Robust Exponential Acceleration in Time-Dependent Billiards

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A class of nonrelativistic particle accelerators in which the majority of particles gain energy at an exponential rate is constructed. The class includes ergodic billiards with a piston that moves adiabatically and is removed adiabatically in a periodic fashion. The phenomenon is robust: deformations that keep the chaotic character of the billiard retain the exponential energy growth. The growth rate is found analytically and is, thus, controllable. Numerical simulations corroborate the analytic predictions with good precision. The acceleration mechanism has a natural thermodynamical interpretation and is applied to a hot dilute gas of repelling particles.

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A simple way to accelerate a classical particle is to place it in a bounded domain where the particle would move inertially between elastic collisions with the boundary, as in billiards [1], and then make the boundary oscillate. The particle may gain or lose energy upon collision, depending on whether or not the particle and the boundary move towards each other [2]. When the balance of consecutive gains and losses is positive, the particle accelerates. Fermi proposed such a model for explaining the origin of cosmic rays [3]. Similar models appear in studies of capacitive discharges in plasmas [4], nuclear fission [5], and mesoscopic devices [6]. In these models the billiard boundary may move randomly or in a smooth fashion—hereafter we consider the smooth case.

In the one-dimensional Fermi-Ulam model, the particle energy remains bounded if the boundary oscillates smoothly [2]. In the two-dimensional case, when the domain's boundary moves so that at every instant the corresponding billiard is chaotic, a nonsaturable ensemble energy growth was numerically observed, suggesting the particle energy may grow up to infinity [7–9]. The existence of orbits with exponential energy growth for any chaotic billiard whose shape oscillates slowly was proven in [10]. However, it was also shown in [10] that the energy averaged over an ensemble of solutions which stay close to any given hyperbolic set of the billiards grows much slower, linearly in the number of collisions, i.e., at best quadratically in time. Numerical and analytical studies show [8] that indeed, for various billiard shapes, the observed ensemble energy growth is close to linear in the number of collisions; the same growth rate was detected for a driven elliptic (integrable) billiard in [11] for large energies (for small initial energies the growth is slower). Yet, for a very special geometry of a nonergodic chaotic billiard (a rectangle with an oscillating horizontal bar in its interior), we found that for typical ensembles the averaged energy grows exponentially [9]. That particular example is not robust: numerics shows that a small deformation of the rectangle restores the standard power-law growth.

Note the difference between the power-law and exponential energy growth: the former is suppressed by weak linear dissipation, while the latter is not. Thus, devising a robust exponential Fermi accelerator is crucial for ensuring an effective energy transfer from the slow subsystem (the moving boundaries) to the fast one (the particle).

In this Letter we propose a new class of accelerators that provide an effective mechanism of the energy transfer, namely, a robust exponential in time energy growth. The machine works as follows. Take a billiard domain $\mathcal{D}$ in the d-dimensional space and deform it slowly. The shape-deformation cycle of a period τ consists of 2 steps [see Fig. 1(a); specific realizations used in the numerical simulations are shown in Figs. 1(b)–1(e)]. In step 1, for $t \in (0, \tau_*)(\mathrm{mod}\,\tau)$, the domain is connected, and at each fixed moment of t the corresponding billiard dynamical system is ergodic and mixing with respect to the standard Liouville measure. At $t = \tau_*$ the billiard domain separates into two connected components, $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$. In step 2, for $t \in (\tau_*, \tau)(\mathrm{mod}\,\tau)$, the two components change their shape while remaining disjoint, and at each fixed t each

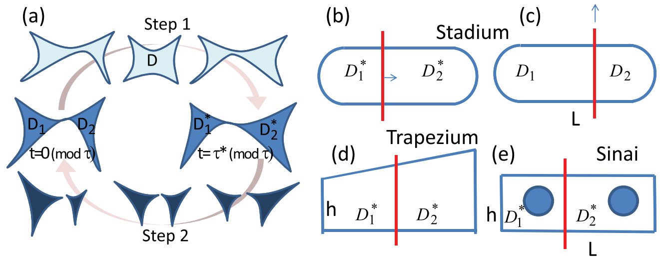

FIG. 1 (color online). General robust accelerator (a), divided stadium at the beginning of the first (b) and second (c) stages of the bar motion, (d) trapezium, (e) "double" Sinai billiard. Here $L = 2h = 2$, the radius of the disks in (e) is $\frac{1}{6}$, the distance between the centers is $\frac{4}{3}$, and the inclination angle of the trapezium is 10°. At the compression stage the bar displacement is 0.2 for trapezium, 0.28 for stadium, 0.18 for Sinai billiard, so the growth rate of Eq. (4) stays the same: $\mathcal{R} \approx 0.08$.

component defines an ergodic and mixing billiard dynamical system. At $t = \tau$ the two components reconnect again and form the same shape as at $t = 0$. Then the process repeats.

Let $V(t)$ be the volume of $\mathcal{D}$ at time t and $V_{1,2}(t) := \lambda_{1,2}(t)V(t)$ be the volumes of the two components in step 2 [so $\lambda_1(t) + \lambda_2(t) = 1$]. Next we show that, provided the ratio of the volumes of the two billiard components is different at the separation and reconnection moments,

$$V_1(0)/V_2(0) \neq V_1(\tau_*)/V_2(\tau_*), \tag{1}$$

a particle of sufficiently large initial energy inside the billiard at $t = 0$ will, on average, gain energy. Indeed, we show that the expectation of the logarithm of the energy gain at the end of the billiard-deformation cycle ($t = \tau$) is strictly positive under condition (1); thus, the process leads to a robust exponential acceleration of particles.

We use the Anosov-Kasuga averaging theory [12]. By this theory, a system with slowly changing in time Hamilton function approximately preserves the Anosov-Kasuga invariant. Namely, given a one-parameter family $H(x, p; \nu)$ of Hamilton functions, assume that for each value of ν the corresponding Hamiltonian system $\dot{x} = H'_p(x, p; \nu)$, $\dot{p} = -H'_x(x, p; \nu)$ is ergodic with respect to the Liouville measure on every energy level. If the parameter is allowed to change slowly with time ($\nu = \varepsilon t$), the energy will no longer be preserved. However, for the majority of orbits, $J(H(x(t), p(t); \varepsilon t), t) \approx J(H(x(0), p(0); 0), 0)$ on time intervals of order ε^{-1}, where the adiabatic invariant $J(E, \nu)$ is defined as the phase space volume bounded by the energy level $H(x, p; \nu) = E$. When the particle energy in the billiard is large (so the particle moves much faster than the boundary does), the Anosov-Kasuga theory is applicable. The phase space of the billiard flow bounded by the energy E is the product of the billiard domain of volume $V(t)$ and the ball in the particle momenta space $p_1^2 + \cdots + p_d^2 \leq 2E$. Hence, $J(E, t) \sim E^{d/2}V(t)$, and we find that for ergodic billiards with slowly moving boundaries the product $I := J^{2/d} = EV^{2/d}$ remains nearly constant for the majority of initial conditions for slow times of order 1. The measure of the set of exceptional trajectories vanishes as the energy grows.

Thus, if a particle at the beginning of the cycle has a sufficiently large energy E, it is expected to have the energy $E' \approx E[V(0)/V(\tau_*)]^{2/d}$ at the separation moment $t = \tau_*$. Similarly, a particle which belongs, after the separation, to the component $\mathcal{D}_i^*$ will have the energy $\bar{E} \approx E'[V_i(\tau_*)/V_i(0)]^{2/d} \approx E[\lambda_i(\tau_*)/\lambda_i(0)]^{2/d}$ at the reconnection moment $t = \tau$. Since the particle moves much faster than the boundary, the particle position is essentially random at the separation moment. Thus, by the ergodicity, the probability to be in the part $\mathcal{D}_i$ at $t = \tau_*$ equals to $\lambda_i(\tau_*)$. Therefore, after a complete cycle, $\ln E$ increases by $\frac{2}{d}\ln\frac{\lambda_i(\tau_*)}{\lambda_i(0)}$ with probability $\lambda_i(\tau_*)$. This gives us the expectation of the random increase in $\ln E$ as

$$\mathbb{E}\left[\ln\frac{\bar{E}}{E}\right] = \frac{2}{d}\left[\alpha\ln\left(\frac{\alpha}{\beta}\right) + (1-\alpha)\ln\left(\frac{1-\alpha}{1-\beta}\right)\right], \tag{2}$$

where $\alpha = \lambda_1(\tau_*)$, $\beta = \lambda_1(0)$. The function on the right-hand side of Eq. (2) achieves its minimum at $\alpha = \beta$ [the second derivative with respect to α is positive at $\alpha \in (0, 1)$, $\beta \in (0, 1)$]. Hence, $\mathbb{E}[\ln(\bar{E}/E)] > 0$ under condition (1).

Since correlations decay fast due to the billiard mixing property, we may further assume that the energy gain E_i/E_{i-1} after the ith cycle is essentially independent of the gain on the previous round. We may thus apply the law of large numbers to the sequence of the gain logarithms. Then we find that for a typical trajectory $\lim_{n\to+\infty}\frac{1}{n}\ln\frac{E_n}{E_0} = \lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^n \ln\frac{E_i}{E_{i-1}} = \mathbb{E}[\ln\frac{\bar{E}}{E}] > 0$; i.e., the majority of the initial conditions experience an exponential in time energy growth, with rate (2).

The expectation of the energy gain after one cycle is

$$\mathbb{E}\left[\frac{\bar{E}}{E}\right] = \alpha^{1+2/d}\beta^{-2/d} + (1-\alpha)^{1+2/d}(1-\beta)^{-2/d}, \tag{3}$$

which also achieves its minimum at $\alpha = \beta$, so $Q = \mathbb{E}[\bar{E}/E] > 1$ under condition (1). Formula (3) provides the rate of the exponential growth of the energy averaged over the (infinite) ensemble of uniformly distributed initial conditions. Indeed, assuming the independence of the energy gains at consecutive cycles, we find $\mathbb{E}[\frac{E_n}{E_0}] = \mathbb{E}[\prod_{i=1}^n \frac{E_i}{E_{i-1}}] = \prod_{i=1}^n \mathbb{E}[\frac{E_i}{E_{i-1}}] = e^{n\ln Q}$.

Note that for finite ensembles the deviation of the ensemble average from the expectation for the energy gain rapidly increases [9]. The standard deviation for the energy gain after n cycles is $\sigma_n = \sqrt{\mathbb{E}[E_n^2/E_0^2] - \mathbb{E}^2[E_n/E_0]} = \sqrt{\mathbb{E}^n[\bar{E}^2/E^2] - \mathbb{E}^{2n}[\bar{E}/E]} = e^{n\ln Q}\sqrt{[1 + (\sigma_1/Q)^2]^n - 1}$, where $\sigma_1 = \alpha(1-\alpha)|(\frac{\alpha}{\beta})^{2/d} - (\frac{1-\alpha}{1-\beta})^{2/d}|$. For an ensemble of K particles, the deviation of the statistical average from the theoretical mean value is $\sim\sigma_n/\sqrt{K}$, so the ensemble rate given by (3) is observed only for a finite number of the boundary-oscillation cycles: $n \ll \frac{\ln K}{\ln[1+(\sigma_1/Q)^2]}$, after which the smaller single-orbit rate (2) is eventually recovered.

We stress the generality of the construction: it is insensitive to the choice of the billiard shape, to the manner by which it is deformed, or to the dimension d. For numerical verification, we take several two-dimensional chaotic billiards. Their overall shape and size do not change in time, while the crucial separation or reconnection of the billiard domain is implemented by inserting a bar inside the billiard. At $t = \tau_*$ the bar divides the billiard into two disjoint regions of areas V_1^* and V_2^*. For $t \in (\tau_*, \tau)$, the dividing bar moves so that the area $V_1^* = V_1(\tau_*)$ expands to $V_1 = V_1(\tau) > V_1^*$ and the area V_2^* compresses to $V_2 < V_2^*$ [as $V_1 + V_2 = V_1^* + V_2^* = V = \text{const}$, condition (1) holds, so the exponential growth is achieved]. At $t = \tau$ the bar is moved vertically, so the two regions connect and the particle mixes in the whole domain. By (3), the expected energy gain is $\frac{\alpha^2}{\beta} + \frac{(1-\alpha)^2}{1-\beta} = 1 + \frac{(\alpha-\beta)^2}{\beta(1-\beta)} > 1$, where $\alpha = V_1^*/V$, $\beta = V_1/V$. In order to mimic a cyclic piston motion in a cylinder, we perform the next stage symmetrically in the opposite direction. The expected energy gain at this stage is $1 + \frac{(\alpha-\beta)^2}{\alpha(1-\alpha)}$. Altogether, the predicted ensemble energy growth rate (per the double-stage cycle) equals

$$\mathcal{R} = \log\left[\left(1 + \frac{(\alpha-\beta)^2}{\beta(1-\beta)}\right)\left(1 + \frac{(\alpha-\beta)^2}{\alpha(1-\alpha)}\right)\right]. \quad (4)$$

Figures 2(a) and 2(b) show the energy growth rate in the three billiard geometries of Figs. 1(b)–1(e), where the oscillating bar follows the above protocol. For ensembles of 10^3–10^4 initial conditions, for 100 rounds of the two-stage adiabatic cycle, the energy growth is indeed exponential. Figure 2(a) corresponds to an abrupt introduction and removal of the bar at the moments of division and reconnection of the billiard. The observed growth rates here are 0.086 ± 0.006 for the trapezium, 0.084 ± 0.006 for the stadium, and 0.082 ± 0.007 for the Sinai billiard (the range of fitted slopes reflects calculation of 10 different ensembles for each case). The rates are quite close to the predicted value $\mathcal{R} \approx 0.08$ for each of the three geometries (see Fig. 1 caption). Similar rates are achieved when the bar is introduced and removed slowly, and the velocity of both vertical and horizontal bar motion is continuous in time: here $\mathcal{R}$ is 0.089 ± 0.009 for the trapezium, 0.085 ± 0.004 for the stadium, and 0.080 ± 0.003 for the Sinai billiard [see Fig. 2(b)]. Finally, Fig. 2(c) ($\log E$ vs $\log t$) shows that the accelerators lose their exponential character if the moving bar does not fully divide the domain into two separated regions: in this case the energy growth is much slower and appears to be power law.

As we see, the numerics conforms to the theoretical predictions. It clearly shows that the choice of a billiard is unimportant for the result. Of real importance is the ergodicity violation during the boundary-oscillation cycle: when the billiard table is connected we have one ergodic component and when it is divided we have two ergodic components (i.e., nonergodicity) on each energy level in the phase space. Indeed, as Fig. 2(c) shows, when the billiard stays ergodic for the whole cycle, the energy growth is significantly slowed. One can explain this as follows. The reflection law in the case of boundary moving with normal velocity u is $\bar{v}_\perp = 2u - v_\perp$, $\bar{v}_\parallel = v_\parallel$, where $(v_\perp, v_\parallel)$ and $(\bar{v}_\perp, \bar{v}_\parallel)$ are the normal and tangent to the boundary components of the particle velocity before and after the collision. This gives $\bar{E} = \frac{m}{2}\bar{v}^2 = E - 2muv_\perp + 2mu^2$, so when $v_\perp/u \gg 1$ the change in the particle's kinetic energy E at the collision is $\sim\sqrt{E}$. For a chaotic billiard, correlations between energy gains and losses at consecutive collisions with the slowly moving boundary decay fast, so we model the change in E by a random walk with independent increments of order $\sqrt{E}$. By denoting $I_n := E_n V(t_n)^{2/d}$, where t_n is the time moment of the nth collision, and E_n is the kinetic energy just after it, we thus have $I_{n+1} - I_n = \sqrt{I_n}\xi_n$, where ξ_n are independent bounded random variables. The approximate preservation of the Anosov-Kasuga invariant means that this random walk is unbiased to the main order, i.e., $\mathbb{E}(\xi_n) \to 0$ as I_n grows. The natural small parameter here is $u/|v| \sim I^{-1/2}$, so we estimate $\mathbb{E}(\xi_n) \sim I_n^{-1/2}$, which leads to $\mathbb{E}(I_n) \lesssim n$. The time intervals between the collisions are of order $|v|^{-1}$, i.e., $t_{n+1} - t_n \sim E_n^{-1/2} \sim I_n^{-1/2} \gtrsim n^{-1/2}$, hence $t_n \gtrsim \sqrt{n}$ and $E(t_n) \lesssim t_n^2$. Thus, the approximate preservation of $EV^{2/d}$ can be responsible for the observed (sub) quadratic energy growth in driven ergodic and mixing billiards. The nonconservation of $EV^{2/d}$ achieved by dividing and reconnecting the billiard is the core of our construction of the exponential accelerator.

Since the construction is valid in any dimension d, the exponential averaged energy growth should also be observed when several repelling particles are put into the accelerator simultaneously. The gas of N elastically

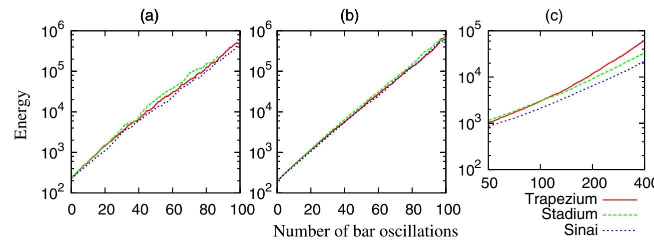

FIG. 2 (color online). Ensemble energy growth (the energy is in the logarithmic scale). (a) Exponential growth for the three geometries of Figs. 1(b)–1(e), where the bar is introduced and removed abruptly. The energy is averaged over ensembles of 2000 initial conditions. (b) The same as (a) with slow and smooth bar motion (the bar velocity is continuous, piecewise linear in time). The graphs are smoother than in (a) since larger ensembles (10^4 initial conditions) are taken. (c) When the bar is 90% of the billiard's height and does not divide the billiard, the energy growth slows down significantly: $E \sim t^{1.55}$ for the stadium and Sinai billiards and $E \sim t^{1.95}$ for the trapezium (5000 initial conditions). The apparent crossover to faster growth rates at higher energies is in agreement with [11].

colliding hard spheres is a billiard in an Nd-dimensional configuration space. For a dilute gas (large N yet the total volume occupied by the particles is a small fraction of the box volume), we may accept the ergodicity of such billiard (Sinai-Boltzmann hypothesis). Then, the above computations of the energy change during the cycle of division and reconnection of the billiard must hold true. The Anosov-Kasuga invariant for this gas is $\sim E^{Nd/2}V^N$, where V is the container volume and E is the total energy of particles. Thus, when the shape of the container is changed slowly, the product $EV^{2/d}$ is approximately preserved. This coincides with the usual law $TV^{2/d} = \text{const}$ of the adiabatic compression for the ideal monatomic gas (where $T = E/N$ is the temperature). We thus may apply standard thermodynamic arguments in the computation of the energy gain for the cycle of the container shape deformation.

Consider the cycle described by Fig. 1 and assume the domain is filled with the ideal monatomic gas, and there is no heat transfer through the boundary. By the adiabatic compression law, if the initial gas temperature is T, then at the end of step 1 the temperature will be $T' = T[V(0)/V(\tau_*)]^{2/d}$. After separation, the fraction of gas particles in D_i^* is given by $\frac{N_i}{N} = \lambda_i(\tau_*)$, where $\lambda_i = V_i/V$. Then the domains are deformed each on its own and their temperatures evolve differently: at the moment before the reconnection the temperature in D_i is given by $\bar{T}_i = T'[V_i(\tau_*)/V_i(0)]^{2/d}$. After the reconnection the system equilibrates at the temperature $\bar{T}_1 N_1/N + \bar{T}_2 N_2/N = \bar{T}_1\lambda_1(\tau_*) + \bar{T}_2\lambda_1(\tau_*)$. Thus, after the complete cycle, the ratio of temperatures is $\bar{T}/T = \lambda_1(\tau_*)^{1+2/d}/\lambda_1(0)^{2/d} + \lambda_2(\tau_*)^{1+2/d}/\lambda_2(0)^{2/d}$, which coincides with (3); i.e., the rate of the energy growth is independent of the number of particles.

Finally, we remark that the billiard in an interval of a straight line is trivially ergodic. Thus, our mechanism includes the exponential energy growth in the one-dimensional situation as well. For example, we obviously achieve an exponential acceleration by repeating the cycle of compressing the interval and then restoring it to its original length abruptly. Yet, this simple mechanism, as well as other one-dimensional examples, cannot be implemented adiabatically. In this Letter, we show that in the higher-dimensional case the effective acceleration can be achieved without relying on the discontinuity in the bar motion. Indeed the results in Fig. 2(b) correspond to a regime where, at every instance, the billiard boundary moves much slower than the particle does (this is important since no resonances that could impede the particle acceleration appear as the particle velocity grows). The proposed mechanism is of a very basic nature: we divide an ergodic system into two components, each one is at equilibrium, then we deform the components in such a way that each part remains at statistical equilibrium, but the effective temperatures change in different ways, so at the end the sum of the two equilibrium states is no longer an equilibrium for the total system. Only after the ergodic components are reconnected the system evolves to a new equilibrium. We established that this process is responsible for the particle gaining energy at an exponential rate on average. We end by noting that this scheme can be applied also to smooth time-dependent Hamiltonian models for particle motion [13]; i.e., these should also admit a similar mechanism of exponential acceleration by separating and reconnecting their ergodic components. In particular, atom-optics billiards [14] may provide a suitable experimental setting for examining the robust exponential particle heating.

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