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On the effect of invisibility of stable periodic orbits at homoclinic bifurcations

S.V. Gonchenko [a], I.I. Ovsyannikov [b], D. Turaev [c,*]

[a] Nizhny Novgorod State University, Institute of Applied Mathematics and Cybernetics, 10 Uljanova St., 603005 Nizhny Novgorod, Russia

[b] Nizhny Novgorod State University, Department of Radiophysics, 23 Prospekt Gagarina, 603950, Nizhny Novgorod, Russia

[c] Imperial College, Department of Mathematics, London SW7 2AZ, UK

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ABSTRACT

We study bifurcations of a homoclinic tangency to a saddle-focus periodic point. We show that the stability domain for single-round periodic orbits which bifurcate near the homoclinic tangency has a characteristic "comb-like" structure and depends strongly on the saddle value, i.e. on the area-contracting properties of the map at the saddle-focus. In particular, when the map contracts two-dimensional areas, we have a cascade of periodic sinks in any one-parameter family transverse to the bifurcation surface that corresponds to the homoclinic tangency. However, when the area-contraction property is broken (while three-dimensional volumes are still contracted), the cascade of single-round sinks appears with "probability zero" only. Thus, if three-dimensional volumes are contracted, chaos associated with a homoclinic tangency to a saddle-focus is always accompanied by stability windows; however the violation of the area-contraction property can make the stability windows invisible in one-parameter families.

1. Introduction

Among nonlocal bifurcations of multidimensional dynamical systems, a special role is played by bifurcations of homoclinic tangencies, i.e. non-transverse intersections of the stable and unstable manifolds of a saddle periodic orbit. Recall that an orbit that lies in the intersection of these manifolds is called homoclinic. When this intersection is transverse, it cannot be removed by any small smooth perturbation of the system. The existence of a transverse homoclinic orbit is an universal criterion of the non-trivial dynamics in the system, i.e. of the dynamical chaos. This is related to the fact that even for an arbitrarily small neighborhood of the closure of the transverse homoclinic orbit the set N of all orbits which lie entirely in this neighborhood (i.e. they never leave it) has a non-trivial structure. Namely, N contains infinitely many periodic, heteroclinic and homoclinic orbits, uncountably many Poisson-stable (non-trivial recurrent) orbits, etc. More precisely, N is a locally-maximal non-trivial hyperbolic set, and admits a complete description by means of symbolic dynamics, see [1].

In the case of non-transverse intersection, i.e. when we deal with a homoclinic tangency, the corresponding problem of the description of the set N becomes much more difficult. In fact, the problem of a "complete description" is unresolvable here. The main source of the difficulty is that arbitrarily small perturbations of any system with the simplest, quadratic homoclinic tangency may lead to the creation of new homoclinic tangencies of arbitrarily high orders, and to the birth of periodic orbits of arbitrarily high orders of degeneracy [2–6]. This means that no finite-parameter unfolding can provide a complete picture of bifurcations of even the simplest homoclinic tangency. Therefore, in studying homoclinic tangencies we have to restrict ourselves with the analysis of most general, interesting and important dynamical properties only.

One of such important classical problems in this topic is the problem of the study of homoclinic bifurcations which lead to the birth of stable periodic orbits ("periodic sinks"). This has a special value from the applied point of view, as homoclinic tangencies present ubiquitously in the "strange attractors" observed in dynamical models from a huge variety of applications. While the periodic attractors which are born from the homoclinic tangencies have quite large periods and small attraction basins, their number may be very large, or even infinite [7], and altogether they may change the character of the chaotic dynamics, see more discussions, e.g., in [7,8].

General criteria for the existence/non-existence of stable periodic orbits in systems close to systems with a homoclinic tangency were obtained in [9,10]. The first result in this direction was established by Gavrilov and Shilnikov in [11] for two-dimensional maps. Here, the answer depends on the so-called saddle value σ. Recall that the saddle value of a saddle periodic orbit is the absolute value of the product of the nearest to the unit circle stable (less than 1 in absolute values) and unstable (greater than 1 in absolute values) multipliers. Thus, for two-dimensional maps, $\sigma = |\lambda\gamma|$, where λ and γ are the stable and,

* Corresponding author.
E-mail addresses: gosv100@uic.nnov.ru (S.V. Gonchenko), ivan.i.ovsyannikov@gmail.com (I.I. Ovsyannikov), d.turaev@imperial.ac.uk, dturaev@imperial.ac.uk (D. Turaev).

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resp., unstable multipliers of the saddle. It is shown in [11] that if a two-dimensional diffeomorphism has a homoclinic tangency to a saddle periodic point with $\sigma < 1$, then stable periodic orbits are born at the bifurcations. Namely, in [11] main bifurcations of periodic orbits were studied for one-parameter families of two-dimensional diffeomorphisms such that at a certain parameter value, μ_0, there is a quadratic homoclinic tangency which unfolds with a non-zero velocity as the parameter μ varies. It was shown, in particular, that if $\sigma < 1$, then an infinite sequence (a "cascade") of disjoint intervals of parameter values for which there exists a stable periodic orbit converges to μ_0. Later the result was called a "theorem on a cascade of periodic sinks".

The stable periodic orbits studied by Gavrilov and Shilnikov are "single-round", i.e. they make only one round near the orbit of homoclinic tangency before closing up. If we consider more complicated stable periodic orbits, the multi-round ones, then infinitely many of them may coexist. Namely, for a two-dimensional diffeomorphism which has a quadratic homoclinic tangency to a saddle with $\sigma < 1$, for any generic one-parameter unfolding there exists a set $\mathcal{B}$ of parameter values for which the system has an infinite set of coexisting multi-round stable periodic orbits. Moreover, this set is dense (and forms a residual subset) in open intervals on the parameter axis [7,12]. This effect is called the "Newhouse phenomenon".[1] A popular conjecture [22] is that this set $\mathcal{B}$, while known to be residual, still has measure zero. This conjecture cannot be true in full generality: for a dense set of one-parameter families the set $\mathcal{B}$ has positive measure [23]. Still it might happen that $\mathcal{B}$ has measure zero for a residual set of one-parameter families, a progress in this direction see in [24].

For the multi-dimensional case, the existence of the cascade of single-round sinks was proven in [25], under conditions that $\sigma < 1$ and that the unstable manifold of the saddle periodic point is one-dimensional. It was, however, shown in [4,15] that the violation of these conditions does not necessarily mean that stable periodic orbits cannot be born. Indeed, even if we restrict ourselves to the three-dimensional case and consider a homoclinic tangency to a saddle periodic orbit O with one-dimensional unstable manifold, then two completely different situations are possible in the case $\sigma > 1$:

(i) the point O is *a saddle*, i.e. its multipliers are real;

(ii) the point O is *a saddle-focus*, i.e. it has multipliers $\lambda_{1,2} = \lambda e^{\pm i\varphi}$, γ such that $0 < \lambda < 1 < |\gamma|$, $0 < \varphi < \pi$.

It was shown in [4,26,10] that in the case of a saddle with $\sigma > 1$, generically[2] no stable periodic orbits can exist in a small neighborhood of the orbit of homoclinic tangency, neither for the map itself, nor for any C^1-close map.

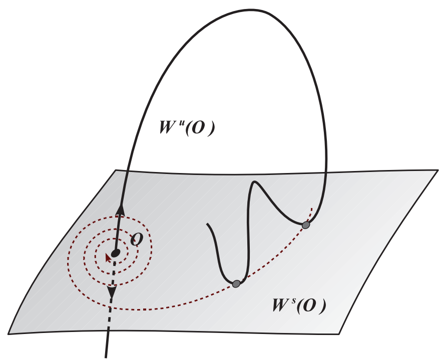

Fig. 1. An example of a homoclinic tangency to a saddle-focus (2,1).

However, in the case of saddle-focus, stable periodic orbits can be born even if $\sigma > 1$ (one should then require $\lambda^2|\gamma| < 1$, the volume-contraction condition). The main bifurcations that lead to the birth of stable periodic points near the homoclinic tangency in this case were studied in [15,10]. Importantly, one needs at least two parameters in order to study these bifurcations even if only single-round periodic orbits are considered. In particular, for the one-parameter unfolding of a homoclinic tangency to a saddle-focus with $\sigma > 1$ the cascade of single-round sinks appears with "probability zero" only (see a discussion in [15,10]). In other words, when we have a chaotic behavior associated with a homoclinic tangency to a saddle-focus, transition from $\sigma < 1$ to $\sigma > 1$ does not destroy the sinks but makes them "invisible".

In this paper we analyze this effect in great detail. Namely, we show that crossing the boundary $\sigma = 1$ leads to a dramatic transformation of the bifurcation diagram, especially in the part which corresponds to the bifurcations of single-round stable periodic orbits. One can get the idea of the structure of the bifurcation diagram from Figs. 2–5, the exact statements are given by Theorems 1 and 2.

2. Statement of the problem and main results

Consider a smooth three-parameter family of C^r-smooth, $r \geq 3$, diffeomorphisms f_ν of an m-dimensional manifold, $m \geq 3$. Here we denote the vector of control parameters as $\nu = (\mu, \sigma, \varphi)$, where μ varies in a small neighborhood of zero, and φ and σ run some compact intervals in $(0, \pi)$ and $R \setminus 0$, respectively. We assume the following conditions are satisfied.

A. For all ν, the diffeomorphism f_ν has a periodic point O with multipliers $\gamma, \lambda_1, \ldots, \lambda_{m-1}$ such that $\max_{i\geq 3}|\lambda_i| < |\lambda_2| = |\lambda_1| = \lambda < 1 < |\gamma|$. The stable leading multipliers $\lambda_{1,2}$ form a complex conjugate pair, such that $\lambda_{1,2} = \lambda e^{\pm i\varphi}$ and $\lambda\gamma = \sigma$, i.e. the control parameter σ is chosen such that its absolute value equals the saddle value of O and the parameter φ is the argument of the complex leading eigenvalues of O.
B. $\lambda^2|\gamma| < 1$ for all ν under consideration.
C. The interior of the range of variation of σ includes the value $\sigma_0 = 1$ (in the case $\gamma > 0$) or $\sigma_0 = -1$ (if $\gamma < 0$).
D. At $\mu = 0$ the stable W^s and unstable W^u invariant manifolds of the point O have a quadratic tangency at the points of a homoclinic orbit Γ (see Fig. 1). As μ changes, the tangency splits with non-zero velocity. In other words, the control parameter μ measures the distance between the manifolds $W^s(O_\nu)$ and $W^u(O_\nu)$ near one of the points of the former homoclinic orbit Γ (for an exact definition see in Section 3).

Condition A implies that $\dim W^u(O) = 1$, $\dim W^s(O) = m - 1$, and that among the stable multipliers λ_i (which correspond to the eigendirections tangent to $W^s(O)$) those largest in the absolute

[1] The multidimensional version of the Newhouse phenomenon was proven in [13,9,14]. Note that infinitely many more complicated attractors may also coexist for different classes of systems from the Newhouse domain. Namely, there may be infinitely many coexisting stable closed invariant curves (tori) [9,4,15,16,10], Henon-like strange attractors [17], pseudohyperbolic wild Lorenz-like attractors [9,18,19], and even hyperbolic (e.g. Plykin) attractors [20,21].

[2] The corresponding conditions of general positions are given in [9] (the conditions of a simple homoclinic tangency) and are analogous to the quasitransversality conditions from [27]. These conditions guarantee [9,26] the existence of a smooth two-dimensional invariant manifold which contains all orbits that stay in a small neighborhood of the homoclinic tangency. For two-dimensional maps, the condition $\sigma > 1$ implies the expansion of areas near the homoclinic orbit, so it automatically prohibits the emergence of stable periodic orbits on the invariant manifold and, hence, in the whole neighborhood of the homoclinic tangency. The genericity (simplicity) conditions, beyond the quadraticity of the tangency, require also that (a) $|\lambda_1| \neq |\lambda_2|$ and (b) the so-called extended unstable manifold $W^{ue}(O)$ is transverse to the leaves of the strong-stable foliation at the points of the homoclinic orbit, see more details in [9,4,26,10]. These conditions are important in the context of this paper, as their violation can lead to the birth of stable periodic orbits even in the case of a saddle with $\sigma > 1$ [28,29].

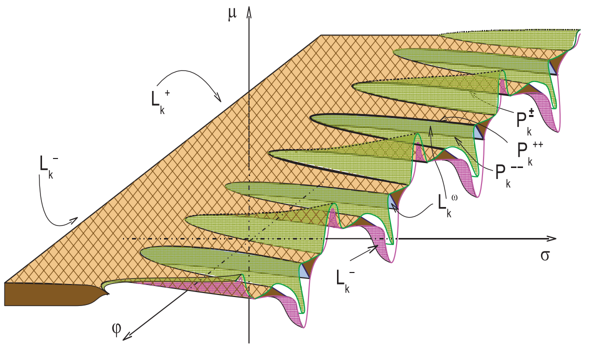

Fig. 2. The comb-like stability domain S_k for the fixed point of the map $T^{(k)}$ in the space of three parameters (μ, σ, φ).

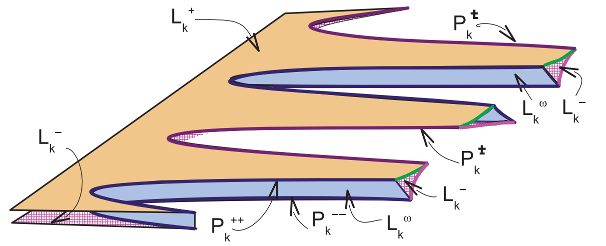

Fig. 3. The boundary of the stability domain in more detail.

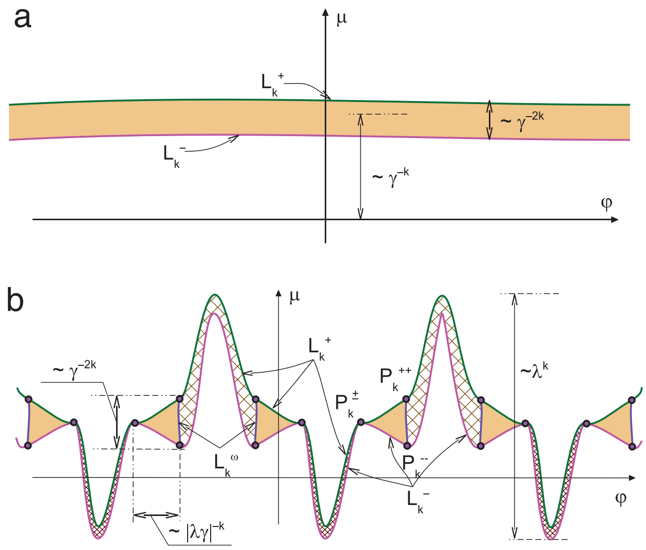

Fig. 4. (a) Section $|\sigma| < 1$. (b) section $|\sigma| > 1$—the zone between L_k^+ and L_k^- is divided by L_k^ω and $P_k^\pm$ into the region where the first-return map $T^{(k)}$ has a stable fixed point (along with a saddle fixed point with a one-dimensional unstable manifold) and the region where $T^{(k)}$ has a pair of saddle fixed points, one with a one-dimensional unstable manifold and the other with a two-dimensional one.

value make a complex-conjugate pair, so the point O is a *saddle-focus of type (2,1)*, in the terminology of [10]. It also follows that O depends smoothly on the parameters ν. Condition B means that the product of any three different multipliers of O is less than 1 in absolute value, so the map f_ν contracts three-dimensional volumes near the orbit of O. Condition D implies that the family f_ν intersects transversely, at $\mu = 0$, a codimension-1 bifurcation surface $\mathcal{H}$ in the space of m-dimensional C^r-diffeomorphisms, which is composed by diffeomorphisms with a quadratic homoclinic tangency to a saddle-focus (2, 1). According to a criterion from [9,10], once conditions A, B and D are satisfied, stable periodic orbits can be born at bifurcations of f_ν.

We will focus on the bifurcations of single-round periodic orbits, which are defined as follows. Let V be a sufficiently small neighborhood of the set $O \cup \Gamma$. Evidently, V is a union of a small neighborhood U of the orbit of the point O and the finite number of small neighborhoods of those points of Γ which do not belong to U. A periodic orbit from V is called single-round, if it visits every connected component of the set $V \setminus U$ exactly once (namely it may stay for an unbounded number of iterations inside U, then it leaves U and, after making a fixed finite number of iterations outside U, closes up immediately as it enters U again).

The way the single-round orbits bifurcate depends strongly on the saddle value, namely on whether $|\sigma|$ is larger or smaller than 1. The saddle value determines the area-contraction properties of f_ν: at $|\sigma| < 1$ the map contracts any two-dimensional area near the orbit of O, so the dynamics and bifurcations are effectively one-dimensional [26,10], while at $|\sigma| > 1$ this property is violated, which leads to drastic changes in dynamics. Thus, condition C means the family f_ν undergoes a jump in the "effective dimension" of its dynamics (cf. [26,10]).

The choice of φ as the third parameter is due to the fact [9,30] that φ is an Ω-modulus for the diffeomorphisms from $\mathcal{H}$ at $|\sigma| > 1$, i.e. no two diffeomorphisms from $\mathcal{H} \cap \{|\sigma| > 1\}$ can be topologically conjugate on the set of nonwandering orbits, if the corresponding values of φ are different. By the very definition of Ω-modulus, any change of its value leads to a change in the set of nonwandering orbits, i.e. to bifurcations of periodic orbits, homoclinic ones, etc. Indeed, the changes in φ are known [9,30] to lead to the bifurcations, at $|\sigma| > 1$, of single-round periodic orbits even if the homoclinic tangency does not split. Therefore, as we want to analyze bifurcations of the single-round orbits, we must take φ as an additional control parameter, cf. [15,10].

In the case of dimension m higher than 3, we need to impose more restrictions on the diffeomorphisms f_ν. Namely, we assume that at $\mu = 0$ the following conditions are satisfied by f_ν for all values of the parameters (σ, φ) under consideration.

E. If $m > 3$, then the extended unstable manifold $W^{ue}(O)$ is transverse to the stable manifold at the points of Γ, and the unstable manifold is not tangent at the points of Γ to the leaves of the strong stable foliation of the stable manifold which pass through these points.

F. If $m > 3$, then the homoclinic orbit Γ does not lie in the strong stable submanifold W^{ss} of the stable manifold.

The strong-stable manifold of condition F is a uniquely defined smooth invariant $(m-3)$-dimensional manifold which is tangent at O to the eigendirections that correspond to the multipliers $\lambda_3, \dots, \lambda_{m-1}$ (i.e. those strictly smaller than λ in absolute values). The extended unstable manifold W^{ue} of condition E is a smooth three-dimensional invariant manifold which is tangent at O to the eigendirections that correspond to the multipliers $\gamma, \lambda_1, \lambda_2$. This manifold is not unique, however: it contains W^u and any two extended unstable manifolds are tangent to each other at every point of W^u. The strong-stable foliation is a uniquely defined smooth invariant foliation of W^s by $(m-3)$-dimensional leaves transverse (in a small neighborhood of O) to W^{ue}_{loc}; the manifold W^{ss} is the leaf of this foliation which passes through O. For more discussions on the extended unstable manifold and strong-stable foliation see e.g. in [31]. Conditions E and F are written in a coordinate form in Section 3, formulas (12), (14) and (11).

Note that condition E is slightly weaker than the condition of transversality of W^{ue} to the strong-stable foliation, which, along with condition F, would guarantee [9,26,31] the existence of a three-dimensional, stable, and persistent to small smooth perturbations, invariant manifold which is transverse to W^{ss} at O and which contains all orbits that stay entirely in a small neighborhood of Γ. However, even in the case where such an invariant manifold exists, we cannot immediately reduce the dimension of the problem by considering the reduction of the map

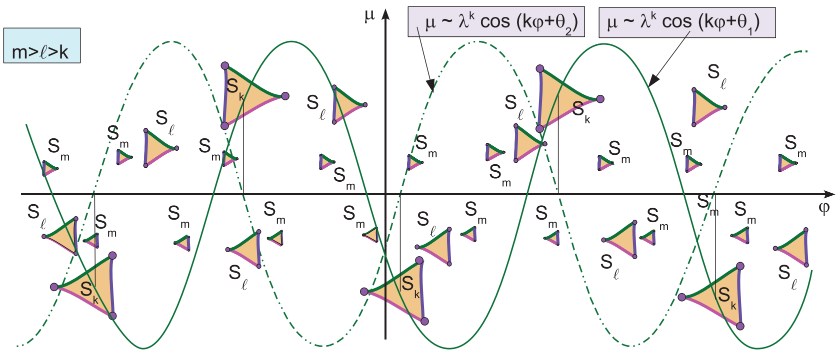

Fig. 5. The stability domains for different single-round periodic orbits at $|\sigma| > 1$. Two auxiliary curves $C_k^1: \ \mu = -K_1\lambda^k\cos(k\varphi+\theta_1)$ and $C_k^2: \ \mu = K_2\lambda^k\cos(k\varphi+\theta_2)$ are also shown, which provide a mnemonic rule for determining the positions of the domains S_k. Namely, the domains S_k are centered around the points $(\varphi=\varphi^*, \mu=\mu^*)$ such that $\cos(k\varphi^*+\theta_2)=0$ and $\mu^* = -K_1\cos(k\varphi^*+\theta_1)$. The same rule holds also for the domains S_l and S_m (one may draw the corresponding curves $C_l^{1,2}$ and $C_m^{1,2}$ and find the approximate positions of the domains S_l and S_m geometrically).

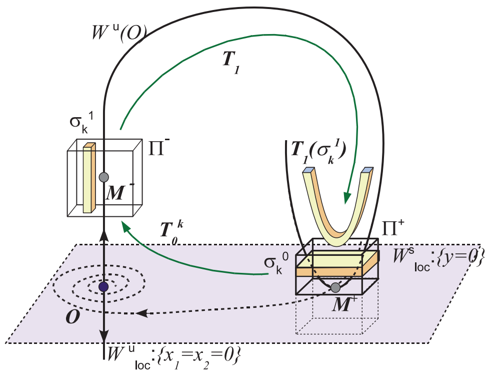

Fig. 6. Geometry of the first return map $T^{(k)} = T_1T_0^k$.

to this manifold, because the manifold is only C^1, in general, and we need more smoothness in our computations (for example, it is difficult to speak about quadratic tangency on the manifold which is not C^2). Thus, while the results here do not depend on m, we have to perform m-dimensional calculations.[3]

Now we may formulate the main result of the paper. Note that any point of a single-round periodic orbit can be obtained as a fixed point of the corresponding first return map defined by the orbits of the diffeomorphism f_ν as follows. Let q be the period of the point O, i.e. $f^qO = O$. Define the so-called "local map" T_0 as $T_0(\nu) \equiv f_\nu^q|_{U_0}$, where U_0 is a small neighborhood of O_ν (the connected component of U which contains the point O). In order to define the "global map" T_1, note that the homoclinic orbit Γ tends to the orbit of O both at forward and backward iterations of f. Thus, in U_0 there are two points, $M^+ \in W^s_{loc}(O)$ and $M^- \in W^u_{loc}(O)$, such that at $\mu = 0$ they both belong to Γ, all the forward iterations of M^+ by T_0 and all the backward iterations of M^- by T_0 stay in U_0, and $T_0M^- \notin U_0$, $T_0^{-1}M^+ \notin U_0$. Since M^- and M^+ are points of the same orbit Γ, there exists $n > 0$ such that $M^+ = f_\nu^nM^-$ at $\mu = 0$. Then, the global map $T_1(\nu)$ is defined as a restriction of f_ν^n to a small neighborhood of M^- (see Fig. 6). By the construction, if M is a point of a single-round periodic orbit near M^+ (i.e. it is the point where this orbit first enters U_0), then M is a fixed point of the first-return map $T^{(k)}(\nu) := T_1T_0^k$, for some positive integer k.

Theorems 1 and 2 below provide a complete description of the stability domain for single-round periodic orbits that may emerge in the V at the bifurcations of the diffeomorphisms f_ν.

Theorem 1 (*Bifurcations of Single-Round Periodic Orbits*)**.** *For all sufficiently small μ, for every sufficiently large k, in the space of the parameters $\nu = (\mu, \sigma, \varphi)$ there exist bifurcation surfaces L_k^+, L_k^- and L_k^ω such that the map $T^{(k)}$ has a fixed point with multiplier $+1$ at $\nu \in L_k^+$, a fixed point with multiplier -1 at $\nu \in L_k^-$, and a fixed point with multipliers $e^{\pm i\psi}$, where $0 < \psi < \pi$, at $\nu \in L_k^\omega$. The equations of these surfaces are*

$$\begin{aligned} L_k^+ :\ \mu &= \lambda^k\xi_k + \frac{1}{4d}\lambda^{2k}(\sigma^{-k} + K_2\cos(k\varphi+\theta_2))^2 \\ &\quad + O(\delta^k(\gamma^{-2k}+\lambda^{2k})) \\ L_k^- :\ \mu &= \lambda^k\xi_k - \frac{3}{4d}\lambda^{2k}(\sigma^{-k} + K_2\cos(k\varphi+\theta_2))^2 \\ &\quad + O(\delta^k(\gamma^{-2k}+\lambda^{2k})) \\ L_k^\omega &: K_2\cos(k\varphi+\theta_2)\sigma^k = 1 + O(\delta^k), \end{aligned} \tag{1}$$

where

$$\xi_k(\sigma,\varphi) = \sigma^{-k}y^- - K_1\cos(k\varphi+\theta_1) + O(\delta^k) \tag{2}$$

is the same for L_k^+ and L_k^-; the coefficients $K_{1,2}$, y^-, d, $\theta_{1,2}$, λ and γ are independent of k and depend on ν smoothly, and δ is a constant less than 1.

It follows from (1) that infinitely many surfaces L_k^ω exist in the region $|\sigma| > 1$ and they disappear (one by one) in the region $|\sigma| < 1$. The surface L_k^ω is bounded by the two lines, P_k^{++} and P_k^{--}, where it adjoins to L_k^+ and L_k^-, respectively. At $\nu \in P_k^{++}$ the fixed point of $T^{(k)}$ has a double multiplier $+1$, and at $\nu \in P_k^{--}$ it has a double multiplier -1. The surfaces L_k^+ and L_k^- touch at the curve $P_k^\pm$ which corresponds to a pair of multipliers $(1, -1)$. These curves are given by the equations

$$\begin{aligned} P_k^{++} &: \begin{cases} \mu = \lambda^k\xi_k + \dfrac{K_2^2}{d}\lambda^{2k}(\cos^2(k\varphi+\theta_2) + O(\delta^k)) \\ K_2\cos(k\varphi+\theta_2)\sigma^k = 1 + O(\delta^k) \end{cases} \\ P_k^{--} &: \begin{cases} \mu = \lambda^k\xi_k - \dfrac{3K_2^2}{d}\lambda^{2k}(\cos^2(k\varphi+\theta_2) + O(\delta^k)) \\ K_2\cos(k\varphi+\theta_2)\sigma^k = 1 + O(\delta^k) \end{cases} \\ P_k^\pm &: \begin{cases} \mu = \lambda^k\xi_k + O(\lambda^{2k}\delta^k) \\ K_2\cos(k\varphi+\theta_2)\sigma^k = -1 + O(\delta^k). \end{cases} \end{aligned} \tag{3}$$

It is seen from here that the curves P_k^{++}, P_k^{--} and $P_k^\pm$ stay in the region $|\sigma| > 1 + o_{k\to\infty}(1)$. As follows from (1)–(3), the surface L_k^ω and lines P_k^{++}, P_k^{--} and $P_k^\pm$ consist each of $\sim k\frac{\Delta\varphi}{\pi}$ connected components (where $\Delta\varphi$ is the range of the values of φ under consideration).

[3] For an example of a homoclinic bifurcation where the low smoothness of the invariant manifold leads to essential dynamical effects see [32].

In Fig. 2, an illustration for the bifurcation set of fixed points of the map $T^{(k)}$ is shown. The surfaces L_k^+, L_k^- and L_k^ω correspond to codimension 1 bifurcations (fold, flip and Andronov–Hopf, respectively) and lines P_k^{++}, P_k^{--} and $P_k^{\pm}$ correspond to codimension 2 bifurcations. These bifurcations can lead to the birth of the stable fixed point for $T^{(k)}$ (i.e. a single-round periodic attractor of period $kq+n$). Therefore, parts of these bifurcation surfaces and lines form boundaries of the stability region for the map $T^{(k)}$. Indeed, the following result holds.

Theorem 2 (*Stability Domains for Single-Round Periodic Orbits*)**.** *The stability domain S_k of the periodic orbit under consideration (i.e. the region of parameters ν for which the map $T^{(k)}$ has a stable fixed point) is bounded by the surfaces L_k^ω, L_k^{+s} and L_k^{-s}, where L_k^{+s} and L_k^{-s} are the subsets of L_k^+ and L_k^- which are bounded by $P_k^\pm$ and P_k^{++} or P_k^{--} (respectively).*

According to Theorem 2, the stability region S_k has a "comb-like" structure, see Fig. 3 where boundaries of S_k are shown (compare with Fig. 2).

At $|\sigma| < 1$, the region S_k is bounded only by the surfaces L_k^+ and L_k^- and looks like a flat layer of thickness of order γ^{-2k} around the plane $\mu = \gamma^{-k}y^-$. Upon crossing to $|\sigma| > 1$, the structure of $S_k \cap \{\sigma = const\}$ changes: at every sufficiently large k this set consists of curvilinear triangles with the size of order γ^{-2k} in the μ-direction and the size of order $|\lambda\gamma|^{-k}$ in the φ-direction, repeated periodically (with period $2\pi/k$) along the φ-axis, see Fig. 4.

In Fig. 5, stability regions S_k, S_l and S_m ($k < l < m$) on a plane of constant σ with $|\sigma| > 1$ are shown, corresponding to different single-round orbits.

The fact that at $|\sigma| > 1$ the size of the individual stability triangles in the φ-direction tends to zero exponentially as $k \to +\infty$ can lead to a perceived disappearance (in one-parameter families) of the infinite cascades of single-round sinks. Indeed, take any smooth one-parameter family F_μ of C^r-diffeomorphisms, such that at $\mu = 0$ the diffeomorphism has a quadratic homoclinic tangency to a saddle-focus (2, 1) satisfying conditions B, E and F and, as μ varies, the tangency is split with non-zero velocity. Let φ_0, θ_{20} and σ_0 be the values of φ, θ_2 and $\sigma = \lambda\gamma$ at $\mu = 0$. We assume $|\sigma_0| > 1$; in particular $\lambda^k \gg |\gamma|^{-k}$ at large k. Since the family is smooth, we have $\varphi(\mu) = \varphi_0 + O(\mu)$, $\sigma(\mu) = \sigma_0 + O(\mu)$, $\theta_2(\mu) = \theta_{20} + O(\mu)$. Thus, by (1), when the family intersects the zone between the surfaces L_k^+ and L_k^-, we have $\varphi = \varphi_0 + O(\lambda^k)$, $\sigma = \sigma_0 + O(\lambda^k)$, $\theta_2 = \theta_{20} + O(\lambda^k)$. Now, we may see from (1), (3) that there exists a positive constant $\delta < 1$ such that, for each sufficiently large k, if

$$K_2|\cos(k\varphi_0 + \theta_{20})| < |\sigma_0|^{-k}(1 - \delta^k), \tag{4}$$

then the family F_μ intersects the stability domain S_k of the fixed point of the first-return map $T^{(k)}$, and if

$$K_2|\cos(k\varphi_0 + \theta_{20})| > |\sigma_0|^{-k}(1 + \delta^k), \tag{5}$$

then F_μ does not intersect S_k.

We say that the family F_μ exhibits an infinite cascade of single-round sinks if there is a sequence of converging to $\mu = 0$ intervals of μ for which F_μ has a single-round stable periodic orbit. By (4) and (5) we obtain the following.

Theorem 3. *There exists $\delta < 1$ such that the existence of an infinite sequence of pairs of integers (j, k) such that*

$$\left| j - k\frac{\varphi_0}{2\pi} - \frac{\theta_{20}}{2\pi} \pm \frac{1}{4} \right| < \frac{1}{2\pi K_2}|\sigma_0|^{-k}(1 + \delta^k) \tag{6}$$

is a necessary condition for the family F_μ to exhibit an infinite cascade of single-round sinks. The existence of an infinite sequence of pairs of integers (j, k) such that

$$\left| j - k\frac{\varphi_0}{2\pi} - \frac{\theta_{20}}{2\pi} \pm \frac{1}{4} \right| < \frac{1}{2\pi K_2}|\sigma_0|^{-k}(1 - \delta^k) \tag{7}$$

is a sufficient condition.

It is well known [33] (see also [34]) that inequality (7) is satisfied by infinitely many integer pairs (j, k) for a dense set of values of φ_0. Thus, a dense set of families F_μ exhibits the infinite cascade of single-round sinks. Since an intersection of F_μ with an open stability domain S_k is an open property, it follows that the set of families with an infinite cascade of single-round sinks is, in fact, residual (i.e. it is an intersection of countably many open and dense sets). However, it is also well known [33,34] that the values of φ_0 and θ_{20} which satisfy inequality (6) for infinitely many integer pairs (j, k) compose a set of zero measure in the (φ, θ)-plane (this is a set of (φ, θ) which admit abnormally good, exponential non-homogeneous approximations by rational fractions: at such φ and θ the pair of straight lines $j = k\frac{\varphi}{2\pi} + \frac{\theta_2}{2\pi} \pm \frac{1}{4}$ in the (j, k)-plane comes to the distance of order $|\sigma|^{-k}$ to the points of the integer-valued lattice). Thus, in one-parameter families transverse to the bifurcation surface of systems with a homoclinic tangency to a saddle-focus with $|\sigma| > 1$, the cascade of infinitely many single-round sinks appears with zero probability[4] (in spite of being a generic phenomenon). Of course, as one can see e.g. from Figs. 4, 5, in any two-parameter family parametrized by μ and φ the infinite sequences of stability domains are always present, provided $\lambda^2|\gamma| < 1$, see also [15,10].

3. Local, global, and first-return maps, and bifurcations in the Henon map

In this section we prove Theorem 1. Recall [34,31,10], that in U_0 one can introduce C^r-coordinates (x, u, y) (where $x \in R^2$, $u \in R^{m-3}$, $y \in R^1$) such that the local map $T_0(\nu)$ for all μ close to 0 takes the following form

$$\begin{aligned}
\bar{x} &= \lambda R_\varphi x + O((x^2 + \|u\|)|y|),\\
\bar{u} &= Bu + O(x^2 + u^2 + \|u\|\ |y|),\\
\bar{y} &= \gamma y + O((\|x\| + \|u\|)y^2),
\end{aligned} \tag{8}$$

where $R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is the rotation matrix to the angle φ, and B is a matrix with the eigenvalues $\lambda_3, \ldots, \lambda_m$, i.e. its spectral radius is strictly less than λ. Note that the coordinate transformation which brings the map to form (8) depends smoothly on parameters. Namely, the transformation itself and its derivatives with respect to coordinates up to the second order are C^{r-2} with respect to the parameters [10]. As our map is at least C^3, it follows that in the new coordinates the map T_0 and its derivatives up to the second order depend smoothly on ν (e.g. λ, γ and B are smooth functions of ν). The same holds true for the map T_1 which we discuss below.

Formula (8) is called "the main normal form" for a smooth map near a saddle fixed point. It is very convenient for the study of homoclinic bifurcations. In particular, in these coordinates, the local stable and unstable manifolds of the point O_ν are straightened: they are given by the equations $W^s_{loc}(O_\nu) : \{y = 0\}$ and $W^u_{loc}(O_\nu) : \{x = 0, u = 0\}$.

[4] Note that it is absolutely unclear whether a similar statement is true for cascades of sinks of an arbitrary, unbounded from above, number of rounds.

At $\mu = 0$ both W^s_{loc} and W^u_{loc} contain infinitely many points of the homoclinic orbit Γ. We take a pair of such points, $M^+ = (x^+, u^+, 0) \in W^s_{loc}$ and $M^- = (0, 0, y^-) \in W^u_{loc}$, so that $M^+ = f^n(M^-)$ for some $n > 0$ at $\nu = \nu_0$. Let Π^+ and Π^- be some small neighborhoods of the points M^+ and M^-, respectively. The global map $T_1(\nu) : \Pi^- \to \Pi^+$ (defined as $T_1 = f^n_\nu|_{\Pi^-}$) is written in the form

$$\begin{aligned}\bar{x} - x^+ &= F(x, u, y - y^-, \nu),\\ \bar{u} - u^+ &= H(x, u, y - y^-, \nu),\\ \bar{y} &= G(x, u, y - y^-, \nu),\end{aligned} \tag{9}$$

where $F(0, 0, 0, \mu = 0) = 0$, $G(0, 0, 0, \mu = 0) = 0$, $H(0, 0, 0, \mu = 0) = 0$.

In coordinates of (8) we have

$$G'_y(0, \mu = 0) = 0, \qquad G''_{yy} := 2d \neq 0.$$

Indeed, these relations mean that the curve

$$\begin{aligned}\bar{x} - x^+ &= F(0, 0, y - y^-, \mu = 0),\\ \bar{u} - u^+ &= H(0, 0, y - y^-, \mu = 0),\\ \bar{y} &= G(0, 0, y - y^-, \mu = 0),\end{aligned}$$

i.e. $T_1(W^u_{loc})$ at $\mu = 0$, has a quadratic tangency with the plane $W^s_{loc} : \bar{y} = 0$ at $(\bar{x}, \bar{u}) = (x^+, u^+)$, which is our condition D.

Now we can write the following Taylor expansion for the functions F, G and H:

$$\begin{aligned}F(x, u, y - y^-, \nu) &= ax + \alpha u + b'(y - y^-)\\ &\quad + O(x^2 + u^2 + (y - y^-)^2),\\ G(x, u, y - y^-, \nu) &= \mu + cx + \beta u + d(y - y^-)^2\\ &\quad + O(x^2 + u^2 + (\|x\| + \|u\|)\\ &\quad \times |y - y^-| + (y - y^-)^3),\\ H(x, u, y - y^-, \nu) &= \tilde{a}x + \tilde{\alpha}u + \tilde{b}(y - y^-)\\ &\quad + O(x^2 + u^2 + (y - y^-)^2);\end{aligned} \tag{10}$$

note that the coefficients here are matrices or vectors, e.g. $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$, $b' = (b_1, b_2)^\top$, $c = (c_1, c_2)$. We should also note that all the coefficients in (10) depend, in general, on the parameters. Namely, using that $d \neq 0$, we choose for every μ close to zero a (uniquely defined) value of y^- such that $G'_y = 0$ at $y = y^-$, i.e. the linear in $(y - y^-)$ term vanishes in the expansion of G. Since such chosen y^- depends (C^{r-2}-smoothly) on ν, all the coefficients of the Taylor expansion at $y = y^-$ are also functions of ν (we expand up to the second order, so the corresponding coefficients are C^{r-2} in ν).

As it is seen from (10), we denote the distance between the curve $T_1 W^u_{loc}$ at $y = y^-$ and W^s_{loc} as μ. We take μ as the first of our control parameters ν (one can always make the first component of the vector ν equal to μ by a smooth transformation in the space of parameters, provided our family is transverse to $\mathcal{H}$).

One may show (see [31,10]) that when the map T_0 is brought to the form (8), the extended unstable manifold W^{ue}_{loc} is tangent to $u = 0$ everywhere on W^u_{loc} and, in particular, at the point M^-; the leaves of the strong-stable foliation are given by $\{x = const, y = 0\}$, and the strong-stable manifold W^{ss}_{loc} is $\{x = 0, y = 0\}$. Thus, condition F (i.e. $M^+ \notin W^{ss}_{loc}$) can be written as

$$x^+ \neq 0. \tag{11}$$

Condition E reads as the transversality of $T_1\{u = 0\}$ to $\bar{y} = 0$ at M^+ and of the absence of tangency (at M^+) between $T_1\{x = 0, u = 0\}$ to $(\bar{y} = 0, \bar{x} = x^+)$. By (9), (10), this is equivalent to

$$c_1^2 + c_2^2 \neq 0, \tag{12}$$

and

$$b_1^2 + b_2^2 \neq 0. \tag{13}$$

By a linear rotation of coordinates $x = (x_1, x_2)$, one can always make the coefficient b_2 vanish. Note that any linear rotation in the (x_1, x_2) coordinates does not change the form (8) of the local map. Therefore, without loss of generality, we can assume that

$$b_2 \equiv 0, \qquad b_1 = b \neq 0 \tag{14}$$

for all μ close to zero.

The advantage of the main normal form (8) is that we have good formulas for its iterations, uniform for arbitrarily large k. Namely, there exist functions $\rho_{1k}, \rho_{2k}, \rho_{3k}$ such that two points (x_0, u_0, y_0) and (x_k, u_k, y_k) from U_0 are related by $(x_k, u_k, y_k) = T_0^k(x_0, u_0, y_0)$ if and only if

$$\begin{aligned}x_k &= \lambda^k R_{k\varphi} x_0 + \hat{\lambda}^k \rho_{1k}(x_0, u_0, y_k, \nu),\\ u_k &= \hat{\lambda}^k \rho_{2k}(x_0, u_0, y_k, \nu)\\ y_0 &= \gamma^{-k} y_k + \hat{\gamma}^{-k} \rho_{3k}(x_0, u_0, y_k, \nu),\end{aligned} \tag{15}$$

where $\hat{\gamma}$ and $\hat{\lambda}$ are sufficiently close to $|\gamma|$ and, resp., λ, and $|\gamma| < \delta\hat{\gamma}$, $\hat{\lambda} < \delta\lambda$ for some constant $\delta < 1$, the same for all ν under consideration, and, as $k \to +\infty$, the functions ρ_k tend to zero, uniformly along with their derivatives up to the order $(r - 2)$ [34,31,10].

Formula (15) represents the maps T_0^k in the so-called "cross-form" (as a relation between (x_0, u_0, y_k) and (x_k, u_k, y_0)). As we see, in the chosen coordinates, the cross-form for arbitrarily large iterations of T_0 is uniformly close to that of a linear map. Thus, we do not rely on unnecessary and restrictive linearizability assumptions which are made some times in order to simplify the study of homoclinic bifurcations.

Using formulas (9)–(15) we derive the following.

Lemma 1 (*Rescaling Lemma*)**.** *Let f_ν be a three parameter family of the diffeomorphisms, as described in Section 2. Then the first-return map $T^{(k)}(\nu)$ can be brought, by a smooth change of the coordinates and parameters, to the following form*

$$\begin{aligned}&\bar{X}_1 = Y + O(\delta^k), \qquad (\bar{X}_2, \bar{U}) = O(\delta^k),\\ &\bar{Y} = M_1 - M_2 X_1 - Y^2 + O(\delta^k),\end{aligned} \tag{16}$$

where the new coordinates (X_1, X_2, U, Y) can take arbitrary finite values as k grows. The $O(\delta^k)$-terms are functions of $(X_1, X_2, U, Y, M_1, M_2)$ which exponentially tend to zero, along with the derivatives up to the order $(r - 2)$, as $k \to +\infty$. The new parameters (M_1, M_2) are related to the original parameters ν as follows:

$$\begin{aligned}M_1 &= -d\gamma^{2k}(\mu - \gamma^{-k}y^- + K_1\lambda^k\cos(k\varphi + \theta_1)\\ &\quad + O(\hat{\lambda}^k + \hat{\gamma}^{-k})),\\ M_2 &= \gamma^k(\lambda^k K_2 \cos(k\varphi + \theta_2) + O(\hat{\lambda}^k)),\end{aligned} \tag{17}$$

where

$$\begin{aligned}&K_1 = \sqrt{(c_1^2 + c_2^2)(x_1^{+2} + x_2^{+2})},\\ &\sin\theta_1 = \frac{c_2 x_1^+ - c_1 x_2^+}{K_1}, \qquad \cos\theta_1 = \frac{c_1 x_1^+ + c_2 x_2^+}{K_1},\\ &K_2 = b\sqrt{c_1^2 + c_2^2}, \qquad \cos\theta_2 = -\frac{c_1}{\sqrt{c_1^2 + c_2^2}},\\ &\sin\theta_2 = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.\end{aligned} \tag{18}$$

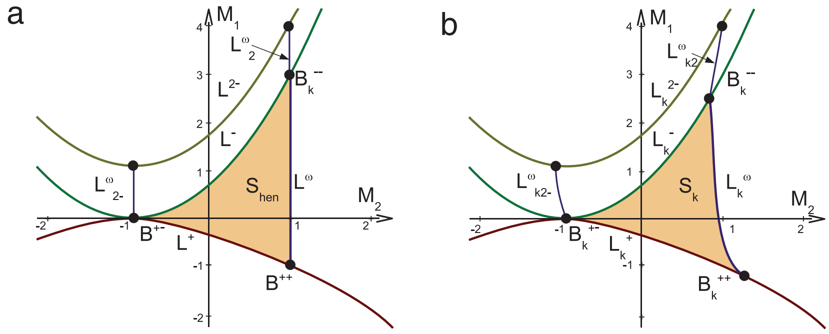

Fig. 7. Elements of the bifurcation diagram (a) for the Henon map (22), (b) for the first-return map $T^{(k)}$.

Proof. By virtue of (9)–(15), we can write the map $T^{(k)} = T_1 T_0^k$ in the following form:

$$\begin{aligned}
&\bar{x}_0 - x^+ = \lambda^k a R_{k\varphi} x_0 + \begin{pmatrix} b \\ 0 \end{pmatrix} (y_k - y^-) + O(\hat{\lambda}^k + (y_k - y^-)^2),\\
&\bar{u}_0 - u^+ = \lambda^k \tilde{a} R_{k\varphi} x_0 + \tilde{b}(y_k - y^-) + O(\hat{\lambda}^k + (y_k - y^-)^2),\\
&\gamma^{-k}\bar{y}_k + \hat{\gamma}^{-k}\rho_{3k}(\bar{x}_0, \bar{u}_0, \bar{y}_k, \nu)\\
&\quad = \mu + \lambda^k c R_{k\varphi} x_0 + d(y_k - y^-)^2\\
&\qquad + O(\hat{\lambda}^k + \lambda^k |y_k - y^-| + |y_k - y^-|^3).
\end{aligned} \tag{19}$$

Let us shift the coordinates: $x = x_0 - x^+ + \eta_{k1}, u = u_0 - u^+ + \eta_{k2}$, $y = y_k - y^- + \eta_{k3}$ (where $\eta_{ki} = O(\lambda^k)$ are some constants, $i = 1, 2, 3$), so that the constant terms in the first and second equations of (19) vanish along with the linear in y term in the third equation. Then system (19) takes the following form:

$$\begin{aligned}
&\bar{x}_1 = by + O(\lambda^k \|x, u, y\| + y^2),\\
&(\bar{x}_2, \bar{u}) = O(\lambda^k \|x, u, y\| + y^2),\\
&\gamma^{-k}\bar{y} + \hat{\gamma}^{-k} O(\|\bar{x}, \bar{u}, \bar{y}\|)\\
&\quad = M + [\lambda^k (c_1 \cos k\varphi + c_2 \sin k\varphi) + O(\hat{\lambda}^k)] x_1 + d\gamma^k y^2\\
&\qquad + O(\lambda^{2k} x_1^2 + \lambda^k (\|x_2, u\| + x_1 y + y^2) + |y|^3),
\end{aligned} \tag{20}$$

where

$$M = \mu - \gamma^{-k} y^- + K_1 \lambda^k \cos(k\varphi + \theta_1) + O(\hat{\lambda}^k + \hat{\gamma}^{-k}). \tag{21}$$

Now, one may check that after introducing the rescaled coordinates $y = -d\gamma^{-k} Y$, $x_1 = -bd\gamma^{-k} X_1$, $(x_2, u) = |\delta\gamma|^{-k}(\lambda^k + |\gamma|^{-k})(X_2, U)$ with $\delta < 1$ sufficiently close to 1, we bring the map to the sought form (16) (we use that $\lambda^2|\gamma| < 1$ by condition B, so the $O(\lambda^{2k}|\gamma|^k \delta^{-k} \|X_2, U\|)$-terms which appear at the rescaling of the last line in (20) can be estimated as $O(\delta^k)$ for δ sufficiently close to 1). □

Due to the Rescaling Lemma, dynamics and main bifurcations of the first return map $T^{(k)}$ can be recovered by studying the limit map obtained from (16) as $k \to \infty$. Namely, we have here the Henon map

$$\begin{aligned}
&\bar{X}_1 = Y,\\
&\bar{Y} = M_1 - M_2 X_1 - Y^2.
\end{aligned} \tag{22}$$

The bifurcation diagram for this map is presented in Fig. 7(a). The peculiarity of this figure is the existence of the stability triangle S_{hen}, with the vertices $B^{++}(1, -1)$, $B^{--}(1, 3)$, $B^{\pm}(-1, 0)$. In S_{hen} the Henon map has a stable fixed point. The boundaries of S_{hen} are formed by the bifurcation curves $L^+, L^- L^\omega$. The curves $L^+ : M_1 = -(1 + M_2)^2/4$ and $L^- : M_1 = 3(1 + M_2)^2/4$ correspond to the moment of a saddle–node bifurcation and period-doubling bifurcation, respectively. The curve $L^\omega : \{M_2 = 1, -1 < M_1 < 3\}$ corresponds to the existence of a fixed point with a pair of complex conjugate multipliers ($e^{\pm i\psi}$, $0 < \psi < \pi$). The points $B^{++}, B^{--}, B^{\pm}$ correspond to a pair of multipliers equal, respectively, to $(1, 1)$, $(-1, -1)$ and $(1, -1)$. The other shown curves, L^{2-} and $L_2^\varphi, L_{2-}^\varphi$, correspond to the existence of period two points with multiplier -1 and with multipliers $e^{\pm i\psi}$ (for $0 < \psi < \pi$), respectively.

At large k, this structure is inherited by map (16); just the bifurcation curves may slightly deform and shift to the distance of order δ^k. Returning to the original parameters (μ, σ, φ) by formulas (17), we immediately obtain Theorems 1 and 2.

We remark that while the Rescaling Lemma is enough for the purposes of proving Theorems 1 and 2, i.e. for establishing the geometry of the stability domain of the fixed point of the first-return map (16), it is not enough for studying the dynamics emerging as the stability boundaries are crossed. The problem is that in the Henon map itself the bifurcations upon crossing the boundary L^ω are *infinitely degenerate*. Indeed, the Henon map has constant Jacobian $J = -M_2$, hence it is area-preserving at $|J| = 1$, e.g. it is area-preserving on the stability boundary L^ω. The $o(1)$-terms in (16) break this property, so in order to determine the character of the Andronov–Hopf bifurcation in the map $T^{(k)}$ upon crossing the stability boundary L_k^ω, one would need to estimate these terms. A similar problem was solved for various cases of homoclinic bifurcations in [35,36,15,37,16,38,29,10]. In the same way one may show that the first-return map $T^{(k)}$ in our case is sufficiently close to the so-called generalized Henon map

$$\bar{X}_1 = Y, \qquad \bar{Y} = M_1 + M_2 X_1 - Y^2 + Q_k X_1 Y$$

where $Q_k = O(\lambda^{2k}|\gamma|^k)$. Its Jacobian is nonconstant and, as it was shown in [15,37], the Andronov–Hopf bifurcation of the fixed point is non-degenerate for this map at $Q_k \neq 0$ (see more about the bifurcations of this map in [37,39,40]). Using this fact, one establishes [15,10] that the Andronov–Hopf bifurcation upon crossing L_k^ω is non-degenerate (a single closed invariant curve is born from the fixed point) for the maps $T^{(k)}$ for all sufficiently large k, provided $\lambda|\gamma|$ is strictly greater than 1. The analysis of the Andronov–Hopf bifurcation in the region where $\lambda|\gamma|$ is close to 1 remains to be done.

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