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Localized solutions in lattice systems and their bifurcations caused by spatial interactions

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Abstract. We demonstrate the scenario showing how the stable spatially localized solutions with nontrivial (periodic, quasiperiodic or chaotic) dynamics may appear in lattice dynamical systems. It is important to mention that bifurcations to such regimes occur when the strength of spatial interactions exceeds some threshold. In fact we first prove the persistence of stationary localized structures in a range of weak interactions and then from this result of the 'anti-integrable limit' type we make the next step to show the existence of bifurcations of these states to the stable spatially localized states with a nontrivial time dynamics. We also show how our approach can be applied to study bifurcations to nonstationary states with spatial structure of general type.

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1. Introduction

Lattice dynamical systems (LDS), that have recently been introduced, have allowed slightly more insight into the dynamics of extended systems. However, thanks to LDS, it became even more clear how very little is known about the spacetime dynamics.

The most essential feature of the dynamics of extended systems, in contrast to non-extended (pointwise) systems, is the presence of spatial interactions between the local (pointwise) subsystems. However, basically, all the mathematical results in the theory of LDS are concerned with situations where the presence of spatial interactions does not, in fact, change the character of dynamics of the collection of *noninteracting* local systems.

However, many papers have been devoted to the numerical studies of coupled map lattices (see, e.g. [10–14, 16, 19, 20, 22]). These studies were concerned with the broad region of spatial interactions and revealed the very rich and beautiful spatio-temporal dynamics of extended lattice systems. However, the rigorous mathematical studies of LDS are, naturally, far behind the numerics and deal eventually with the region of weak spatial interactions.

Indeed, the two main activities that have emerged in the mathematical studies of LDS deal exactly with such situations. (Needless to say, the range of spatial interactions in these studies is usually very narrow and, what is very important, it always contains zero, i.e. the case where there is no space time dynamics at all because there are no spatial interactions.)

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One of these activities is the construction of various types of solutions and the study of their stability in the range of weak spatial interactions. The procedure is as follows. (1) Pick some solutions of the local system. (2) 'Glue' them together by considering the collection of all these solutions at all sites of the lattice (certainly, such collection of solutions for local systems gives a formal solution for LDS without spatial interactions). (3) Find an appropriate Banach space and prove there an implicit function theorem to ensure that the corresponding solution persists (exists and is stable) for weak (but nonzero!) spatial interactions.

This program (which is sometimes called the concept of anti-integrability [4, 17]) proved to be very efficient and allowed to construct various nontrivial solutions for many interesting and important lattice dynamical systems (see, e.g. [1, 3–5, 17–19, 23–27]).

Another active area of mathematical research, started in [9], is an attempt to understand the phenomenon of the spacetime chaos. Again, it starts in the situation where there are no spatial interactions at all. However, it deals with local systems which have strongly chaotic dynamics. Therefore, we consider an ensemble of trajectories of local systems which are chaotic and therefore cannot be explicitly written or described (unlike to the anti-integrable limit approach where solutions under consideration are fairly simple). The problem to study in this approach is proving that to each sufficiently small strength of spatial interactions there corresponds a unique natural invariant measure in an (infinite-dimensional) phase space of LDS which is spacetime mixing [9]. Again, the existence and uniqueness of such measure in an absence of space interactions follows trivially from the known facts in the theory of pointwise (finite-dimensional) dynamical systems. Instead of simple solutions of local systems one must now 'glue' together invariant measures of these chaotic local systems, i.e. just to take their direct product. The problem is to show that under weak spatial interactions this measure, which generates identically zero spatial correlations, will be transformed into a measure which generates spatial correlations that decay with a distance along a lattice and also preserves a decay of time correlations. This program was realized for various LDS (see, e.g. [6, 7, 9, 20]).

However, the problems that are intrinsically related to the main features of extended dynamical systems must deal with the situations where the spatial interactions essentially influence the spacetime dynamics.

One of these problems on chaos-order transition has been formulated in [9]. It investigates the mechanism of the appearance of coherent structures from chaos, when the strength of spatial interactions increases. There is to date very little progress in this area (see, e.g. [6, 8]). Another problem, first formulated in [8], addresses the opposite situation when a complex behaviour of LDS is generated by spatial interactions while the dynamics of local systems is fairly simple.

It is important to mention that neither of these problems are of a kind that can be handled by a perturbation theory but go into the ranges of spatial interactions that are bounded away of zero. Thus, the first problem is concerned with the appearance of coherent structures from the spacetime chaos while the second one with the mechanisms that generate the spacetime chaos via spatial interactions. In particular, the second problem goes beyond the range of weak spatial interactions where the concept of anti-integrability might work.

Thus, the second problem deals with various (spacetime) bifurcations that increase the complexity of spacetime dynamics and appear when the strength of spatial interactions increases. One of such bifurcations, called the peak-crossing, has been studied in [10, 11].

In this paper we study the rather general class of bifurcations which lead to spatially localized solutions with nontrivial time dynamics. Besides breathers these solutions include those with a quasiperiodic or chaotic dynamics.

This paper is organized as follows. In section 2 we describe the class of LDS under study: here the pointwise subsystem is a map with two fixed points, one is exponentially stable and one is on the boundary of stability. This is a degenerate situation, so we assume that the local map depends on a number of parameters (the structural parameters) which unfold the degeneracy. We give a centre manifold theorem for the LDS composed of such maps with a weak spatial interaction. In section 3 we exploit this theorem to show that for small values of interaction parameter, single-pulse spatially localized stationary solutions undergo essentially the same bifurcations as in the uncoupled system. In fact, our results are two-fold. On one hand, it is a standard consideration of the case of weak spatial interaction. On the other hand, we show that the interplay between the structural parameters and the parameter of spatial interaction for the reduced map on the centre manifold leads to the following phenomenon: for the frozen values of the structural parameters of the local subsystem, slightly below critical, there is a threshold. The originally stable localized stationary state bifurcates to a more complex localized solutions when the strength of spatial interactions exceeds the threshold. Thus, the degeneracies of the local subsystems of the LDS can be responsible for the nontrivial temporal behaviour of the LDS with nonzero spatial interaction.

2. Centre manifold for stationary localized states

Consider a one-dimensional lattice dynamical system

$$\bar{x}_i = f(x_i; \gamma, \varepsilon) + \varepsilon \mathcal{F}(x_{i-s}, \dots, x_i, \dots, x_{i+s}; \gamma, \varepsilon) \tag{1}$$

where $i \in \mathbb{Z}$, each x_i belongs to R^k, the function f is C^r ($r \geqslant 1$) with respect to all its arguments, γ is a vector of real parameters ('the structural parameters'), ε is the 'interaction' parameter which is supposed to be small, and $\mathcal{F}$ is a C^r function. Since the 'interaction term' $\mathcal{F}$ depends on a finite number $(2s + 1)$ neighbouring values of x, it is a finite-range LDS.

We suppose that at $\gamma = 0, \varepsilon = 0$ *the local subsystem*

$$\bar{x} = f(x; \gamma, \varepsilon) \tag{2}$$

has two fixed points $x = 0$ and $x = x^*$ (i.e. $f(0) = 0$, $f(x^*) = x^*$) such that $x = 0$ is exponentially stable (i.e. $\|f'(0)\| < 1$ hence with no loss of generality we may assume that $x = 0$ is the fixed point of the local map at all small γ) and $x = x^*$ is asymptotically stable, though having all k multipliers on the unit circle.

We allowed for the dependence of the local map on ε for greater generality. In particular, we may then assume that

$$\mathcal{F}_{x=0} = 0. \tag{3}$$

Take any finite set $I = \{i_1^0, \dots, i_m^0\}$ of integers and consider the sequence

$$X^0 = \{x_i^0\} : \begin{cases} x_i^0 = x^* & \text{at } i \in I \\ x_i^0 = 0 & \text{at } i \notin I. \end{cases}$$

This is a stationary, spatially localized solution of (1) at $\varepsilon = 0, \gamma = 0$. The following variant of the centre manifold theorem describes the time evolution of solutions starting in a small neighbourhood of X^0 (in the uniform norm $\|X\| = \max \|x_i\|$) for small ε and γ. Denote $x_I = \{x_{i_1^0}, \dots, x_{i_m^0}\}$. For any i, let $d(i; I)$ be the minimal integer greater than or equal to $|i - i_n^0|/s$ where i_n^0 denotes here the element of I nearest to the given i (s is the radius of interaction).

Theorem 1. *There exists a small $\delta > 0$ such that at all small ε and γ the iterations of any initial sequence X, δ-close to X^0 in the uniform norm, tend to an invariant centre manifold W^c of the form*

$$x_i = \Psi_i(x_I; \varepsilon, \gamma) \tag{4}$$

where

$$\Psi_i = (K\varepsilon)^{d(i;I)} \Phi_i(x_I; \varepsilon, \gamma) \tag{5}$$

for some constant K and some C^r-functions Φ uniformly bounded along with all their derivatives.

Proof. Let Y denote the vector composed of differences $\{x_i - x^*\}_{i \in I}$ and the parameters ε and γ; we denote by Z the rest of the x-variables. The map (1) is a C^r-smooth map of the space (Y, Z) with the uniform norm. The origin $(Y, Z) = (0, 0)$ is a fixed point of this map. Schematically, the map (1) can be written as

$$\begin{cases} \bar{Y} = AY + G_1(Y, Z) \\ \bar{Z} = BZ + CY + G_2(Y, Z) \end{cases} \tag{6}$$

where $G_{1,2}$ denote small nonlinear terms, A is a finite-dimensional matrix whose eigenvalues, according to our assumptions on the fixed points of the local map, are equal to unity in absolute value, B is a linear operator such that $\|B\| < 1$ and C is a bounded linear operator which is just the derivative of the right-hand sides of (1) with respect to the parameters ε and γ at $X = X^0$, $\varepsilon = 0$, $\gamma = 0$. Since Y is a finite-dimensional vector, the centre manifold theorem applies immediately to such map.

In terms of the original notation it reads as follows. For some small δ, for all small ε and γ, in the δ-neighbourhood of X^0 there exists an invariant manifold W^c of the type (4) such that for any initial sequence X such that $\|X - X^0\| \leqslant \delta$, if the forward iterations of X by the map (1) stays all in the δ-neighbourhood, then they tend to W^c. Note that by our assumption the fixed point X^0 is asymptotically stable for the map (1) at $\varepsilon = 0$, $\gamma = 0$. Therefore, it follows that for any δ any forward orbit starting in the δ-neighbourhood of X^0 stays there forever, provided ε and γ are sufficiently small. Thus, any forward orbit starting close to X^0 tends to the centre manifold.

It remains to prove estimates (5). As is well known, the centre manifold is found as the limit of the iterations of the surface $Z = 0$ by the map

$$\begin{cases} \bar{Y} = AY + \chi(\|Y/\delta\|)G_1(Y, Z) \\ \bar{Z} = BZ + CY + \chi(\|Y/\delta\|)G_2(Y, Z) \end{cases} \tag{7}$$

where $\chi(u)$ is a smooth function equal identically to 1 at $|u| \leqslant 1$ and to 0 at $|u| \geqslant 2$. This map is defined for all Y, not necessary small now, and it coincides with the original map at $\|Y\| \leqslant \delta$. The initial surface $Z = 0$ satisfies (5). Thus, to complete the proof it remains to check that if some surfaces have the form (5) where the C^r-norm of the functions Φ_i is uniformly bounded by some appropriate constant, its image by (7) has the same form with the norm of the new functions Φ_i bounded by the same constant.

One can see (since $\frac{\partial \bar{Y}}{\partial Y}$ in (7) has bounded inverse uniformly for all Y and Z from the domain of definition) that it is equivalent to verifying that after rescaling $Z \to (K\varepsilon)^D Z_{\text{new}}$: $\{x_i \to (K\varepsilon)^{d(i;I)} x_{i,\text{new}}\}_{i \notin I}$ the map (7) remains smooth and contracting in Z_{new} in the uniform norm. But the latter obviously follows from the structure of the map (1) provided (3) holds and K is chosen large enough. □

The theorem above shows that the loss of stability of the spatially localized stationary states leads to creation of the new localized solutions. The temporal behaviour of such solutions is finite dimensional, respective to the dimension of the centre manifold. Even if it is one dimensional, the new solutions may be temporally periodic, and for the greater dimensions they may be quasiperiodic or even temporally chaotic.

Note that, obviously, this result is not quite so sensitive to the choice of the norm in the space of infinite sequences $\{x_i\}$. Indeed, the theorem shows that any uniformly small perturbation relaxes to a (exponentially) self-localized state. Thus, all the bifurcations will be the same, despite whether the problem studied in the uniform norm, or, say, in l_2.

In the next section we give the simplest example of the application of theorem 1 to the study of the loss of stability of single-pulse localized solutions for the case of linear diffusive interaction.

3. Loss of stability of single-pulse stationary states

Let the interactions in the map (1) be linear and of radius 1. Namely, we consider a particular case

$$\bar{x}_i = f(x_i; \gamma) + \varepsilon \mathcal{D}(x_{i-1} - 2x_i + x_{i+1}) \tag{8}$$

where $\mathcal{D}$ is a $(k \times k)$ diagonal matrix: $\mathcal{D} = \mathrm{diag}(d_1, \ldots, d_k)$, and the local map f, as assumed above, has an exponentially stable fixed point $x = 0$ and, at $\gamma = 0$, a degenerate fixed point $x = x^*$ which is asymptotically stable but has all the multipliers on the unit circle.

We are interested in the behaviour of the single-pulsed solutions—i.e. those which satisfy $\|x_0 - x^*\| \leqslant \delta$; $\|x_i\|_{i \neq 0} \leqslant \delta$ for some small δ. By theorem 1 the iterations of any such initial state tend to a localized solution for which

$$\|x_i\| \leqslant (K\varepsilon)^{|i|}$$

and the evolution of the coordinate x_0 is given by a k-dimensional map (the restriction of (8) onto the centre manifold)

$$\bar{x} = f(x; \gamma) - 2\varepsilon \mathcal{D} x + \mathrm{O}(\varepsilon^2). \tag{9}$$

If the fixed point $x = x^*$ of the local map does not have a multiplier equal to 1 at $\gamma = 0$, then at all small γ and ε the map (9) has a close to x^* fixed point

$$x^*_{\varepsilon,\gamma} = x^*_\gamma - 2\varepsilon(1 - f'(x^*_\gamma))^{-1} \mathcal{D} x^*_\gamma + \mathrm{O}(\varepsilon^2) \tag{10}$$

where x^*_γ is the fixed point of the local map (at $\varepsilon = 0$).

The linearization matrix of the map (9) at the fixed point is

$$f'(x^*_\gamma) - 2\varepsilon(f''(x^*_\gamma)(1 - f'(x^*_\gamma))^{-1} \mathcal{D} x^*_\gamma - \mathcal{D}). \tag{11}$$

Given γ, the critical value of ε can be found (up to the terms of order ε^2) from the characteristic equation

$$\det[\lambda I - f'(x^*_\gamma) - 2\varepsilon(f''(x^*_\gamma)(1 - f'(x^*_\gamma))^{-1} \mathcal{D} x^*_\gamma + \mathcal{D})] = 0 \tag{12}$$

subject to the requirement that the given number of the roots λ lie on the unit circle. By assumption, all the eigenvalues of $f'(x^*)$ lie on the unit circle at $\gamma = 0$. Therefore, in a general position, for any fixed small γ, at appropriately chosen diffusion ratios $d_1, \ldots, d_k$ one can find from (12) the threshold value of ε which corresponds to any given number of multipliers on the unit circle.

Exceeding the threshold leads to a bifurcation to a nontrivial temporal dynamics. If only one multiplier (-1) crosses the unit circle, then the localized stationary state bifurcates to a solution of temporal period 2. When two multipliers cross the unit circle, an invariant torus is born, i.e. the solution may become quasiperiodic. The passage of three multipliers across the unit circle may give rise to a chaotic behaviour (see [2, 28]) of the localized solution.

As an example of calculations by formula (12), let us consider the LDS (8) with the local map

$$f(x) = x + x(a(1-x) - b(1-x)^2)$$

where $x \in R^1$. The fixed points under consideration are $x = 0$ and $x = x^* = 1$. Here $f'(0) = 1 + a - b$, $f'(1) = 1 - a$. Thus, $x = 0$ is stable at $b \in (a, a+2)$.

The fixed point $x = 1$ is stable at $a \in (0, 2)$, and at $a = 2$ the multiplier becomes equal to -1. We have $f''(1) = -2a - 2b$, $f'''(1) = -6b$. The first Lyapunov value at the bifurcational moment $a = 2$ is $-(f'''(1) + \frac{3}{2}f''(1)^2) = -6(b + (b+2)^2)$. It is negative, therefore the fixed point $x = 1$ is asymptotically stable at $a = 2$. Theorem 1 and formulae (9)–(12) are thus applied, which gives that if $a < 2$ but it is close to 2, then at $b \in (a, a+2)$ and at all $\varepsilon \in (0, \varepsilon_c)$ the LDS under consideration has a stable localized stationary state $X^*_{a,b,\varepsilon}$ such that for some K

$$|x_i^*| \leqslant (K\varepsilon)^{|i|} \qquad \text{at } i \neq 0$$

and

$$x_0^* = 1 - 2\varepsilon a^{-1} + \mathrm{O}(\varepsilon^2)$$

(we assume $\mathcal{D} = 1$ here).

The critical value ε_c corresponds to the loss of stability of this stationary state. By (12)

$$\varepsilon_c \approx \frac{(2-a)a}{2(a+2b)}.$$

When ε exceeds the threshold $\varepsilon = \varepsilon_c$ the solution bifurcates to a spatially localized solution of temporal period 2.

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References

[1] Afraimovich V S, Glebsky L Y and Nekorkin V I 1994 Stability of stationary states and topological spatial chaos in multidimensional lattice dynamical systems *Rand. Comput. Dynam.* **2** 287–303

[2] Arneodo A, Coullet P, Spiegel E and Tresser C 1985 Asymptotical chaos *Physica* D **14** 327–47

[3] Aubry S 1997 Breathers in nonlinear lattices: existence, linear stability and quantization *Physica* D **103** 201–50

[4] Aubry S and Abramovici G 1990 Chaotic trajectories in the standard map: the concept of anti-integrability *Physica* D **43** 199–219

[5] Bambusi D 1996 Exponential stability of breathers in Hamiltonian networks of weakly coupled oscillators *Nonlinearity* **9** 433–57

[6] Bricmont T and Kupiainen A 1996 High temperature expansions and dynamical systems *Commun. Math. Phys.* **178** 703–32

[7] Bricmont T and Kupiainen A 1997 Infinite dimensional SRB measures *Physica* D **103** 18–33

[8] Bunimovich L A 1995 Coupled map lattices: One step forward and two steps back *Physica* D **86** 248–55

[9] Bunimovich L A and Sinai Ya G 1988 Spacetime chaos in coupled map lattices *Nonlinearity* **1** 491–516

[10] Bunimovich L A and Venkatagiri S 1996 Onset of chaos in coupled map lattices via the peak-crossing bifurcation *Nonlinearity* **9** 1281–98

[11] Bunimovich L A and Venkatagiri S 1997 On one mechanism of transition to chaos in lattice dynamical system *Phys. Rep.* **290** 81–100

[12] Chaté H and Manneville P 1988 Spatiotemporal intermittency in coupled map lattices *Physica* D **32** 409–23

[13] Chaté H and Manneville P 1989 Role of defects in transition to turbulence via spatiotemporal intermittency *Physica* D **37** 33–41

[14] Chaté H and Manneville P 1989 Coupled map lattices as cellular automata *J. Stat. Phys.* **56** 357–70

[15] Chaté H and Manneville P 1992 Collective behaviour in spatially extended systems *Prog. Theor. Phys.* **87** 1–60

[16] Crutchfield J and Kaneko K 1987 Phenomenology of the spacetime chaos *Directions in Chaos* ed Hao-Bai Lin (Singapore: World Scientific) pp 272–353

[17] Defontaines A D, Pomeau Y and Rostand B 1990 Chain of coupled bistable oscillators: a model *Physica* D **46** 201–16

[18] Flach S 1995 Existence of localized exitations in nonlinear Hamiltonian lattices *Phys. Rev.* E **51** 1503–7

[19] Flach S and Willis C R 1992 Localized exitations in a discrete Klein–Gordon system *Phys. Lett.* A **181** 232–8

[20] Jiang M 1995 Equilibrium states for lattice models of hyperbolic type *Nonlinearity* **8** 631–59

[21] Kaneko K (ed) 1993 *Theory and Applications of Coupled Map Lattices* (New York: Wiley)

[22] Kaneko K (ed) 1993 Focus issue on coupled map lattices *Chaos* **2** 279–460

[23] Livi R, Spicci M and MacKay R S 1997 Breathers on a diatomic FPU chain *Nonlinearity* **10** 1421–34

[24] MacKay R S 1996 Dynamics of networks: features which persist from the uncoupled limit *Stochastic and Spatial Structures in Dynamical Systems* ed S T van Strien and S M Verduyn Lunel (Amsterdam: North-Holland) pp 81–104

[25] MacKay R S and Sepulchre T-A 1995 Multistability in networks of weakly coupled bistable units *Physica* D **82** 243–54

[26] Nekorkin V I and Makarov V A 1995 Spatial chaos in a chain of coupled bistable oscillators *Phys. Rev. Lett.* **74** 4819–22

[27] Sepulchre T-A and MacKay R S 1997 Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators *Nonlinearity* **10** 679–713

[28] Shilnikov A L, Shilnikov L P and Turaev D V 1993 Normal forms and Lorenz attractors *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **3** 1123–39