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On models with non-rough Poincaré homoclinic curves

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The possibility of an a priori complete description of finite-parameter models including systems with structurally unstable Poincaré homoclinic curves is studied. The main result reported here is that systems having a countable set of moduli of Ω-equivalence and systems having infinitely many degenerate periodic and homoclinic orbits are dense in the Newhouse regions of Ω-non-stability. We discuss the question of correctly setting a problem for the analysis of models of such type.

1. Introduction

It is well-known that in three-dimensional smooth dissipative systems, two types of strange attractors may occur: those of the Lorenz type and the quasi-attractors. The latter ones may have non-rough Poincaré homoclinic curves in addition to rough[#1] homoclinic cures. Due to this fact, either the system itself or a nearby system may have a countable set of stable periodic orbits together with a countable set of saddle-type periodic orbits [1,2]. The dynamics of such systems turns out to be very sensitive to small perturbations of the parameters. This sensitive dependence manifests itself in numerical experiments. Quasi-attractors can be observed in many different systems such as the Lorenz model[#2], the Hénon map, systems with spiral chaos [4], systems with destruction of two-dimensional tori [5,6], etc. In this regard, the following question becomes of special importance: in principle, is it possible to have a complete description of the dynamics of such systems within the framework of finite-parameter families of differential equations as in traditional models of dynamical phenomena?

The setting up of a problem dealing with model description goes back to Andronov. It consists of: (1) dividing a parameter space into regions of roughness and finding the bifurcation set; (2) dividing the bifurcation set into connected components corresponding to the same phase portraits in the sense of topological equivalence. Therefore, a good model must possess a sufficient number of parameters in order to render possible the bifurcation analysis of the equilibrium points, of the periodic, heteroclinic and homoclinic orbits, etc.

As a formal basis for further considerations, let us introduce the following

Definition. A family X_μ of differential equations, which smoothly depends on a set of parameters μ belong to some region $D \subseteq \mathbb{R}^n$, will be called *good* if, for some neighbourhood of the family X_μ in the space of dynamical systems, there exists a continuous foliation with basis D and with Banach leaf of codimension m such that: (1) all leaves satisfy the Lipschitz condition with a common constant; (2) X_μ intersects each leaf at a unique point, transversally to all smooth manifolds of the same Lipschitz constant, and close

[#1]This term was originally proposed by A.A. Andronov and is equivalent to "structurally stable".

[#2]In the Lorenz system, the non-rough Poincaré homoclinic curves are contained in an attractor and they emerge near the boundary of the existence domain of the Lorenz attractor at $\sigma = 10$, $b = 8/3$, and $r > 31$ [3].

enough to the leaf; (3) any two systems belonging to the same leaf are topologically equivalent and, moreover, the homeomorphism realizing the equivalence is close to identity when the systems are close enough.

We shall suppose that it is reasonable to set up a problem of complete description only for good models in the sense of the definition given above. If some model does not satisfy this requirement then we can increase the number of parameters or, in the case when such a procedure turns out to be ineffective, make the equivalence relation less restrictive. For instance, if the details of the transitional processes are not interesting we may consider the Ω-equivalence i.e. the topological equivalence on a non-wandering set.

It is evident that in a good model all bifurcations of periodic, homo- and heteroclinic orbits have a finite codimension and that the model is transversal to all bifurcation submanifolds. In the case of differential equations in the plane these requirements are sufficient for a model to be good (at least it is so in the case when all bifurcations have low codimensions). In the multidimensional case, fundamental differences already arise in the class of systems with trivial dynamics, i.e. without Poincaré homoclinic curves. We consider two examples. The first one is a diffeomorphism of the plane with non-rough heteroclinic orbits (the simplest non-roughness being related to non-transversal intersection of the invariant manifolds of some periodic points). As shown by Palis [7], the set of such systems can be divided into a continuum of classes of topological equivalence which can be distinguished by continuous topological invariants, the so-called moduli.

Definition. We say that a system X possesses a *modulus* if X lies in a Banach manifold M on which a continuous functional h is defined which is not locally constant and such that the systems X_1 and X_2 belonging to M are not equivalent if $h(X_1) \neq h(X_2)$. We say that X possesses *m-moduli*, if X lies in a Banach manifold M on which m independent moduli are defined and we say that X has a *countable set of moduli* if X has any finite number of moduli.

Let a diffeomorphism of the plane have two periodic points P_1 and P_2 of saddle type such that the unstable manifold of P_1 touches the stable manifold of P_2. In this case [7,8], the modulus is the quantity $\alpha = -\ln|\lambda_1|/\ln|\gamma_2|$ where λ_1 is a multiplier of P_1 and γ_2 is a multiplier of P_2 ($|\lambda_1| < 1$, $|\gamma_2| > 1$). The presence of moduli is the main reason why one-parameter families, intersecting transversally the "pellicle" of diffeomorphisms with non-rough heteroclinic curves, are not good. In principle, if a heteroclinic non-roughness generates only a finite number of moduli the model can be enlarged to a good one by increasing the number of parameters. The conditions under which there may be a countable set of moduli in a given class were given in [9]. In the last case, there are no good models from the point of view of topological equivalence. However, if we restrict ourselves to the Ω-equivalence then systems of this type turn out to be rough ones, and, consequently, every finite-parameter family is good.

Our second example is the bifurcation at the emergence of a two-dimensional torus from a periodic orbit with multipliers $e^{\pm 2\pi i\omega}$. Using ref. [10], we can prove that, in a space of smooth dynamical systems, there exist systems having arbitrarily degenerate periodic orbits with winding numbers close to ω in the neighbourhood of systems possessing a two-dimensional invariant torus with irrational winding number ω. It is evident that, for such a model, any finite-parameter family is not good even in the sense of Ω-equivalence. Nevertheless, it is worth mentioning that, in this case, we can reasonably weaken the relation of equivalence in such a way that we obtain good models. Let us choose some positive integer q_0 and consider that two flows on a smooth two-dimensional torus are equivalent (we take flows having a global cross-section) if:

(1) they have the same winding numbers; (2) they are topologically equivalent for a rational winding number with denominator less then q_0. In other words, we shall distinguish systems among a finite number of the fundamental resonance zones only with the aid of a ratio of frequencies, which is quite natural from a practical viewpoint[#3]. With such an equivalence relation, it is sufficient to take two-parameter models for the investigation of a given bifurcation.

In the above examples, the values of the parameters for which a model is not good form a nowhere dense set in D. The situation is much more complicated in the class of systems with strange attractors because non-rough systems may fill up some regions. Two examples of dense Ω-non-roughness are best known: (1) models with Lorenz attractors and (2) models with non-rough Poincaré homoclinic curve. For systems with Lorenz attractors the pair of kneading invariants [12,13] is a complete invariant of the Ω-equivalence. This fact enables us to carry out the investigation of the dynamics using two-parameter models. In the second case, the situation is less trivial. In particular, this is due to a remarkable phenomenon discovered by Newhouse. It turns out [14] that any model transversal to a bifurcation "pellicle" of codimension one corresponding to some homoclinic orbit with quadratic tangency intersects regions where the systems with non-rough Poincaré homoclinic curves are dense. The existence of Newhouse regions is a characteristic feature of systems with quasi-attractors. The results given below (theorems 2–5) show that systems with arbitrarily degenerate periodic and homoclinic orbits as well as systems with a countable set of moduli of Ω-equivalence are dense in these regions. Therefore, there is no good model in the sense of Ω-equivalence in the Newhouse regions. Moreover, we show that our attempts to weaken the equivalence relation in a reasonable way, as we have done for the flows on a torus, are vain in this case (theorem 6).

[#3] In particular systems, q_0 can take very different values. For instance q_0 takes values of several hundreds in ref. [11].

2. The types of systems with non-rough homoclinic curve

Let us consider a C^r-smooth $(3 \le r \le \infty)$ flow X_0 defined on a three-dimensional manifold M. Let X_0 satisfy the following conditions:

(A) X_0 has a periodic orbit L_0 of saddle type with multipliers λ, γ where $|\gamma| > 1 > |\lambda|$;

(B) the saddle value $\sigma = |\lambda\gamma| < 1$;

(C) $W^s(L_0)$ and $W^u(L_0)$ have a quadratic tangency along a homoclinic curve Γ_0.

Let U be a small neighbourhood of the contour $L_0 \cup \Gamma_0$. U consists of a solid torus V containing L_0 to which a handle containing a piece of the curve Γ_0 is glued. We denote by N the set of orbits of the flow X_0 entirely lying in U. Let $S \subset U$ be a smooth cross-section of L_0. It is convenient to study orbits belonging to the set N using a Poincaré map of section S. This map can be represented as iterates of the map T_0: $S \to S$, acting along orbits close to L_0, combined with iterates of the map T_1: $S \to S$, acting along orbits belonging to a neighbourhood $U \backslash V$ of the piece of Γ_0. We denote that the map T_0 can be written as [15]

$$\bar{x} = \lambda x + f(x, y)\, x^2 y \,,$$
$$\bar{y} = \gamma y + g(x, y)\, x y^2 \,, \qquad (1)$$

in some C^{r-1}-coordinates (x, y). Taking this fact into account, the equations of the manifolds $W^s_{loc}(L_0) \cap S$ and $W^u_{loc}(L_0) \cap S$ have the forms $y = 0$ and $x = 0$ respectively. Let $M^+(x^+, 0)$ and $M^-(0, y^-)$ be the pair of homoclinic points in the section S. Without loss of generality, we can suppose that $x^+ > 0$ and $y^- > 0$. Let the domains Π_0 and Π_1 of S be lying in some neighbourhoods of the homoclinic points $M^+(x^+, 0)$ and $M^-(0, y^-)$ which are small enough so that $T_0(\Pi_i) \cap \Pi_i = \varnothing$, $i = 0, 1$. The map T_1 can be written as

$$\bar{x} - x^+ = F(x, y - y^-)\,, \quad \bar{y} = G(x, y - y^-)\,, \tag{2}$$

where $F(0,0) = G(0,0) = 0$. In virtue of condition (B), we have

$$\frac{\partial G(0,0)}{\partial y} = 0\,, \quad \frac{\partial^2 G(0,0)}{\partial y^2} = 2d \neq 0\,.$$

Therefore, the following identity is valid [16]:

$$G(x, y - y^-) = D(x, y)\,[y - y^- - \phi(x)]^2 + C(x)\,x\,,$$

$$D(0, y^-) = d\,, \quad \phi(0) = 0\,, \quad C(0) = c \neq 0\,.$$

The domain of the map $T_0': \Pi_0 \to \Pi_1$, acting along the orbits of the flow X_0 passing in the neighbourhood of L_0, is a set σ_0 [17] consisting of a countable set of strips

$$\sigma_k^0 \subset \Pi_0,\ k = \bar{k}, \bar{k} + 1, \ldots,$$

$$\text{with } \sigma_k^0 = \Pi_0 \cap T_0^{-k}\Pi_1\,,$$

(see fig. 1). The range of the map T_0' is $\sigma^1 = \bigcup_{k=\bar{k}}^{\infty} \sigma_k^1$ where $\sigma_k^1 = T_0^k \sigma_k^0$. Here $\bar{k}$ is some sufficiently large integer (the smaller the sizes of Π_0 and Π_1, the greater the integer $\bar{k}$). As $k \to \infty$, the strips σ_k^0 and σ_k^1 accumulate to $W^s_{\text{loc}} \cap \Pi_0$ and $W^u_{\text{loc}} \cap \Pi_1$ respectively. In [17], it was shown that the map T_0^k: $\sigma_k^0 \to \sigma_k^1$ has the following form:

$$x_1 = \lambda^k x_0 (1 + \gamma^{-k} \varphi_k(x_0, y_1) x_0 y_1)\,,$$

$$y_0 = \gamma^{-k} y_1 (1 + \gamma^{-k} \psi_k(x_0, y_1) x_0 y_1)\,,$$

where $(x_0, y_0) \in \Pi_0$, $(x_1, y_1) \in \Pi_1$ and $\varphi_k x_0^2 y_1$ and $\psi_k x_0 y_1^2$ are functions having $(r-2)$ continuous derivatives and being uniformly bounded with respect to k.

The structure of the set N significantly depends on the nature of the intersections of the strips $T_1 \sigma_i^1$ and σ_j^0 for various i and j. The nature of these intersections is mainly determined by the signs of λ, γ, c, and d [16]. We can distinguish three types of non-rough Poincaré homoclinic curves [1] (see fig. 2 where the corresponding diffeomorphisms of the plane are given for $\lambda > 0$, $\gamma > 0$). In the case of a curve of the first type ($\gamma > 0$, $d < 0$) the set N has a trivial structure: $N = \{L_0, \Gamma_0\}$ [16]. This is related to the fact that the intersection $T_1 \sigma_i^1 \cap \sigma_j^0$ may be non-empty only for $j > i$ because the strip σ_j^0 lies

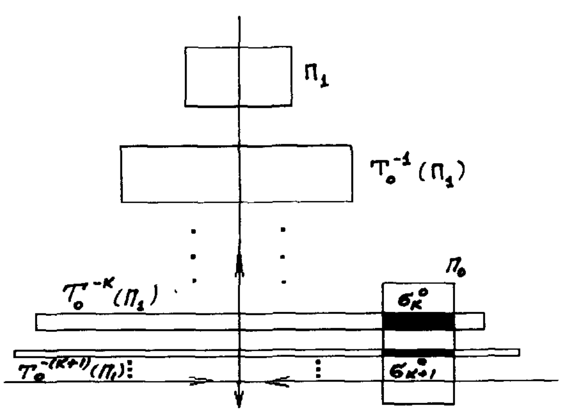

Fig. 1.

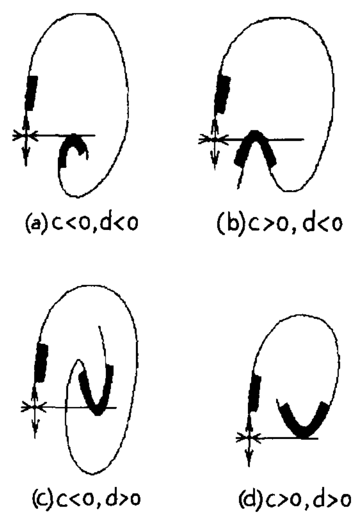

Fig. 2.

at a distance of order γ^{-j} from $W^s_{\text{loc}} \cap \Pi_0$, whereas the top of the strip $T_1\sigma_i^1$ lies at a distance of order $|\lambda|^i \ll \gamma^{-i}$ (see fig. 3a). We remark that, for $\lambda > 0, \gamma > 0, c < 0, d < 0$, the strips $T_1\sigma_i^1$ and σ_j^0 (for all $i, j \geq \bar{k}$) lie on different sides of $W^s_{\text{loc}} \cap \Pi_0$, and $T_1\sigma_i^1 \cap \sigma_j^0 = \varnothing$ (see fig. 3b).

In the case of a curve of the second type ($\lambda > 0, \gamma > 0, c < 0, d > 0$), the intersection of $T_1\sigma_i^1$ and σ_j^0 is regular for any $i, j \geq \bar{k}$, i.e. it consists of two connected components (fig. 3c). The preimage of each component under $T_1T_0^i$: $\sigma_i^0 \to \sigma_j^0$ is a strip $\sigma_{ij}^{0\alpha} \subset \sigma_i^0$ ($\alpha = 1, 2$). In this case, it turns out that the map $T_1T_0^i$ on $\sigma_{ij}^{0\alpha}$ is of saddle type in the sense of ref. [17]. Therefore, due to the lemma about a saddle-type fixed point in a direct product of spaces [17], the set N is equivalent to a suspension over a quotient system constructed from a Bernoulli scheme on three symbols $\{0, 1, 2\}$ after identification of the two homoclinic orbits $(\ldots, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$ and $(\ldots, 0, \ldots, 0, 2, 0, \ldots, 0, \ldots)$ [16]. Here, all the orbits belonging to $N \backslash \Gamma_0$ are of saddle type.

In the case of a curve of the third type (all remaining combinations of signs of λ, γ, c, and d correspond to this type), the set N contains non-trivial hyperbolic subsets [16], but, in general, these subsets do not exhaust the set N. The reason is that there may be irregular intersections besides the regular ones (see figs. 3d). In the case $\lambda > 0, \gamma > 0$, a necessary condition for the existence of an intersection between $T_1\sigma_i^1$ and σ_j^0 is the inequality[#4] [2,19]

$$j < i\theta - \tau_0 + s\gamma^{-\bar{k}/2} . \tag{3}$$

A sufficient condition for a regular intersection of $T_1\sigma_i^1$ with σ_j^0 is the inequality

$$j < i\theta - \tau_0 - s\gamma^{-\bar{k}/2} . \tag{4}$$

Here $s > 0$ is some constant independent of i and j while

$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|} , \quad \tau_0 = \frac{1}{\ln|\gamma|} \ln\left|\frac{cx^+}{y^-}\right| (\operatorname{mod}(\theta - 1)) .$$

In the case where θ is rational, i.e. $\theta = p/q$ and $q\tau_0 \not\in \mathbb{Z}$, the inequalities (3) and (4) admit the same integers for solutions (all these integers are greater than some constant $\bar{k}(\theta, \tau_0)$). Therefore, in this case, all the trajectories belonging to $N \backslash \Gamma_0$ are of saddle type [1,18]. If θ is irrational the inequalities (3) and (4) are not anymore equivalent and, as a consequence, non-saddle-type orbits can emerge in the set N. Moreover, flows with non-rough periodic orbits, flows with a countable set of stable periodic motions, and flows with non-rough homoclinic counters [1,2] are dense in the set of flows having non-rough homoclinic curves of the third type. The necessity of studying systems with curves of the third type within the analysis of systems with quasi-hyperbolic behaviour can be explained using the following.

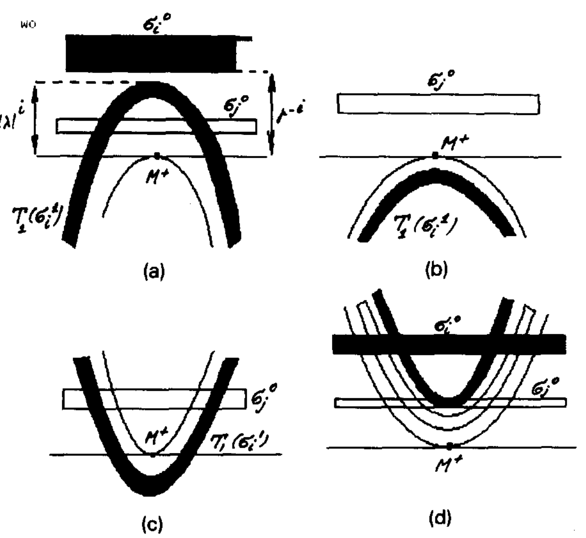

Fig. 3.

[#4] For the other combinations of signs of λ and γ, these conditions are given by similar inequalities [18]. We omit them here to avoid overloading the text.

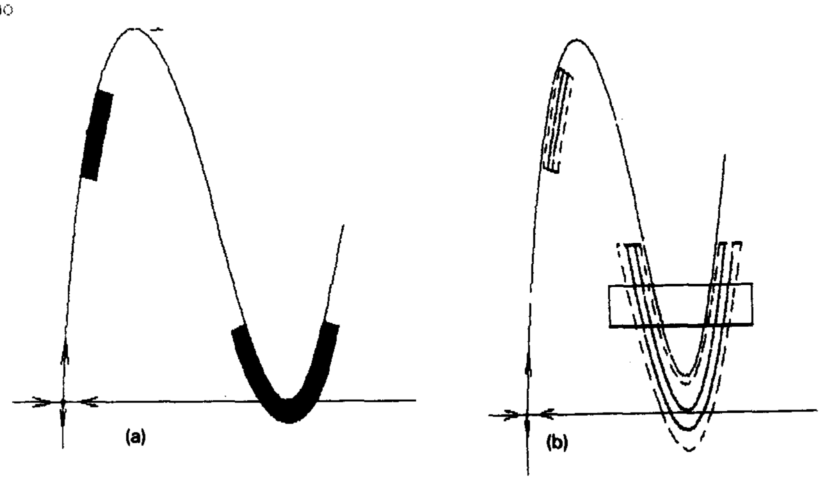

Fig. 4.

Theorem 1. In any neighbourhood (in the C^r-topology, $r \leq 3$) of a system with a non-rough Poincaré homoclinic orbit, there exist systems with orbits of the third type.

An illustration of this theorem is given in fig. 4a for the case where X_0 has a homoclinic curve of the second type. We choose a flow $\tilde{X}$ close to the flow X_0 such that the manifold $T_1(W^{\mathrm{u}}_{\mathrm{loc}} \cap \Pi_1)$ of $\tilde{X}$ lies above $W^{\mathrm{s}}_{\mathrm{loc}} \cap \Pi_0$ and intersects the strip σ^0_k along the two components W^1_k and W^2_k whereas the strip $T_1\sigma^1_k$ intersects $W^{\mathrm{s}}_{\mathrm{loc}} \cap \Pi_0$. It is obvious that $\tilde{X}$ can be chosen in such a way that $T_1T^k_0(W^2_k)$ would touch $W^{\mathrm{s}}_{\mathrm{loc}} \cap \Pi_0$. It is easy to convince oneself that a non-rough homoclinic curve of the third type corresponds to this tangency (fig. 4b).

3. Infinite degeneracies in a class of systems with a homoclinic orbit of the third type

We suppose that Γ_0 is a curve of the third type. The set of flows X which are C^r-close to X_0 and which have a single-circuit homoclinic orbit Γ of the third type close to Γ_0 form a smooth submanifold H_r of codimension 1 in the space of C^r-flows on M.

Theorem 2. A set B such that any flow belonging to B has a countable set of periodic orbits of saddle type each having a non-rough homoclinic curve of the third type is dense in H_r.

We give only a sketch of the proof of theorem 2. The proof is divided into three stages. The first stage consists in proving that flows with homoclinic contours formed by one rough and one non-rough homoclinic orbits of some single-circuit[#5] periodic orbits of saddle-type are dense in H_r. For any flow $X_1 \in H_r$, let p_i be a fixed point of the map $T_1T^i_0$, $p_i \in \sigma^0_i$. We denote by W^{s}_i and W^{u}_i respectively the connected pieces of the intersections with σ^0_i of the stable and unstable manifolds of the point p_i[#6]. For arbitarily small $\delta > 0$ and for any $\bar{k}$, there exists a pair of integers (i, j), $j > i$, greater than $\bar{k}$ and satisfying the inequalities

$$j \geq i\theta(X_1) - \tau_0 + s\gamma^{-\bar{k}/2}\,, \tag{5}$$

$$j < i(\theta(X_1) + \delta) - \tau_0 - s\gamma^{-\bar{k}/2}\,. \tag{6}$$

[#5]A periodic orbit is called single-circuit periodic orbit if it corresponds to some fixed point of the maps $T_1T^i_0$.

[#6]Since the intersection $T_1T^i_0\sigma^0_i \cap \sigma^0_i$ is of saddle type due to (4) p_i is also of saddle type.

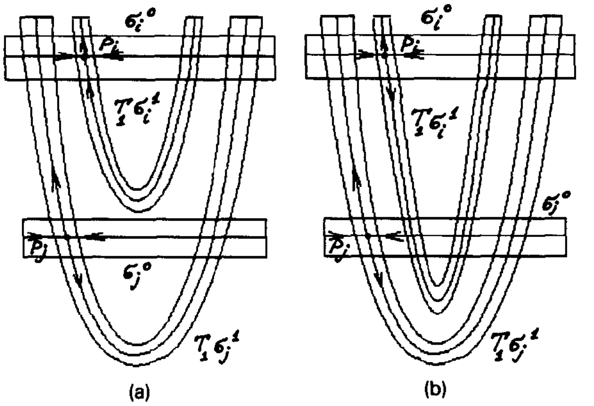

Fig. 5.

Because (5) has the opposite inequality with respect to (3) we have $T_1\sigma_i^1 \cap \sigma_j^0 = \varnothing$ whereupon $T_1 T_0^i W_i^{\mathrm{u}} \cap W_j^{\mathrm{s}} = \varnothing$ for the flow X_1 (fig. 5a). Now, staying in H_r, let us perturb the flow X in such a way that $\theta(X)$ changes from $\theta(X_1)$ to $\theta(X_1)+\delta$. If $\theta(X_2)=\theta(X_1)+\delta$, $T_1 T_0^i W_i^u$ intersects transversally W_j^{s} at two points for the flow X_2 because the inequality (6) coincides with (4) in this case (fig. 5b). Consequently, there is a flow X between X_1 and X_2 such that $T_1 T_0^i W_i^{\mathrm{u}}$ touches W_j^{s} along some non-rough heteroclinic orbit (fig. 6a). Because the intersection of $T_1\sigma_j^1$ with σ_i^0 is always of saddle-type due to (4) the flow X has a rough heteroclinic orbit corresponding to the intersection of $T_1 T_0^j W_j^{\mathrm{u}}$ with W_i^{s}. These two heteroclinic orbits together with the orbits p_i and p_j form the desired homoclinic contour. This is the simplest possible non-rough contour and we denote it by C_{ij}.

At the second stage, we show that the flows having a countable set of the simplest non-rough homoclinic contours are dense in H_r. Let X belong to H_r. Then, in any neighbourhood U_δ of the flow X in H_r, there is a flow X_1 having some countour $C_{i_1 j_1}$. According to the construction, the perturbation leading to the formation of a non-rough contour may be localized in any small neighbourhood of the orbit L_0. Therefore, it is possible to achieve one more non-rough contour using some arbitrarily small perturbation of the flow X_1 while keeping the tangency between the manifolds $W^{\mathrm{u}}(p_1)$ and $W^{\mathrm{s}}(p_{j_1})$. Consequently, there is a flow X_2 in U_δ having contours $C_{i_1 j_1}$ and $C_{i_2 j_2}$. Repeating this procedure, we eventually obtain a flow $X^* \subset U_\delta$ having a countable set of the simplest contours $C_{i_1 j_1}, \ldots, C_{i_n j_n}, \ldots$ where $i_n, j_n \to \infty$ as $n \to \infty$. These flows form a dense set in H_r.

At the third stage, it is necessary to remark that we can achieve a non-rough homoclinic curve of P_i using arbitrarily small perturbation of a non-rough homoclinic contour C_{ij}. Such an "operation" may be carried out independently for all contours of the flow X^* and the statement of theorem 2 follows from this fact.

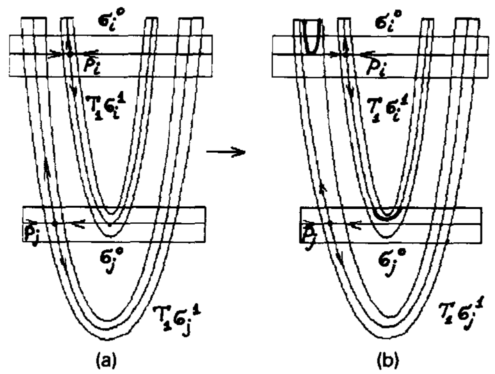

Fig. 6.

We have to keep in mind that we have constructed flows with non-rough homoclinic orbits having a sufficiently large number of circuits. Indeed, we can show that the flows with the simplest non-rough homoclinic orbits of single-circuit periodic orbits are dense in H_r (the orbits are simplest in the sense that already the curves W_i^{s} have a tangency with $T_1 T_0^j[T_1 T_0^i(W_i^{\mathrm{u}})]$ (fig. 6b).

In [1,2], it was noted that the structure of the set N in the case of a homoclinic orbit of the third type is essentially determined by arithmetical properties of the invariants θ and τ_0. We illustrate this feature with the following example. In order for a system to have the simplest non-rough contour C_{ij} or the simplest non-rough homoclinic curve, it is necessary that the intersection of $T_1\sigma_i^1$ with σ_j^0 is not of the saddle type and, as a consequence of (3) and (4), the following inequality must be satisfied:

$$|j - i\theta + \tau_0| < s\gamma^{-i/2}, \tag{7}$$

(We have replaced the integer $\bar{k}$ by i in the right-hand side because the intersections with S of the orbits making a contour totally lie in the strips labelled with the numbers i and j.) Thus, in order for a system to have a countable set of simplest non-rough homoclinic contours or simplest non-rough homoclinic orbits it is necessary to satisfy a countable set of inequalities of the type (7) with i and j going to infinity. We come to the conclusion that θ and τ_0 must be numbers allowing "supernormal" non-homogeneous approximations by rational fractions.

One more example of this kind is given by flows of H_r having a countable set of stable double-circuit periodic orbits [1,2]. As shown in [1,2], a set B_s consisting of flows for which the invariants θ and τ_0 satisfy a countable set of inequalities similar to (7),

$$\nu_{ij}^1 < j - i\theta + \tau_0 - s_1\gamma^{-i/2} < \nu_{ij}^2, \tag{8}$$

is dense in H_r. Here s_1 is some constant determined by the parameters of the homoclinic curve, $\nu_{ij}^1 < \nu_{ij}^2$, $\nu_{ij}^\alpha = \mathcal{O}(|\lambda|^i|\gamma|^{-i/2})$, $\alpha = 1, 2$. For such flows, there exists a stable double-circuit periodic orbit in U corresponding to a fixed point of the map $T_1T_0^iT_1T_0^j$ for any pair (i, j) satisfying the inequalities (8). We note that the only necessary condition for the existence of a countable set of simplest non-rough contours is the abnormally good ability to approximate θ and τ_0, whereas it is necessary and sufficient to satisfy the countable set of inequalities (8) for the existence of a countable set of stable double-circuit periodic orbits.

Thus, we may conclude that "good" arithmetical properties of the invariants θ and τ_0 lead to "bad" properties of the orbits belonging to N. On the other hand, in the case of "bad" ability to approximate θ and τ_0 we may expect that the properties of the orbits belonging to N may turn out to be "good" (or at least that "bad" orbits will have multiple circuits).

Let us consider a flow X^* belonging to the set B mentioned in theorem 2. Let $(L_k^*)_{k=0}^{k=\alpha}$ be a sequence of a single-circuit periodic orbits of the saddle type having non-rough homoclinic orbits Γ_k^* of the third type. It is obvious that X^* belongs to the intersection of a countable set of smooth Banach submanifolds M_m such that any flow belonging to M_m has m periodic orbits $L_1, \ldots, L_m$ close to $L_1^*, \ldots, L_m^*$ together with non-rough homoclinic curves $\Gamma_1, \ldots, \Gamma_m$ of the third type where all L_k and Γ_k smoothly depends on the flow. Let λ_k, γ_k ($|\gamma_k| > 1 > |\lambda_k|$) be the multipliers of L_k. It follows from [15,19,20] that the quantities $\theta_k = -\ln|\lambda_k|/\ln|\gamma_k|$ are the moduli of Ω-equivalence on M_m. (Note that the phase space U has a nontrivial fundamental group in this case and, from now on, we shall consider the equivalence of non-wandering sets with the aid of a homeomorphism which is homotopic to the identity.) As a result we come to the following

Theorem 3. Flows having a countable set of smooth moduli of Ω-equivalence are dense in H_r.

Definition. Let some C^k-smooth flow ($1 \le k \le \infty$) on M have a periodic orbit L of saddle type. Assume that $W^s(L)$ and $W^u(L)$ have a tangency along a homoclinic orbit Γ. Since Γ is a non-rough homoclinic curve, the functions F and G appearing in eq. (2) satisfy $F(0,0) = G(0,0) = 0$ and $\partial^i G(0,0)/\partial y^i = 0$, $i = 1, 2, \ldots, n$ for some n ($1 \le n \le \infty$). We say that a *homoclinic tangency is of order* $n < k$ if $\partial^{n+1}G(0,0)/\partial y^{n+1} \ne 0$ and that a *homoclinic tangency is of indefinite order* if $n = k$.

Theorem 4. Flows with homoclinic tangencies of any order (both definite and indefinite) are dense in H_r.

The proof of this theorem is based on the following lemma.

Lemma 1. Let O_1, O_2, O_3 be periodic orbits of saddle-type of some three-dimensional flow. Let $W^u(O_1)$ and $W^s(O_2)$ have a tangency of order n and let $W^u(O_2)$ and $W^s(O_3)$ have a quadratic tangency. Then an arbitrarily small C^r-perturbation which is localized in a small neighborhood of the orbits of tangency can make $W^u(O_1)$ and $W^s(O_3)$ have a tangency of order $n+1$.

A proof of this lemma is not given here. We note only that it is gometrically evident in the case $n=1$. Fig. 7 shows how to achieve a cubic tangency of $W^u(O_1)$ and $W^s(O_3)$. First, one should obtain an intersection of $W^u(Q_1)$ with $W^s(O_2)$. Then, let us take some connected piece K_0 of $W^u(O_1)$ lying beneath $W^s(O_2)$ and let K_i be the ith iteration of K_0. For some i, K_i does not intersect $W^s(O_3)$ and lies slightly above it (fig. 7b). Furthermore, with a small perturbation, K_i can have an intersection with $W^s(O_3)$ at four points (fig. 7c). Then a particular deformation from fig. 7b to fig. 7c gives the required cubic tangency (figs. 7d, e).

To prove theorem 4, let us note that flows possessing n non-rough homoclinic contours are dense in H_r for any n, as follows from the proof of theorem 2. Let $C_{i_1j_1}, C_{i_2j_2}, \ldots, C_{i_nj_n}$ $(j_n > i_n > \cdots > j_1 > i_1)$ be such a family of contours. Because of (4), the intersection of $T_1\sigma_k^1$ with σ_{k+1}^0 is of saddle type for any sufficiently large k. Therefore, we have successively: a quadratic tangency between $W^u(p_{i_1})$ and $W^s(p_{j_1})$, transverse intersections between $W^u(p_{j_1})$ and $W^s(p_{j_1}+1)$, $W^u(p_{j_1}+1)$ and $W^s(p_{j_1}+2), \ldots,$ $W^u(p_{i_2-1})$ and $W^s(p_{i_2})$, a quadratic tangency between $W^u(p_{i_2})$ and $W^s(p_{j_2})$, transverse intersections between $W^u(p_{j_2})$ and $W^s(p_{j_2+1})$, etc. and, finally, a quadratic tangency between $W^u(p_{i_n})$ and $W^s(p_{j_n})$ and a transverse intersection between $W^u(p_{j_n})$ and $W^s(p_{i_1})$ (see fig. 8 for the case $n=2$). A small perturbation of such a contour generates quadratic tangencies between $W^u(p_{i_1})$ and $W^s(p_{i_2})$, $W^u(p_{i_2})$ and $W^s(p_{i_3}), \ldots, W^u(p_{i_n})$ and $W^s(p_{j_n})$, while the intersection of $W^u(p_{j_n})$ with $W^s(p_{i_1})$ obviously remains transverse. Furthermore, using lemma 1, we can achieve a cubic tangency between $W^u(p_{i_1})$ and $W^s(p_{i_3})$, thereafter a quartic tangency between $W^u(p_{i_1})$ and $W^s(p_{i_4})$, etc. till producing a heteroclinic contour where $W^u(p_{i_1})$ has a tangency of order $n+1$ with $W^s(p_{j_n})$ while $W^u(p_{j_n})$ has a transverse intersection with $W^s(p_i)$. Now, a homoclinic curve with a tangency of the $(n+1)$th order can be obtained for p_{i_1} using an arbitrarily small perturbation of that contour.

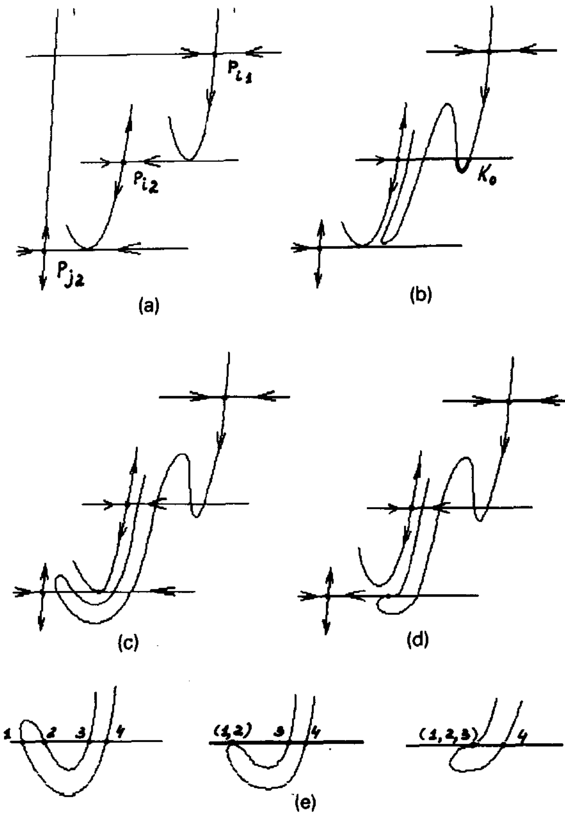

Fig. 7.

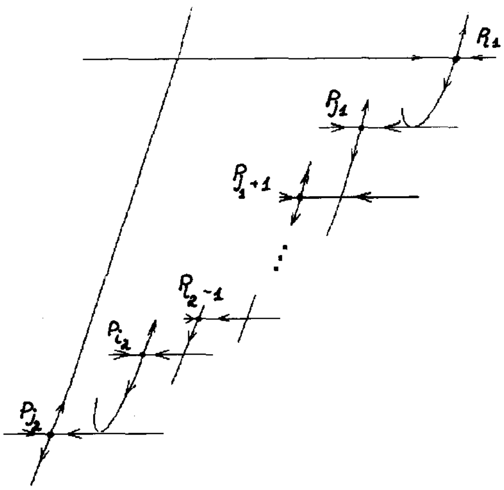

Fig. 8.

Remark. Systems with any finite order of homoclinic tangency were studied in [1]. An one-parameter analysis of bifurcation was carried out in [16] for the case of a quadratic tangency while a two-parameter analysis was performed in [25] for the case of a cubic tangency. This last work has shown, among other results, the existence of an infinite number of cusp bifurcation points in the parameter plane. Apparently, the proximity of cubic tangencies also implies the phenomenon [26] of violation of the "natural" bifurcation order in one-parameter families of plane diffeomorphisms close to a system with a quadratic tangency.

Homoclinic tangencies can also lead to bifurcations of periodic orbits with higher degeneracies.

Definition. Let some C^k-smooth flow $(k \geq 1)$ have a periodic orbit L with a multiplier $\nu = \pm 1$ while the absolute values of the other multipliers remain different from unity. Then a restriction of the Poincaré map to the central manifold can be written in the form

$$\bar{y} = y + l_n y^n + \cdots \quad \text{when } \nu = 1\,,$$

$$\bar{y} = -y - l_n y^{2n+1} + \cdots \quad \text{when } \nu = -1\,,$$

(where $l_n \neq 0$ is the nth Lyapunov value with $1 \leq n \leq k-1$ when $\nu = 1$ and with $1 \leq n \leq (k-1)/2$ when $\nu = -1$) or, if all the Lyapunov values are equal to zero, in the form

$$\bar{y} = \nu y + o(y^k)\,.$$

In the first case, the periodic orbit L will be said to be *n-degenerate*. Otherwise, we shall say that the periodic orbit has *a degeneracy of indefinite order*.

Lemma 2. Let the invariant manifolds of some periodic orbit O in a three-dimensional C^k-smooth flow X have a tangency of order $n < k$ and let the saddle value be less than unity. Consider an arbitrary generic C^k-smooth n-parameter family M containing X. Then M contains flows with n-degenerate periodic orbits with $\nu = 1$ and flows with $[(n+1)/2]$-degenerated periodic orbits with $\nu = -1$.

The following theorem is a corollary of this lemma and of theorem 4.

Theorem 5. Flows having periodic orbits of any order of degeneracy (definite and indefinite) both with multiplier $\nu = 1$ and $\nu = -1$ are dense in H_r.

As was noted above, any family transversal to H_r intersects some Newhouse regions in which systems with non-rough Poincaré homoclinic orbits are dense. It follows from [16] and [21] that systems with a countable set of stable periodic orbits are dense in these regions[#7]. From theorems 2–5, we infer

Corollary. Flows having a countable set of non-rough homoclinic orbits, a countable set of moduli of Ω-equivalence, non-rough homoclinic or-

[#7] This result can also be derived from the fact that systems with homoclinic orbits of the third type are dense in the Newhouse regions (theorem 1) and systems with a countable set of stable periodic motions are dense in the "pellicle" of systems of such type [1,2].

bits of any order of tangency, and non-rough periodic orbits of any order of degeneracy both with multiplier $\nu = 1$ and $\nu = -1$ are dense in Newhouse regions[#8].

We come to the conclusion that models with non-rough Poincaré homoclinic curves cannot be good from the viewpoint of the Ω-equivalence. One can try to weaken the equivalence relation, for instance, to give up the consideration of the local behaviour of orbits. This can be done as follows. For every orbit belonging to N, let us consider its code, i.e. an infinite sequence of symbols 0 and 1 where 0 corresponds to a passage of the orbit near L_0 and 1 corresponds to a passage of the orbit near $\Gamma \backslash V$. We say that any two systems close to X_0 are C-equivalent if the sets of their codes coincide. It should be emphasized that we used a basic construction of the symbolic dynamics [22], namely:

(1) the phase space of a smooth dynamical system is partitioned into a finite number of domains;

(2) a code is established for each orbit according to the sequence of passages in these domains;

(3) the shift map acting on the set of codes is studied.

This construction is the main tool for studying systems with complex dynamics. One of the reasons at the basis of this assertion is the fact that the relation of C-equivalence that we introduced above practically coincides with the Ω-equivalence for systems with hyperbolic behaviour and for systems with Lorenz attractors after a suitable choice of phase space partition. It is also obvious that the relation of C-equivalence is very natural when used in applications. For instance, the relation of equivalence that we introduced above for flows on torus can be considered as some variant of C-equivalence.

If we are interested only in stable motions, we may restrict ourselves to the study of the set of codes for the orbits belonging to the attractor. It is unknown what the structure of the attractor would be in the case of systems with non-rough homoclinic curves, but it is clear that all the stable periodic orbits necessarily belong to the attractor. Accordingly, we introduce the following

Definition. We say that two flows are C^+-equivalent if their sets of codes for stable periodic orbits coincide.

The following theorem shows that there do not exist good models in Newhouse regions for this equivalence relation either.

Theorem 6. The systems with a countable set of moduli of C-equivalence and the systems with a countable set of moduli of C^+-equivalence are dense in the Newhouse regions[#9].

Therefore, for systems with quasi-attractors, even the complete study of the set of stable periodic orbits cannot be achieved. How to correctly set the problem of studying such systems is an extremely difficult question. Probably we have to give up the ideology of complete description and restrict oneself to the study of some particular features and properties of a system. The properties which are worth studying must essentially depend on the specific problem under consideration.

5. On bifurcation of single-circuit periodic orbits in systems with indefinite order of tangency

The following question was considered in the aforementioned paper by Robinson [21]: do there exist stable periodic orbits in one-parameter families of systems with non-rough homo-

[#8] We can show even more, for instance, that systems with a countable set of arbitrarily degenerate periodic and homoclinic orbits are dense in the Newhouse regions.

[#9] Here as in theorem 2, we consider that the moduli are the quantities θ_k.

clinic orbits formed by a one-sided tangency of the manifolds of a periodic orbit of saddle-type? The results of [21] guarantee the existence of a countable set of intervals of stable periodic orbits for analytic three-dimensional models where non-rough homoclinic orbit with one-sided tangency of the stable and unstable manifolds can be found, for instance, using computer calculations. Use is made of the fact that the tangency between the manifolds has always some finite order for analytic flows. This result is well-known in the case of quadratic tangency [16,23,24]. Moreover, a multidimensional case was considered in [23,24] and several first bifurcations of period doubling for stable periodic orbits have been studied therein. As established by us above, infinitely degenerate tangencies are also possible in the case of smooth (i.e., non-analytic) flows. In this connection, we consider the existence of stable periodic orbits for multidimensional systems in the case of one-sided tangency without the assumption of finite order of tangency.

Consider a continuous one-parameter family X_μ of vector fields of class C^r, $r \geq 2$, given on a $(n+2)$-dimensional smooth manifold M. Let us suppose that the following condition is satisfied:

(1) At $\mu = 0$, the flow X_μ has a non-rough homoclinic orbit Γ_0 associated with a periodic orbit L_0 of saddle type with multipliers $\lambda_1, \ldots, \lambda_n, \gamma$ such that $|\lambda_i| < 1 < |\gamma|$ and $|\lambda_i \gamma| < 1$, $i = 1, \ldots, n$.

In this case and for μ small enough, the system X_μ has a periodic orbit L_μ of saddle type which is close to L_0 and which has multipliers $\lambda_{i\mu}$, γ_μ such that $\lambda_{i\mu} \to \lambda_i$, $\gamma_\mu \to \gamma$ when $\mu \to 0$. Let S be some $(n+1)$-dimensional smooth local cross-section of L_0. Then the map $T_{0\mu}: S \to S$ along the orbits of X_μ belonging to a neighbourhood of the orbit L_μ can be written in the following form in some appropriate coordinates

$$\bar{x} = A(\mu)x + f(x, y, \mu)\, x\,,$$

$$\bar{y} = \gamma_\mu y + g(x, y, \mu)\, y\,,$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^1$, $A(\mu)$ is a $(n \times n)$ matrix with eigenvalues $\lambda_{1\mu}, \ldots, \lambda_{n\mu}$, $f, g \in \mathbb{C}^{r-1}$, $f(0, 0, \mu) = 0$, $g(0, 0, \mu) = 0$. The manifolds $W^s_{loc}(L_0) \cap S$ and $W^u_{loc}(L_0) \cap S$ are given by the equations $y = 0$ and $x = 0$, respectively. For μ small enough, we can define the map $T_{1\mu}: S \to S$ along orbits close to a piece of Γ_0 and lying outside some small neighbourhood of L_0. This map – from a small neighbourhood of the homoclinic point $P^-(x = 0, y = y^-) \in W^u_{loc}(L_0) \cap S$ onto another small neighbourhood of the homoclinic point $P^+(x = x^+, y = 0) \in W^s_{loc}(L_0) \cap S$ – can be written in the form

$$\bar{x} - x^+ = F(x, y - y^-, \mu)\,, \quad \bar{y} = G(x, y - y^-, \mu)$$

where $F(0, 0, 0) = G(0, 0, 0) = 0$ and $\partial G(0, 0, 0)/\partial y = 0$ because of condition (1). We suppose that the following condition is satisfied:

(2) At $\mu = 0$, the tangency between $W^s(L_0)$ and $W^u(L_0)$ along Γ_0 is one-sided.

This condition means that there exist an $\varepsilon > 0$ and a constant α which is equal to either $+1$ or -1 such that $\operatorname{sign} G(0, \varepsilon, 0) = \operatorname{sign} G(0, -\varepsilon, 0) = \alpha$ and such that the inequality $\alpha G(0, y - y^-, 0) \geq 0$ is satisfied for $|y - y^-| \leq \varepsilon$. In particular, condition (2) is valid when the tangency of $W^s(L_0)$ with $W^u(L_0)$ along Γ_0 is of even order. However, in general, Γ_0 may also be a non-isolated single-circuit homoclinic curve but the case where $W^s(L_0)$ and $W^u(L_0)$ partially coincide is not expected.

We shall also assume that the following condition is satisfied:

(3) When $\alpha\mu < 0$, there exists no single-circuit homoclinic curve associated with the orbit L_0 in a neighbourhood $U(L_0 \cup \Gamma_0)$ and, when $\alpha\mu > 0$, there is a non-removable single-circuit homoclinic curve in U.

Condition (3) means that the curve $T_{1\mu}W^{\mathrm{u}}_{\mathrm{loc}}(L_\mu) \cap S$ lies strictly above or below the surface $W^{\mathrm{s}}_{\mathrm{loc}}(L_\mu) \cap S$ when $\alpha\mu < 0$ and that it has points lying on different sides of $W^{\mathrm{s}}_{\mathrm{loc}}(L_\mu) \cap S$ when $\alpha\mu > 0$.

Theorem 7. Let a family X_μ satisfy conditions (1)–(3) for $|\mu| \le \mu_0$. Then, in the interval $[-\mu_0, +\mu_0]$, there exists a countable set of intervals $\Delta_k = (\mu_k^1, \mu_k^2)$ ($\mu_k^i \to 0$ when $k \to \infty$, $i = 1, 2$) such that for $\mu \in \Delta_k$ the system X_μ has a stable single-circuit periodic orbit. All the μ_k^i are positive if $\gamma > 0$ but the values μ_k^i and μ_{k+1}^i have opposite signs if $\gamma < 0$.

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