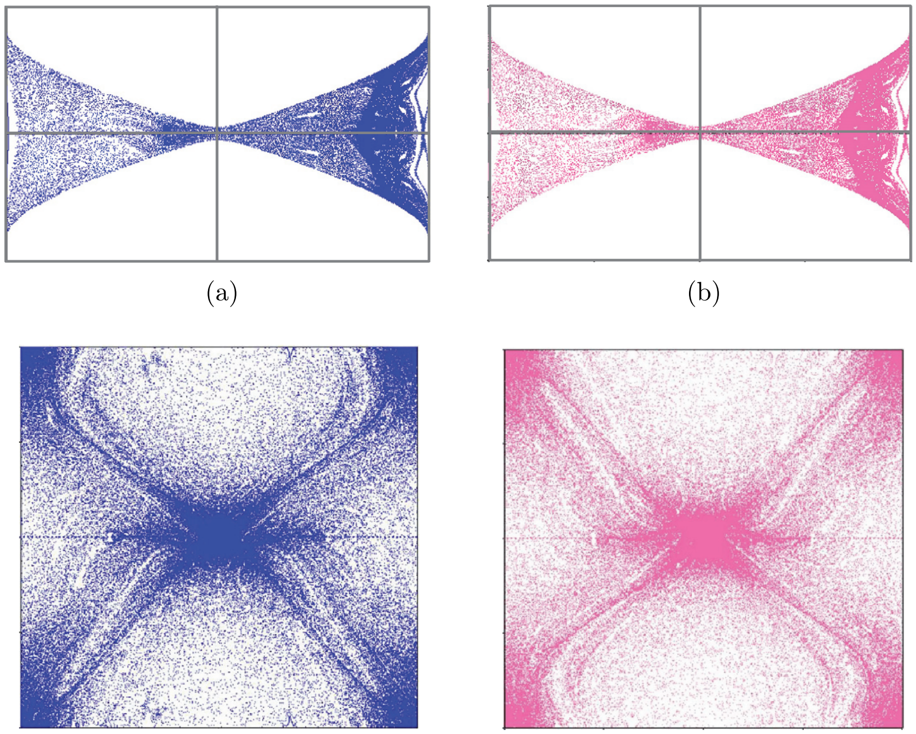
As we have attractors, repellers, and a kind of conservative object (reversible core) unseparated from each other, we speak about mixed dynamics near the core. The numerical detection of a reversible core can be based on the following

**Theorem 2.** *If a CRH attractor has a non-empty intersection with a CRH repeller, they must coincide and form a reversible core.*

**Proof.** Since the attractor $A$ and the repeller $R$ are both chain transitive and have a non-empty intersection, their union is also chain transitive. Therefore, every point of $A \cup R$ is attainable by $\varepsilon$-orbits that start in $A$ for every $\varepsilon > 0$, which means that $A = A \cup R$ (because $A$, by definition, is a stable set, meaning that $\varepsilon$-orbits that start in $A$ cannot get far from $A$). Similarly, $R = A \cup R$, so $A$ and $R$ coincide. Now, since $A$ is a CRH repeller, no $\varepsilon$-orbit that starts at a distance from $A$ bounded away from zero can get close to $A$ if $\varepsilon > 0$ is sufficiently small. Thus, $A = R$ is a reversible core, and both forward and backward $\varepsilon$-orbits never leave its small neighborhood. $\square$

By Theorem 2, if we have a reversible core, then in numerical simulations we would see that the attractor and repeller occupy roughly the same region in the phase space. However, we will not see that the attractor coincides exactly with the repeller: as numerics add some small noise, it makes sense to expect that a numerical forward orbit will shadow some absorbing domain around the core, while the backward numerical orbit will shadow some absorbing domain for the inverse map; such domains cannot completely coincide. Thus, a numerical indication of the mixed dynamics would be a numerically obtained attractor which would not coincide with a numerically obtained repeller but would have a sizable intersection with it and the difference between these two sets would appear small.

**Remark.** The question therefore arises: do Figs. 1b and 1d indicate the presence of a large reversible core? Note that the systems whose phase portraits are depicted in Fig. 1 are not covered by the theory developed in the next sections, as they cannot have generic elliptic points (the dimension of $\mathrm{Fix}(g)$ is less than half the dimension of the phase space; for example, in the model of the Chaplygin ball (Figs. 1c and 1d), the involution $g$ that takes the attractor to the repeller is the central symmetry, so it is orientation-preserving and $\mathrm{Fix}(g)$ is just one point). On the other hand, there are various examples of reversible systems having symmetric generic elliptic points (see, e.g., [13, 27, 28, 36]). In particular, in the Suslov model [28] and in the Pikovsky–Topaj model [36], the involution $g$ of the corresponding two-dimensional Poincaré map is $x \to x$, $y \to -y$ and, thus, the attractor and



(a) (b)

(c) (d)

**Fig. 5.** Attractor (a) and repeller (b) for the Suslov model (the pictures are taken from [28]). Attractor (c) and repeller (d) for the Pikovsky–Topaj model (the pictures are taken from [13]).

repeller are always symmetric with respect to the axes $y = 0$ (see Fig. 5). The question of whether the "large" intersection of the attractor and repeller observed in Figs. 1 and 5 is a reversible core, whether it is related to elliptic orbits in the cases of Fig. 5, and what it is related to in the cases of Fig. 1 remains open.

## 3. GENERIC REVERSIBLE CORES IN TWO-DIMENSIONAL REVERSIBLE MAPS

We now show that reversible cores exist for a large class of dynamical systems. We restrict ourselves to reversible maps, as they are known to provide examples of mixed dynamics in abundance, as we mentioned in the Introduction. Moreover, in this section we consider only two-dimensional maps. Thus, let $f$ be a $C^r$ diffeomorphism $(r = 1, \dots, \infty)$ of a two-dimensional orientable manifold and assume that $f$ is reversible, i.e.,

$$f^{-1} = g \circ f \circ g, \tag{3.1}$$

where $g$ is a $C^r$-smooth involution (a map such that $g \circ g = \mathrm{id}$).

A periodic orbit $\{x_0, \dots, x_m\}$ of $f$ is called *symmetric* if it is invariant with respect to $g$; namely, $gx_0 = f^j x_0$ for some $j \le m$ (then, by (3.1), $gfx_0 = f^{j-1}x_0$, and so on). It is easy to see that for a symmetric periodic orbit at least one of its points is either a fixed point of $h = g$ or a fixed point of $h = f \circ g$. Let $x_0$ be such point; we call it a symmetric periodic point. By (3.1), $h$ is an involution and

$$T^{-1} = h \circ T \circ h,$$

where $T = f^m$ is the first return map near $x_0$ (so $Tx_0 = x_0$).

For a symmetric periodic orbit,

- if $\lambda$ is a multiplier, then $\lambda^{-1}$ is also a multiplier.

Indeed, let $x_0$ be a symmetric periodic point, i.e., $Tx_0 = x_0$ and $hx_0 = x_0$. Denote by $A = T'$ the derivative of $T$ at $x_0$. By the Bochner theorem [5], we can always choose coordinates near $x_0$ such that the involution $h$ is linear. By the equality $T^{-1} = h \circ T \circ h$, we have $A^{-1} = h \circ A \circ h$. If $A\mathbf{e} = \lambda\mathbf{e}$, i.e., $\mathbf{e}$ is an eigenvector of $A$ with the eigenvalue (multiplier) $\lambda$, then it follows from the equality $A^{-1} = h \circ A \circ h$ that $A^{-1}h\mathbf{e} = \lambda h\mathbf{e}$, i.e., $h\mathbf{e}$ is also an eigenvector of $A$ with the eigenvalue $\lambda^{-1}$.

Note that if $\lambda^2 \neq 1$, i.e., the multipliers $\lambda$ and $\lambda^{-1}$ are different, then the eigenvectors $\mathbf{e}$ and $h\mathbf{e}$ are not collinear. Thus, the involution $h$ interchanges a pair of non-collinear vectors, which means that $h$ reverses the orientation. We will further assume that the map $f$ is orientation-preserving, so the original involution $g$ must be orientation-reversing in this case (recall that $g = h$ or $g = f^{-1} \circ h$). This will be our standing assumption from now on.

Since the linearization matrix $A$ is real, if $\lambda$ is an eigenvalue of $A$, then the complex conjugate $\lambda^*$ must also be an eigenvalue. So, if a symmetric periodic orbit has a complex (not real) multiplier $\lambda$, then $\lambda^* = \lambda^{-1}$, i.e., both multipliers must lie on the unit circle. This means that there exists $\omega \in (0, \pi)$ such that the multipliers of the periodic orbit are $e^{\pm i\omega}$. We call the symmetric periodic orbit *elliptic* in this case.

It is well known [35] that a symmetric elliptic periodic orbit of a two-dimensional reversible $C^r$ diffeomorphism remains elliptic under $C^r$-small perturbations which keep the map reversible. So, systems with elliptic orbits form an open subset in the space of $g$-reversible $C^r$ diffeomorphisms. Empirical evidence suggests that this open set should be quite large (cf. [13]). In particular, elliptic orbits are born [8, 9, 30] at bifurcations of reversible maps with heteroclinic cycles described in the Introduction and Subsection 2.2. Therefore, systems with elliptic periodic orbits form a dense (and open) subset of the absolute Newhouse domain in the space $\mathcal{R}^r_g$ of two-dimensional $g$-reversible $C^r$ diffeomorphisms. On the other hand, bifurcations near elliptic orbits lead to creation of ultimately wild hyperbolic sets [17], which means that the $C^r$ closure of the set of systems from $\mathcal{R}^r_g$ with elliptic orbits coincides with the $C^r$ closure of the absolute Newhouse domain in $\mathcal{R}^r_g$.

A natural conjecture (a similar conjecture for area-preserving maps can be found, e.g., in [32]) would be that if the map has a chaotic attractor but is not uniformly hyperbolic, then homoclinic tangencies or cycles with heteroclinic tangencies can be created by $C^r$-small perturbations. No mathematical technique is currently available for proving such a statement. Still it is reasonable to conjecture that if a chaotic attractor is observed in a given $g$-reversible map, then most probably the attractor is not uniformly hyperbolic and contains a wild hyperbolic set (unless we deal with an Anosov map on a torus), and if the attractor intersects its own image under the involution $g$, then the wild hyperbolic set is ultimately wild (i.e., it contains periodic orbits with Jacobians both greater and less than 1). Thus, such a map can be suspected to be in the absolute Newhouse domain and, in the case of orientation-reversing involution $g$, this means that elliptic periodic orbits should be expected.

According to [35], most of the neighborhood of a generic[7] elliptic point of a reversible map is occupied by invariant KAM curves, as in the conservative case. However, as shown in [17], a generic elliptic point of a two-dimensional non-conservative reversible map is also a limit of an infinite sequences of periodic attractors and periodic repellers (that are born in the resonant zones). Here, we strengthen this result and prove the following

**Theorem 3.** *All symmetric elliptic periodic orbits of a $C^r$-generic two-dimensional $g$-reversible map are reversible cores.*

[7] The genericity conditions in this case include the absence of strong resonances, i.e., $\omega \neq \pi/2, 2\pi/3$, and the non-vanishing of one of the Birkhoff coefficients in the case of irrational $\omega/\pi$ or of one of the first Birkhoff coefficients in the case of rational $\omega/\pi$. In the conservative case the generic elliptic point is KAM stable [4]. In the reversible case, the KAM stability, of course, holds, but, as Theorem 3 shows, elliptic points of a generic non-conservative reversible map are stable in a stronger sense.

**Proof.** Since a periodic orbit is a chain transitive set, it suffices to prove that each elliptic orbit is (generically) surrounded by a sequence of nested absorbing domains both for the map $f$ itself and for its inverse. We start with proving that given an elliptic orbit $P$ and an open neighborhood $U$ of $P$, one can make an arbitrarily small $C^r$ perturbation of the map (within the class $\mathcal{R}_g^r$ of reversible systems) such that the perturbed map $f$ and its inverse $f^{-1}$ would each have an absorbing domain lying inside $U$ and containing $P$.

Let $x_0$ be a symmetric point on $P$ and $T$ be the first return map near $x_0$. We can introduce a complex coordinate $z$ near $x_0$ such that the map $T$ will be the period map of the time-reversible non-autonomous flow [29] defined by the differential equation

$$\dot{z} = i\omega z + F(z, z^*, t), \tag{3.2}$$

where the $C^r$ function $F$ is 1-periodic, i.e., $F(z, z^*, t) = F(z, z^*, t+1)$, and it vanishes at $z = 0$ along with its derivative with respect to $z$ and $z^*$, so $z = 0$ is the elliptic fixed point of $T$. The reversibility means here that $F(z, z^*, t) = F^*(z^*, z, -t)$; i.e., this system is invariant with respect to the transformation $\{t \to -t,\ z \leftrightarrow z^*\}$, so here the involution $h$ described at the beginning of this section is the complex conjugation operation $\{z \to z^*\}$.

One can add an arbitrarily small perturbation to (3.2) such that $\omega/\pi$ would become irrational. Then the normal form theory for reversible maps [29] will ensure the existence of a $C^r$ coordinate transformation which brings the map $T$ to the form

$$T = R_\omega \circ \mathcal{T}, \tag{3.3}$$

where $R_\omega \colon (z, z^*) \mapsto (e^{i\omega} z, e^{-i\omega} z^*)$ is the rotation through the angle $\omega$ and $\mathcal{T}$ is the time 1 map induced by the flow of

$$\dot{z} = i\Omega(|z|^2)z + o(|z|^r), \tag{3.4}$$

where $\Omega$ is a real polynomial such that $\Omega(0) = 0$; all time-dependent terms are now in the $o(|z|^r)$ term (which vanishes at $z = 0$ along with all derivatives with respect to $z$ and $z^*$ up to the order $r$). The next step is to remove the $o(|z|^r)$ term in (3.4), i.e., add a $C^r$-small perturbation after which the map $\mathcal{T}$ will coincide in a sufficiently small neighborhood of zero with the time 1 map of the reversible autonomous flow given by

$$\dot{z} = i\Omega(|z|^2)z. \tag{3.5}$$

By an additional small perturbation we can always ensure that $\Omega_1 = \Omega'(0) \neq 0$.

By a $C^r$-small perturbation of the map $f$ within the class of reversible maps, we can make $\omega = 2\pi p/q$ where $p$ and $q$ are coprime integers (we will also assume that $q \geq r$), while keeping the map $T$ in the form (3.3). We can also change equation (3.5) in such a way that in a small neighborhood of $z = 0$

$$\dot{z} = -i\mu z + i\Omega(|z|^2)z + i\delta(z^*)^{q-1} + iBz^{q+1} + iCz(z^*)^q, \tag{3.6}$$

where $\delta$ and $\mu$ are small real parameters and $B$ and $C$ are some real constants. Note that all coefficients on the right-hand side of (3.6) are pure imaginary, so the equation is time-reversible, as it should.

Note that equation (3.6) is the truncated normal form for the bifurcations of a symmetric elliptic point with a degenerate resonance corresponding to the multiplier $\lambda = e^{2\pi ip/q}$ and the coefficient of the first non-trivial resonant term $(z^*)^{q-1}$ vanishing at the moment of bifurcation. For $C = B(q+1)$, equation (3.6) is Hamiltonian. However, varying $B$ and $C$ may break the conservativity: the equation may have, e.g., asymptotically stable and unstable equilibria, as we will see below.

The equation is symmetric with respect to the rotation through the angle $2\pi p/q$. This means that the maps $R_{2\pi p/q}$ and $\mathcal{T}$ in (3.3) commute, so the map $T^q$ is the time $q$ map of (3.6). In particular, every equilibrium state of (3.6) is a period $q$ point of $T$; asymptotically stable equilibria correspond to asymptotically stable periodic points. In this way, based on the analysis of stability of equilibrium states of (3.6), it was shown in [17] that stable periodic orbits can be born at bifurcations of elliptic periodic points in reversible maps. Here, we need a more detailed investigation of the dynamics of equation (3.6).

To this aim, introduce polar coordinates: $z = \sqrt{\rho}e^{i\phi/q}$. The equation takes the form

$$\begin{aligned} \dot\rho &= 2\rho^{q/2}(\delta - (B-C)\rho)\sin\phi, \\ \dot\phi &= q(\Omega(\rho) - \mu) + q\rho^{(q-2)/2}(\delta + (B+C)\rho)\cos\phi. \end{aligned} \tag{3.7}$$

Note that this system is invariant with respect to the transformation $\{t \to -t,\ \phi \to -\phi\}$.

Assume $B \neq 0$ and $B \neq C$. Choose sufficiently small $\rho_0 > 0$, put

$$\mu = \Omega(\rho_0), \tag{3.8}$$

and consider the behavior of the system for $\rho$ close to $\rho_0$. We do this by scaling

$$\rho = \rho_0 - \rho_0^{q/2}\frac{2B}{\Omega_1}V, \tag{3.9}$$

where the range of values of $V$ can be as large as we want if $\rho_0$ is small enough (the coefficient $\Omega_1 \neq 0$ is equal to $\Omega'(0)$). We also scale the small parameter

$$\delta = (B-C)\rho_0 + \frac{2B}{\Omega_1}(B-C)\rho_0^{q/2}D, \tag{3.10}$$

where the rescaled parameter $D$ is no longer small and can take arbitrary finite values, and introduce the new time $s = 2(C-B)\rho_0^{q/2}t$. The system will take the form

$$\begin{aligned} \dot V &= (D+V)\sin\phi + O\big(\rho_0^{(q-2)/2}\big), \\ \dot\phi &= \frac{Bq}{B-C}(V - \cos\phi) + O\big(\rho_0^{(q-2)/2}\big). \end{aligned} \tag{3.11}$$

The limit of this rescaled system as $\rho_0 \to 0$ is

$$\dot V = (D+V)\sin\phi, \qquad \dot\phi = \beta(V - \cos\phi), \tag{3.12}$$

where $\beta = qB/(B-C)$. This is a time-reversible system on the cylinder parameterized by $(V,\phi)$. Importantly, this system can be solved. Thus, it is easy to see that the phase curves of this system satisfy

$$\cos\phi + D = (D+V)\frac{\beta}{\beta-1} + K|D+V|^{\beta}$$

with indefinite constants $K$. At $\beta = 1$ this formula should be replaced by

$$\cos\phi + D = K(D+V) - (D+V)\ln|D+V|.$$

With these formulas, one can construct the phase portrait of system (3.12) on the cylinder for different values of $D$ and $\beta$, as shown in Fig. 6.

In particular, for $\beta > 0$ and $|D| < 1$ (see Fig. 6a) this system has two symmetric (with respect to the involution $\phi \to -\phi$) saddle equilibria $O_+(1,0)$ and $O_-(-1,\pi)$ and two asymmetric equilibria

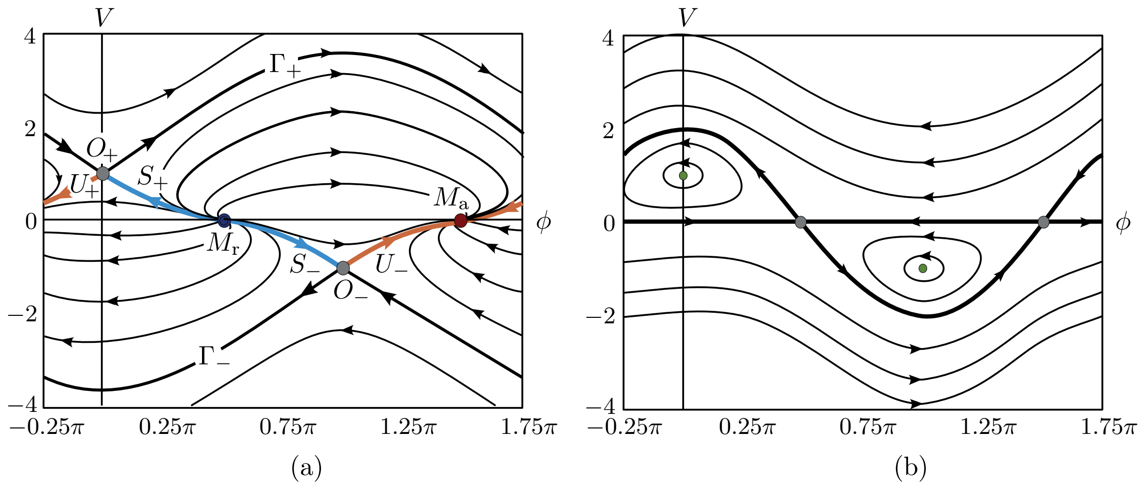



**Fig. 6.** The phase portrait of system (3.12) on the cylinder (a) for $D = 0$, $\beta = 1$ and (b) for $D = 0$, $\beta = -1$.

$M_{\mathrm{a,r}} = (-D, \phi_{\mathrm{a,r}})$, where $\cos\phi_{\mathrm{a,r}} = -D$, with the equilibrium $M_{\mathrm{a}}$ (with $\sin\phi_{\mathrm{a}} < 0$) being asymptotically stable and $M_{\mathrm{r}}$ (with $\sin\phi_{\mathrm{r}} > 0$) being asymptotically unstable. Two of the separatrices of $O_+$ coincide and form a homoclinic loop $\Gamma_+$, while of the two other separatrices the unstable separatrix $U_+$ tends to $M_{\mathrm{a}}$ as $t \to +\infty$ and the stable separatrix $S_+$ tends to $M_{\mathrm{r}}$ as $t \to -\infty$. The same is true for $O_-$: two of its separatrices form a homoclinic loop $\Gamma_-$ and the other two separatrices tend one ($U_-$) to $M_{\mathrm{a}}$ as $t \to +\infty$ and the other ($S_-$) to $M_{\mathrm{r}}$ as $t \to -\infty$. In the invariant annulus bounded by $\Gamma_+$ and $\Gamma_-$, all the orbits in its interior, except for the repeller $M_{\mathrm{r}}$ and the separatrices $S_\pm$, tend to $M_{\mathrm{a}}$ as $t \to +\infty$, while all the orbits except for the attractor $M_{\mathrm{a}}$ and the separatrices $U_\pm$ tend to $M_{\mathrm{r}}$ as $t \to -\infty$.

It follows that if we remove a small neighborhood of $\Gamma_+ \cup S_+ \cup M_{\mathrm{r}}$ from the phase cylinder, then the connected component that contains $M_{\mathrm{a}}$ is an absorbing domain. Note that it contains the part of the cylinder corresponding to $V \to -\infty$. Similarly, one obtains an absorbing domain corresponding to $V \to +\infty$; this is the connected component of the cylinder minus a small neighborhood of $\Gamma_- \cup S_- \cup M_{\mathrm{r}}$ which contains $M_{\mathrm{a}}$ (see Fig. 7a). In the same way, by removing a small neighborhood of $\Gamma_+ \cup U_+ \cup M_{\mathrm{a}}$ or a small neighborhood of $\Gamma_- \cup U_- \cup M_{\mathrm{a}}$ from the cylinder and taking the connected component that contains the repeller $M_{\mathrm{r}}$, we obtain a pair of absorbing domains (with the boundaries $B_{\mathrm{r}+}$ and $B_{\mathrm{r}-}$, see Fig. 7a) for the system obtained from (3.12) by the time reversal.

The absorbing domains do not disappear under small perturbations of the system, so they persist for system (3.11) for all small $\rho_0$. System (3.11) is obtained from (3.6) by rescaling the coordinates. In the non-rescaled coordinates, $z = 0$ corresponds either to very large positive $V$ or to very large negative $V$, so in any case we find that for appropriately chosen values of $\mu$, $\delta$, $B$, and $C$, both system (3.6) and the system obtained from it by the time reversal have an absorbing domain containing the equilibrium at $z = 0$. Note that $\mu$ and $\delta$ can be made as small as we want by taking $\rho_0$ small (see (3.8) and (3.10)), as required. Note also that the absorbing domains containing points with large negative values of $V$ do not extend beyond $\Gamma_+$ and the absorbing domains containing points with large positive values of $V$ do not extend beyond $\Gamma_-$. As these boundaries correspond to bounded values of $V$, it follows that in the non-rescaled coordinates $z$ these boundaries correspond to $|z|$ close to $\rho_0$; i.e., the pair of absorbing domains around the point $z = 0$ (for system (3.6) and for its time reversal) lies entirely in an $O(\rho_0)$-neighborhood of this point (see Fig. 7b).

The time $q$ map of system (3.6) is the map $T^q$. Thus, we have constructed a $C^r$-small perturbation of the map $f$ such that the map $T^q = f^{qm}$ (where $m$ is the period of the periodic orbit $P$

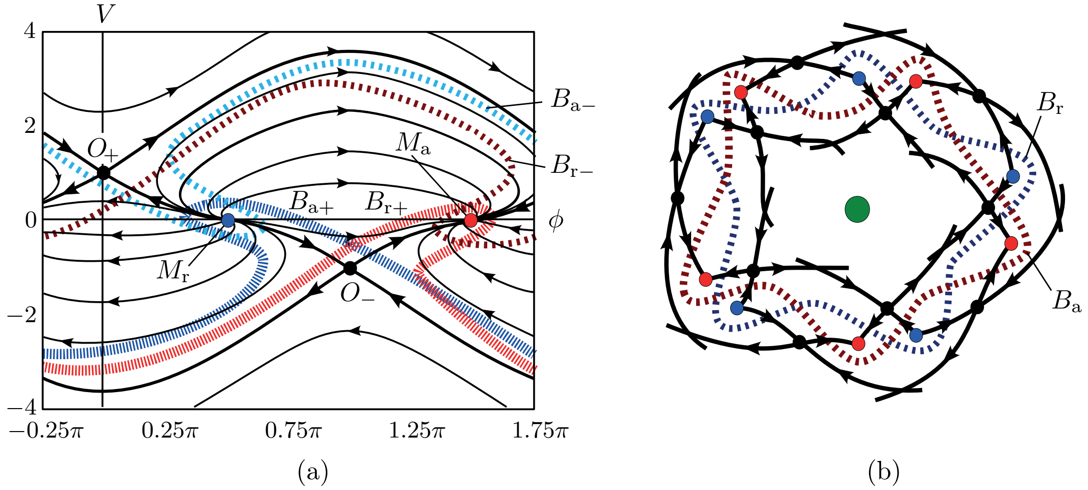



**Fig. 7.** (a) Boundaries of absorbing domains for system (3.12) and its inverse (with $t$ replaced by $-t$). The domains with boundaries $B_{a+}$ and $B_{r+}$ contain, respectively, the attractor $M_a$ and the repeller $M_r$ and the upper part of the cylinder. The domains with boundaries $B_{a-}$ and $B_{r-}$ contain, respectively, $M_a$ and $M_r$ and the lower part of the cylinder. (b) A pair of absorbing domains with boundaries $B_a$ and $B_r$ around the point $z = 0$ for system (3.6) and for its time reversal.

under consideration) and the map $T^{-q}$ have each an absorbing domain ($\mathcal{D}_a$ and $\mathcal{D}_r$, respectively) in a small (as small as we want) neighborhood of some point on $P$. Obviously, a small open neighborhood of the closure of $\bigcup_{i=0}^{qm-1} f^i\mathcal{D}_a$ is an absorbing domain for the map $f$ which contains the whole orbit $P$ and is contained in a small neighborhood of $P$. A small open neighborhood of the closure of $\bigcup_{i=0}^{qm-1} f^{-i}\mathcal{D}_r$ is an absorbing domain for the map $f^{-1}$; it contains the whole orbit $P$ and is contained in a small neighborhood of $P$.

We can now finish the proof of the theorem. First, we recall the well-known fact that if a two-dimensional $g$-reversible map with an orientation-reversing involution $g$ has a symmetric orbit $L$ of period $m$ with the multipliers $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$, then by an arbitrarily small $C^r$ perturbation of the map one can ensure that in a small neighborhood of $L$ all orbits of the same period $m$ will be either elliptic or hyperbolic (i.e., $|\lambda_{1,2}| \neq 1$).

Now, take a countable base of balls $U_s$, $s = 1, \dots, \infty$ (so every open set is a union of some sequence of the balls $U_s$). Choose one of these balls and take an integer $m \geq 1$. As we just mentioned, by an arbitrarily small perturbation of the map $f$ within the class $\mathcal{R}_g^r$ one can ensure that all symmetric orbits of period at most $m$ that intersect the chosen ball $U_s$ are either elliptic or hyperbolic. There are no other orbits of the same or smaller period in a neighborhood of an elliptic or hyperbolic periodic orbit of period $m$, nor can such orbits be born under a $C^r$-small perturbation, so the number of points of symmetric elliptic orbits of period $\leq m$ in $U_s$ is finite and this property holds for an open and dense set of maps from $\mathcal{R}_g^r$. As we proved above, by an additional $C^r$-small perturbation we can create a pair of absorbing domains (one for the map $f$ and the other for the map $f^{-1}$) around each of the elliptic orbits of period $\leq m$ that intersect $U_s$, and these absorbing domains lie inside $\bigcup_{i=-mr}^{mr} f^i(U_s)$. Absorbing domains persist under small perturbations of the map, so the set $\mathcal{E}_{s,m}$ with this property is open and dense in $\mathcal{R}_g^r$. The intersection $\mathcal{E}^*$ of the sets $\mathcal{E}_{s,m}$ over all $U_s$ and all integer $m$ is a countable intersection of open and dense sets, so it is a residual set and every map from this intersection is, by definition, $C^r$-generic. By construction, for every map from $\mathcal{E}^*$ and for every elliptic orbit of it, in any neighborhood of this orbit there exists a pair of absorbing domains (one for the map itself and one for the inverse map) such that both domains contain this orbit. Thus every elliptic orbit of every map from $\mathcal{E}^*$ is simultaneously a CRH attractor and a CRH repeller; i.e., it is a reversible core. □

## 4. UNIVERSAL DYNAMICS NEAR ELLIPTIC ORBITS IN REVERSIBLE SYSTEMS

The richness of dynamics near a reversible core does not need to be exhausted by simple periodic attractors and repellers. In particular, as we show in this section, near elliptic orbits of reversible maps, the dynamics can be as complicated and diverse as possible for the given dimension of the phase space. We will not restrict our consideration to two-dimensional maps here, so we start with recalling the classification of periodic orbits of $n$-dimensional reversible maps.

Let $\mathcal{M}$ be an $n$-dimensional manifold. Let $g\colon \mathcal{M} \to \mathcal{M}$ be a $C^r$ involution, so $g \circ g = \mathrm{id}$. A diffeomorphism $f\colon \mathcal{M} \to \mathcal{M}$ is called a reversible map if it is conjugate by $g$ to its own inverse. A periodic orbit of $f$ is called symmetric if it is invariant with respect to $g$; at least one point $x_0$ of a symmetric periodic orbit is a fixed point of an involution $h$, where $h = g$ or $h = f \circ g$. Let $T$ be the first return map near $x_0$, so $Tx_0 = x_0$ and $hx_0 = x_0$. By reversibility

$$T^{-1} = h \circ T \circ h. \tag{4.1}$$

Denote by $A$ the derivative of $T$ at $x_0$. We can always choose $C^r$ coordinates near $x_0$ such that the involution $h$ is linear [5]. By (4.1), we have

$$A^{-1} = h \circ A \circ h. \tag{4.2}$$

If $A\mathbf{e} = \lambda\mathbf{e}$, i.e., $\mathbf{e}$ is an eigenvector of $A$ with the eigenvalue (multiplier) $\lambda$, then it follows from (4.2) that $A^{-1}h\mathbf{e} = \lambda h\mathbf{e}$, i.e., $h\mathbf{e}$ is also an eigenvector of $A$ with the eigenvalue $\lambda^{-1}$.

Note that if an eigenvector of $A$ corresponds to a multiplier which is not $\pm 1$ (i.e., $\lambda \neq \lambda^{-1}$), then this eigenvector is not an eigenvector of the involution $h$. On the other hand, every eigenvector of $A$ which corresponds to $\lambda = \pm 1$ is a linear combination of eigenvectors which are, at the same time, eigenvectors of $h$.[8] One can also show that if $A$ has an eigenvector $\mathbf{e}$ with the eigenvalue $\pm 1$ and this eigenvector is not an eigenvector of $h$, then, without destroying the reversibility, one can add an arbitrarily small perturbation to the map $T$ such that, for the perturbed $T$, we would have $Ay = \widetilde{\lambda}\mathbf{e}$ and $A(h\mathbf{e}) = \widetilde{\lambda}^{-1}h\mathbf{e}$ for some $\widetilde{\lambda}$ different from $\pm 1$. This means that for a generic symmetric periodic orbit, exactly those multipliers that correspond to the eigenvectors of $A$ which are not eigenvectors of $h$ are different from $\pm 1$.

One can also show that the linearization matrix $A$ for a generic symmetric periodic orbit has no Jordan blocks. Thus, the invariant subspace $I_+$ of $A$ which corresponds to the eigenvalue $+1$ is spanned by eigenvectors of $A$, and either all of them satisfy $h\mathbf{e} = \mathbf{e}$ or all of them satisfy $h\mathbf{e} = -\mathbf{e}$. The same is true for the invariant subspace $I_-$ which corresponds to the multiplier $-1$. However, we can always, if necessary, replace the involution $h$ by the involution $T \circ h$ (identity (4.1) would not change) and ensure that the involution is identity on $I_-$. After this choice is made, it can be shown (see the remark after formula (4.6) in the proof of Theorem 5 below) that if $h = -\mathrm{id}$ on $I_+$, then the symmetric periodic point can be made to disappear by an arbitrarily small perturbation of the map. Therefore, for the generic symmetric periodic orbit we have

$$h|_{I_+ \oplus I_-} = \mathrm{id}. \tag{4.3}$$

It can be shown that condition (4.3) is necessary and sufficient for the symmetric periodic orbit to persist under small smooth perturbations which preserve the reversibility. Below, such periodic orbits will be called *regular*.

If the matrix $A$ has some hyperbolic eigenvalues, i.e., multipliers not on the unit circle, then the map $T$ near the periodic point $x_0$ has an invariant center manifold $W^c$, which is an intersection of the

[8] If $A\mathbf{e} = \lambda\mathbf{e}$ with $\lambda^2 = 1$ and $h\mathbf{e} \neq \pm\mathbf{e}$, then $\mathbf{e} + h\mathbf{e}$ and $\mathbf{e} - h\mathbf{e}$ are linearly independent eigenvectors of $A$. Indeed, since $h\mathbf{e}$ is an eigenvector of $A$ with the eigenvalue $\lambda^{-1} = \lambda$, we have $A(\mathbf{e} \pm h\mathbf{e}) = A\mathbf{e} \pm Ah\mathbf{e} = \lambda\mathbf{e} \pm \lambda^{-1}h\mathbf{e} = \lambda(\mathbf{e} \pm h\mathbf{e})$.

center-stable manifold $W^{\mathrm{cs}}$ and the center-unstable manifold $W^{\mathrm{cu}}$. The invariant manifold $W^{\mathrm{cs}}$ is tangent to the invariant space of $A$ which corresponds to all the multipliers smaller than or equal to 1 in absolute value (i.e., the multipliers on the unit circle or inside it), while the invariant manifold $W^{\mathrm{cu}}$ is tangent to the invariant space of $A$ which corresponds to all the multipliers whose absolute value is greater than or equal to 1 (the multipliers on the unit circle and outside it). We can choose $W^{\mathrm{cu}} = h(W^{\mathrm{cs}})$; then the invariant center manifold $W^{\mathrm{c}} = W^{\mathrm{cs}} \cap h(W^{\mathrm{cs}})$ will be $h$-invariant. The center manifold $W^{\mathrm{c}}$ is smooth (of class $C^r$ for any finite $r$) and persists under $C^r$-small perturbations. For every map close to $f$, all the orbits that never leave a small neighborhood of the periodic orbit under consideration must belong to $W^{\mathrm{c}}$. The dynamics transverse to $W^{\mathrm{c}}$ is hyperbolic (trivial): the orbits not from $W^{\mathrm{c}}$ either leave a small neighborhood of the periodic orbit both under forward and backward iterations of $f$, or lie in $W^{\mathrm{cs}}$ and exponentially approach $W^{\mathrm{c}}$ under forward iterations of $f$ (so they leave a small neighborhood of the periodic orbit under backward iterations), or lie in $W^{\mathrm{cu}}$ and exponentially approach $W^{\mathrm{c}}$ under backward iterations of $f$ (and leave a small neighborhood of the periodic orbit under forward iterations). The dynamics in $W^{\mathrm{c}}$ is, generically, very non-trivial. In particular, we have the following result.

**Theorem 4.** *Consider a regular symmetric periodic orbit of a $g$-reversible $C^r$-smooth map $f$. Assume the orbit has at least one pair of complex multipliers on the unit circle, i.e., $\lambda = e^{\pm i\omega}$ with $\omega \in (0, \pi)$. Then for a $C^r$-generic $g$-reversible map sufficiently close to $f$, the periodic orbit is a limit of an infinite sequence of uniformly hyperbolic attractors and uniformly hyperbolic repellers of all topological types possible for a smooth map of a $d$-dimensional disc, where $d$ is the dimension of the center manifold for this orbit.*

This theorem is an immediate consequence of a more general statement (Theorem 5) that employs the notion of a *universal* map from [40, 42]. Given a $C^r$-smooth diffeomorphism $F$ of an $n$-dimensional manifold $\mathcal{M}$, we consider the set of the so-called renormalized iterations of $F$, defined as follows. Take a unit ball $\mathcal{B}_n \subset \mathbb{R}^n$. Let $\psi$ be a $C^r$ map $\mathbb{R}^n \to \mathcal{M}$ which is a diffeomorphism between $\mathbb{R}^n$ and its image $\psi(\mathbb{R}^n)$. Take the ball $\psi(\mathcal{B}_n)$ and suppose that its image $F^k \circ \psi(\mathcal{B}_n)$ lies inside $\psi(\mathbb{R}^n)$ for some positive integer $k$. Then the $C^r$ diffeomorphism $F_{k,\psi}\colon \mathcal{B}_n \to \mathbb{R}^n$ defined as $F_{k,\psi} = \psi^{-1} \circ F^k \circ \psi|_{\mathcal{B}_n}$ is a renormalized iteration of $F$.

- The map $F$ is called *$d$-universal* if the set of its renormalized iterations is $C^r$-dense in the set of all orientation-preserving $C^r$ diffeomorphisms from $\mathcal{B}_n$ to $\mathbb{R}^n$.

By the definition, iterations of any $d$-universal map approximate arbitrarily well all dynamics possible in a $d$-dimensional ball. Therefore, every $C^r$-robust phenomenon occurring in any $d$-dimensional diffeomorphism is also present in each $d$-universal map. In particular, every $d$-universal map has, simultaneously, uniformly hyperbolic attractors and repellers of all topological types possible in the $d$-dimensional ball. Thus, Theorem 4 is a direct consequence of the following result, which we obtain by using the theory from [42] on smooth perturbations of the identity.

**Theorem 5.** *Consider a regular symmetric periodic orbit of a $g$-reversible $C^r$-smooth map $f$. Assume the orbit has at least one pair of complex multipliers on the unit circle. Then any $C^r$-generic $g$-reversible map sufficiently close to $f$, restricted to the local center manifold in an arbitrarily small neighborhood of the periodic orbit under consideration, is $d$-universal, where $d$ is the dimension of the center manifold.*

**Proof.** Take a regular symmetric periodic point $x_0$ and consider a small piece of the local center manifold $W^{\mathrm{c}}$ that contains $x_0$. We will show below that by an arbitrarily small perturbation of the map within the class of $C^r$-smooth $g$-reversible maps one can create a *periodic spot* in $W^{\mathrm{c}}$, i.e., an open $d$-dimensional ball in $W^{\mathrm{c}}$ for which all points are periodic *with the same period.* After that is done, we will just need an easy adaptation of the result of [42] to the reversible case in order to make the map universal.

Thus, we have a periodic orbit with multipliers $e^{\pm i\omega_1}, \dots, e^{\pm i\omega_m}$ ($0 < \omega_j < \pi$, $m \ge 1$), with $s$ multipliers equal to $+1$, and with $k$ multipliers equal to $-1$, so $2m + k + s = d$ multipliers lie on the unit circle and the remaining $n - d$ multipliers are not on the unit circle. As it was explained before, we may generically assume that there are no Jordan blocks in the linearization matrix $A$ of the period map $T$ (for the multipliers $\pm 1$ this is due to (4.3); for the other multipliers we assume that they are all simple). Also, by a small perturbation, we can make the values of $\omega_j$, $j = 1, \dots, m$, jointly rationally independent together with $\pi$.

Let $\sigma$ be the linear map that changes the sign of the coordinates in the invariant subspace of the matrix $A$ corresponding to all real negative multipliers. Then, the normal form of the map $T$ near $x_0$ is equal [29] to $\sigma$ times the time 1 map induced by the flow of a system of differential equations of the form

$$
\begin{aligned}
\dot z_j &= i\Omega_j(Z,u,v)z_j + O(y) + o(\|x - x_0\|^r), \qquad j = 1, \dots, m,\\
\dot u &= F(Z,u,v) + O(y) + o(\|x - x_0\|^r),\\
\dot v &= G(Z,u,v) + O(y) + o(\|x - x_0\|^r),\\
\dot y &= H(Z,u,v,y)y + o(\|x - x_0\|^r),
\end{aligned}
\tag{4.4}
$$

where $x = x_0 + (z,u,v,y)$, the variables $z_j$ are complex (projections to the eigenspaces corresponding to the multipliers $e^{\pm i\omega_j}$), $Z = (|z_1|^2, \dots, |z_s|^2)$, $u \in \mathbb{R}^s$ (these are the projections to the eigenspace corresponding to the multipliers $+1$), $v \in \mathbb{R}^k$ (these are the projections to the eigenspace corresponding to the multipliers $-1$), and $y \in \mathbb{R}^{n-d}$ (projections to the eigenspace corresponding to the multipliers not on the unit circle); the functions $F$ and $G$ have zero linear parts, $\Omega_j(0,0,0) = \omega_j$, and $H(0,0,0,0)$ is a matrix with eigenvalues outside the imaginary axis. The $o(\|x - x_0\|^r)$ terms can be time-dependent; they vanish at zero along with the derivatives up to the order $r$. The $O(y)$ terms stand for the functions vanishing at $y = 0$. The normal form system is also $\sigma$-equivariant, i.e., $\Omega_j$ and $F$ are even functions of $v$, while $G$ is an odd function of $v$. The same normalizing transformation brings the involution $h$ from (4.1) and (4.2) to the form $h(z,u,v,y) = (z^*, u, v, \widehat{h}y)$ or $h(z,u,v,y) = (z^*, -u, v, \widehat{h}y)$, where $z^*$ is complex conjugate to $z$ and $\widehat{h}$ is some linear involution in the $y$-space.

By adding a $C^r$-small perturbation to $f$, one can make the $o(\|x - x_0\|^r)$ terms in (4.4) vanish in a sufficiently small neighborhood of $x_0$. Then, the map $T$ near $x_0$ will be equal to $\sigma$ times the time 1 map induced by the flow of the autonomous system

$$
\begin{aligned}
\dot z_j &= i\Omega_j(Z,u,v)z_j + O(y), \qquad j = 1, \dots, m,\\
\dot u &= F(Z,u,v) + O(y),\\
\dot v &= G(Z,u,v),\\
\dot y &= H(Z,u,v,y)y.
\end{aligned}
\tag{4.5}
$$

The center manifold for this map is given by $y = 0$ (as it is invariant and is tangent to the eigenspace of the matrix $A$ corresponding to all multipliers on the unit circle). Thus the restriction of $T$ onto $W^c$ equals $\sigma$ times the time 1 map induced by the flow of

$$
\begin{aligned}
\dot z_j &= i\Omega_j(Z,u,v)z_j, \qquad j = 1, \dots, m,\\
\dot u &= F(Z,u,v),\\
\dot v &= G(Z,u,v).
\end{aligned}
\tag{4.6}
$$

Recall that either $h|_{W^c}\colon (z,u,v)\to(z^*,u,v)$ or $h|_{W^c}\colon (z,u,v)\to(z^*,-u,v)$. In the first case, the reversibility requires that the functions $F$ and $G$ must be identically zero and $\Omega_j$ must be real. In the second case, the reversibility allows $F(0,u,0)$ to be an arbitrary even function of $u$. Thus, generically, $F(0,u,0)=au^2+O(u^4)$ with $a\neq 0$, so by adding an arbitrary constant term to $F$ we can destroy the fixed point. It follows that the only generic case where a symmetric periodic orbit may have multipliers equal to $+1$ corresponds to $h$ being the identity on the $u$-space. The flow normal form (4.6) is then rewritten as

$$\dot z_j = i\Omega_j(Z,w)z_j,\quad j=1,\dots,s,\qquad \dot w=0, \tag{4.7}$$

where $w=(u,v)$ and $\Omega_j$ is a real function. In the polar coordinates $z_j=\sqrt{Z_j}\,e^{i\phi_j}$ the system takes the form

$$\dot\varphi=\varphi+\Omega(Z,w),\qquad \dot Z=0,\qquad \dot w=0, \tag{4.8}$$

where $\varphi=(\varphi_1,\dots,\varphi_s)$ and $\Omega=(\Omega_1,\dots,\Omega_s)$. In these coordinates the involution $h$ is given by $h(\varphi,Z,w)=(-\varphi,Z,w)$.

We further assume that $\det(\partial\Omega/\partial Z)\neq 0$. Then, arbitrarily close to $Z=0$ there exists a value of $Z=Z^*$ such that the corresponding vector of frequencies $\Omega$ is a rational multiple of $\pi$, i.e., $\Omega(Z^*,0)=\pi(p_1/q,\dots,p_s/q)$, where $p_j$ and $q>0$ are integers. It follows that every point in the torus ($Z=Z_j$, $w=0$) is $2q$-periodic. Take a non-symmetric periodic orbit on this torus and let $M=(\varphi^*,Z^*,0)$ be a point of this orbit. We can choose the frequencies $\Omega-\Omega(Z^*,0)$ as new $Z$-coordinates near $M$ (so $M$ will be given by ($\varphi=\varphi^*$, $Z=0$, $w=0$) in the new coordinates). The map $T|_{W^c}^{2q}$ in these coordinates is given by

$$\overline\varphi=\varphi+2qZ,\qquad \overline Z=Z,\qquad \overline w=\sigma^{2q}w=w.$$

By a small perturbation localized in a small neighborhood of the point $f^{-1}M$, we can bring the map $T|_{W^c}^{2q}$ near $M$ to the form

$$\overline\varphi=\varphi+2qZ,\qquad \overline Z=Z(1-2q\varepsilon)-\varepsilon(\varphi-\varphi^*),\qquad \overline w=w, \tag{4.9}$$

where $\varepsilon>0$ is a small parameter. The perturbation is localized in a small neighborhood of a non-symmetric point $f^{-1}M$, so by adding an appropriate perturbation localized in a small neighborhood of the point $gf^{-1}M$ we can keep our map in the class of reversible systems; the map $T|_{W^c}^{2q}$ near the point $M$ would keep its form, as the orbit of $M$ is not symmetric and hence does not enter the small neighborhood of $gf^{-1}M$. Now note that the map (4.9) is, at each level $w=\text{const}$, a linear rotation near $M$. Therefore, for an appropriate choice of $\varepsilon$, all the points in a neighborhood of $M$ are periodic with the same period; i.e., we have created a periodic spot on $W^c$ in the small neighborhood of the periodic point $x_0$.

It is shown in [42] that arbitrarily close in $C^r$ to a map $f$ with a periodic spot there exists a universal map $\widehat f$ such that the perturbation $\widehat f-f$ is supported in the periodic spot and, moreover, any $C^r$-generic map close enough to $\widehat f$ is universal too. In our situation, we have a periodic spot, which we denote by $Q$, on the center manifold $W^c$; note that it is away from the set $\mathrm{Fix}(g)$ of the fixed points of $g$. By [42], we can perturb the map $f|_{W^c}$ to make it $d$-universal; the perturbation can be extended outside $W^c$ in such a way that it will be supported in a small neighborhood of $Q$. We then make the perturbed map $\widehat f$ reversible (i.e., we ensure that $\widehat f^{-1}=g\circ\widehat f\circ g$) by adding an appropriate small perturbation to $f$ in a neighborhood of $gQ$ (we can do it as $Q$ and $gQ$ do not intersect).

Thus, let $\mathcal U$ be an open set in the space of $g$-reversible $C^r$ diffeomorphisms such that the generic symmetric periodic orbit $P$ under consideration persists for every map $f$ from $\mathcal U$. We have shown that for any $\delta > 0$, for a residual subset $\mathcal{U}_\delta$ of $\mathcal{U}$, the restriction of any map from $\mathcal{U}_\delta$ to the local center manifold $W^c(P)$ in the $\delta$-neighborhood of $P$ is $d$-universal, where $d = \dim(W^c(P))$. Every map which belongs to the intersection of the sets $\mathcal{U}_{\delta_j}$ over a sequence $\delta_j$ tending to zero is $C^r$-generic in $\mathcal{U}$ and has the desired property: its restriction to $W^c(P)$ in an arbitrarily small neighborhood of $P$ is $d$-universal. $\square$

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