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Review

Analytical search for homoclinic bifurcations in the Shimizu-Morioka model

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ABSTRACT

The existence of a homoclinic butterfly to a saddle with zero saddle value is established analytically for the Shimizu-Morioka model.

In this paper we consider the Shimizu–Morioka system [1]

$$\dot{x} = y, \qquad \dot{y} = (1 - z)x - \lambda y, \qquad \dot{z} = -\alpha(z - x^2).$$

It emerges as an asymptotic normal form for bifurcations of triply degenerate equilibrium states and periodic orbits in systems with certain types of symmetry [2,3]. The system was extensively studied in [4–7,3], where it was, in particular, shown that there exists a region of positive values of (α, λ) for which the system has a Lorenz attractor [8]. This conclusion was based on one of the criteria proposed in [9]: a Lorenz attractor is born at bifurcations of a homoclinic butterfly to a saddle with zero saddle value and separatrix value A such that $|A| < 2$ (see a proof in [10]; for a definition of the saddle and separatrix values see e.g. [11]). The existence of such homoclinic butterfly is a codimension-2 bifurcation. It was shown numerically that this bifurcation indeed occurs at some (α, λ).

In this paper we provide an analytic (free of computer assistance) derivation of the existence of the homoclinic loops with zero saddle value in this system. Unfortunately, we did not obtain estimates for the corresponding separatrix value A (we explain the setup of the problem in the last section). Thus, the Lorenz attractor is not immediately given by our results, though they can be considered as a step forward towards a fully analytical proof of the existence of the Lorenz attractor.

The homoclinic butterfly here is a pair of homoclinic loops symmetric to each other by $(x, y, z) \to (-x, -y, z)$. By the symmetry of the system, it is enough to establish the existence of only one of the loops. The loop is an orbit which tends to the saddle equilibrium state $O(0, 0, 0)$ both as the time t tends to $+\infty$ and to $-\infty$. The saddle value σ is the sum of the positive characteristic exponent at O with the nearest to the imaginary axis negative one. In order to apply the criterion from [9], we need $\sigma = 0$, which is equivalent to $\lambda = (1 - \alpha^2)/\alpha$, as one can easily see. From now on we impose this restriction on α and λ. After scaling the time $t \to t/\alpha$ and $y \to \alpha y$, the system takes the form

$$\dot{x} = y, \qquad \dot{y} = (a + 1)(1 - z)x - ay, \qquad \dot{z} = -z + x^2, \tag{1}$$

where $a = \lambda/\alpha = -1 + 1/\alpha^2$.

Theorem 0.1. *There exists a value $a_0 > 0$ such that the system (1) at $a = a_0$ has a homoclinic loop to the saddle point $O(0, 0, 0)$.*

The proof occupies the next two sections. We search for the homoclinic loop by means of a refined version of the "method of comparison systems", developed in [12,13] for the analytic proof of the existence of homoclinic loops in the Lorenz model. Roughly speaking, the method consists in finding the traces which the stable and unstable manifolds of the saddle leave on a Poincaré section, the surface $z = x^2$ in our case. Once certain bounds are found for the position of the traces, we show that when the parameter a varies the two traces meet one another, this event corresponds to a formation of the sought homoclinic loop.

We also note that numerical investigations reveal indeed a unique point $a_0 \simeq 1.718$ for which the system possesses a homoclinic orbit (cf. [4–6]).

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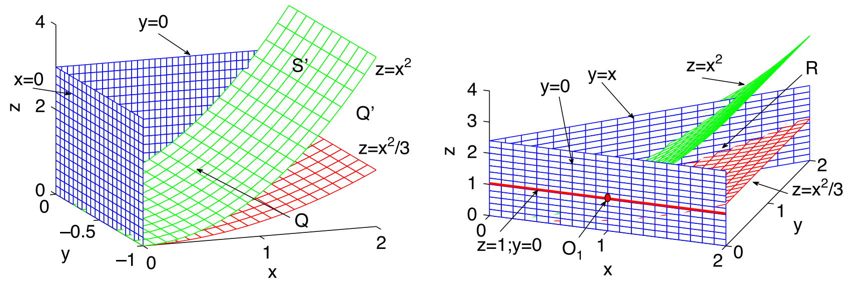

Fig. 1. Surfaces and regions related to Lemma 1.1.

1. Estimating the unstable manifold

The equilibrium points of system (1) are $O(0,0,0)$, $O_1(1,0,1)$ and $O_2(-1,0,1)$. The Jacobian matrix associated to the system at $O(0,0,0)$ has the eigenvalues $(1,-1,-a-1)$ and the corresponding eigenvectors $(1,1,0),(0,0,1),\left(-\frac{1}{a+1},1,0\right)$. As we see, $O(0,0,0)$ is a saddle with a one-dimensional unstable manifold W_0^u and a two-dimensional stable manifold W_0^s, and with zero saddle value indeed (we assume $a \geq 0$).

The one-dimensional tangent space to W_0^u at O is given by

$$TW_0^u = \{(x,y,z) : x = y, z = 0\}.$$

The curve W_0^u is divided by O into two branches, the separatrices. Denote by W_+^u the separatrix which leaves O towards $x > 0$. The equation of W_+^u in a small neighborhood of the origin can be written as

$$y = x + b_2x^2 + b_3x^3 + \cdots$$
$$z = c_2x^2 + c_3x^3 + c_4x^4 + \cdots. \tag{2}$$

By plugging this Taylor expansion in (1) and equating coefficients (recall that W_+^u is invariant with respect to the system) we find that the equation of W_+^u is

$$y = x - \frac{(a+1)}{3(a+4)}x^3 + o(x^3),$$
$$z = \frac{1}{3}x^2 + \frac{2(a+1)}{45(a+4)}x^4 + o(x^4). \tag{3}$$

So, for x small enough, W_+^u lies in the region $R := \{(x,y,z) : 0 < y < x, \frac{1}{3}x^2 < z < x^2\}$.

Lemma 1.1. *For every $a \geq 0$, the separatrix W_+^u leaves R at a finite moment of time by intersecting transversely the surface $S := \left\{y = 0, z > 1, \sqrt{z} < x < \sqrt{3z}\right\}$. After crossing S, the separatrix intersects transversely the surface $S' := \left\{x > 0, y < 0, z = x^2\right\}$. After crossing S', it either stays in the region $Q := \{x > 0, y < 0, z > x^2\}$ forever, and then it tends to O and forms a homoclinic loop, or it leaves Q by either transversely intersecting the plane $x = 0$, or transversely intersecting the plane $y = 0$ at $x < 1$, Fig. 1.*

Proof. The surface $S_1 := \{y = x, x > 0, z > 0\}$ is a surface without contact for the orbits of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}(y-x)\Big|_{y=x} = -(a+1)xz < 0.$$

This implies that no orbit from R can leave R by crossing S_1 (we have $y < x$ inside R, so if the orbit comes to $y = x$ from R, then there must be $\frac{\mathrm{d}}{\mathrm{d}t}(y-x) \geq 0$ at the moment of contact, a contradiction). The same is true for the surface $S_2 := \{z = x^2/3, y < x, x > 0\}$: we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(z - \frac{1}{3}x^2\right)\Big|_{z=x^2/3} = \frac{2}{3}x(x-y) > 0,$$

so no orbit can come to S_2 from the side $z > x^2/3$.

Let us check that leaving R by an intersection with $S_3 := \{y = 0, z \leq 1, \sqrt{z} \leq x \leq \sqrt{3z}\} \setminus O_1(1,0,1)$ is also impossible. Indeed,

$$\frac{\mathrm{d}y}{\mathrm{d}t}\Big|_{y=0,z<1} = (a+1)x(1-z) > 0,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\Big|_{y=0,z=1,x>1} = (a+1)x(1-x^2) < 0,$$

so y must be an increasing function of t before the orbit intersects S_3, i.e. the orbit may come to S_3 only from the side of negative y, and not from R (where $y > 0$). Analogously, exiting R across $S_4 := \{z = x^2, y \geq 0\} \setminus O_1(1,0,1)$ is also forbidden, since

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 - z)\Big|_{z=x^2,y>0,x>0} = 2xy > 0,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x^2 - z)\Big|_{z=x^2,y=0,z>1} = 2(a+1)x^2(1-z) < 0,$$

which means that any orbit that intersects S_4 must approach it from the side $z > x^2$, i.e. not from R.

As the points $O_1(1,0,1)$ and $O(0,0,0)$ are equilibria, the orbit from inside R cannot pass through these points. Thus, since the boundary of R is contained in $S \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup O_1 \cup O$, the separatrix W_+^u must either leave R by intersecting S, or stay in R forever. Let us show that the latter case is ruled out.

Note that if W_+^u stays in R for all times, it cannot tend to infinity. Indeed, the function $H(x,y) = \frac{y^2}{2} - \frac{a+1}{2}x^2 + \frac{a+1}{12}x^4$ is decreasing along the orbits in R:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(x,y) = y\dot{y} - (a+1)x(1-x^2/3)\dot{x}$$
$$= -(a+1)xy(z - x^2/3) - ay^2 < 0,$$

so $x(t)$ and $y(t)$ must stay bounded, which implies the boundedness of $z(t)$ as well (since $z < x^2$ in R).

Thus, the ω-limit set Ω of W_+^u must be non-empty and lie in the closure of R in this case. Since the function $H(x(t),y(t))$ is monotone along W_+^u, it has to be constant on Ω : $H(x,y)|_{(x,y,z)\in\Omega} = \lim_{t\to+\infty} H(x(t),y(t))$. Therefore, since Ω is an invariant set, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}H = 0$ on Ω, i.e. Ω must lie in $y = 0$. Similarly, as $\dot{z} > 0$ in R, it follows that $z > 0$ and $\dot{z} = 0$ on Ω, i.e. the ω-limit set of W_+^u must be an invariant subset of the line $\{y = 0, z = x^2, x > 0\}$. The only such set is the equilibrium O_1.

Let us show that W_+^u cannot tend to O_1 without leaving R at a finite time. We first note that the equilibrium point O_1 is exponentially stable at $a > 2$, and is unstable (a saddle-focus) at $0 \leq a < 2$. The linearization matrix of system (1) at the point O_1 is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & -(a+1) \\ 2 & 0 & -1 \end{pmatrix},$$

the characteristic equation is

$$p^3 + (a+1)p^2 + ap + 2(a+1) = 0.$$

It is easy to see that at $a \geq 0$ one of the roots, p_*, is real and strictly negative (in fact $p_* < -1$), and the other two roots are complex. If (at some $a \geq 2$) the separatrix W_+^u tends to O_1 spiraling along the two-dimensional eigenplane that corresponds to the pair of complex eigenvalues, it must inevitably leave R at some finite time moment. Therefore, if W_+^u tends to O_1 while remaining in R, it must be tangent at O_1 to the eigendirection corresponding to the real negative root p_*. This eigenvector is given by $\left(1, p_*, \frac{2}{1+p_*}\right)$, hence near O_1 the curve W_+^u will be given by

$$x = 1 + s + o(s), \qquad y = p_* s + o(s),$$
$$z = 1 + \frac{2}{1+p_*} s + o(s), \quad s < 0$$

(we take the small parameter s negative, in order to ensure that $y > 0$ as it should be for $W_+^u \subset R$). Now, as $p_* < -1$, we find that $x^2 - z = \frac{2p_*}{1+p_*} s + o(s) < 0$ on W_+^u, which means that W_+^u cannot stay in R when approaching O_1.

We have proven that W_+^u must, for every $a \geq 0$, intersect the surface S at a finite moment of time t_0. The intersection is transverse since

$$\left.\frac{dy}{dt}\right|_{y=0, z>1, x>0} = (a+1)x(1-z) < 0. \qquad (4)$$

Right upon crossing S, the orbit occurs in the region $Q' : \{x > \sqrt{z} > 1, y < 0\}$. Everywhere in this region $\dot{z} > 0$, so z is monotonically increasing function of time, which implies that $z > z(t_0) > 1$ and, hence, $x > 1$, all the time the orbit remains in Q'. In particular, $x(1-z)$ stays bounded away from zero, which implies that $\dot{y}(=(a+1)x(1-z) - ay)$ is strictly negative for small y, which implies that y remains bounded away from zero upon the orbit enters Q'. It follows that $\frac{d}{dt}(x^2 - z)(=2xy - \dot{z})$ is negative and bounded away from zero all the time the orbit stays in Q', which immediately implies that the orbit indeed must intersect the surface $S' : \{z = x^2, y < 0, x > 0\}$ transversely at a finite moment of time.

After the intersection, the separatrix enters the region Q. If it does not remain in Q forever, it must leave by crossing the boundary of Q. The boundary is formed by the surfaces $z = x^2$, $x = 0$ and $y = 0$. Since

$$\left.\frac{d}{dt}(x^2 - z)\right|_{z=x^2, y<0, x>0} = 2xy < 0,$$

the separatrix W_+^u cannot leave Q by crossing the boundary $x^2 - z = 0$. So W_+^u leaves Q either by crossing $x = 0$, or by crossing $y = 0$. As the line $\{x = 0, y = 0\}$ is invariant with respect to system (1), the separatrix W_+^u cannot intersect this line. Hence, if it crosses the surface $\{x = 0\}$, it does it at $y < 0$, and the required transversality of the intersection follows as $\frac{dx}{dt} = y \neq 0$.

Let us show that when W_+^u leaves Q across $y = 0$, the intersection is also transverse. First, we note that by virtue of (4) it is impossible to cross from Q (where $y < 0$ and $x > 0$) to the region $\{y > 0\}$ at $z > 1$. Thus, when W_+^u leaves the region Q via the surface $\{y = 0\}$ the corresponding intersection point satisfies $1 \geq z \geq x^2 > 0$. In fact, $x^2 \leq z < 1$ at the intersection point: since

$$\left.\frac{dy}{dt}\right|_{y=0, z=1} = 0, \qquad \left.\frac{d^2y}{dt^2}\right|_{y=0, z=1, 0<x<1} = (a+1)x(z - x^2) > 0,$$

no orbit can approach the line $\{y = 0, z = 1, 0 < x < 1\}$ from the side of negative y, so the only possibility for the orbit from Q to intersect $\{y = 0, z = 1\}$ would be to come to the point $(1, 0, 1)$, but the latter is an equilibrium state. Now, as the intersection point of W_+^u with $\{y = 0\}$ satisfies $z < 1$ and $x > 0$, we immediately obtain the sought transversality of the intersection since

$$\left.\frac{dy}{dt}\right|_{y=0, z<1, x>0} = (a+1)x(1-z) \neq 0.$$

To finish the lemma, it remains to consider the case where the separatrix does not leave Q in a finite time. As $\dot{x}(=y) < 0$ and $\dot{z}(=x^2 - z) < 0$ everywhere in Q, the x- and z- coordinates will stay bounded in this case for all times. Once the boundedness of x is established, the boundedness of y follows immediately, since $x = \int y dt$ and y keeps constant sign.

Thus, the orbit W_+^u stays in the bounded subset of Q, therefore its ω-limit set is bounded and lies in the closure of Q. As x decays monotonically along W_+^u, the coordinate x stays constant on this ω-limit set, i.e. $\dot{x} = y = 0$ everywhere on it. This means that W_+^u tends to a compact invariant subset of $\{y = 0, x \geq 0\}$, and the only two such subsets are the equilibria O and O_1. To complete the lemma, we must prove that W_+^u cannot tend to O_1 without leaving the region Q.

Note, first, that the separatrix cannot spiral towards O_1 along the two-dimensional eigen-plane that corresponds to the pair of complex eigenvalues of O_1—in this case W_+^u would inevitably leave Q at some finite time moment. Thus, W_+^u could approach O_1 only along the eigen-vector $\left(1, p_*, \frac{2}{1+p_*}\right)$ corresponding to the real negative root p_* (this is similar to the discussion above where we showed that W_+^u cannot tend to O_1 without leaving the region R). We have that near O_1 the curve W_+^u would then be given by

$$x = 1 + s + o(s), \qquad y = p_* s + o(s),$$
$$z = 1 + \frac{2}{1+p_*} s + o(s), \quad s > 0,$$

where the small parameter s is taken positive, in order to ensure that $y < 0$, as it should be when W_+^u approaches O_1 from the region Q. Now, as $p_* < -1$, we find that $x^2 - z = \frac{2p_*}{1+p_*} s + o(s) > 0$ on W_+^u, which contradicts to W_+^u lying in Q. □

This lemma allows us to establish the existence of the sought homoclinic loop at some $a_0 > 0$ in the following way. Let A_1 be the set of parameters $a \geq 0$ for which the separatrix W_+^u leaves the region Q by crossing $x = 0$, and A_2 be the set of parameters $a \geq 0$ for which the separatrix W_+^u leaves Q by crossing $y = 0$, Fig. 2. Because of the transversality of the intersection of W_+^u with either plane, the sets A_1 and A_2 are open (as subsets of $[0, +\infty)$). Thus, if we prove that both these sets are nonempty, i.e. there exist $a_1 \geq 0$ such that W_+^u leaves Q by crossing $x = 0$ and $a_2 \geq 0$ such that W_+^u leaves Q by crossing $y = 0$, we will immediately obtain that there exists $a_0 \in (a_1, a_2)$ which belongs to neither of the two sets. By the lemma, at $a = a_0$, the separatrix W_+^u forms the homoclinic loop.

2. Behavior at small and large *a*

As we see, in order to prove our theorem, it is enough to show that at large positive a the separatrix W_+^u stays in the region $x > 0$ for all times, while at $a = 0$ it leaves this region at finite t (it stays, first, at $x > 0$, $y > 0$, then crosses to $y < 0$, and then intersects the plane $x = 0$ without returning to $y > 0$).

We consider the case of large a first. At large $a > 0$ system (1) is slow-fast, with fast y-variable and slow (x, z). Therefore, given any, arbitrarily large $L > 0$, for all a sufficiently large the system in the ball $U_L : \{\|x, y, z\| \leq L\}$ has an attractive invariant manifold $\mathcal{M}$ which is $O(a^{-1})$-close in this ball to the slow manifold $y = x(1-z)$ obtained by formally taking the limit $a = +\infty$ in the y-equation of (1). Moreover, as t grows, every orbit from U_L tends to $\mathcal{M}$ or leaves U_L, which implies that W_+^u lies in $\mathcal{M}$ until W_+^u stays in U_L.

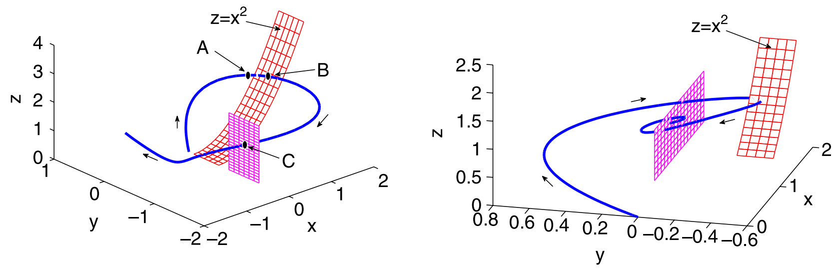

Fig. 2. The separatrix W_+^u for small and large a. For $a = 1$ (left), W_+^u crosses first $y = 0$ at $A(1.79, 0.00, 1.73)$, enters $Q : x > 0, y < 0, z > x^2$ through $B(1.46, -1.30, 2.14)$ and leaves Q by $x = 0$ at $C(0.00, -1.12, 1.13)$, while for $a = 10$ (right), after crossing $y = 0$ and entering Q, W_+^u leaves Q by $y = 0$ at $(0.97, 0.00, 0.99)$.

By plugging in (1) the equation $y = x(1 - z) + O(a^{-1})$ for the invariant manifold $\mathcal{M}$, we find the system on $\mathcal{M}$:

$$\dot{x} = x(1 - z) + O(a^{-1}), \qquad \dot{z} = -z + x^2. \tag{5}$$

The origin $O(0, 0)$ is a saddle equilibrium of this system. The curve W_+^u is an unstable separatrix of O. It is easy to check that the bounded region $\left\{0 \le x \le \sqrt{3}, \frac{1}{3}x^2 \le z \le 3\right\}$ is forward-invariant for the limit system ($a = +\infty$)

$$\dot{x} = x(1 - z), \qquad \dot{z} = -z + x^2.$$

Thus, the separatrix of O stays in this region forever. Since the divergence of the vector field is strictly negative here, there can be no limit cycles or separatrix cycles, hence the ω-limit set of the separatrix is the exponentially stable equilibrium $O_1(1, 1)$. This picture is structurally stable, so for all a large enough the unstable separatrix of the saddle O of system (5) also tends to O_1, stays in a uniformly bounded region of the (x, z)-plane and does not intersect $x = 0$. As this separatrix is W_+^u, we thus have shown that W_+^u stays in the region $x > 0$ for all times if a is sufficiently large. In terms of Lemma 1.1 this means that the separatrix leaves the region $Q = \{x > 0, y < 0, z > x^2\}$ by intersecting the plane $y = 0$.

Let us now consider the case $a = 0$. By Lemma 1.1, the separatrix W_+^u leaves the region R by intersecting the plane S at a point $(x^*, 0, z^*)$ with $x^* > \sqrt{z^*} > 1$. Denote

$$C^* := z^*/(x^*)^2 < 1.$$

Let us show that before crossing $y = 0$ the separatrix satisfies

$$z < C^* x^2, \qquad y > 0. \tag{6}$$

In order to do this, let us denote $y = kx$ and $z = Cx^2$. System (1) at $a = 0$ will take the form

$$\begin{cases} \dot{x} = kx \\ \dot{k} = 1 - k^2 - Cx^2 \\ \dot{C} = 1 - C(1 + 2k). \end{cases} \tag{7}$$

The separatrix here is the solution which tends to $x = +0, k = 1, C = 1/3$ as $t \to -\infty$. By (3), the separatrix satisfies

$$C = \frac{1}{3} + \frac{1}{90}x^2 + o(x^2), \qquad k = 1 - \frac{1}{12}x^2 + o(x^2),$$

hence at small x it lies in the region

$$\dot{C} > 0, \qquad \dot{k} < 0. \tag{8}$$

Let us show that the orbit cannot leave this region at positive k. First, note that $\dot{C} > 0$ at $C = 0$ and $\dot{C} < 0$ at $C = 1, k > 0$, so we have $C \in (0, 1)$ all the time. Next, we note that at the moment the orbit leaves this region there should be either

$$\dot{k} = 0, \qquad \dot{C} \ge 0, \qquad \ddot{k} \ge 0, \tag{9}$$

or

$$\dot{C} = 0, \qquad \ddot{C} \le 0, \qquad \dot{k} < 0. \tag{10}$$

However, as it follows from (7),

$$\ddot{k} = (C - 1)x^2 \quad \text{if } \dot{k} = 0,$$

and

$$\ddot{C} = -2C\dot{k} \quad \text{if } \dot{C} = 0,$$

so neither (9) nor (10) can happen at $1 > C > 0$ and $k > 0$. Thus, we have shown that (8) is fulfilled all the time the separatrix stays at $k > 0$, i.e. at $y > 0$. In particular, $\dot{C} > 0$, which means that $C \equiv z/x^2$ is an increasing function of time on the separatrix at $y > 0$, which proves (6).

After the separatrix W_+^u crosses to $\{y < 0, z < x^2\}$, the variable x is decreasing and z is increasing (until W_+^u intersects the surface $S' = \{x > 0, y < 0, z = x^2\}$, see Lemma 1.1). This immediately gives that W_+^u satisfies

$$z > C^* x^2, \qquad y < 0 \tag{11}$$

all the time before the separatrix crosses the surface S'.

Now, let us define a function

$$V = \frac{y^2}{2} - \frac{x^2}{2} + C^* \frac{x^4}{4}.$$

By virtue of (1) at $a = 0$,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = xy(C^* x^2 - z),$$

hence, by (11), (6), we find that $\dot{V} \ge 0$ on the separatrix all the time before the intersection with S'. Thus, V is a non-decreasing function along the separatrix, i.e. at the point of the separatrix intersection with S' we have $V \ge V(0, 0, 0) = 0$. In other words, the point $W_+^u \cap S'$ lies in the region $y \le -\sqrt{x^2 - C^* x^4/2}$. As $C^* < 1$, it follows that $\Phi(x, y) := \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ is strictly positive at this point.

After the intersection with S', the orbit enters the region Q : $\{y < 0, x > 0, z > x^2\}$. In this region, at $a = 0$, the derivative $\dot{\Phi} = xy(x^2 - z)$ is positive, so $\frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ is positive and bounded away from zero all the time the orbit stays in Q. Thus, the separatrix cannot stay in Q and tend to O (as Φ is zero at O), nor can it leave Q by intersecting the surface $\{y = 0, x \in (0, 1)\}$ (as $\Phi(x, 0) = x^4/4 - x^2/2 < 0$ there). It follows, by Lemma 1.1, that W_+^u enters the region $x < 0$ at a finite time moment.

We see that the separatrix behavior is different at small and large a. As we explained in the end of the previous section, this gives us the sought existence of a homoclinic loop at some $a_0 > 0$, see Fig. 3.

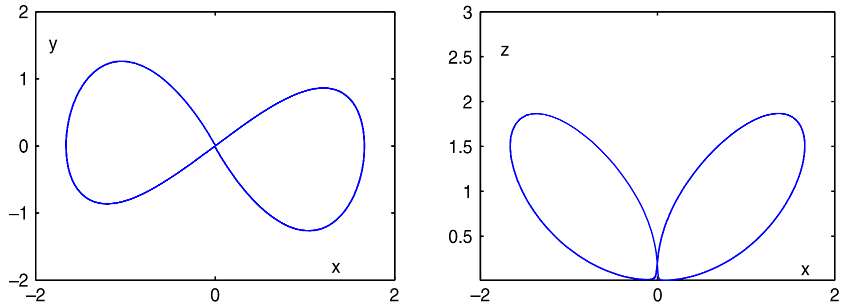

Fig. 3. Two homoclinic orbits (homoclinic butterfly) at $a \cong 1.718$.

3. Separatrix value

As we mentioned, one could obtain an analytic proof of the existence of a Lorenz attractor in the Shimizu–Morioka system (1) for an open set of parameter values adjoining the point (α_0, λ_0) for which the system has the homoclinic butterfly to the saddle O with the zero saddle value, should we prove that the separatrix value satisfies $0 < A < 2$ at $(\alpha, \lambda) = (\alpha_0, \lambda_0)$. The separatrix value A can be defined in our case as follows. Let $(x_0(t), y_0(t), z_0(t))_{t=(-\infty,+\infty)}$ be the equation of the homoclinic loop (the two loops in the butterfly are symmetric, so we can take any of them; for instance, let us take the loop which corresponds to positive $x_0(t)$). The linearized system obtained by the differentiation of (1) at the points of the loop is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\\z\end{pmatrix} = B(t)\begin{pmatrix}x\\y\\z\end{pmatrix} \tag{12}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 \\ (a_0+1)(1-z_0(t)) & -a_0 & -(a_0+1)x_0(t) \\ 2x_0(t) & 0 & -1 \end{pmatrix}. \tag{13}$$

Let ξ_1, ξ_2 be any two vectors and let $\eta = \xi_1 \times \xi_2$ be their vector product. If the evolution of ξ_1 and ξ_2 is defined by Eq. (12), then the evolution of η is governed by

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -(B^\top - \mathrm{tr}(B)I)\eta,$$

where I is the (3×3) identity matrix. Thus, we have the following equation for the evolution of infinitesimal two-dimensional areas near the homoclinic loop:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \begin{pmatrix} -(a_0+1) & -(a_0+1)(1-z_0(t)) & -2x_0(t) \\ -1 & -1 & 0 \\ 0 & (a_0+1)x_0(t) & -a_0 \end{pmatrix}\eta. \tag{14}$$

Since $x_0(t)$ and $z_0(t)$ exponentially tend to zero as $t \to \pm\infty$, the asymptotic behavior of the solutions of (14) is determined by the limit matrix

$$\begin{pmatrix} -(a_0+1) & -(a_0+1) & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -a_0 \end{pmatrix}.$$

Its eigenvalues are $0, -a_0 < 0$ and $-(a_0+2) < 0$, so every solution of (14) tends, as $t \to +\infty$, to a constant times the eigenvector that correspond to the zero eigenvalue (this is the vector $\eta^* = (1, 1, 0)^\top$). Note that only one solution tends to η^* as $t \to -\infty$. We take this solution $\eta(t)$, denote

$$\lim_{t\to+\infty} \eta(t) := A\eta^*,$$

then A is the sought separatrix value. One can see that

$$|A| = \sup \lim_{t\to+\infty} \frac{\|\eta(t)\|}{\|\eta(-t)\|}$$

where the supremum is taken over all the solutions of (14). So, the absolute value of A gives the coefficient of expansion of infinitesimal areas by the system near the homoclinic loop (the sign of A describes the orientability of the loop; one may check that this definition of the separatrix values coincides with that of [8,10,11]).

It was numerically checked [4–6] that $0 < A < 1$ in our case, i.e. conditions of [9,10] are fulfilled. We do not have an analytic proof for this fact, which is the remaining obstacle for an analytic proof of the Lorenz attractor in this system.

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