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ON THE CLASSIFICATION OF SELFLOCALIZED STATES OF ELECTROMAGNETIC FIELD WITHIN NONLINEAR MEDIUM

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The problem of identifying and classifying soliton-type states for nonintegrable equations is one of essential problems in nonlinear physics. The difficulty of its solution is explained by the fact that from the mathematical point of view this problem is reduced to the analysis of nonintegrable conservative systems (in particular, Hamiltonian ones) with two or more degrees of freedom : the selflocalized states are related to the homoclinic loops of saddle equilibrium states of dynamical systems.

The electrodynamics' equations for nonlinear nondissipative medium in the approximation of a high-frequency potential (i.e. we take into account the self-action but neglect the excitation by multiple frequences) for the flat geometry of wave-guide states of the field lead to the following dynamical system with two degrees of freedom [1] :

$$\frac{dP}{dx}=(k_z^2-k^2\mathcal{E}(\omega,E^2))E_y\ ,\quad \frac{dE_y}{dx}=P$$

$$k_z\frac{dE_z}{dx}=(k_z^2-k^2\mathcal{E}(\omega,E^2))E_x\ ,\quad k_z\frac{dE_x}{dx}=\Omega(E,P)/(\mathcal{E}+2E_x^2\frac{d\mathcal{E}}{d(E^2)}) \qquad (1)$$

Here $\Omega \equiv (k_z^2\mathcal{E}-2(k_z^2-k^2\mathcal{E})E_x^2\frac{d\mathcal{E}}{d(E^2)})E_z-2k_zE_xE_yP\frac{d\mathcal{E}}{d(E^2)}$,

$\mathcal{E}(\omega,E^2)$ is the dielectric permittivity; k_z is a wave vector; $k=\omega/c$; ω is the frequency of the electromagnetic wave of the type $E_j(x)\exp(ik_zz)$, $j=x,y,z$. The system (1) has a first integral

$H=P^2-k_z^2(E_x^2+E_y^2)+k^2\int_0^{E^2}\mathcal{E}(\omega,s)ds+(k_z^2-k^2\mathcal{E})^2E_x^2/k_z^2$. For $k_z^2-k^2\mathcal{E}(\omega,0)>0$, homoclinic loops of the singular point $O(E_x=E_y=E_z=P=0)$ in the phase space R^4 on the level of the first integral $H=0$ correspond to selflocalized waveguide states of the field, for which $\lim_{x\to\pm\infty}E=0$, $\lim_{x\to\pm\infty}P=0$. Note that O is a saddle with multiple characteristic exponents $(\lambda, \lambda, -\lambda, -\lambda)$ where $\lambda = (k_z^2 - k^2 \mathcal{E}(\omega, 0))^{1/2}$. For the TE- and TM-type fields described by electrical vectors $(0, E_y, 0)$ and $(E_x, 0, E_z)$, the image of known [2,3] selflocalized states is given by the four homoclinic loops S_1, S_2, S_3, S_4 lying in invariant planes : S_1 and S_2 in (E_y, P), and S_3 and S_4 in (E_x, E_z).

A numerical study has been performed for $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_2 E^2$, where $\mathcal{E}_0(\omega) > 0$, $\mathcal{E}_2(\omega) > 0$. Simple scaling transformations of the electrical vector and the independent variable show that the system contains the only structural parameter $\gamma^2 = k_z^2/(k^2 \mathcal{E}_0(\omega))$. The computations have been done for $\gamma^2 = 2$. Below the results of qualitative and numerical analysis are outlined.

1) Numerically detected have been a saddle periodic motion L and a heteroclinic trajectory g_1 connecting L with 0. The presence of discrete symmetries in the system (1) show that the system contains a second heteroclinic curve g_2 connecting 0 with L. In addition, a curve g_L homoclinic to L has been found. We have checked that W_L^u with W_0^s, W_0^u with W_L^s and W_L^u with W_L^s intersect along, respectively g_1, g_2 and g_L in a stable way. Here we assert the following [4,5]: within each neighbourhood of the homoclinic contour $L \cup g_1 \cup g_2 \cup g_L \cup 0$ there exists a countable set of homoclinic loops to 0 being in one-to-one correspondence with the set of all finite words composed of symbols L and g_L (in accordance with the principles of symbolic dynamics, the symbol L stands for one rotation near the trajectory L, and the symbol g_L stands for a passage of the loop in a neighbourhood of the trajectory g_L).

2) Numerically found has been a series of homoclinic loops performing a lot of rotations in the vicinity of TE- and TM-loops; namely, the following loops have been found : $S_3 S_2 g_{21} S_2$, $S_3 S_2 g_{21} S_2 g_{22} S_2$, $S_1 g_{11} S_1 g_{12} S_1$, $S_3 S_2 g_{21} S_2 S_3$, $S_1 S_3 S_1 g_{11} S_1 S_3$, $S_1 g_{12} S_1 S_4 S_2 g_{22} S_2$. Here the symbols S_1, S_2 code the parts of the trajectory that are close to the TE-loops, and the symbols S_3, S_4 code the parts close to the TM-loops. The symbols g_{ij} denote the trajectory parts that, in projection to the plane (E_x, E_z), are close to S_4 (symbols g_{i1}) or to S_3 (symbols g_{i2}) but

the coordinate E_y is somewhat biased from zero: for g_{2j} — to the positive side, and for g_{1j} — to the negative side.

PROPOSITION 5. Let a dynamical system X in R^4 be symmetric with respect to transformations
$(u_1,u_2,v_1,v_2) \leftrightarrow (-u_1,u_2,-v_1,v_2)$, $(u_1,u_2,v_1,v_2) \leftrightarrow (u_1,-u_2,v_1,-v_2)$ and have a smooth first integral $H=u_1v_1-u_2v_2+\dots$ (Here (u_1,u_2,v_1,v_2) are coordinates in R^4). Suppose a saddle equilibrium state O with multiple characteristic exponents lies in the level H=0. Let O have, in invariant planes $(u_1=v_1=0)$ and $(u_2=v_2=0)$, two homoclinic eights (S_1,S_2) and, resp., (S_3,S_4), along which W_0^u and W_0^s intersect transversely. Suppose also that the coefficients β_1 and β_2 of the terms $u_2^2v_1$ and $u_1^2v_2$ in the normal form of the system in the saddle are different from zero. Then the bunch $B=S_1\cup S_2\cup S_3\cup S_4$ possesses two-dimensional stable and unstable manifolds W_B^s and W_B^u.

As $x \to +\infty$ ($x \to -\infty$) all trajectories from W_B^s (resp. W_B^u) tend to B. For $\gamma^2=2$ in system (1), the manifolds $W_B^u\setminus B$ and $W_B^s\setminus B$ consist of two connected components W_1^u, W_2^u and W_1^s, W_2^s; the motion near $W_1^{u(s)}$ and $W_2^{u(s)}$ is a repetitive passage along the loop groups $S_1S_3S_1S_4$ and $S_2S_3S_2S_4$, respectively.

As $x \to +\infty$ all the trajecories from a small neighbourhood V of B that do not lie in W_B^s leave V along W_B^u. The neighbourhood V evidently has no other homoclinic loops but S_i. As system (1) has homoclinic loops close, in their initial and final parts, to the loops of B this makes it possible to suppose the existence of homoclinic to B trajectories along which the manifolds W_B^u and W_B^s intersect in a stable way: $\{g_{i1},g_{i2}\} \subseteq W_i^u\cap W_i^s$.

Note that a homoclinic trajectory to a bunch of homoclinic loops is a new object for the theory of dynamical systems.

PROPOSITION 5. In any neighbourhood of the contour $B \bigcup_{i,j} g_{ij} \cup 0$ there exists a countable set of homoclinic loops being in one-to-one correspondence to the set of the words of the type $U_{i_1}^{p_1} g_{i_1 j_1} S_{i_1}^{q_1} U_{i_2}^{p_2} g_{i_2 j_2} \dots g_{i_n j_n} S_{i_n}^{q_n}$ .Here the symbol g_{ij} denotes a passage of the loop near the trajectory g_{ij} ; $j_s \in \{1,2\}$; S_i^q and U_i^p are groups of symbols S_1, S_2, S_3, S_4 denoting respectively q and p rotations near W_i^s and W_i^u ,

S_i^q start with S_1 or S_2, and U_i^p terminate with S_3 or S_4 ; p_s and q_s are sufficiently large integers,all even except, perhaps,for p_1 and q_n; q_s can be taken arbitrary, while p_{s+1} is uniquely determined by q_s ; for each given i_s the freedom in the choice of the i_{s+1} is restricted by the only condition that the group $S_{i_s}^{q_s} U_{i_{s+1}}^{p_{s+1}}$ cannot contain the following quadruplets of symbols : $S_1S_3S_2S_3$, $S_1S_4S_2S_4$, $S_2S_3S_1S_3$, and $S_2S_4S_1S_4$.

Similar problems of classifying the soliton states and identifying the basic series of solitons arise for the system of two coupled Shrödinger equations.Here,in particular, bifurcations can occur that produce vector solitons from the polarized solitons.

REFERENCES

1.Eleonsky V.M.,Oganesjantz L.G. and Silin V.P.,Radiophifika, 17,1812-1816,1974

2.Chiao R.,Garmire E. and Townes C.,Phys.Rev.Lett.,13,479,1964

3.Litvak A.G.,Radiophizika,9,675-680,1966

4.Turaev D.V.,Shil'nikov L.P.,DAN SSSR,304,811-815,1989

5.Eleonsky V.M.,Kulagin N.E.,Turaev D.V. and Shil'nikov L.P., Methods of qualitative theory and theory of bifurcations : USSR,Gorky,1988,49-57