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Elliptic islands appearing in near-ergodic flows

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Abstract. It is proved that periodic and homoclinic trajectories which are tangent to the boundary of any scattering (ergodic) billiard produce elliptic islands in the 'nearby' Hamiltonian flows i.e. in a family of two-degrees-of-freedom smooth Hamiltonian flows which converge to the singular billiard flow smoothly where the billiard flow is smooth and continuously where it is continuous. Such Hamiltonians exist; indeed, sufficient conditions are supplied, and thus it is proved that a large class of smooth Hamiltonians converges to billiard flows in this manner. These results imply that ergodicity may be lost in the physical setting, where smooth Hamiltonians which are arbitrarily close to the ergodic billiards, arise.

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1. Introduction

The behaviour of a point particle travelling with a constant speed in a region, undergoing elastic collisions at the region's boundary, is known as the billiard problem. This system has been extensively studied both in its classical and quantized formulation. Numerous applications lead to the study of such a model problem. First, there exist direct mechanical realizations of this model. For example, the motion of N rigid d-dimensional spheres in a d-dimensional box may be reduced to a billiard problem, possibly in higher dimensions [31, 32, 9, 17]. See also [2, 8] for the inelastic case. Second, it serves as an idealized model for the motion of charged particles in a potential, a model which enables the examination of the relation between classical and quantized systems, see [18, 34] and references therein. Finally, and most importantly, this model has been suggested [31] as a first step for substantiating the basic assumption of statistical mechanics—the ergodic hypothesis of Boltzmann (see especially the discussion and references in [32, 35]).

In all the applications of this model, in particular that mentioned above, of special interest are so-called *scattering* billiards, i.e. billiards in a complement to the union of a finite number of convex regions, see figure 1. For example, the two-dimensional idealization of the Lorenz gas in the form of a lattice of rigid disks produces a scattering billiard ('the Sinai billiard'). The motion in a scattering billiard is highly unstable and thus produces strong mixing in the phase space. More precisely, it has been shown [31, 13, 3] that the

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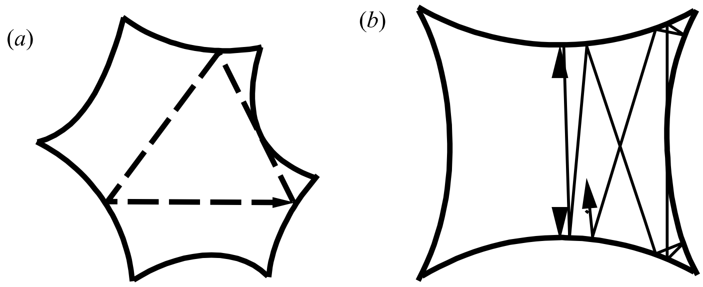

Figure 1. Tangent trajectories in scattering billiards. (*a*) Singular (tangent) periodic trajectory. (*b*) - - - - Non-singular periodic trajectory, —— tangent homoclinic trajectory to the periodic orbit.

corresponding dynamical system is (non-uniformly) hyperbolic, it is ergodic with respect to the natural invariant measure and it possesses the K-property. Based on this theory, statistical properties of various scattering systems have been analysed (see [7, 6]).

Do small perturbations ruin the ergodicity property of a scattering billiard? Here we consider the perturbation caused by smoothening of the billiard flow. The influence of such smoothening is a non-trivial question, since the dynamical system associated with the billiard we consider (in the simplest setting, this is a two-dimensional area-preserving mapping [31]) is singular. In particular, as explained more precisely in section 2.1, singularities appear near trajectories which are tangent to the billiard's boundary—like the ones shown in figure 1. Thus, even though the scattering billiard is hyperbolic almost everywhere, the singular set (e.g. singular periodic orbits) might produce stability islands under small perturbation. While such a phenomenon seems to be quite common, a general theory does not exist. Indeed, it is clear that the results are not straightforward—namely, it is not true that all smooth systems approaching a singular hyperbolic and mixing system have stable periodic orbits, nor is the converse—that they have the same ergodic properties as the singular system. (As an example, consider an analogous problem for one-dimensional maps; for a family of tent maps of an interval which is known to be ergodic and mixing, the ergodicity property may be easily destroyed in an arbitrarily close smooth family: if the maximum of the interval image produces a periodic orbit, it is clearly stable. However, the smooth one-dimensional map does not always possess stable periodic orbits: there may be a positive measure set of parameter values for which the smooth maps are ergodic and mixing [20]).

In this paper we prove (theorems 1 and 2) that, indeed, *a perturbation of a scattering billiard to a smooth Hamiltonian flow may create stability islands near singular periodic and homoclinic orbits of the billiard*.

More precisely, we consider smooth Hamiltonian flows which converge to the singular billiard flow, smoothly where the billiard flow is smooth and continuously where it is continuous (see section 2). For such approximations, we propose two mechanisms for the appearance of elliptic islands which destroy the ergodic properties of scattering billiards; one mechanism is controlled by the existence, in the billiard flow, of a singular periodic orbit and another mechanism is controlled by the existence of a singular homoclinic orbit. First, we study the phase-space structure of the local Poincaré map near such orbits, showing that locally these create a 'sharp' horseshoe. Embedding the billiard in a one-parameter family of billiards in which the boundary of the billiard table slightly moves with the parameter γ near the tangent point, we show that the horseshoe unravels as γ varies (see figure 6). Then, considering a two-parameter family of smooth Hamiltonian flows $h_t(\epsilon;\gamma)$ which approaches the family of billiards as $\epsilon \to 0$, we establish that for sufficiently small ϵ there exist a series of bifurcations associated with the disappearance of a Smale's horseshoe. It is well established that generically elliptic islands are created in such a process. Thus, it follows that for each sufficiently small ϵ there exist intervals of γ values for which elliptic islands exist.

We expect that singular homoclinic and periodic orbits are, in fact, unavoidable in scattering billiards; we conjecture that systems possessing such orbits are dense among all scattering billiards. We provide a numerical example which supports such a conjecture regarding the density of billiards with singular homoclinic orbits. A proof of this conjecture combined with the results presented here would imply that *for any given scattering billiard on a plane, there exists a close smooth Hamiltonian flow possessing elliptic islands.*

Furthermore, we establish sufficient conditions on the potentials of natural Hamiltonian systems so that their corresponding flows indeed converge to the billiard flows as assumed above. Surprisingly, the connection between the billiard model and the smooth Hamiltonian flows with steep potentials was not previously formalized. However, by our current results, the problem of relating the statistics manifested by the billiard dynamical systems to actual physical applications must inevitably include the study of the smoothening of the billiard potential. In many works, this connection has been implicitly assumed, see [19] and references therein. Nevertheless, our analysis reveals non-trivial requirements on smooth potentials approaching the step-function (billiard) potential, which are essential for the dynamics of the corresponding Hamiltonian system to follow the dynamics of the billiard flow.

In [27] a more general question of the behaviour of the symplectic structure when a family of smooth Hamiltonians approaches a singular limit is studied and related to the general study of distributions on manifolds. In this setting, it is shown that some properties of the smooth Hamiltonians are preserved by the singular one. For example, it is proved that if a family of Hamiltonians is uniformly mixing, then the mixing property carries to the singular system as well. Here we investigate the other direction of the above result: given a singular system which is mixing—what can be said on families of smooth Hamiltonian which appropriately approach this limiting system?

Finite-range potentials supported on a finite number of disks were extensively studied, see for example [30, 24, 25, 1, 23, 10] and references therein. In these works, the form of the potentials on each disk is taken to be radially axisymmetric, thus locally integrable. In such systems, the effect of the potential is to produce a finite-length travel $\Delta\theta(\phi)$ along the scattering disk, thus the study of such systems elegantly reduces to the study of the 'generalized Sinai billiard' with the reflection law $\phi \to -\phi, s \to s + \Delta\theta(\phi)$ (mod 2π), where (ϕ, s) correspond to the incidence angle and position on the disk boundary respectively. In [30, 24, 25, 1, 11] such potentials producing ergodic systems were sought. In [1, 23] non-ergodic behaviour was proved and studied for step-function potentials (where $\Delta\theta'(\phi) = \text{constant} < 2$). However, the *billiard limit* has not been studied in these works. In [10] it has been shown that for certain types of potentials, $\Delta\theta(\phi)$ produces focusing shifts near tangent trajectories and thus, that for any given energy level (high energies correspond, roughly, to the billiard limit) there exists an arrangement of the disks for which elliptic islands exist (see in particular theorems 5.3 and 5.4 of [10]). More closely related, in [19], it has been noted that the diamagnetic Kepler problem near singular homoclinic orbits of the four-disk billiard system (which has similar spatial structure) may produce elliptic islands by homoclinic tangencies.

Here, a completely different approach is taken, which in particular, does not assume any specific geometry of the scatterers nor that the potential is of a finite-range or locally axisymmetric. Most importantly, in the limit $\epsilon \to 0$, our Hamiltonian flows do approach the billiard flow, a necessary property for establishing meaningful asymptotic results.

The general scheme of the paper is as follows. In 2.1 we introduce the billiard flow in a general domain, and describe its nature near regular and tangent collision points and its relation to the standard billiard map. In 2.2, we define the smooth Hamiltonian approximation of the billiard flow and state some immediate consequences of this definition. In section 3 we prove the existence of elliptic islands in Hamiltonian flows which approximate scattering (Sinai) billiards. In section 4 the appearance of persistent singular homoclinics and singular (tangent) periodic orbits for scattering billiards is conjectured and the former is numerically demonstrated. Section 5 is devoted to a discussion on the implication of these results. In the appendix we formulate conditions on smooth Hamiltonians and prove these are sufficient to insure that the Hamiltonians flows approximate properly the corresponding billiard flows.

2. Billiards and their smooth Hamiltonian approximations

2.1. Billiard flow

Consider an open bounded region D on a plane with a piecewise smooth (C^{r+1}, $r \geqslant 2$) boundary S. On S there is a finite set C of so-called *corner points* $c_1, c_2, \ldots$ such that the arc of the boundary that connects two neighbouring corner points is C^{r+1}-smooth. Let us call these arcs *the boundary arcs* and denote them by $S_1, S_2, \ldots$. The set C includes all the points where the boundary loses smoothness and all the points where the curvature of the boundary vanishes. Thus, the curvature has a constant sign on each of the arcs S_i. Being equipped with the field of inward normals, the arc is called *convex* if its curvature is negative (with respect to the chosen equipment) and it is called *concave* if its curvature is positive (see figure 2).

Consider *the billiard flow* on $\bar{D}$. The phase space of the flow is co-ordinatized by $q \equiv (x, y, p_x, p_y)$ where (x, y) is the position of the particle in $\bar{D}$ and (p_x, p_y) is the (non-zero) velocity vector:

$$\dot{x} = p_x \qquad \dot{y} = p_y. \tag{2.1}$$

Henceforth, we reserve the term 'orbit' for the orbits in the phase space and the term 'trajectory' for the projection of an orbit to the (x, y)-plane. The velocity vector (p_x, p_y) is constant in the interior, and at the boundary it changes by the elastic reflection rule so $p_x^2 + p_y^2 =$ constant and the angle of reflection equals the angle of incidence with the opposite sign. Taking the point of reflection as the origin of the coordinate frame and the boundary's normal at that point as the y-axis, the reflection rule is simply

$$p_x \to p_x, \qquad p_y \to -p_y; \tag{2.2}$$

namely, the angle of incidence ϕ is $\arctan p_y/p_x$. This law is well defined only when the normal can be well defined: it is invalid at the corners (including inflection points). Generally, the incidence angle ϕ belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\phi = \pm\frac{\pi}{2}$, which corresponds to a trajectory tangent to S (figure 2) may be attained only when the boundary arc is concave.

Denote the time t map of the billiard flow as $b_t : q_0(x_0, y_0, p_{x0}, p_{y0}) \mapsto q_t(x_t, y_t, p_{xt}, p_{yt})$. By writing $q_t = b_t q_0$, we mean, in particular, that the piece of trajectory that connects (x_0, y_0) and (x_t, y_t) is on a finite distance of the corner set C, though it may have one or more points of tangency with concave components of S.

A point q in the phase space is called *an inner point* if $(x, y) \notin S$, and *a collision point* if $(x, y) \in (S\backslash C)$. Obviously, if q_0 and $q_t = b_t q_0$ are inner points, then q_t depends

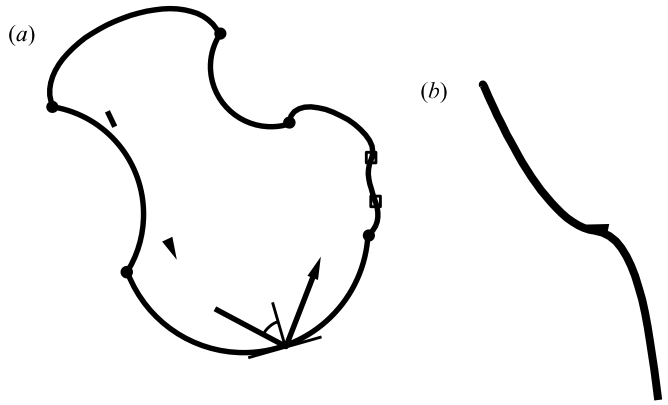

Figure 2. Billiard trajectories. (a) ● standard corner points, □ inflection corner points, $S_{1,3,5}$ concave boundary arcs, $S_{2,4,6,7}$ - convex arcs. —— Regular reflection, - - - - tangent trajectory. (b) — · — Tangent trajectory terminated at an inflection point.

continuously on q_0 and t. Otherwise, if q_t is a (non-tangent) collision point, the velocity vector undergoes a jump; denoting by $q_{t-0} = b_{t-0}q_0$ and $q_{t+0} = b_{t+0}q_0$ the points just before and just after the collision, it follows that (p_{xt+0}, p_{yt+0}) and (p_{xt-0}, p_{yt-0}) are related by the elastic reflection law. To avoid ambiguity we assume that at a collision point the velocity vector is oriented inside D; thus, we put $b_t \equiv b_{t+0}$.

Further, if q_t is an inner point and if the piece of trajectory that connects (x_0, y_0) and (x_t, y_t) does not have tangencies with the boundary, then q_t depends C^r-smoothly on q_0 and t. However [31], the map b_t loses smoothness at any point q_0 whose trajectory is tangent to the boundary at least once on the interval $[0, t]$. Indeed, choosing coordinates so that the origin is a point on a concave boundary arc S_i, the y-axis is the normal to S_i and the x-axis is tangent to S_i, the arc is locally given by the equation

$$y = -x^2 + \cdots.$$

It follows that for small $\delta > 0$ the time $t = \delta$ map of the slanted line $(x_0 = -\delta/2 + ay_0, p_{x0} = 1, p_{y0} = 0)$ has a square-root singularity in the limit $y_0 \to -0$ which corresponds to the tangent trajectory (see figure 3; $a \neq 0$ for graphical purposes):

$$\begin{aligned}(x_\delta, y_\delta, p_{x\delta}, p_{y\delta}) &= (\tfrac{1}{2}\delta + ay_0, y_0, 1, 0) \qquad \text{at } y_0 \geqslant 0\\ &= (\tfrac{1}{2}\delta + ay_0 + \mathrm{O}(\delta y_0), 2\sqrt{-y_0}\delta\\ &\quad + \mathrm{O}(\delta y_0), 1 + \mathrm{O}(y_0), 2\sqrt{-y_0} + \mathrm{O}(y_0)) \text{ at } y_0 \leqslant 0\end{aligned}$$

If q_0 and $q_t = b_t q_0$ are inner points, then for two arbitrary small cross sections in the phase space, one through q_0 and the other through q_t, *the local Poincaré map* is defined by the orbits of the billiard flow. If no tangency to the boundary arcs is encountered between q_0 and q_t, then *the Poincaré map is locally a C^r-diffeomorphism*.

One can easily prove that the same remains valid if q_0, q_t, or both are collision points, provided the corresponding cross sections are composed of the nearby collision points. In fact, the collision set (the surface $(x, y) \in S$ in the phase space) provides a *global* cross section for the billiard flow. The corresponding Poincaré map relating consecutive collision points is called *the billiard map*. A point on the surface is determined by the position s on the boundary S and by the reflection angle ϕ which yields the direction of the outgoing velocity vector (the absolute value of the velocity does not matter). The initial conditions,

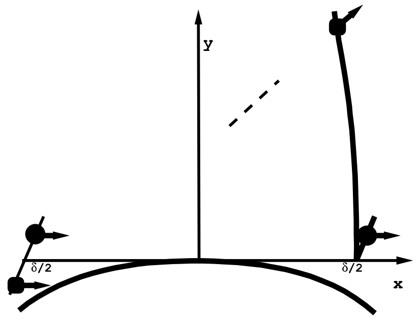

Figure 3. Singularity near a tangent trajectory.

corresponding to a trajectory directed to a corner or tangent to a boundary arc at the moment of the next collision, form *the singular set* on the (s, ϕ)-surface. Generically, the singularity set is a collection of smooth curves which may be glued at some points. The billiard map is a C^r-diffeomorphism outside the singular set; it may be discontinuous at the singular points. Near a singular point corresponding to the tangent trajectory the continuity of the map can be restored locally by taking two iterations of the map on half of the neighbourhood of the singular point (see figure 3). The obtained map will, nevertheless, be non-smooth at the singular point, having the square-root singularity described above.

If a trajectory has exactly one tangency to the billiard boundary and does not approach the corner points it is called a *simple singular* trajectory (and its corresponding orbit *simple singular orbit*). For periodic orbits, the same definition applies per period.

2.2. Smooth Hamiltonian approximation

Formally, the billiard flow may be considered as a Hamiltonian system of the form

$$H_b = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V_b(x, y) \tag{2.3}$$

where,

$$V_b(x, y) = \begin{cases} 0 & (x, y) \in D \\ +\infty & (x, y) \notin D. \end{cases} \tag{2.4}$$

Clearly, this is an approximate model of the motion of a pointwise particle in a *smooth* potential which stays nearly constant in the interior region and grows very fast near the boundary. However, it is not obvious immediately when (and in which sense) this motion is indeed close to the billiard motion. We say that a family of C^r smooth Hamiltonian flows $h_t(\epsilon)$ *r-converges* to the billiard flow if the following assumption holds.

$\mathbf{A_r}$. *If q_0 and $q_t = b_t q_0$ are inner phase points, and if the billiard trajectory of q_0 has no tangencies to the boundary for the time interval $[0, t]$, then, as $\epsilon \to 0$, the time t map $h_t(\epsilon)$ of the smooth Hamiltonian flow limits to the map b_t in the C^r-topology in a small neighbourhood of q_0. However, if a tangency occurs, then $h_t(\epsilon) \to b_t$ in the C^0 sense.*

Obviously, one needs to show that the above definition is not vacuous.

Lemma 2.1. *For any billiard domain D there exist families of Hamiltonian flows satisfying assumption* $\mathbf{A_r}$.

Indeed, in the appendix we consider the family of Hamiltonian systems associated with

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y; \epsilon) \tag{2.5}$$

where the potential $V(x, y; \epsilon)$ tends to zero inside the region D as $\epsilon \to 0$ and it tends to infinity outside. We prove that there exists a large class of smooth (C^∞) potentials for which assumption $\mathbf{A_r}$ holds for any finite r. For example, the potentials which are of the following form near the boundary (where Q denotes, roughly, the distance from the boundary):

$$\frac{\epsilon}{Q^\beta}, \ (1 - Q^\beta)^{\frac{1}{\epsilon}}, \ \epsilon e^{-\frac{1}{Q^\beta}}, \ \epsilon |\ln Q|^\beta, \ \epsilon \ln \ldots |\ln Q|, \ \beta > 0 \tag{2.6}$$

produce Hamiltonian flows which satisfy $\mathbf{A_r}$ for all finite r.

Moreover, it is proved that adding any C^r smooth function $H_\epsilon(x, y, p_x, p_y; \epsilon)$ which is uniformly small in $\bar{D}$ as $\epsilon \to 0$ still produces a family of flows satisfying $\mathbf{A_r}$.

Lemma 2.2. *For any billiard domain D there exist families of Hamiltonian flows in general position which satisfy* $\mathbf{A_r}$.

Assumption $\mathbf{A_r}$ implies that the Poincaré maps defined by the billiard and Hamiltonian flows are close. Let q_0 and $q_t = b_t q_0$ be inner phase points. The local Poincaré sections through q_t and q_0 are three dimensional, and are foliated by equi-energy two-dimensional surfaces. For sufficiently small ϵ, similar foliation exists for the smooth Hamiltonian flow, thus a reduced two-dimensional Poincaré map is well defined.

Corollary 2.1. *Provided the billiard trajectory between q_0 and q_t does not have tangencies to the boundary of the billiard domain, as $\epsilon \to 0$ the reduced Poincaré map of the smooth Hamiltonian flow satisfying* $\mathbf{A_r}$ *converges, in C^r-topology, to a Poincaré map of the billiard flow as does the flight time. If the tangency does occur, the convergence is only C^0.*

Corollary 2.1 allows us to utilize persistence theorems regarding two-dimensional area-preserving diffeomorphisms (e.g. see [16, 21]) in order to establish relations between periodic orbits of the billiard flow and of the Hamiltonian flows under consideration. For a non-singular periodic orbit, and a cross section through an inner point on it, the reduced Poincaré map of the billiard flow is locally a diffeomorphism, and the intersection of the periodic orbit with the cross section in the phase space is a fixed point of the diffeomorphism. Generally, the fixed point is either hyperbolic or elliptic (for the scattering billiards it is hyperbolic). Fixed points of both types are preserved under small smooth perturbations in the class of area-preserving diffeomorphisms.

Corollary 2.2 (Persistence of periodic orbits). *If a non-singular periodic orbit L_0 of the billiard flow is hyperbolic or elliptic, then at ϵ sufficiently small the Hamiltonian flow $h_t(\epsilon)$ satisfying* $\mathbf{A_r}$ *with $r \geqslant 1$ has a unique continuous family of hyperbolic or, respectively, elliptic periodic orbits L_ϵ in the fixed energy level of L_0 which limit to L_0 as $\epsilon \to 0$.*

If L_0 is hyperbolic, the local stable ($W^s_{\rm loc}(L_\epsilon)$) and unstable ($W^u_{\rm loc}(L_\epsilon)$) manifolds of L_ϵ depend continuously on ϵ (as smooth manifolds) and limit to $W^s_{\rm loc}(L_0)$ and $W^u_{\rm loc}(L_0)$ respectively. The global stable and unstable manifolds -$W^u(L_\epsilon)$ and $W^s(L_\epsilon)$—are obtained as the continuation of $W^s_{\rm loc}(L_\epsilon)$ and $W^u_{\rm loc}(L_\epsilon)$ by the orbits of the flow. Note that for the billiard flow, by applying the continuation process, tangencies to the boundary and corner points are bound to be encountered by some points belonging to the manifolds. Using local cross sections as above, it is easy to see that the following result holds.

Corollary 2.3 (Extensions of stable and unstable manifolds). *Under the same assumptions as corollary 2.2, and assuming L_0 is hyperbolic, any piece K_0 of $W^u(L_0)$ or $W^s(L_0)$ obtained as a time $t > 0$ shift of some region in $W^u_{\rm loc}(L_0)$ (respectively, a time $t < 0$ shift of some region in $W^s_{\rm loc}(L_0)$) is a C^0, or if no tangencies to the boundary are encountered in the continuation process, C^r-limit of a family of surfaces $K_\epsilon \subset W^u(L_\epsilon)$ (resp. $K_\epsilon \subset W^s(L_\epsilon)$).*

The above persistence results apply only to non-singular periodic orbits; near the singular periodic orbits, which are studied next, the billiard flow is non-smooth and the standard theory is not valid.

3. Elliptic islands

Hereafter, consider the case of the so-called *scattering* billiards. Scattering billiards are billiards composed of concave arcs with the curvature bounded away from zero, and non-zero angles between the arcs at the corner points. Then, the billiard flow is hyperbolic whence all non-singular periodic orbits are hyperbolic. We, nevertheless, show that the *simple singular* periodic orbits give rise to stable (elliptic) periodic orbits in the Hamiltonian systems limiting to the scattering billiards.

3.1. Structure near singular periodic orbits

The hyperbolic structure of the phase space of the scattering billiards plays a crucial role in the understanding of the behaviour near a singular periodic orbit. For the billiard map B, the presence of hyperbolic structure implies that for almost every point $P(s, \phi)$ in the phase space there exist stable and unstable directions E^u_P and E^s_P, depending continuously on P. The system of stable and unstable directions is invariant with respect to the linearized map: $d_P B E^{s(u)} = E^{s(u)}_{BP}$, which is uniformly expanding along the unstable direction and uniformly contracting along the stable direction: if $v \in E^u$ ($v \in E^s$), then $||d_P B v|| \geqslant e^{\lambda\tau}||v||$ (resp. $||d_P B v|| \leqslant e^{-\lambda\tau}||v||$) in a suitable norm; here, τ is the flight time from P to BP, the uniformity means that the value $\lambda > 0$ is independent of P (see details in [5]).

Equivalently, there is an invariant family of stable and unstable cones: the unstable cone at a point P is taken by the linearized map $d_P B$ into the unstable cone at the point BP; the image is stretched in the unstable direction and shrinks in the stable direction. Similar behaviour appears for the stable cone under backward iterations. There is an explicit geometrical description of these cones for scattering billiards [36]. Consider a point (s, ϕ) in the phase space and a small curve passing through this point. Taking two points on this curve defines two inward directed rays emanating from the billiard boundary near s (see figure 4). If these rays intersect, then the tangent direction to this curve belongs to the stable cone of (s, ϕ); otherwise, it belongs to the unstable cone (in other words, the unstable cones are given by $ds \cdot d\phi > 0$ and the stable cones by $ds \cdot d\phi < 0$). Moreover, it can also be shown that if the intersection of the rays with each other occurs before the first intersection of the rays with the billiard boundary, then the tangent direction to the forward image of the small curve under consideration belongs to the unstable cone of the image of (s, ϕ).

It follows that the tangent to a line of singularity (the line composed of the points whose trajectories are tangent to the billiard boundary) at any point lies in the stable cone, and the tangent to any iteration of the singularity line by the billiard map lies in the corresponding unstable cone. In particular, the intersections of the singularity lines with their images are always transverse.

Utilizing these observations, we find the normal form of the first return map of the billiard map near a simple singular periodic orbit. Consider a periodic orbit L with the

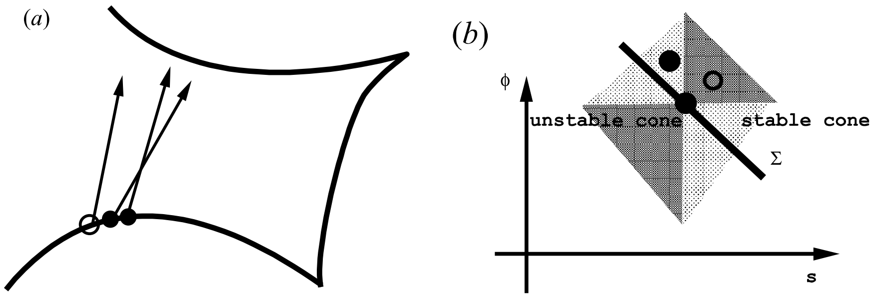

Figure 4. Hyperbolic structure—the stable and unstable cones. (a) Geometrical interpretation of stable/unstable directions. (b) Phase space structure.

corresponding sequence of collision points $P_i(s_i, \phi_i)$ ($i = 0, \ldots, n-1$): $P_{i+1} = BP_i$ where $P_n = P_0$. Let $P \equiv P_0$ belong to the singular set (so $|\phi_1| = \pi/2$). Take a small neighbourhood U of P and denote as Σ the line of singular points in U.

Proposition 3.1. *Given a simple singular periodic orbit L as above, the local return map near P_0 may be reduced to the form,*

$$\begin{cases} \bar{u} = v \\ \bar{v} = \xi(v - \sqrt{\max(v, 0)}) - u + \cdots \end{cases} \tag{3.1}$$

where $v = 0$ gives the singularity line, $u = 0$ is its image, and $|\xi| > 2$.

Proof. Consider the local structure in U, near the singularity line Σ. The line Σ divides U into two parts, U_r and U_s; the orbits starting on U_r (e.g. P_0'' in figure 5) do not hit the boundary near s_1 and approach it near the point s_2, the orbits starting on U_s (e.g. P_0' in figure 5) have a nearly tangent collision with the boundary in a neighbourhood of s_1. Without loss of generality we assume that Σ is locally a straight line $(s-s_0)+k(\phi-\phi_0) = 0$, where $k > 0$ because Σ must lie in the stable cone $(s-s_0)(\phi-\phi_0) < 0$, and that U_r is given by $(s-s_0)+k(\phi-\phi_0) < 0$ and U_s by $(s-s_0)+k(\phi-\phi_0) \geqslant 0$.

Consider *the first return map* $\bar{B}$ defined on U. The map $\bar{B}$ equals $B_{n-1}\ldots B_2B_1B_0$ on U_s and $B_{n-1}\ldots B_2B_0$ on U_r where B_i is a restriction of the billiard map on a small neighbourhood of P_i. According to section 2.1.1, $\bar{B}$ is a continuous map which loses its smoothness on Σ. Namely, the restriction B_{0s} of B_0 on U_s exhibits the square-root singularity described in section 2.1.1 whereas the map $B|_{U_r}$ is regular and it can be continued onto the whole U as a smooth map B_{0r}: erasing a small piece of the boundary containing the tangency point s_1, B_{0r} will simply be the billiard map from U to a small neighbourhood of P_2 (see the action of B_{0r} on P_0' in figure 5). Obviously, $B_{0r}\Sigma = B_1B_{0s}\Sigma$, therefore the first return map $\bar{B}$ is continuous. One may represent the map $\bar{B}$ as a superposition of regular and singular maps:

$$\bar{B} = B^{(r)} \cdot B^{(s)}$$

where

$$B^{(r)} = B_{n-1}\ldots B_2B_{0r}$$

and

$$B^{(s)} = \begin{cases} \text{id} & \text{on } U_r \\ B_{0r}^{-1}B_1B_{0s} & \text{on } U_s. \end{cases}$$

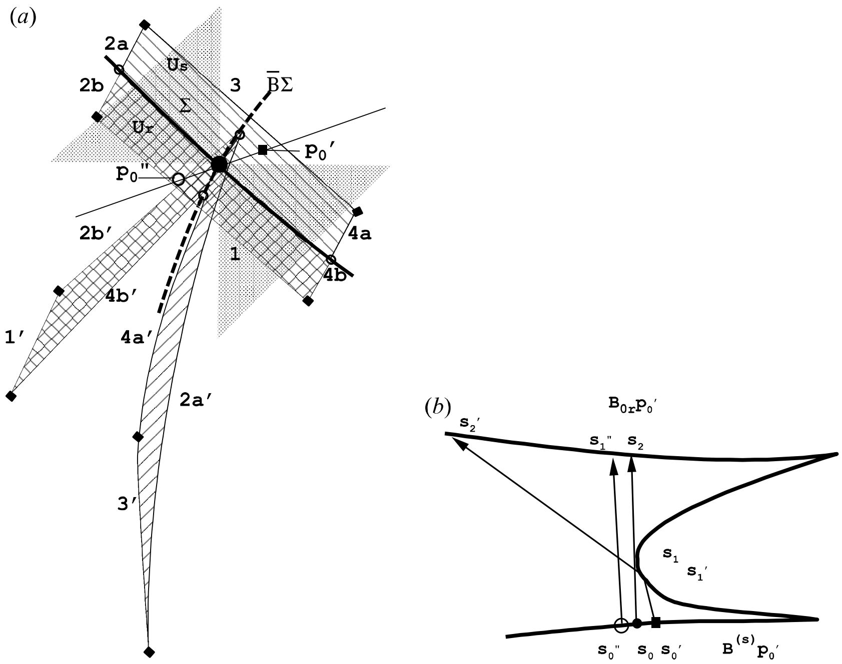

Figure 5. Structure near singular periodic orbit. (*a*) Phase-space structure near singular periodic orbit: 1234 is mapped onto $1'2'3'4'$. (*b*) Action of billiard map near a singular segment of trajectory.

The singular part $B^{(s)} : U \to U$ may be obtained by inverted reflection near the tangency point s_1 (see the action of $B^{(s)}$ on P_0' in figure 5). It is not too difficult to calculate that $B^{(s)}$ is given by

$$\begin{cases} S' = S + k\sqrt{\max(S + k\Phi, 0)} + \cdots \\ \Phi' = \Phi - \sqrt{\max(S + k\Phi, 0)} + \cdots \end{cases}$$

where $S = s - s_0$, $\Phi = \phi - \phi_0$ are coordinates in U, and the dots stand for the quantities infinitely small in comparison with S, Φ or $\sqrt{\max(S + k\Phi, 0)}$ as $S, \Phi \to 0$.

The regular part $B^{(r)}$ is, by definition, the first return map for the auxiliary billiard obtained by pushing the boundary near the tangency point s_1 slightly aside from the trajectory of L. The point P is a fixed point for $B^{(r)}$ (as well as for the map $\bar{B}$). Since the auxiliary billiard is still scattering, the point P is a hyperbolic fixed point for $B^{(r)}$. Moreover, the unstable cone $S \cdot \Phi \geqslant 0$ must be mapped inside itself by the linearization of $B^{(r)}$ at P. If $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is the corresponding linearization matrix, the last condition is equivalent to the requirement that all b_{ij} are of the same sign. Recall that $B^{(r)}$ is an area-preserving diffeomorphism, so

$$b_{11}b_{22} - b_{12}b_{21} = 1.$$

Superposition of $B^{(r)}$ and $B^{(s)}$ gives, to leading order in S, Φ and $\sqrt{\max(S + k\Phi, 0)}$, the following formula for the map $\bar{B}$:

$$\begin{cases} \bar{S} = b_{11}S + b_{12}\Phi - (b_{12} - b_{11}k)\sqrt{\max(S + k\Phi, 0)} + \cdots \\ \bar{\Phi} = b_{21}S + b_{22}\Phi - (b_{22} - b_{21}k)\sqrt{\max(S + k\Phi, 0)} + \cdots . \end{cases} \tag{3.2}$$

Provided inequalities (3.3) are satisfied, as proved in the lemma below, the normal form (3.1) is obtained from the above expression by changing to the new coordinates u, v where u is aligned with the singularity line ($v \propto S + k\Phi$) and v is aligned with its image. From the calculation, it follows that the quantity ξ is $(b_{11} + b_{22})$, i.e. it is the trace of the linearization matrix of the first return map $B^{(r)}$ of the auxiliary billiard about the periodic orbit. Since the auxiliary billiard is scattering, its regular periodic orbits are hyperbolic, hence $|\xi| > 2$. □

Lemma 3.1. *The coefficients b_{ij} in (3.2) obey the inequalities,*

$$(b_{12} - b_{11}k)(b_{22} - b_{21}k) > 0, \qquad |b_{12}| < |b_{11}|k, \qquad |b_{22}| < |b_{21}|k. \tag{3.3}$$

Proof. Since the image $\bar{B}\Sigma$ of the singularity line $S + k\Phi = 0$ must lie in the unstable cone $\bar{S} \cdot \bar{\Phi} > 0$, (3.2) implies the first inequality in (3.3).

For a small piece l of a straight line through P which lies in the unstable cone, i.e. for which the increase of s is followed with the increase of ϕ (see figure 5—imagine a line going through P_0'', P_0, P_0') the image of $l \cap U_r$ by B_0 and the image of $l \cap U_s$ by $B_1 B_0$ both lie to one side of the point P_2 (or the point s_2 when projected to the configuration plane). Namely, these images both belong to the same half of the unstable cone of P_2 corresponding to a definite sign of $(s - s_2)$. Since the linearization of each of the maps B_i preserves the decomposition into the stable and unstable cones, it follows that the image of l by $\bar{B}$ is a folded line with the vertex at P which divides $\bar{B}l$ into two parts both belonging to the same half of the unstable cone of P; i.e. $\bar{S}$ and $\bar{\Phi}$ have the same sign on $\bar{B}(l \cap U_r)$ and $\bar{B}(l \cap U_s)$. By (3.2), it is equivalent to the condition that the sign of $(b_{12} - b_{11}k)$ is opposite to the sign of b_{12} and b_{11} and the sign of $(b_{22} - b_{21}k)$ is opposite to the sign of b_{22} and b_{21} (recall that all b_{ij} have the same sign). Thus, the second and third inequalities in (3.3) hold. □

Now, embed the billiard in a one-parameter family of scattering billiards $b_t(\cdot; \gamma)$ for which all arcs depend smoothly on the parameter γ, while the corner points are held fixed; suppose the billiard with the simple singular periodic orbit L is realized at $\gamma = 0$. The regular part $B^{(r)}$ of the first return map of U depends smoothly on γ, hence its hyperbolic fixed point $P_\gamma^{(r)}$ is also a smooth function of γ. The same is valid for the position of the singularity line Σ_γ. For a *generic* family of billiards, the parametrization by γ may be chosen so that the distance between $P_\gamma^{(r)}$ and Σ_γ is proportional to γ (it is true if, for instance, one changes the billiard boundary locally, near the tangency point s_1 only: such a perturbation moves the singularity line but the map $B^{(r)}$ and the position of its fixed point remain unchanged). Assume, with no loss of generality, that $P_\gamma^{(r)} \in U_r$ for $\gamma > 0$ and that $P_\gamma^{(r)} \in U_s$ for $\gamma < 0$. Therefore, by the definition of $B^{(r)}$, its fixed point is a fixed point of $\bar{B}$ for $\gamma > 0$, and its fixed point is imaginary when $\gamma < 0$.

For such a family of billiards, the normal form (3.1) of the first return map $\bar{B}$ is rewritten as

$$\begin{cases} \bar{u} = v \\ \bar{v} = \xi(\gamma + v - \sqrt{\max(v, 0)}) - u + \cdots . \end{cases} \tag{3.4}$$

This map looks similar to the Hénon map (though it has a singular nonlinearity).

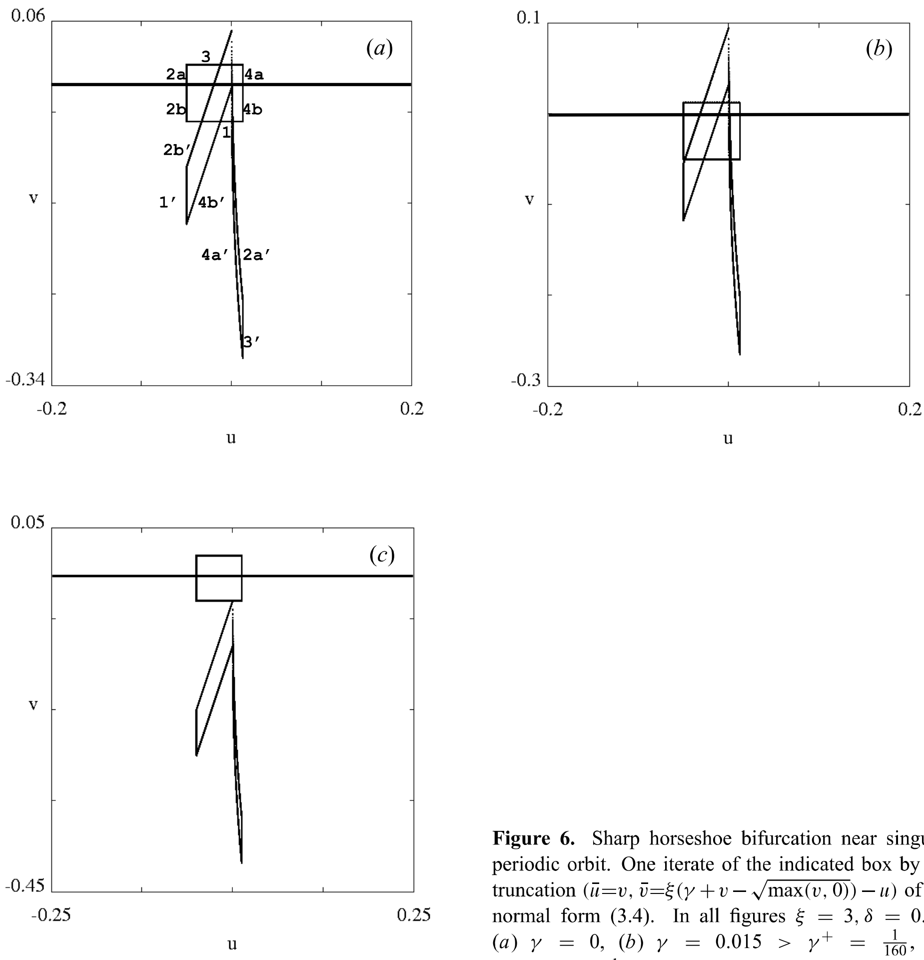

Figure 6. Sharp horseshoe bifurcation near singular periodic orbit. One iterate of the indicated box by the truncation ($\bar{u}=v$, $\bar{v}=\xi(\gamma+v-\sqrt{\max(v,0)})-u$) of the normal form (3.4). In all figures $\xi = 3, \delta = 0.05$. (a) $\gamma = 0$, (b) $\gamma = 0.015 > \gamma^+ = \frac{1}{160}$, ($c$) $\gamma = \gamma^- = -\frac{1}{30}$.

Proposition 3.2. *Consider the map (3.4). For a small fixed neighbourhood U of the origin, let Ω_γ be the set of all orbits of $\bar{B}_\gamma$ which never leave U. Then there exist small $\gamma^\pm$ values such that $\Omega_\gamma = \emptyset$ for $\gamma = \gamma^- < 0$, and if $\gamma = \gamma^+ > 0$, then Ω_γ is in one-to-one correspondence with the set of all sequences composed of two symbols (r, s): 'r' corresponds to entering U_r and 's' corresponds to entering U_s.*

Proof. Take a small $\delta > 0$ and let the neighbourhood U be a rectangle $\{-\delta < u < \kappa\delta, -\delta < v < \kappa\delta\}$ where $\kappa = \frac{1}{2}(\frac{1}{2}|\xi| - 1) > 0$ (recall that $|\xi| > 2$). Let $\gamma^+ = (\frac{1}{2} - \frac{1}{\xi})\delta > 0$ and $\gamma^- = -\frac{2}{|\xi|}\delta$. Then, for sufficiently small δ, one may check that for the given choice of U the map (3.4) takes the horizontal boundaries of U (marked 1 and 3 in figure 6) on a finite distance of U for all $\gamma \in [\gamma^-, \gamma^+]$. The images of the vertical boundaries 2 and 4 which intersect the singularity line, fold as indicated in figure 6: the segments 2a,4a are mapped to 2a',4a' and the segments 2b,4b are mapped to 2b',4b'. The folded lines 2',4' may intersect U but they lie on a finite distance of their pre-images (the boundaries 2 and 4) for all $\gamma \in [\gamma^-, \gamma^+]$. Thus, the image of U by $\bar{B}_\gamma$ has a specific shape of a sharp horseshoe.

Changing γ shifts the horseshoe along the v-axis, so at $\gamma = \gamma^+$ the intersection of the horseshoe with U consists of two distinct connected components (figure 6(b)). On each component the map $\bar{B}_\gamma$ is smooth and hyperbolic. The statement regarding the one-to-one correspondence to Bernoulli shift on two symbols follows as in the standard construction of the horseshoe map [33, 26]. In particular, it implies that each of the two components has a hyperbolic fixed point. Moreover, one of the fixed points has two positive multipliers and the other two negative multipliers. On the other hand, at $\gamma = \gamma^-$ the intersection of $\bar{B}_\gamma U$ with U is empty (figure 6(c)). □

Note the following three conclusions from the proof of the above proposition. First, that there exist $\gamma^\pm$ values such that for γ^+ two hyperbolic fixed points exist and for γ^- no fixed points exist in the square region U near the intersection of the singularity line with its image. Second, that $\gamma^\pm$ may be chosen arbitrarily small (by taking smaller U). Third, no fixed points can pass through the boundary of U as γ varies from γ^- to γ^+ because the image of the horizontal boundaries of U never intersects the boundary of U and the image of the vertical boundaries U may intersect only the horizontal parts of the boundary.

Now, take a *two-parameter* family of smooth Hamiltonian flows $h_t(\cdot;\epsilon,\gamma)$ which approach, uniformly with respect to γ, the family of billiard flows $b_t(\cdot;\gamma)$ as $\epsilon \to 0$, as in assumption $\mathbf{A_r}$. Note that for the billiard flow, the structure of the Poincaré map of an arbitrary small cross section ω through an *inner* point on the simple singular periodic orbit L is absolutely the same as described above (see section 2.1). Due to the C^0-closeness of the billiard flow and the smooth Hamiltonian flow it follows that for ϵ sufficiently small the corresponding Poincaré map $\Pi_{\epsilon\gamma}$ for the Hamiltonian system transforms a rectangle $U' \subset \omega$ (analogous to the rectangle U) to a horseshoe shape (which is now smooth because the Hamiltonian system is smooth at all $\epsilon > 0$). At $\gamma = \gamma^-$ the intersection $\Pi_{\epsilon\gamma}U' \cap U'$ is empty for small ϵ whence $\Pi_{\epsilon\gamma^-}$ has no fixed points in U'. Moreover, no fixed points can pass through the boundary of U' as γ varies from γ^- to γ^+ because the fixed points of the first return billiard map stay a finite distance from the boundary of U' for all $\gamma \in [\gamma^-, \gamma^+]$.

The two fixed points of the Poincaré map of the billiard flow which exist at $\gamma = \gamma^+$ are hyperbolic and do not belong to the singularity line. Thus, by corollary 2.2, each of these hyperbolic fixed points exists for the map $\Pi_{\epsilon\gamma^+}$ at all sufficiently small ϵ, moreover the multipliers of one of the fixed points are negative as for the billiard. Now, fixing any ϵ small enough, a fixed point of $\Pi_{\epsilon\gamma^+}$ changes continuously as γ decreases, until it merges with some other fixed point (as we mentioned, the fixed point must disappear before $\gamma = \gamma^-$ and it cannot leave U' via crossing the boundary). Since fixed points may disappear only when their multipliers are equal to 1, it follows that the fixed point with the negative multipliers at $\gamma = \gamma^+$ must become elliptic for some interval of γ values before the moment of disappearance. Thus we have proved the following.

Proposition 3.3. *Consider a one-parameter family of scattering billiards which has a simple singular periodic orbit L for the parameter value* $\gamma = 0$*. Consider a two-parameter family of* C^r, $r \geqslant 1$ *smooth Hamiltonian flows* $h_t(\epsilon,\gamma)$ *satisfying* $\mathbf{A_r}$ *uniformly in* γ*. Then, for any small* ϵ *there exists an interval of* γ *on which elliptic periodic orbits exist in the energy level of L. As* $\epsilon \to 0$ *these intervals accumulate to zero and the elliptic periodic orbits limit to the singular periodic orbit.*

In a *generic* family of sufficiently smooth (C^r, $r \geqslant 5$ [29]) two-degrees-of-freedom Hamiltonian systems non-resonant elliptic periodic orbits are stable, and in particular they are surrounded by KAM tori, creating the so-called elliptic islands.

Theorem 1. *Consider a scattering billiard which has a simple singular periodic orbit L. Then, there exists a one parameter family of C^r, $r \geqslant 5$ smooth Hamiltonian flows $\bar{h}_t(\epsilon)$, which r-converges to the billiard flow as $\epsilon \to 0$, and for which there exists a sequence of intervals of ϵ converging to 0 where elliptic islands exist.*

Proof. Embed the billiard in a one-parameter family of scattering billiards as in the above proof. Consider a two-parameter family of Hamiltonian flows $h_t(\epsilon, \gamma)$ which r-converge to the family of billiards as $\epsilon \to 0$, uniformly in γ. Such families exist by lemma 2.1. By proposition 3.3 there exists a path $(\epsilon, \gamma(\epsilon))$ which intersects regions where elliptic periodic orbits exist. Since, by lemma 2.2, $h_t(\epsilon, \gamma)$ may always be put in a general position, its elliptic periodic orbits are generic and hence stable. □

In fact, it is desirable to state the above for *natural Hamiltonian systems* (systems of the form (2.5)). For that, we need to show that some coefficient in the Birkhoff normal form is non-zero for generic potentials. This obviously seems to be correct (otherwise non-resonant elliptic periodic orbits of natural systems would not be generically stable). However, we failed to find the corresponding reference.

3.2. Singular homoclinic orbits

Consider a *non-singular* hyperbolic periodic orbit L_0 of the billiard flow. Suppose, its stable and unstable manifolds intersect along some orbit Γ. This is a *homoclinic* orbit; i.e. it asymptotes L_0 exponentially as $t \to \pm\infty$. Assume that Γ is *simple singular* which means that its trajectory has one point of tangency with the billiard's boundary (see figure 1(*b*)).

Let $P(s, \phi)$ and $\bar{P}(\bar{s}, \bar{\phi})$ be collision points on Γ: P is the last before the tangency and $\bar{P}$ is the first after the tangency. By definition, $\bar{P} = B^2 P$ where B is the billiard map. Consider, in the (s, ϕ) plane, the local segment W^u of the unstable manifold of L_0 to which P belongs. Since the tangent to W^u at P belongs to the unstable cone, it must intersect the singularity line transversely at P. Thus, as explained in the proof of lemma 3.1, the image of W^u in a neighbourhood of $\bar{P}$ by the billiard map folds with a sharp square root singularity at $\bar{P}$, see figure 7. Now, the point $\bar{P}$ belongs to the stable manifold as well. Since the tangent to W^s belongs to the stable cone, it follows that the folded image of W^u lies to one side of W^s, so a sharp homoclinic tangency is created at $\bar{P}$, as shown in figure 7.

In a generic family of scattering billiards (as in section 3.1), two transverse homoclinic intersections appear at $\gamma > 0$ and none at $\gamma < 0$. For the corresponding two-parameter Hamiltonian family, arguments analogous to those in the proof of proposition 3.3 show that *generically, for any ϵ sufficiently small there exists $\gamma^*(\epsilon)$ for which a quadratic homoclinic tangency occurs.*

Recall that the occurrence of homoclinic tangencies is a well known mechanism for the creation of elliptic islands [28] for smooth Hamiltonian flows. Thus, using the same arguments as in theorem 1 we have established the following.

Theorem 2. *If a scattering billiard has a simple singular homoclinic orbit Γ, then there exists a one-parameter family of C^r, $r \geqslant 5$ smooth Hamiltonian flows $\bar{h}_t(\epsilon)$ which r-converges to the billiard flow as $\epsilon \to 0$ and for which there exists a sequence of intervals of ϵ values converging to zero for which elliptic islands exist in the energy level of Γ.*

The period of the elliptic periodic orbits mentioned in theorem 2 goes to infinity as $\epsilon \to 0$. In fact, in the two-parameter family of smooth Hamiltonians elliptic periodic orbits of bounded period limit, as $\epsilon \to 0$, to singular periodic orbits corresponding to $\gamma \neq 0$. Thus,

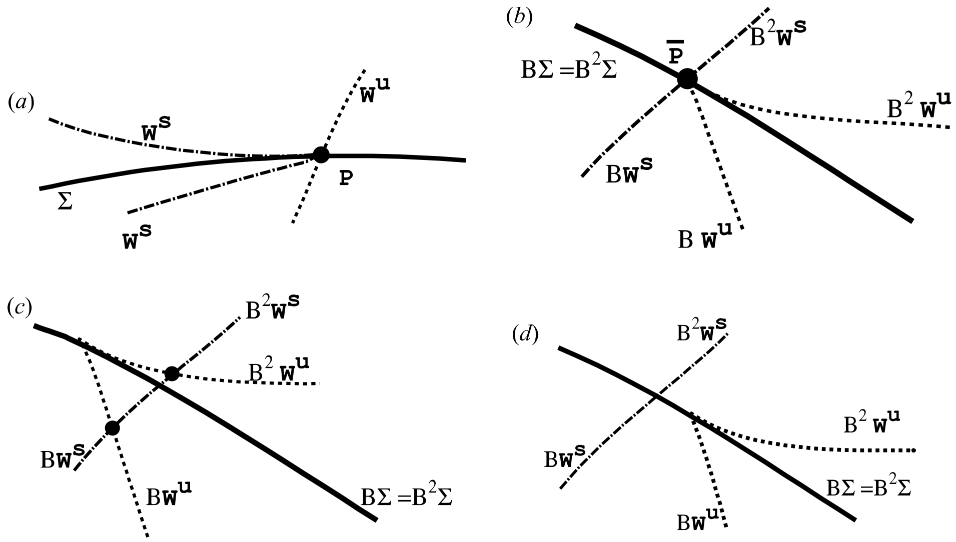

Figure 7. Bifurcation of singular homoclinic orbit. (*a*) $\gamma = 0$ near Σ, (*b*) $\gamma = 0$ near Σ's image, (*c*) $\gamma > 0$, (*d*) $\gamma < 0$, near Σ's image, ● homoclinic points.

theorems 1 and 2 are very much related. Indeed, like the appearance of stable periodic orbits near a homoclinic tangency is proved in smooth situations (see [14, 28, 15]), one may show that *in a generic family of scattering billiards having a sharp homoclinic tangency at* $\gamma = 0$ *there is a sequence of values of* γ *accumulating at* $\gamma = 0$ *for which singular periodic orbits exist.*

Now the reference to theorem 1 gives another proof of theorem 2.

4. On the genericity of the elliptic islands creation

It is well known [22, 4, 5] that for scattering billiards the hyperbolic non-singular periodic orbits are dense in the phase space. The stable/unstable manifolds of such orbits cover the phase space densely and the orbits of their homoclinic intersections also form a dense set.

It follows that the periodic orbits and the homoclinic orbits get arbitrarily close to the singularity set. It seems thus intuitively clear that for any scattering billiard very small smooth perturbations may be applied to place a specific periodic orbit or a specific homoclinic orbit exactly on the singularity line, so that theorem 1 and 2 may be applied. Proving these intuitive statements turns out to be quite a delicate issue, thus we formulate these as conjectures.

Conjecture 1. *Any scattering billiard may be slightly perturbed to a scattering billiard for which a singular (tangent) periodic orbit exists.*

Conjecture 2. *Any scattering billiard may be slightly perturbed to a scattering billiard for which there exists a non-singular hyperbolic periodic orbit which has a singular homoclinic orbit.*

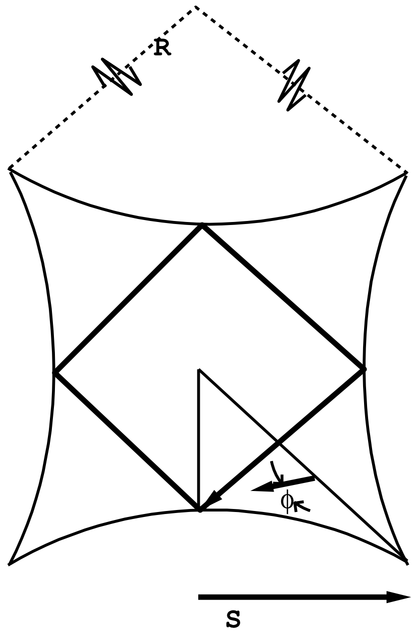

Figure 8. Billiard between four disks.

4.1. Numerically produced singular homoclinic orbits

To examine the appearance of singular homoclinic orbits we consider the billiard in a domain bounded by four symmetrical circles

$$x^2 + \left(y \pm \frac{1}{\gamma}\right)^2 = R^2; \qquad (x \pm \frac{1}{\gamma})^2 + y^2 = R^2$$

where $R^2 = 1 + (1 - \frac{1}{\gamma})^2$. The quantity γ (which is, approximately, the curvature of the circles) serves as the free parameter for unfolding the singularity. We found explicitly the corresponding billiard map, and using DSTOOL package [12], we found numerically hyperbolic periodic orbits of this mapping and their stable and unstable manifolds. The billiard map is found on the fundamental domain of the billiard—a triangular region cut by an arc as shown in figure 8. We find the return map to the slanted side of the triangle, which is parametrized by s, the horizontal coordinate, and by ϕ, the outgoing angle to the normal vector $(-1, -1)$, see figure 8. We choose an arbitrary value of γ and the simplest hyperbolic non-singular periodic orbit, as shown in the figure (the fixed point of the return map to the slanted side of the reduced domain). Then, we construct the stable and unstable manifolds for this periodic orbit. We examine how these manifolds vary by small variation of γ, until we find a value of γ for which a singular homoclinic orbit appears. The success (see figure 9 and 10) of the very crude search for such a delicate phenomena, near every γ value we have chosen, supports conjecture 2 regarding the density of systems for which such orbits exist. In fact we have found, by such a search near $\gamma_i = i * 0.05, i = 1, \ldots, 10$, eleven sharp homoclinics to this specific periodic orbit (at $\gamma \approx 0.0837$, 0.101 65, 0.1018, 0.153, 0.2077, 0.2552, 0.292 45, 0.3329, 0.3832, 0.4143, 0.4692).

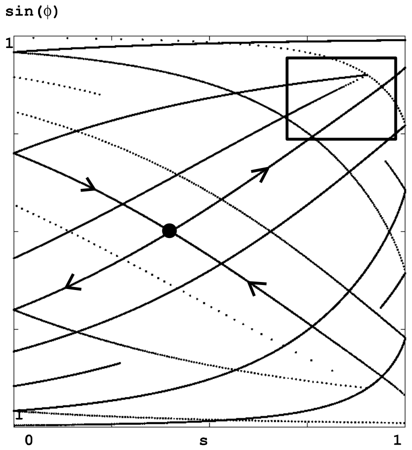

Figure 9. Numerically produced sharp homoclinics.

5. Conclusion

The main result of this paper is that we have established that if a scattering billiard (we use the particular hyperbolic structure associated with such billiards) has a singular periodic orbit or a singular homoclinic orbit, then arbitrarily close to it smooth Hamiltonian flows may possess elliptic islands, hence these are not ergodic (theorem 1 and 2). Moreover, we have conjectured, and have provided numerical support to these conjectures, that billiards with singular periodic orbits and singular homoclinic orbits are dense among scattering billiards (conjectures 1 and 2 of section 4). If these conjectures are correct, then our results will imply that arbitrarily close to any scattering billiard there exists a family of non-ergodic smooth Hamiltonian flows.

Such statements imply that *ergodicity and mixing results concerning two-dimensional non-smooth systems cannot be directly applied to the smooth dynamics they model*. Whether the same holds for higher-dimensional systems, e.g. three-dimensional billiards or multi-particle billiards, is yet to be studied.

On the other hand, even though stability islands may appear in smooth billiard-like problems, the size of an individual island is expected to be very small. Thus, without doubt, while the smooth flow may be non-ergodic, it will 'seem' to be ergodic for a very long time. Statistics (e.g. correlation functions) which are based upon finite-time realizations may appear to behave as in the scattering billiards (e.g. fall off quasi-exponentially [7]). Whether longer realizations will reveal very different statistical properties, depends on the number of elliptic islands, the total area they cover in the phase space and in the parameter space, and on the 'typical' period of the islands. Thus, estimates of the islands sizes, their periods, and of the real potential steepness (the 'physical ϵ') are necessary to supply estimates on the time scale for which the mixing property will appear to hold.

We may try to estimate the periodicity of the elliptic periodic orbits of smooth flows approaching generic scattering billiards, by very naive arguments. Indeed, since stable periodic orbits are generated from singular periodic orbits of the billiard, one may expect (if conjecture 1 is correct) that the least period of stable periodic orbits of a smooth Hamiltonian system which is ϵ-close to the billiard is of the order of the Poincaré return time to an ϵ-neighbourhood of the singularity surface for the billiard flow. Note that the billiard flow

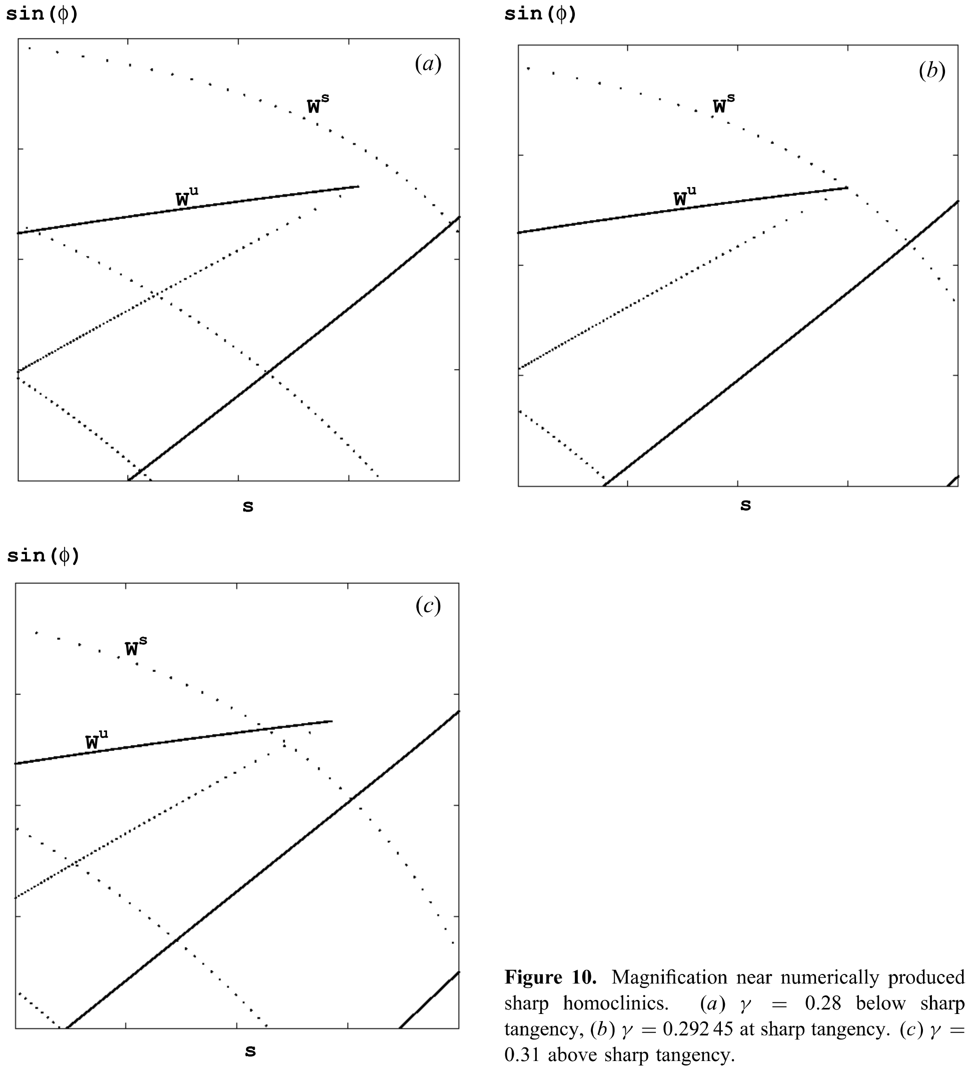

Figure 10. Magnification near numerically produced sharp homoclinics. (*a*) $\gamma = 0.28$ below sharp tangency, (*b*) $\gamma = 0.292\,45$ at sharp tangency. (*c*) $\gamma = 0.31$ above sharp tangency.

is a hyperbolic system; therefore, the return time in the billiard and, correspondingly, the typical period of the stable periodic motions in its smooth approximation must, essentially, be *logarithmic* in ϵ and not of a power-law type. Namely, very small ϵ values, corresponding to very steep potentials, may still produce stability islands which are observable on physical timescales.

Acknowledgments

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Appendix A. Class of smooth Hamiltonians

We prove the following.

Theorem 3. *Consider the Hamiltonian systems associated with,*

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y; \varepsilon) + H_\epsilon(x, y, p_x, p_y; \epsilon). \tag{A.1}$$

*If the potential $V(x, y; \varepsilon)$ satisfies conditions **I–IV** stated below, and $H_\epsilon(x, y, p_x, p_y; \epsilon)$ tends to zero as $\epsilon \to 0$ uniformly in some neighbourhood of $\bar{D}$ along with all its derivatives, then the Hamiltonian flow (A.1) r-converges to the billiard flow in D.*

Note that, in particular, H_ϵ, which is introduced for greater genericity, may be taken to be identically zero.

*A.1. Conditions **I–IV** on $V(x, y; \varepsilon)$*

Condition I. *For any compact region $K \subset D$ the potential $V(x, y; \varepsilon)$ diminishes along with all its derivatives as $\varepsilon \to 0$:*

$$\lim_{\varepsilon \to +0} ||V(x, y; \varepsilon)|_{\{(x,y)\in K\}}||_{C^{r+1}} = 0. \tag{A.2}$$

The growth of the potential to infinity across the boundary is a more delicate issue. We assume that V is evaluated along the level sets of some *finite* function near the boundary. Namely, suppose that in a neighbourhood of $(\bar{D}\backslash C)$ (C is the set of corner points) there exists *a pattern function* $Q(x, y; \varepsilon)$ which is C^{r+1} with respect to (x, y) and it depends continuously on ε (in C^{r+1}-topology) at $\varepsilon \geqslant 0$ (it has, along with all derivatives, a proper limit as $\varepsilon \to 0$). Assume that:

Condition IIa. *The billiard boundary is composed of level lines of $Q(x, y; 0)$:*

$$Q(x, y; \varepsilon = 0)|_{(x,y)\in S_i} \equiv Q_i = \text{constant}. \tag{A.3}$$

For each boundary component S_i, for Q close to Q_i, let us define *a barrier function* $W_i(Q; \varepsilon)$ which does not depend explicitly on (x, y) and assume that:

Condition IIb. *There exists a small neighbourhood N_i of the arc S_i in which*

$$V(x, y; \varepsilon)|_{(x,y)\in N_i} \equiv W_i(Q(x, y; \varepsilon); \varepsilon) \tag{A.4}$$

and

Condition IIc. ∇V *does not vanish in a finite neighbourhood of the boundary arcs, thus:*

$$\nabla Q|_{(x,y)\in N_i} \neq 0 \tag{A.5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}Q} W_i(Q; \varepsilon) \neq 0. \tag{A.6}$$

Now, the rapid growth of the potential across the boundary may be described in terms of the barrier functions W_i alone. Choose any of the arcs S_i and henceforth suppress the index i. Without loss of generality assume $Q = 0$ on S. By (A.5), the pattern function Q is monotonically increasing across S and assume Q is positive inside D near S and negative outside (otherwise, change inequalities in (A.7) to the opposite ones). Assume the following.

Condition III. *As $\varepsilon \to +0$ the barrier function increases from zero to infinity across the boundary S_i:*

$$\lim_{\varepsilon \to +0} W(Q; \varepsilon) = \begin{cases} +\infty & Q < 0 \\ 0 & Q > 0. \end{cases} \tag{A.7}$$

To formulate the final condition on the potential, note that by (A.6) the value of Q may be considered as a function of W (and ε) near the boundary arc. At small ε, a finite change in W corresponds to a small change in Q (by **III**). Therefore, the following condition makes sense.

Condition IV. *As $\varepsilon \to +0$, for any finite, strictly positive W_1 and W_2, the function $Q(W; \varepsilon)$ tends to zero uniformly on the interval $[W_1, W_2]$ along with all its $(r + 1)$ derivatives.*

A few remarks are now in order.

We study the limiting behaviour (as $\varepsilon \to +0$) of the smooth Hamiltonian system (A.1) in a neighbourhood of a given billiard orbit, thus near a *fixed* non-zero energy level $H =$ constant. The conservation of energy implies that all trajectories stay in the region $W \leqslant H^*$ for any ε. It follows, in particular, that the symbol $+\infty$ in (A.7) may be replaced by any value greater than H^*.

Clearly, if the potential V satisfies condition **I**, the particle moves in the interior of D with essentially constant velocity along a straight line until it reaches a thin layer near the boundary S where the potential runs from small to very large values (the smaller the value of ε, the thinner the boundary layer). By **III**, if the particle enters the layer near an interior point of some boundary arc (corner points are not considered in this paper), it is either reflected, exiting the boundary layer near the point where it entered, or it might, in principle, stick into the layer, travelling along the boundary far away from the entrance point. Conditions **II** formalize the natural requirement that the reaction force must be normal to the boundary, so they guarantee that the reflection will be of the right character, approximately preserving the tangential component (p_x) of the momentum and changing the sign of the normal component (p_y). However, conditions **I**–**III** are insufficient for preventing the existence of non-reflecting trajectories; adding condition **IV** with $r = 1$ guarantees that the travel distance along the boundary vanishes asymptotically.

Moreover, condition **II** guarantees a correct reflection law only in the C^0-topology and not in the C^r-topology. To explain this statement, take the same initial conditions $(x_0, y_0, p_{x0}, p_{y0})$ for an orbit of the Hamiltonian system (A.1) and for a billiard orbit. Consider a time interval t for which the billiard orbit collides with the boundary S only once, at some point (x_c, y_c) (see figure A.1). Here, the incidence angle ϕ^{in} is the angle between the vector $(x_0 - x_c, y_0 - y_c)$ and the inward normal to S at the point (x_c, y_c); the reflection angle ϕ^{out} is the angle between the vector $(x_t - x_c, y_t - y_c)$ and the normal, where (x_t, y_t) is the point reached by the billiard trajectory at the time t. Define the incidence and reflection angles for the trajectory of the Hamiltonian system in the same way where (x_c, y_c) is set by the billiard trajectory and $(x_t(\varepsilon), y_t(\varepsilon))$ is defined by the Hamiltonian flow

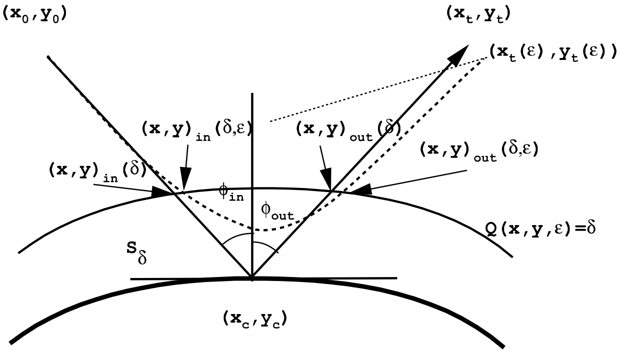

Figure A.1. Reflection by Hamiltonian flow.

(see figure A.1). We expect the trajectory of the Hamiltonian system to be close to the billiard trajectory; in particular, it should demonstrate a correct reflection law

$$\phi^{\text{in}}(\varepsilon) + \phi^{\text{out}}(\varepsilon) \approx 0$$

for sufficiently small ε. Note, however, that $(\phi^{\text{in}}+\phi^{\text{out}})$ is a function of the initial conditions, and to satisfy assumption $\mathbf{A_r}$ this function must be close to zero *along with all the derivatives*.

Conditions **I–IV** are in fact quite general; for the pattern function, consider any smooth function Q depending on two variables (x, y). Corners are created at the singularities of the level sets and at the points of inflection. For the barrier function $W(Q, \varepsilon)$ many 'classical' monotonically decreasing functions satisfy **I–IV**, see the list (2.6). Moreover, one may easily produce more examples as there is no restriction on the growth rate: given any potential V satisfying conditions **I–IV** the potential $\psi(V)$ also satisfies these conditions provided ψ is a smooth, non-singular, strictly monotonic function of $V \in [0, \infty)$ such that $\psi(0) = 0$, $\psi(\infty) = \infty$.

Proof of theorem 3. We should prove that assumption $\mathbf{A_r}$ is satisfied for any inner point q_0 whose time t billiard trajectory does not enter the corner points. It is enough to consider the case where the billiard trajectory hits the boundary only once on the time interval under consideration. The two different cases of tangent and non-tangent trajectories are considered.

We use the term *the smooth* orbit of q_0 for the orbit of the flow defined by the Hamiltonian (A.1). Since the Hamiltonian flow is C^r-close to the billiard flow outside an arbitrarily small boundary layer (by virtue of **I**), we only need to consider the behaviour of the smooth orbit in the boundary layer $N_\delta = \{|Q(x, y; \varepsilon) - Q(x_c, y_c; \varepsilon)| \leqslant \delta\}$ where (x_c, y_c) is the collision point for the billiard trajectory of q_0. The quantity δ slowly tends to zero as $\varepsilon \to +0$.

For small ε, the smooth trajectory enters N_δ at some time $t^{\text{in}}(\delta, \varepsilon)$ at a point $(x^{\text{in}}(\delta, \varepsilon), y^{\text{in}}(\delta, \varepsilon))$ close to (x_c, y_c) with the velocity $(p_x^{\text{in}}(\delta, \varepsilon), p_y^{\text{in}}(\delta, \varepsilon))$ close to (p_{x0}, p_{y0}). We denote the moment of exiting the boundary layer as $t^{\text{out}}(\delta, \varepsilon)$ and the corresponding phase point is denoted as $q^{\text{out}}(\delta, \varepsilon)$. Our aim is to prove that in the limit $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}$ in the C^r (resp. C^0) topology for the non-tangent (resp. tangent) case,

$$\begin{aligned}
&(x^{\text{out}}, y^{\text{out}}, t^{\text{out}}) - (x^{\text{in}}, y^{\text{in}}, t^{\text{in}}) \to 0\\
&(p_x^{\text{out}})^2 + (p_y^{\text{out}})^2 - (p_x^{\text{in}})^2 - (p_y^{\text{in}})^2 \to 0\\
&(p_x^{\text{out}} - p_x^{\text{in}})Q_y(x^{\text{in}}, y^{\text{in}}) - (p_y^{\text{out}} - p_y^{\text{in}})Q_x(x^{\text{in}}, y^{\text{in}}) \to 0
\end{aligned} \tag{A.8}$$

which coincides with the billiard reflection law as $\delta \to 0$. The second line of (A.8) is clearly correct; since $Q^{\text{out}} = Q^{\text{in}}$, and $H_\epsilon \to 0$ uniformly in $\bar{D}$, the conservation of energy implies that the total momentum is asymptotically conserved.

Move the origin of the coordinate system in D to the reflection point, so $(x_c, y_c) = (0, 0)$; without loss of generality assume $Q = 0$ at the origin. By condition **IIa**, the boundary arc passing through the point of reflection is $Q(x, y; 0) = 0$. Let the interior of D correspond to positive values of $Q(x, y; 0)$. Let the x-axis be tangent to the level line $Q(x, y; \varepsilon) = 0$ and the y-axis be inward normal to it. Thus, the partial derivatives Q_x and Q_y satisfy

$$Q_x|_{(0,0;\varepsilon)} = 0, \qquad Q_y|_{(0,0;\varepsilon)} = 1. \tag{A.9}$$

By (A.1), near the boundary the equations of motion have the form,

$$\begin{aligned} \dot{x} &= p_x + \mathrm{o}(1) \qquad & \dot{p}_x &= -W'(Q)Q_x + \mathrm{o}(1) \\ \dot{y} &= p_y + \mathrm{o}(1) \qquad & \dot{p}_y &= -W'(Q)Q_y + \mathrm{o}(1) \end{aligned} \tag{A.10}$$

where the o(1) terms correspond to the partial derivatives of $H_\epsilon(x, y, p_x, p_y; \epsilon)$ which are assumed to be uniformly small.

Lemma A.1. *There exists $\xi(\delta, \epsilon)$ which diminishes to zero as $\delta \to 0, \epsilon \to 0$ such that*

$$t^{\text{out}} - t^{\text{in}} \leqslant \xi \tag{A.11}$$

and for any $t \in [t^{\text{in}}, t^{\text{out}}]$

$$x(t) = x^{\text{in}} + \mathrm{O}(\xi), \qquad y(t) = y^{\text{in}} + \mathrm{O}(\xi) \tag{A.12}$$

$$p_x(t) = p_x^{\text{in}} + \mathrm{O}(\xi). \tag{A.13}$$

$$\frac{p_y^2(t)}{2} + W(Q(x(t), y(t)); \epsilon) = \frac{p_y^2(t^{\text{in}})}{2} + W(\delta; \epsilon) + \mathrm{O}(\xi). \tag{A.14}$$

Proof. First we prove that (A.12)–(A.14) are valid for any $t \in [t^{\text{in}}, t^{\text{in}} + \xi]$ for any ξ (provided $\xi \geqslant \mathrm{o}(1)$ terms coming from H_ϵ). Indeed, (A.12) follows since $\dot{x}$ and $\dot{y}$ are uniformly bounded by the energy constraint. From (A.9) and (A.12) it follows that

$$Q_x(x, y; \varepsilon) = \mathrm{O}(\xi), \qquad Q_y(x, y; \varepsilon) = 1 + \mathrm{O}(\xi). \tag{A.15}$$

at $t \in [t^{\text{in}}, t^{\text{in}} + \xi]$. Divide this time interval into two regions: $I_<$ where $|W'(Q)| < 1$ and $I_>$ where $|W'(Q)| \geqslant 1$. In $I_<$, the change in p_x is obviously $\mathrm{O}(\xi)$. In $I_>$, since $Q_y \neq 0$ (see (A.15)), $\dot{p}_y$ does not vanish, hence $\dot{p}_x$ may be divided to $\dot{p}_y$ in (A.10):

$$\frac{\mathrm{d}p_x}{\mathrm{d}p_y} = \frac{Q_x}{Q_y} + \mathrm{o}(1). \tag{A.16}$$

By (A.15), this implies that the change in p_x is $\mathrm{O}(\xi)$ times the total variation in p_y. The latter is uniformly bounded; indeed, p_y is a uniformly bounded smooth function of time, the time interval under consideration is finite, and the derivative $\dot{p}_y$ is bounded from below. Thus, (A.13) is proved. The approximate conservation law (A.14) follows from (A.13) and the exact conservation of energy H.

To complete the proof, we take $\xi \gg \sqrt{\delta}$, and prove that for sufficiently small ϵ (for which (A.12)–(A.14) are satisfied), the trajectory which enters the boundary layer N_δ at $t = t^{\text{in}}$ must exit it before $t^{\text{in}} + \xi$.

Consider first the case of nearly tangent trajectory, where $p_y^{\rm in} \approx 0$. By (A.14), since inside N_δ the value of $W(Q)$ is bigger than $W^{\rm in} = W(\delta)$, it follows that $|p_y(t)| = |p_y^{\rm in}| + \mathrm{O}(\sqrt{\xi})$. Thus, $|p_y|$ stays small unless the trajectory leaves N_δ or $t - t^{\rm in}$ becomes large. For positive energy level, the smallness of $|p_y^{\rm in}|$ implies that $|p_x^{\rm in}| > 0$. By (A.13) it follows that $p_x(t)$ is bounded away from zero, on the same time interval. Hence, the trajectory is close to a straight line parallel to the x-axis.

We assume the curvature of the boundary arcs is non-zero; i.e. $Q_{xx} \neq 0$. Thus, a straight line must exit the boundary layer $|Q| \leqslant \delta$ at a distance $\mathrm{O}(\sqrt{\delta})$ from the entrance point. Since $|p_x(t)|$ is bounded away from zero, the time spent by the nearly tangent trajectory in the boundary layer is $t^{\rm out} - t^{\rm in} = \mathrm{O}(\sqrt{\delta})$, i.e. it is indeed less than the chosen ξ.

Now consider the case of non-tangent trajectory, so that $p_y^{\rm in}(\delta, \varepsilon)$ is bounded away from zero (it is negative). Since the value of $W^{\rm in} = W^{\rm out} = W(Q = \delta)$ vanishes as $\varepsilon \to +0$, it follows from (A.14) that the normal momentum $p_y(t)$ stays bounded away from zero unless the potential $W(Q)$ reaches some finite value.

Therefore, if we take some sufficiently small ν and consider the part $N^{(1)}$ of the boundary layer N_δ which corresponds to *small* values of W: $W(Q;\varepsilon) \leqslant \nu$, then the value of $\frac{\mathrm{d}}{\mathrm{d}t}Q(x,y) = p_x Q_x + p_y Q_y$ is bounded away from zero in $N^{(1)}$ (because Q_x is small and p_y is non-zero). Thus, the trajectory entering $N^{(1)}$ must approach the inner part $N^{(2)} : W(Q;\varepsilon) \geqslant \nu$ at time which is proportional to the width of $N^{(1)}$ (it is $O(\delta)$). Moreover, if the trajectory leaves $N^{(2)}$ after some time, it must have positive p_y, hence, by the same arguments, it must leave the whole boundary layer N_δ after an additional time of order δ. It follows that $t^{\rm out} - t^{\rm in} = \mathrm{O}(\delta)$+the time spent in $N^{(2)}$. The latter, in turn, vanishes as $\epsilon \to 0$. Indeed, since $|p_y|$ is bounded from above, this time must be bounded by constant $\times$ $((\min_{N^{(2)}} W'(Q))^{-1})$ (see (A.10)). Now note that in $N^{(2)}$ the value of the potential is bounded away from zero (and it is bounded from above by the initial value of H) whence, according to condition **IV**, $W'(Q)(\equiv Q'(W)^{-1}) \to \infty$ as $\epsilon \to 0$.

Thus, we have shown that as $\epsilon \to 0$, the total collision time is $\mathrm{O}(\delta)$ in the non-tangent case. This completes the lemma. □

This lemma proves the C^0-version of the theorem (indeed, cf (A.11)–(A.14) with (A.8) and note that in our coordinate frame $Q_x \to 0$ as $(x,y) \to (x_c, y_c)$). Thus, it remains to prove the C^r-convergence for the non-tangent case.

As in the lemma above we divide N_δ into two parts $N^{(1)} : W \leqslant \nu$ and $N^{(2)} : W \geqslant \nu$. There is a freedom in the choice of ν and we consider the limit $\lim_{\delta\to 0}\lim_{\nu\to 0}\lim_{\epsilon\to 0}$.

In $N^{(1)}$, the value of $\dot{Q}$ is non-zero. Thus, we divide the equations of motion (A.10) to $\dot{Q}$:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}Q} &= \frac{p_x}{Q_x p_x + Q_y p_y} + \mathrm{o}(1) \qquad & \frac{\mathrm{d}p_x}{\mathrm{d}Q} &= -W'(Q)\frac{Q_x}{Q_x p_x + Q_y p_y} + \mathrm{o}(1) \\
\frac{\mathrm{d}y}{\mathrm{d}Q} &= \frac{p_y}{Q_x p_x + Q_y p_y} + \mathrm{o}(1) \qquad & \frac{\mathrm{d}p_y}{\mathrm{d}Q} &= -W'(Q)\frac{Q_y}{Q_x p_x + Q_y p_y} + \mathrm{o}(1) \\
\frac{\mathrm{d}t}{\mathrm{d}Q} &= \frac{1}{Q_x p_x + Q_y p_y} + \mathrm{o}(1)
\end{aligned} \tag{A.17}$$

or, in the integral form,

$$
\begin{aligned}
&x(Q_2) - x(Q_1) = \int_{Q_1}^{Q_2} X\,\mathrm{d}Q \qquad y(Q_2) - y(Q_1) = \int_{Q_1}^{Q_2} Y\,\mathrm{d}Q \\
&p_x(Q_2) - p_x(Q_1) = -\int_{W(Q_1)}^{W(Q_2)} P_{11}\,\mathrm{d}W(Q) + \int_{Q_1}^{Q_2} P_{12}\,\mathrm{d}Q \\
&p_y(Q_2) - p_y(Q_1) = -\int_{W(Q_1)}^{W(Q_2)} P_{21}\,\mathrm{d}W(Q) + \int_{Q_1}^{Q_2} P_{22}\,\mathrm{d}Q \\
&t(Q_2) - t(Q_1) = \int_{Q_1}^{Q_2} T\,\mathrm{d}Q
\end{aligned}
\tag{A.18}
$$

where X, Y, P_{ij}, T denote, schematically, some functions of (x, y, p_x, p_x) which are uniformly bounded along with all derivatives (see (A.17), the boundedness follows since $\dot{Q} \equiv Q_x p_x + Q_y p_y$ is bounded away from zero in $N^{(1)}$).

In the region under consideration, the change in W is bounded by the small ν and the change in Q is bounded by the small δ. Thus, the integrals in the right-hand side are small. It follows (applying, say, the successive approximation method) that as $\delta \to 0$, $\nu \to 0$, the Poincaré map from $Q = Q_1$ to $Q = Q_2$ which is found as the solution of (A.18), limits to the identical map, along with all derivatives with respect to initial conditions.

Thus, only the region $N^{(2)}$ gives a non-trivial contribution to the Poincaré map defined by the Hamiltonian flow. It is convenient to evaluate the Poincaré map in $N^{(2)}$ for the cross section in the phase space defined by fixing the absolute value of p_y rather than the corresponding value of W (by (A.14), it does not make a great difference).

In $N^{(2)}$, as $\epsilon \to 0$, the value of the potential is bounded away from zero and infinity ($\nu \leqslant W \leqslant H$). Thus, according to condition **IV**, the value of the pattern function Q may be considered as a function of the value of the potential $Q = Q(W; \varepsilon)$ and this function is uniformly small along with all derivatives.

In particular, $W'(Q) \equiv Q'(W)^{-1}$ is bounded away from zero. Thus, we may divide the equations of motion (A.10) to $\frac{\mathrm{d}}{\mathrm{d}t} p_y$ and take p_y as a new time variable. We obtain

$$
\frac{\mathrm{d}x}{\mathrm{d}p_y} = -Q'(W)\frac{p_x}{Q_y} + \mathrm{o}(1) \qquad \frac{\mathrm{d}p_x}{\mathrm{d}p_y} = \frac{Q_x}{Q_y} + \mathrm{o}(1). \tag{A.19}
$$

Here Q_x and Q_y are known functions of x and y and the value of y is uniquely determined by the values of x and Q (since $Q_y \neq 0$). The value of Q is considered as the function of the potential W and the value of W is found from the conservation of energy:

$$
W = H - \tfrac{1}{2}(p_x^2 + p_y^2) - H_\epsilon. \tag{A.20}
$$

Thus, equations (A.19) and (A.20) are self-consistent and define the orbit completely. According to condition **IV**, equations (A.19) have the following system as the C^r-limit as $\varepsilon \to +0$:

$$
\frac{\mathrm{d}x}{\mathrm{d}p_y} = 0 \qquad \frac{\mathrm{d}p_x}{\mathrm{d}p_y} = \frac{Q_x}{Q_y}\Big|_{Q(x,y;0)=0}. \tag{A.21}
$$

The solution of this system is the C^r-limit of the solution of (A.19) (because the change in p_y—i.e. the interval of integration—is finite). This, in fact, finishes the proof of the theorem, because the solution of (A.21) gives exact billiard reflection law: by the first equation, x is constant whence $x^{\mathrm{out}} = x^{\mathrm{in}} = x_c$ and the same is true for y and t, and plugging $(x, y) = (x^{\mathrm{in}}, y^{\mathrm{in}})$ in the right-hand side of the second equation gives the last equation of (A.8). $\square$

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