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On dynamical properties of multidimensional diffeomorphisms from Newhouse regions: I

S V Gonchenko[1], **L P Shilnikov**[1] **and D V Turaev**[2]

[1] Institute for Applied Mathematics and Cybernetics, 10 Ul'janov Street, Nizhny Novgorod 603005, Russia
[2] Department of Mathematics, Ben Gurion University, Beer Sheva 84105, Israel

E-mail: gosv100@uic.nnov.ru, lpshilnikov@mail.ru and turaev@math.bgu.ac.il

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Abstract
The phenomenon of the generic coexistence of infinitely many periodic orbits with different numbers of positive Lyapunov exponents is analysed. Bifurcations of periodic orbits near a homoclinic tangency are studied. Criteria for the coexistence of infinitely many stable periodic orbits and for the coexistence of infinitely many stable invariant tori are given.

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Introduction

Homoclinic tangency is a tangency between stable and unstable invariant manifolds of a saddle periodic orbit L. If the stable and unstable manifolds of L are tangent at some point, the orbit of such a point belongs to both the manifolds, so it is homoclinic to L, and at each point of this orbit the stable and unstable manifolds of L have a tangency. Typically, the tangency is quadratic. This is a codimension-1 bifurcation: in a generic one-parameter unfolding, the quadratic tangency of the stable and unstable invariant manifolds at a given point is either removed or transformed into a pair of transverse intersections. Still, as discovered by Newhouse [1, 2], there exist *open regions in the space of dynamical systems where systems with homoclinic tangencies are dense* (in the C^r-topology with any $r \geqslant 2$); moreover these regions exist in any neighbourhood of any two-dimensional diffeomorphism with a homoclinic tangency. If $\sigma \neq 1$ (the saddle value σ is the absolute value of the product of the multipliers of L), the Newhouse regions exist in any one-parameter family of diffeomorphisms which unfolds the quadratic homoclinic tangency generically [2]. These results were extended to the multidimensional case in [3–5] (a conservative version is proven in [6, 7]).

Since Newhouse regions exist near any system with a homoclinic tangency, they can be found in the space of parameters of virtually any dynamical model demonstrating chaotic behaviour in the absence of uniform hyperbolicity, in particular in popular examples such as the Hénon map, Chua circuit and Lorenz model (outside the region of existence of the Lorenz attractor, see [8]). Moreover, as numerics shows (see, e.g., [9]), Newhouse regions can be quite large.

The basic feature of chaotic dynamics of systems from these regions is its extreme richness. Indeed, systems having homoclinic tangencies of arbitrarily high orders—in fact, infinitely many coexisting homoclinic tangencies of all possible orders, are dense in the Newhouse regions, as well as systems having arbitrarily degenerate periodic orbits [10–13]. This means [14, 15] that any attempt to give a complete description of the dynamics and bifurcations in the Newhouse regions will fail. Here, one has to restrict the analysis to some particular details or some most general features only.

As the most important such general property of systems in the Newhouse regions, we select the *coexistence of many periodic orbits of different stability types* (i.e. with different numbers of positive/negative Lyapunov exponents). Thus, it has been known since [16] that coexisting stable and saddle periodic orbits are born at the bifurcations of two-dimensional maps with a quadratic homoclinic tangency to a saddle periodic orbit with $\sigma < 1$. In fact, in the case $\sigma < 1$, a generic map from the Newhouse region has infinitely many stable periodic orbits whose closure may include a non-trivial hyperbolic set with infinitely many saddle periodic orbits within [17].

These stable periodic orbits are born at the saddle-node bifurcations which occur in any generic one-parameter unfolding of a quadratic homoclinic tangency [16]. If $\sigma > 1$, there can be no stable periodic orbits near the homoclinic tangency, and the saddle-nodes of [16] disintegrate into one saddle and one completely unstable periodic orbit. In this case, the Newhouse construction gives infinitely many coexisting unstable periodic orbits (periodic repellers). In [18] we considered two-dimensional maps having a non-transverse heteroclinic cycle with two saddles and showed that if $\sigma < 1$ at one saddle and $\sigma > 1$ at the other saddle, then, in the corresponding Newhouse intervals, a generic map simultaneously possesses infinitely many periodic attractors and infinitely many periodic repellers, and the closures of the set of attractors and the set of repellers intersect along a non-trivial hyperbolic set.

The goal of this paper is to investigate the problem of coexistence of different types of periodic orbits near homoclinic tangencies in the multidimensional case as well as to study main bifurcations here. We show that the cases of a saddle and a saddle-focus are pretty much different. A saddle periodic orbit is called simply *a saddle* when both its stable and unstable leading multipliers (i.e. those nearest to the unit circle) are real, but it is called *a saddle-focus* if there is a pair of complex conjugate numbers among the leading multipliers[3]. Generically, i.e. for the so-called *simple homoclinic tangencies* (see section 1.1), bifurcations near a homoclinic tangency to a multidimensional saddle follow the same pattern as in the two-dimensional case. However, we have here additional (non-leading) stable and unstable directions. Therefore, instead of bifurcations of saddle-node periodic orbits, there may occur bifurcations of saddle-saddles which give rise to a pair of saddle periodic orbits with different dimensions of unstable manifolds. Thus, in the multidimensional case, there may exist Newhouse regions where systems have neither stable nor completely unstable periodic orbits, but systems with infinitely many coexisting saddles with different dimensions of unstable manifolds are dense there. This phenomenon, also discussed in [5, 19], was used in the construction of a wild spiral attractor in [20].

In the case of a saddle-focus we show that there can be more complicated bifurcations than saddle-nodes and period-doublings. Indeed, in this case periodic orbits can be born with *more than one* multiplier on the unit circle (theorem 1).

[3] We use the term saddle periodic orbit when we do not distinguish between a saddle and a saddle-focus.

Based on this, we show (theorem 3) that even when there are no non-leading directions there may exist Newhouse regions where systems with infinitely many coexisting periodic orbits with *more than two* different indices of instability are dense. For example, consider a four-dimensional diffeomorphism with a homoclinic tangency to a fixed point for which all the multipliers are complex (we call this point a saddle-focus $(2, 2)$). Then, under certain conditions on the multipliers, in the corresponding Newhouse regions, maps with infinitely many coexisting sinks and saddles with one-dimensional, two-dimensional and three-dimensional unstable manifolds are dense.

In fact, theorem 3 describes all possible types of hyperbolic periodic orbits which may exist in systems close to a system with a simple homoclinic tangency. Here, as in [19], we compute the so-called *effective dimension of the problem*, d_e, which is determined by the relations between the leading multipliers. It may take values 1, 2 or 3. We show that in the corresponding Newhouse regions a generic system has *infinitely many coexisting periodic orbits with* $(d_e + 1)$ *different indices of instability*.

We pay special attention to the birth of stable periodic orbits. Namely, we prove (theorem 5) that *if there are no unstable non-leading multipliers and the absolute value* J *of the product of all the leading multipliers is less than* 1*, then generic systems from the corresponding Newhouse regions have infinitely many coexisting sinks* (for partial results, see [4, 17, 21–23]). Note that if the conditions of theorem 5 are violated, namely, *if* $J > 1$ *or if there exist unstable non-leading multipliers, then no stable periodic orbits can be born* at the bifurcations of the corresponding simple homoclinic tangency at all, as follows from results of [19] on uniform partial hyperbolicity near non-transverse heteroclinic/homoclinic cycles (see theorem 4).

We also give conditions for the coexistence of infinitely many *non-trivial attractors* for multidimensional maps from the Newhouse regions. In the case of a homoclinic tangency to a saddle-focus with $d_e \geqslant 2$, theorem 1 gives us periodic orbits with two or three multipliers on the unit circle. Analysis of bifurcations of these periodic orbits allowed us to show (theorem 6) that if $d_e \geqslant 2$, then under conditions of theorem 5 in the corresponding Newhouse regions *generic maps have infinitely many stable invariant closed curves*.

Also note that in the case $d_e = 3$, e.g. in the above four-dimensional example with a saddle-focus $(2, 2)$, bifurcations of triplets of unit multipliers can lead to the birth of spiral and Lorenz-like attractors [24–28], so one may expect here *infinitely many coexisting chaotic attractors* also.

We stress the fact that we are speaking about Newhouse regions in finite-parameter families of diffeomorphisms. The number of parameters that we need equals d_e (theorem 1 gives periodic orbits with d_e unit multipliers, so the number of bifurcation parameters must not be less than that). The first parameter, μ, controls the splitting of the stable and unstable manifolds near a point of the homoclinic tangency. Other parameters are the arguments φ and ψ of complex leading multipliers at the saddle-focus. Our choice of φ and ψ as additional governing parameters is dictated by the fact [13, 29, 30] that they are invariants of the local Ω-conjugacy[4] (the so-called Ω-moduli) for the systems with a homoclinic tangency to a saddle-focus. Moreover, even when the homoclinic tangency is not split, any change in the values of these Ω-moduli leads to changes in the structure of the set of single-round periodic orbits [29, 30], so choosing the Ω-moduli as bifurcation parameters is only natural.

Periodic orbits which lie in a neighbourhood U of a homoclinic orbit Γ to a saddle periodic orbit L are called *single-round* if they leave a small neighbourhood U_0 of L only once; after

[4] Topological conjugacy on the set of non-wandering orbits lying entirely in a small neighbourhood of the given homoclinic tangency.

that they follow $U \backslash U_0$, return to U_0 and close up. The corresponding Poincaré map is called *a first-return map*; fixed points of first-return maps correspond to single-round periodic orbits. Note that there is an infinite sequence of the first-return maps T_k defined near the homoclinic orbit Γ. Here, the integer k runs over all sufficiently large values, and it is equal to the number of iterations that the orbit makes within the small neighbourhood of L before it makes an excursion along Γ. Obviously, large k correspond to the starting points close to the stable manifold of L.

In the rescaling lemmas of section 1.4 we show that one can rescale the coordinates and parameters in such a way that the first-return maps near an orbit of homoclinic tangency take a particular form which is asymptotically, as $k \to +\infty$, C^r-close to a certain special quadratic map. Essentially, lemmas 1–3 say that when the absolute value J of the product of the leading multipliers is less than 1, the dynamics near single-round periodic orbits is described by iterations of one of the maps below[5]

(i) *parabola map* $\bar{y} = M - y^2$—in the case $d_e = 1$;

(iia) *Hénon map* $\bar{x} = y$, $\bar{y} = M - Bx - y^2$ or

(iib) *generalized Hénon map* $\bar{x} = y$, $\bar{y} = M - Bx - y^2 + Q_k xy$—in the case where $d_e = 2$ and the stable leading multipliers form a complex-conjugate pair while the unstable leading multiplier is real and single;

(iii) *Mira map* $\bar{x} = y$, $\bar{y} = M - Cy - x^2$—when $d_e = 2$ and L is a saddle-focus with a complex-conjugate pair of unstable leading multipliers;

(iv) *a three-dimensional Hénon map* $\bar{x} = y$, $\bar{y} = z$, $\bar{z} = M - Bx - Cz - y^2$—when $d_e = 3$ and both the stable and unstable leading multipliers are complex.

Essentially, M is the rescaled splitting parameter μ, while B and C are rescaled deviations of the governing parameters φ and ψ from their initial values. The rescaling factors tend to infinity as $k \to +\infty$; therefore, arbitrarily small changes in μ, φ and ψ cause variations of M, B and C within an arbitrarily large range, provided k is large enough. The non-zero coefficient Q_k in the generalized Hénon map is small and depends only on k and on some invariants of the homoclinic structure.

The fact that parabola map (i) appears as the rescaled first-return map near a quadratic homoclinic tangency is known since the paper [31] where the two-dimensional case was considered. Our lemma 4 shows that the same is true in the multidimensional case. However, if $d_e \geqslant 2$, then in certain regions of parameter values one can make a rescaling to d_e-dimensional quadratic maps (ii)–(iv).

Map (iib) (the generalized Hénon map) was constructed in [32] as the rescaled first-return map for two-dimensional diffeomorphisms with a quadratic homoclinic tangency to a saddle of neutral type (i.e. with $\sigma = 1$). The same map appears as the rescaled first-return map near a non-transverse heteroclinic cycle [33, 34]. It was also shown in [35–37] that maps (iia), (iib) and (iii) emerge near a *non-simple* quadratic homoclinic tangency.

Most of the paper is occupied by the proof of rescaling lemmas. Other results are deduced from them based on the analysis of bifurcations in maps (i)–(iv). We use here the fact [3] that in the Newhouse regions under consideration parameter values are dense which correspond to a simple homoclinic tangency to the *same* saddle periodic orbit L; hence bifurcations of single-round periodic orbits near any such homoclinics are described by the same rescaled map.

Many of the results of this paper were announced in [38], cases without non-leading multipliers were considered in [15, 39, 40] (see, e.g., [13]). While in this paper we restrict ourselves mostly to the phenomenon of the coexistence of periodic orbits of different types and to their bifurcations, other interesting effects such as high order homoclinic

[5] When $J > 1$, the inverse to the first-return map should be considered.

tangencies, coexistence of infinitely many strange attractors (hyperbolic, Lorenz-like), persistent heteroclinic connections between different types of periodic orbits will be considered in forthcoming papers.

1. Setting the problem and main results

1.1. Simple homoclinic tangency

Let f be a C^r-smooth, $r \geqslant 2$, diffeomorphism of an $(m+n)$-dimensional, $m \geqslant 1, n \geqslant 1$, smooth manifold. Suppose that f has a saddle periodic orbit L such that $\dim W^s(L) = m$, $\dim W^u(L) = n$ and, besides, the stable $W^s(L)$ and unstable $W^u(L)$ invariant manifolds of L intersect non-transversely at the points of some homoclinic orbit Γ_0.

Let $\lambda_1, \ldots, \lambda_m, \gamma_1, \ldots, \gamma_n$ be the multipliers of L ordered so that $|\gamma_n| \geqslant \cdots \geqslant |\gamma_1| > 1 > |\lambda_1| \geqslant \cdots \geqslant |\lambda_m|$. The multipliers inside the unit circle (i.e. λ_i) are called *stable* and those outside the unit circle (i.e. γ_j) are called *unstable*. Denote $\lambda = |\lambda_1|$, $\gamma = |\gamma_1|$. Those multipliers which are equal in absolute value to λ or γ are called *leading multipliers*, and the rest are called *non-leading*. Denote as n_s and n_u the numbers of the leading stable and, respectively, unstable multipliers and assign the type (n_s, n_u) to L. Suppose that the following condition holds.

A. The leading multipliers of L are simple, and L is of one of the four following types:

(1, 1) λ_1 and γ_1 are real, and $\lambda >| \lambda_2 |, \gamma <| \gamma_2 |$;
(2, 1) $\lambda_{1,2} = \lambda e^{\pm i\varphi}(\varphi \neq 0, \pi)$, γ_1 is real and $\lambda >| \lambda_3 |, \gamma <| \gamma_2 |$;
(1, 2) λ_1 is real, $\gamma_{1,2} = \gamma e^{\pm i\psi}(\psi \neq 0, \pi)$ and $\lambda >| \lambda_2 |, \gamma <| \gamma_3 |$;
(2, 2) $\lambda_{1,2} = \lambda e^{\pm i\varphi}$, $\gamma_{1,2} = \gamma e^{\pm i\psi}(\varphi, \psi \neq 0, \pi)$ and $\lambda >| \lambda_3 |, \gamma <| \gamma_3 |$.

L is called a saddle in the first case and a saddle-focus in the other cases.

Define $J = \lambda^{n_s}\gamma^{n_u}$, i.e. J is the absolute value of the product of the leading multipliers. A system in the general position satisfies either of the two conditions:

B. $J < 1$, and $\lambda\gamma \neq 1$ in case (2, 1) or $\lambda\gamma^2 \neq 1$ in case (2, 2) or

B′. $J > 1$, and $\lambda\gamma \neq 1$ in case (1, 2) or $\lambda^2\gamma \neq 1$ in case (2, 2).

In fact, by considering diffeomorphism f^{-1} instead of f, condition B is transformed to B′ and vice versa. Therefore, it suffices to consider only the case where B holds.

The meaning of the quantity J is quite simple: if L has no non-leading multipliers, then J is the Jacobian of the Poincaré map at L, so the volumes are contracted near L if $J < 1$ and expanded if $J > 1$.

We will need more information about the volume-contraction properties near L. Assume condition B holds and introduce an 'effective dimension' d_e [19]:

$d_e = 1$—in case (1, 1) and in case (2, 1) at $\lambda\gamma < 1$;
$d_e = 2$—in case (2, 1) at $\lambda\gamma > 1$, in case (1, 2), and in case (2, 2) at $\lambda\gamma^2 < 1$;
$d_e = 3$—in case (2, 2) at $\lambda\gamma^2 > 1$.

By construction, since we assume $J < 1$, it follows that if L has no non-leading multipliers, then $(d_e + 1)$-dimensional volumes are exponentially contracted near L while d_e-dimensional volumes may be expanded by the iterations of f.

Let p be a period of L, i.e. L is a set of p points $\{O, f(O), \ldots, f^{p-1}(O)\}$ and $f^p(O) = O$. Denote as T_0 the restriction of the Poincarè map f^p onto a small neighbourhood U_0 of O. We call T_0 *the local map*; O is its fixed point of T_0. By putting the origin of the coordinate system to O, the map T_0 takes the following form

$$\bar{x} = A_1 x + \cdots, \qquad \bar{u} = A_2 u + \cdots, \qquad \bar{y} = B_1 y + \cdots, \qquad \bar{v} = B_2 v + \cdots, \tag{1.1}$$

where $x \in R^{n_s}, y \in R^{n_u}, u \in R^{m-n_s}, v \in R^{n-n_u}$; the dots stand for nonlinear terms; the eigenvalues of A_1 and B_1 are, respectively, the stable and unstable leading multipliers of O; the eigenvalues of A_2 and B_2 are, respectively, the stable and unstable non-leading multipliers of O. Accordingly, we will say that x and y are the leading (stable and unstable, respectively) coordinates and u and v are the non-leading ones. Note that if λ_1 is real, then $A_1 = \lambda_1$ and x is a scalar; if λ_1 is complex, then $\lambda_1 = \bar{\lambda}_2 = \lambda e^{\pm i\varphi}$, $x = (x_1, x_2)$ and $A_1 = \lambda\begin{pmatrix} \cos\phi & -\sin\varphi \\ \sin\phi & \cos\varphi \end{pmatrix}$. Analogously, if γ_1 is real, then $B_1 = \gamma_1$ and y is a scalar; if γ_1 is complex, then $\gamma_1 = \bar{\gamma}_2 = \gamma e^{\pm i\psi}$, $y = (y_1, y_2)$ and $B_1 = \gamma\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$.

The intersection points of the homoclinic orbit Γ_0 with U_0 belong to $W^s_{\rm loc}$ and to $W^u_{\rm loc}$ and converge to O at the forward or, respectively, backward iterations of T_0. Let $M^+ \in W^s_{\rm loc}$ and $M^- \in W^u_{\rm loc}$ be two points of Γ_0. Since these are points of the same orbit, there exists a positive integer k_0 such that $M^+ = f^{k_0}(M^-)$. Let Π^+ and Π^- be some small neighborhoods of M^+ and M^-, respectively. We will call the map $T_1 \equiv f^{k_0} : \Pi^- \to \Pi^+$ *the global map*.

By assumption, the n-dimensional surface $T_1(W^u_{\rm loc})$ is tangent to the m-dimensional surface $W^s_{\rm loc}$ at the point M^+. We suppose that the tangency is *simple* in the sense that conditions C, D and E are fulfilled. We formulate these conditions following [3, 38]; in essence, they represent a version of conditions of quasi-transversal intersection from [41]. Denote as $\mathcal{T}_M W$ the tangent space to a manifold W at the point M. We assume that

C. The surfaces $T_1(W^u_{\rm loc})$ and $W^s_{\rm loc}$ have, at the point M^+, a unique common tangent vector, i.e. $\dim(\mathcal{T}_{M^+}W^s \cap \mathcal{T}_{M^+}(T_1 W^u_{\rm loc})) = 1$.

D. The tangency of $T_1 W^u_{\rm loc}$ and $W^s_{\rm loc}$ at the point M^+ is quadratic.

These conditions mean (see [41]) that one can introduce coordinates ($z_1 \in R^1, z_2 \in R^1, w_1 \in R^{n-1}, w_2 \in R^{m-1}$) with the origin at M^+, such that the equation of $W^s_{\rm loc}$ will become ($z_1 = 0, w_1 = 0$) and the equation of $T_1 W^u_{\rm loc}$ will be ($z_1 = \Psi(z_2), w_2 = 0$) where $\Psi(0) = 0$, $\Psi'(0) = 0$, $\Psi''(0) \neq 0$. Note that the coordinates in which the manifolds $W^s_{\rm loc}$ and $T_1 W^u_{\rm loc}$ have such a form near M^+ can be introduced for any close system also; moreover the conditions $\Psi'(0) = 0$, $\Psi''(0) \neq 0$ will hold. Then, the tangency of $W^s_{\rm loc}$ and $T_1 W^u_{\rm loc}$ is split if and only if *the splitting parameter* $\mu \equiv \Psi(0) \neq 0$.

In the cases where the point O has no non-leading multipliers, conditions C and D are the only conditions for a homoclinic tangency to be simple. However, if O does have non-leading multipliers, we need one more assumption. Recall that if O has stable and unstable non-leading multipliers, then the manifolds $W^s_{\rm loc}$ and $W^u_{\rm loc}$ contain strong-stable and strong-unstable invariant C^r-smooth submanifolds: $W^{ss}_{\rm loc} \subset W^s_{\rm loc}$ and $W^{uu}_{\rm loc} \subset W^u_{\rm loc}$, where $\dim W^{ss}_{\rm loc} = m - n_s$, $\dim W^{uu}_{\rm loc} = n - n_u$. When the map T_0 is in form (1.1), $W^u(O)$ and $W^s(O)$ are tangent at O to the coordinate spaces (y, v) and (x, u), respectively; the manifold $W^{ss}_{\rm loc}$ is tangent at O to the space $u = 0$ and $W^{uu}_{\rm loc}$ is tangent at O to the space $v = 0$. It is also well known (see, e.g., [42]) that on $W^u_{\rm loc}$ there exists an invariant C^r-smooth foliation consisting of $(n - n_u)$-dimensional leaves transverse to the y-subspace. This strong-unstable foliation, which we denote as F^{uu}, is defined uniquely (by the condition of transversality to the y-subspace). Note that the leaf of F^{uu} that contains O is exactly $W^{uu}_{\rm loc}$. Analogously, on $W^s_{\rm loc}$ there exists a uniquely defined C^r-smooth strong-stable invariant foliation F^{ss} consisting of $(m - n_s)$-dimensional leaves transverse to the x-subspace; $W^{ss}_{\rm loc}$ is the leaf of F^{ss} which contains O. Another fact we use (see, for example, [19, 41, 42, 44]) is that $W^u(O)$ is a part of the so-called extended unstable manifold W^{ue}. It is an invariant $(n + n_s)$-dimensional smooth (at least C^1) manifold which is tangent, at O, to the (x, y, v)-space. Although the manifold W^{ue} is not defined uniquely, any two such manifolds contain W^u_{loc} and are tangent to each other at the points of W^u_{loc}. Analogously, $W^s_{\mathrm{loc}}(O)$ lies in an $(m+n_u)$-dimensional extended stable manifold W^{se}, which is tangent to the (x, y, u)-space at O. Again, such a manifold is not unique, but all of them are tangent to each other at the points of W^s_{loc}. Thus, at the homoclinic points M^+ and M^- the tangent spaces $H_u = \mathcal{T}_{M^+}(W^{ue}_{\mathrm{loc}})$ and $H_s = \mathcal{T}_{M^-}(W^{se}_{\mathrm{loc}})$ are defined uniquely. We suppose that

E1. $T_1(H_u)$ is transverse to the leaf l^{ss} of F^{ss} which passes through M^+ and
E2. $T_1^{-1}(H_s)$ is transverse to the leaf l^{uu} of F^{uu} which passes through M^-.

Counting dimensions shows that these conditions are well posed. Since the manifolds and foliations involved are invariant, conditions E1 and E2 are independent of the choice of the homoclinic points M^+ and M^- (as well as conditions C and D).

1.2. Bifurcation parameters. Newhouse regions

Let f be a diffeomorphism with a homoclinic tangency, satisfying conditions A–E. Diffeomorphisms which are close to f and have a non-transverse homoclinic orbit close to Γ form, in the space of C^r-diffeomorphisms, a smooth bifurcational surface $\mathcal{H}$ of codimension 1. In this paper we consider bifurcations in parametric families f_ε which are transverse to $\mathcal{H}$ at $\varepsilon = 0$. The number of parameters we need is equal to the effective dimension d_e. The first parameter is the so-called splitting parameter μ which measures the distance between W^s_{loc} and the fold in $T_1 W^u_{\mathrm{loc}}$ near the point M^+. Formally speaking, μ is a smooth functional defined for any diffeomorphism close to f, and the bifurcational surface $\mathcal{H}$ is given by the equation $\mu = 0$. The family f_ε is transverse to $\mathcal{H}$ if and only if $\frac{\partial}{\partial\varepsilon}(\mu(f_\varepsilon)) \neq 0$ at $\varepsilon = 0$. This condition allows us simply to take $\varepsilon = \mu$ in the case $d_e = 1$ or, if the number of parameters is greater than 1, to take μ as the first component of the vector of parameters ε.

It is known since [16] in the two-dimensional case and [21] in the multidimensional case that near a simple homoclinic tangency to a saddle (1, 1) as well as to a saddle-focus (2, 1) with $\lambda\gamma < 1$, *single-round* periodic orbits can undergo only the simplest saddle-node and period-doubling bifurcations. One parameter is enough to analyse these bifurcations, and we indeed consider only one-parameter families f_μ in this case. In the remaining cases, it is necessary to consider at least two- or three-parameter families f_ε, because single-round periodic orbits with two or three unit multipliers can appear (see theorem 1). This is connected with the existence of the Ω-*moduli*: if two systems on $\mathcal{H}$ are Ω-conjugate, then the values of Ω-moduli must coincide for both systems. In the case of a saddle-focus, the angular arguments φ and ψ of the leading multipliers λ_1 and, respectively, γ_1 are such Ω-moduli. *By definition*, any change in the value of an Ω-modulus causes bifurcations in the non-wandering set. Moreover, as shown in [15, 29, 30], any change in the values of the moduli φ or ψ leads to a change in the structure of (i.e. to the bifurcations in) the set of single-round periodic orbits[6]. Therefore, in addition to the transversality to $\mathcal{H}$, we require the family f_ε to be *transverse, at* $\varepsilon = 0$*, to the surface* $\varphi = \mathrm{const}$ *in cases* (2, 1) *with* $\lambda\gamma > 1$ *and* (2, 2) *with* $\lambda\gamma^2 > 1$ *and to the surface* $\psi = \mathrm{const}$ *in cases* (1, 2) *and* (2, 2). We will call the families that satisfy these transversality conditions *proper*.

[6] When we consider double- or triple-round periodic orbits, an analogous situation occurs in cases (1, 1) and (2, 1) with $\lambda\gamma < 1$ also. Here, any change in the value of the modulus $\theta = -\ln\lambda/\ln\gamma$ leads to bifurcations of such orbits [13, 45, 46]. We cannot have here more than one multiplier on the unit circle (because $d_e = 1$), but there may appear an additional degeneracy in the nonlinear terms. Thus, cusp bifurcations of triple-round periodic orbits were found in [47].

The transversality conditions allow one to choose μ, $\varphi - \varphi_0$ and $\psi - \psi_0$ as bifurcation parameters (where φ_0 and ψ_0 are the values of φ and ψ at $\varepsilon = 0$). In other words, one may set

(1) $\varepsilon = \mu$ in the case (1, 1), and in the case of saddle-focus (2, 1) with $\lambda\gamma < 1$;
(2) $\varepsilon = (\mu, \varphi - \varphi_0)$ in the case of saddle-focus (2, 1) with $\lambda\gamma > 1$;
(3) $\varepsilon = (\mu, \psi - \psi_0)$ in the case of saddle-focus (1, 2), as well as in the case of saddle-focus (2, 2) with $\lambda\gamma^2 < 1$;
(4) $\varepsilon = (\mu, \varphi - \varphi_0, \psi - \psi_0)$ in the case of saddle-focus (2, 2) with $\lambda\gamma^2 > 1$.

One of the general results on the proper families f_ε is the existence of *Newhouse regions* in their parameter space. Recall the following result from [3].

Theorem on Newhouse intervals. *Let f_μ be a one-parameter family of C^r-smooth ($r \geqslant 2$) diffeomorphisms, transverse to the bifurcational surface $\mathcal{H}$ of diffeomorphisms satisfying conditions A–E*[7]*. Then, in any neighbourhood of the point $\mu = 0$ there exist Newhouse intervals such that (1) in these intervals values of μ are dense which correspond to the existence of a simple homoclinic tangency to O; (2) the family f_μ is transverse to the corresponding bifurcational surfaces.*

Note that the Newhouse intervals that are constructed in this theorem depend continuously on the family f_μ. Also note that in item 1 we do not speak about all homoclinic tangencies to O and select only those which are quadratic and for which the transversality property of item 2 holds (while other homoclinic tangencies can also be encountered [10, 48], we ignore them). Moreover, the transversality to the bifurcational surfaces that correspond to the selected tangencies holds uniformly for every one-parameter family close to the given family f_μ. Thus, since every finite-parameter family f_ε transverse to $\mathcal{H}$ is foliated by one-parameter families transverse to $\mathcal{H}$, we obtain the following result.

Newhouse regions in finite-parameter families. *In the space of parameters ε there exists a sequence of open regions δ_j, converging to $\varepsilon = 0$, such that in δ_j values of ε are dense which correspond to the existence of an orbit of simple homoclinic tangency to O. Moreover, the family f_ε is transverse to the corresponding bifurcational surfaces.*

1.3. Main results

Our first result about the dynamics in the Newhouse regions δ_j is concerned with possible types of non-hyperbolic periodic orbits. We call a set of non-zero complex numbers *admissible* if for any number ν from this set its complex-conjugate ν^* also belongs to it. The set of multipliers of any periodic orbit of a real map is always admissible.

Theorem 1. *In the Newhouse regions δ_j, parameter values are dense for which f_ε has a periodic orbit with d_e multipliers on the unit circle. Moreover, given any admissible set of d_e numbers $\{\nu_1, \ldots, \nu_{d_e}\}$, values of ε are dense in δ_j, for each of which a periodic orbit exists having $\{\nu_1, \ldots, \nu_{d_e}\}$ among its multipliers.*

Here, d_e is the effective dimension. Thus, theorem 1 says, in particular, that in the case $d_e = 3$ (a saddle-focus (2, 2) with $\lambda\gamma^2 > 1$), for any triplet (ν_1, ν_2, ν_3) from the set $\{(1, 1, 1), (-1, -1, -1), (-1, 1, 1), (-1, -1, 1), (1, e^{i\omega}, e^{-i\omega}), (-1, e^{i\omega}, e^{-i\omega})\}$ (with any $\omega \in (0, \pi)$) diffeomorphisms with periodic orbits having (ν_1, ν_2, ν_3) among the multipliers are dense in the Newhouse regions δ_j. In the case $d_e = 2$ (a saddle-focus (2, 1) with $\lambda\gamma < 1$, a saddle-focus

[7] In fact, instead of condition B, we need only $\lambda\gamma \neq 1$ (see [3]). Note that our condition B always includes this requirement.

$(1, 2)$, a saddle-focus $(2, 2)$ with $\lambda\gamma^2 < 1$), the unit multipliers (ν_1, ν_2) may form any pair from the set $\{(1, 1), (-1, -1), (-1, 1), (e^{i\omega}, e^{-i\omega})\}$, while in the case $d_e = 1$ (a saddle $(1, 1)$ and a saddle-focus $(2, 1)$ with $\lambda\gamma < 1$), the non-hyperbolic periodic orbits may have either a multiplier equal to +1 or a multiplier equal to -1.

In [19], estimates were obtained for the number of unit multipliers for periodic orbits born at homoclinic bifurcations. These estimates imply that *diffeomorphisms close to a diffeomorphism with a simple homoclinic tangency cannot have periodic orbits with more than d_e multipliers on the unit circle* (see remark after theorem 3). Thus, theorem 1 shows that the estimates of [19] are sharp in our case.

Note that when conditions B and E of the simplicity of the tangency are violated, periodic orbits with a greater number of unit multipliers may appear. For example, in the class of two-dimensional diffeomorphisms with $J \neq 1$ there can be no periodic orbits having two unit multipliers, because we have either contraction (at $J < 1$) or expansion (at $J > 1$) of areas. However, in the codimension-2 case where $J = 1$ at the moment of homoclinic tangency, periodic orbits with two unit multipliers are born [32,49,50]. The same phenomenon occurs in three-dimensional diffeomorphisms of type $(1, 1)$ when condition E is violated at the moment of tangency [35,36,37].

Theorem 1 is a consequence of the following theorem.

Theorem 2. *For any admissible set of complex numbers $\nu_1, \ldots, \nu_{d_e}$ there exists a sequence of parameters $\varepsilon_k \to 0$ such that the map f_ε has, at $\varepsilon = \varepsilon_k$, a single-round periodic orbit with $(n - n_u)$ multipliers outside the unit circle, $(m + n_u - d_e)$ multipliers inside the unit circle and d_e multipliers equal exactly to $\nu_1, \ldots, \nu_{d_e}$.*

The proof is based on the rescaling lemmas 1 and 2 of section 1.4. We complete the proof of theorem 2 in section 1.5. Theorem 1 is deduced from theorem 2 as follows.

Proof of theorem 1. Arbitrarily close to any ε from the Newhouse regions δ_j there are values which correspond to homoclinic tangencies to O that satisfy conditions A–E. By theorem 2, perturbation of any of these tangencies (within the same family f_ε) creates a periodic orbit with any of the sets of multipliers claimed by theorem 1. □

Another implication of theorem 2 is the following theorem.

Theorem 3. *In the Newhouse regions δ_j there exists a dense (residual) subset of values of ε such that for each ε from this subset the corresponding diffeomorphism f_ε has, for any $d = 0, \ldots, d_e$, infinitely many hyperbolic periodic orbits with an $(m + n_u - d)$-dimensional stable manifold.*

Proof. Arbitrarily close to any parameter value from δ_j we have a value of ε for which the map f_ε has a homoclinic tangency to O satisfying conditions A–E. Arbitrarily close to this value, for any $d = 0, \ldots, d_e$, by putting (in theorem 2) d of the multipliers $\nu_1, \ldots, \nu_{d_e}$ outside the unit circle and the rest of them inside, we find a parameter value for which the system has a periodic orbit with exactly $(m + n_u - d)$ multipliers inside the unit circle and $(n - n_u + d)$ multipliers outside the unit circle. It is a hyperbolic periodic orbit, so it exists in some region in the parameter space around the found parameter value. Repeating the arguments, we may find inside this region a smaller region which corresponds to the existence of another hyperbolic periodic orbit with $(m + n_u - d)$ multipliers inside the unit circle, with the same d or with any other d from the range $0, \ldots, d_e$, etc. After repeating this procedure infinitely many times for every $d = 0, \ldots, d_e$, we obtain a nested sequence of open regions such that the values of ε in the intersection of these regions correspond to the coexistence of infinitely many hyperbolic periodic orbits with the dimension of the stable manifold equal to $(m + n_u - d)$, for all d from 0 to d_e. By construction, the set of such obtained values of ε is an intersection of a countable number of open and dense sets. □

It was shown in [19] that conditions A, B and E impose some *a priori* restrictions on possible types of hyperbolic periodic orbits which can be born at the bifurcation of the homoclinic tangency. In short, the arguments of [19] are as follows. Conditions A and E imply that $\Gamma \cup O$ (i.e. the closure of the homoclinic orbit) is a uniformly partially hyperbolic set, i.e. every point of this set has a strongly stable leaf and a strongly unstable leaf, the leaves depend continuously on the point, both the set of strongly stable leaves and the set of strongly unstable leaves are invariant, the map is expanding along the strongly unstable leaves and contracting along the strongly stable leaves and the contraction or expansion in the directions transverse to the leaves is essentially weaker than in the directions tangential to the leaves. In our case the strong-stable and strong-unstable leaves are, respectively, the leaves of the foliations F^{ss} and F^{uu}. The uniform partial hyperbolicity is inherited by the set of orbits lying (entirely) in a small neighbourhood U of $\Gamma \cup O$ and it holds for all close maps. Therefore, every orbit which stays in U has strong-stable and strong-unstable leaves also, of the same dimension as the leaves of $\Gamma \cup O$, i.e. the dimension of the strongly unstable leaf is $(n - n_u)$ and the dimension of the strongly stable leaf is $(m - n_s)$. It follows that for the map itself, and for every C^1-close map, for any periodic orbit L from U the following estimates must hold: $\dim W^s(L) \geqslant m - n_s$, $\dim W^u(L) \geqslant n - n_u$. Now, condition B implies that in the directions transverse to the strongly stable and strongly unstable leaves the $(d_e + 1)$-dimensional volumes are contracted, so we cannot have more than d_e expanding directions other than directions tangential to the strongly unstable leaf. This gives us, finally, the following estimate [19]:

$$n - n_u \leqslant \dim W^u(L) \leqslant n - n_u + d_e. \tag{1.2}$$

Thus, theorem 3 shows that *hyperbolic periodic orbits of every possible type allowed by the* a priori *restriction* (1.2) *can indeed be born at the bifurcation of a simple homoclinic tangency; moreover, all of them can coexist*[8].

In what follows we deal with the problem of the existence of stable periodic orbits and other attractors in the vicinity of a homoclinic tangency. Let us, first, formulate the following criterion of the absence of stable periodic orbits.

Theorem 4. *Suppose a diffeomorphism f satisfies A, C, D and E, and one of the following conditions holds:* (1) $n > n_u$, *i.e. O has non-leading unstable multipliers; or* (2) $J > 1$. *Then, any diffeomorphism that is C^1-close to f has no stable periodic orbits in a small neighbourhood U of $\Gamma \cup O$.*

Proof. As follows from (1.2), for every map C^1-close to f, if $n > n_u$, then $\dim W^u > 0$ for every periodic orbit that lies in U, which obviously prevents stability. In case $J > 1$ volumes are expanded in the directions transverse to the strongly stable leaves, which implies at least one positive Lyapunov exponent for every orbit in U. □

Thus, in theorems 5 and 6 we deal with the situation where the conditions of theorem 4 do not hold.

Theorem 5. *Let f have an orbit of simple homoclinic tangency to a saddle periodic orbit O with $J < 1$ (i.e. it satisfies conditions A–E). Assume that O has no unstable non-leading multipliers (i.e. $n = n_u$). Then, in the Newhouse regions δ_j the values of ε are dense (and form a residual set) for which f_ε has infinitely many coexisting stable periodic orbits.*

[8] Recall that we assume condition B here, i.e. $J = \lambda^{n_s}\gamma^{n_u} < 1$. If $J > 1$, then in order to determine possible types of coexisting hyperbolic periodic orbits one should apply theorem 3 to the map f_ε^{-1}.

Being a particular case of theorem 3, theorem 5 follows from theorem 2 in the same way as theorem 3 does. One should just put all the multipliers $\nu_1, \ldots, \nu_{d_e}$ in theorem 2 inside the unit circle and see that *given any diffeomorphism f_0 with a simple homoclinic tangency to a saddle periodic orbit O with no unstable non-leading multipliers and with $J < 1$, in the d_e-parameter family f_ε there exists a sequence of regions in the parameter space which accumulate at $\varepsilon = 0$ and in each of which the system has a stable single-round periodic orbit.*

Note that if $d_e = 1$ (i.e. in the case of a saddle (1,1) or a saddle-focus (2,1) with $\lambda\gamma < 1$), this statement gives us intervals of the existence of stable single-round periodic orbits in any one-parameter family transverse to the bifurcational surface $\mathcal{H}$. This result is known as the existence of 'a cascade of periodic sinks' [4,16,17,21]. Note that one can show (see section 3.7) that if the following condition of the general position holds: $M^+ \notin W^{ss}_{\mathrm{loc}}$, $M^- \notin W^{uu}_{\mathrm{loc}}$, then the intervals of the existence of stable single-round periodic orbits do not intersect and accumulate on $\mu = 0$ monotonically: from one side if $\gamma_1 > 0$ and from both sides if $\gamma_1 < 0$. Thus, the situation is completely analogous to the two-dimensional case [16][9].

In the case $d_e \geqslant 2$, such cascades are not typical for one-parameter families f_μ: it is important to have more bifurcational parameters in order to surely detect stable periodic orbits. Indeed, single-round stable periodic orbits of large period k correspond to (stable) fixed points of first-return maps which are given by lemma 1 in the next section. As one can see from formulae (1.4)–(1.6), the fixed points may be stable only for bounded values of the coefficients B and C. This corresponds to $\cos(k\varphi)$ and, respectively, $\cos(k\psi)$ tending to a certain finite limit exponentially as $k \to +\infty$ (see (1.7)). The values of φ_0 and ψ_0 for which such exponentially good approximations are possible form a zero measure set. Therefore, the existence of a cascade of single-round stable periodic orbits in the one-parameter perturbations of a system with a simple homoclinic tangency is a 'probability zero' event in the case $d_e \geqslant 2$, while it is a generic phenomenon for the d_e-parameter perturbations.

Theorem 5 is obtained by the analysis of stable periodic points of the first-return maps. The following result about the coexistence of an infinite number of more complicated attractors in the Newhouse regions is obtained by the analysis of bifurcations of periodic points, see section 3.7.

Theorem 6. *Let the hypothesis of theorem 5 hold, and let $d_e \geqslant 2$. Then, in the Newhouse regions δ_j the values of parameters are dense (and form a residual set) for which f_ε has infinitely many coexisting stable closed invariant curves.*

Remarks.

(1) When conditions B (B′) or E are violated, closed invariant curves can be born in case (1, 1) also. Thus, bifurcations leading to closed invariant curves were studied in [32,49,50] for the case of a homoclinic tangency with $J = 1$ and in [35–37] for the case of violation of condition E.

(2) In the case of two-dimensional diffeomorphisms, bifurcations of homoclinic tangencies with $J \neq 1$ cannot lead to closed invariant curves. However, if there is a heteroclinic cycle containing at least two saddles, one with $J < 1$ and another with $J > 1$, both stable and completely unstable closed invariant curves can be born. Moreover, near systems with such heteroclinic cycles there exist Newhouse regions where diffeomorphisms that have

[9] Note that cascades of periodic sinks exist in generic one-parameter families of two-dimensional diffeomorphisms with non-transverse heteroclinic cycles [18, 51]. However, the intervals of the existence of stable single-round periodic orbits can, in this case, intersect, and, moreover, infinitely many of them can contain point $\mu = 0$, i.e. a diffeomorphism with a non-transverse heteroclinic cycle can possess infinitely many stable single-round periodic orbits simultaneously [51].

simultaneously infinitely many of both stable and completely unstable closed invariant curves are dense [13, 18, 33, 34].

(3) Birth of invariant tori from heteroclinic cycles with two saddle equilibria was studied in [19, 52, 53].

1.4. Rescaling lemmas

The proof of the above theorems is based on the study of bifurcations of single-round periodic orbits near an orbit of a simple homoclinic tangency. Such an orbit intersects the small neighbourhood Π^+ of the homoclinic point M^+ at a single point which is a fixed point of the first-return map $T^{(k)} = T_1 T_0^k$ for some sufficiently large k. Recall that T_0 is a local Poincaré map in the small neighbourhood U_0 of the periodic point O and T_1 is a global map defined by the orbits close to the piece of the homoclinic orbit Γ which lies outside U_0. The global map is defined in the small neighbourhood Π^- of the homoclinic point M^-. Therefore, the domain of definition of the map $T^{(k)}$ on Π^+ is $\sigma_k^0 = \Pi^+ \cap T_0^{-k}\Pi^-$. These domains are non-empty for all sufficiently large k, and they accumulate at $W^s_{\rm loc} \cap \Pi^+$ as $k \to +\infty$ (see section 2).

The following lemmas show that the first-return maps $T^{(k)}$ can be brought, for all large k, to a certain standard form.

Lemma 1. *Let f_0 be a C^r-diffeomorphism ($r \geqslant 2$) satisfying conditions A–E, embedded into a proper d_e-parameter family f_ε. Assume that the saddle point O has no unstable non-leading multipliers (i.e. $n = n_u$). Then, in the space of parameters there is a sequence of regions Δ_k, accumulating at $\varepsilon = 0$, such that at $\varepsilon \in \Delta_k$ there exists a C^r-smooth transformation of coordinates on σ_k^0 which brings the first-return map $T^{(k)} : (x, u, y) \mapsto (\bar{x}, \bar{u}, \bar{y})$ to one of the following forms:*

(i) in the case (1,1) and in the case (2,1) with $\lambda\gamma < 1$—

$$\bar{y} = M - y^2 + o(1), \qquad (\bar{x}, \bar{u}) = o(1); \tag{1.3}$$

(ii) in the case (2,1) with $\lambda\gamma > 1$—

$$\begin{aligned} &\bar{x}_1 = y, \\ &\bar{y} = M - y^2 - Bx_1 + o(1), \qquad (\bar{x}_2, \bar{u}) = o(1) \end{aligned} \tag{1.4}$$

(iii) in the case (1,2) and in the case (2,2) with $\lambda\gamma^2 < 1$—

$$\begin{aligned} &\bar{y}_1 = y_2, \\ &\bar{y}_2 = M - Cy_2 - y_1^2 + o(1), \qquad (\bar{x}, \bar{u}) = o(1); \end{aligned} \tag{1.5}$$

(iv) in the case (2,2) with $\lambda\gamma^2 > 1$—

$$\begin{aligned} &\bar{x}_1 = y_1, \qquad \bar{y}_1 = y_2, \\ &\bar{y}_2 = M - Cy_2 - Bx_1 - y_1^2 + o(1), \qquad (\bar{x}_2, \bar{u}) = o(1), \end{aligned} \tag{1.6}$$

where the $o(1)$-terms tend to zero, as $k \to \infty$, along with all the derivatives up to the order r with respect to the coordinates and up to the order $(r-2)$ with respect to the rescaled parameters M, B, C, uniformly on any bounded set of (x, y, u, M, B, C). Here, $x \in R^{n_s}$, $y \in R^{n_u}$, $u \in R^{m-n_s}$; the domain of definition of the map $T^{(k)}$ in these coordinates is an asymptotically large region which, as $k \to +\infty$, covers all finite values of (x, y, u). The rescaled parameters M, B, C are functions of ε (i.e. of (μ, φ, ψ)):

$$\begin{aligned} M \sim \gamma^{2n_u k}(\mu + O(\gamma^{-k} + \lambda^k)), \qquad &B \sim (\lambda^{n_s-1}\gamma^{n_u})^k \cos(k\varphi + \alpha_k(\varepsilon)), \\ C \sim \gamma^k \cos(k\psi + \beta_k(\varepsilon)), & \end{aligned} \tag{1.7}$$

where α_k, β_k are C^{r-2}-functions of ε, tending to a finite limit as $k \to +\infty$, uniformly with all the derivatives. When ε varies within the region Δ_k, the parameters M, B and C run over asymptotically large regions which, as $k \to +\infty$, cover all finite values.

In the general case, when O does have unstable non-leading multipliers (i.e. when $n > n_u$), we have the following result.

Lemma 2. *In the space of parameters there is a sequence of regions Δ_k, accumulating at $\varepsilon = 0$, such that the map $T^{(k)}$ has, at $\varepsilon \in \Delta_k$, a repelling $(m + n_u)$-dimensional invariant C^r-smooth manifold $\mathcal{M}_k^u \subset \sigma_k^0$, and the restriction of $T^{(k)}$ onto $\mathcal{M}_k^u$ is written, in appropriately chosen coordinates (x, y, u), in the form given by (1.3)–(1.6).*

Recall that we formulate these lemmas for the case $J \equiv \lambda^{n_s}\gamma^{n_u} < 1$; in the case $J > 1$ the same results hold true for the inverse maps $(T^{(k)})^{-1}$. The lemmas were announced in [38]; the proof occupies section 3 of this paper. Case (1, 1) was considered in many papers, see e.g. [3–5, 31]; for cases (2, 1), (1, 2) and (2, 2) without non-leading multipliers, see [13,15,39,40].

For a proof of theorem 6, we need a more accurate account of the asymptotically small terms in map (1.4), which leads us to the following result (section 3.3).

Lemma 3. *In case (2,1) with $\lambda\gamma > 1$, when $\varepsilon \in \Delta_k$ and when the corresponding value of B is bounded away from zero, the map $T^{(k)}$ (if $n > n_u$—the map $T^{(k)}|_{\mathcal{M}_k^u}$) has a two-dimensional attracting invariant C^r-smooth manifold $\mathcal{M}_k^s \subset \sigma_k^0$. In the rescaled coordinates of lemma 1, the manifold $\mathcal{M}_k^s$ is the graph of a function (x_2, u) versus (x_1, y) such that $(x_2, u) = o(1)$ as $k \to \infty$.*

When $r \geqslant 3$, the map $T^{(k)}|_{\mathcal{M}_k^s}$ is written in the form

$$\bar{x}_1 = y, \qquad \bar{y} = M - y^2 - Bx_1 - \frac{2J_1}{B}(\lambda^2\gamma)^k(x_1 y + o(1)), \tag{1.8}$$

where $J_1 \neq 0$ is some constant (J_1 is the Jacobian of the map $T_1|_{W^{se}_{\mathrm{loc}} \cap W^{ue}_{\mathrm{loc}}}$ at the homoclinic point M^- at $\varepsilon = 0$, see (3.21)).

In the case $r = 2$ the map $T^{(k)}|_{\mathcal{M}_k^s}$ is written in the form

$$\bar{x}_1 = y, \qquad \bar{y} = M - y^2 - Bx_1 + \vartheta_k(y) - \frac{2J_1}{B}(\lambda^2\gamma)^k(x_1 y + o(1)), \tag{1.9}$$

where $\vartheta_k(y) = o(y^2)$ and tends to zero as $k \to \infty$.

The maps of the form (1.8) are called generalized Hénon maps. They were introduced in [32,49] where it was shown, in particular, that they undergo a non-degenerate Andronov–Hopf bifurcation and have a stable closed invariant curve for the values of parameters (M, B) from some open regions (see section 3.7).

Another rescaling result (cf [3,5]) stresses the special role of the parabola map.

Lemma 4. *Let f_0 be a C^r-diffeomorphism ($r \geqslant 2$) satisfying conditions A, C, D, E, embedded into a one-parameter family f_μ transverse to the bifurcational surface $\mathcal{H}$ of diffeomorphisms with a homoclinic tangency. Assume that $\lambda\gamma < 1$ at $\mu = 0$. Then, there exist a sequence $k_j \to \infty$ and a sequence of intervals I_{k_j}, accumulating at $\mu = 0$, such that at $\mu \in I_{k_j}$ the map $T^{(k_j)}$ has a repelling $(m+1)$-dimensional invariant C^r-smooth manifold $\mathcal{W}_{k_j}^u \subset \sigma_{k_j}^0$, and the restriction of $T^{(k_j)}$ onto $\mathcal{W}_{k_j}^u$ is written, in appropriately chosen coordinates (y, z), where $y \in R^1$, $z \in R^{m+1}$, in the form*

$$\bar{y} = M - y^2 + o_{k\to\infty}(1), \qquad \bar{z} = o_{k\to\infty}(1), \tag{1.10}$$

where

$$M \sim \gamma^{2k}(\mu + O(\gamma^{-k})). \tag{1.11}$$

Thus, the parabola map appears in one-parameter unfoldings of a simple homoclinic tangency in all four cases: (1, 1), (2, 1), (1, 2) and (2, 2). This was used in [3–5] in order to extend the two-dimensional Newhouse result [2] to the multidimensional case. Note that we do not require condition B, but the condition $\lambda\gamma < 1$ is crucial in lemma 4. However, this excludes the case of symplectic maps, as $\lambda\gamma = 1$ for such maps automatically. The case where $\lambda\gamma = 1$ at $\mu = 0$ is also interesting by itself, without a connection to symplectic dynamics (see [13, 32, 49, 50]). As the following lemma shows (cf [7, 32, 33, 54] for two-dimensional maps and [55] for three-dimensional flows), the role of the parabola map is played by the Hénon map here.

Lemma 5. *Let f_0 be a C^r-diffeomorphism ($r \geqslant 2$) satisfying conditions A, C, D, E, embedded into a one-parameter family f_μ transverse to the bifurcational surface $\mathcal{H}$ of diffeomorphisms with a homoclinic tangency. Assume that $\lambda\gamma = 1$ at $\mu = 0$. Then, there exist a sequence $k_j \to \infty$ and a sequence of intervals I_{k_j}, accumulating at $\mu = 0$, such that at $\mu \in I_{k_j}$ the map $T^{(k_j)}$ has a two-dimensional C^r-smooth invariant 'centre' manifold $\mathcal{W}^c_{k_j} \subset \sigma^0_{k_j}$, and the restriction of $T^{(k_j)}$ onto $\mathcal{W}^c_{k_j}$ is written, in appropriately chosen coordinates (x, y), in the form*

$$\bar{x} = y + o_{k\to\infty}(1), \qquad \bar{y} = M - y^2 - B_k x + o_{k\to\infty}(1), \tag{1.12}$$

where M is given by (1.11), and the coefficient B_k, bounded away from zero and infinity, is given by (3.44), (3.48), (3.53) and (3.55).

The proof of lemmas 4 and 5 is given in section 3.6.

1.5. Proof of theorem 2

Theorem 2 is an almost immediate consequence of lemma 2. Indeed, a single-round periodic orbit corresponds to fixed points of the first-return map $T^{(k)} = T_1 T_0^k$. When O has non-leading unstable multipliers, the map $T^{(k)}$ is expanding in directions transverse to the invariant manifold $\mathcal{M}^u_k$. Hence a single-round periodic orbit starting on $\mathcal{M}^u_k$ has $(n - n_u)$ multipliers outside the unit circle. The rest of the multipliers are found by the linear analysis near the fixed point of the corresponding map (1.3)–(1.6). Such an analysis is quite straightforward. Thus, for any of these maps, for any admissible set $\nu_1, \ldots, \nu_{d_e}$ (where $d_e = 1$ for map (1.3), $d_e = 2$ for maps (1.4) and (1.5) and $d_e = 3$ for map (1.6)) there exist values of parameters $M = M_k, B = B_k, C = C_k$ bounded by a constant independent of k, such that for all sufficiently large k the map has, at the given parameter values, a fixed point with multipliers $\nu_1, \ldots, \nu_{d_e}$ and the rest of the multipliers are $o(1)$ (see exact formulae in section 3.7).

According to (1.7), bounded values of parameter M correspond to $\mu \to 0$, as $k \to +\infty$. Any bounded values of B and C can be obtained by arbitrarily small changes in φ and ψ. Thus, theorem 2 indeed follows. □

2. Local and global maps

In order to study the properties of the first-return maps $T^{(k)} = T_1 T_0^k$, we need appropriate formulae for the maps T_1 and T_0. We give such formulae in the lemmas below.

One can always consider such C^r-coordinates in U_0 that the fixed point O_ε of the local map T_0 is at the origin for all small ε. Moreover, we write $T_0(\varepsilon)$ in the form (1.1), i.e. the linear part has a block-diagonal form with the blocks A_1, B_1 and A_2, B_2 corresponding to the leading and non-leading coordinates. Note that in lemmas 6 and 7 we do not assume condition A, e.g. we do not require that the leading multipliers are simple. Indeed, we prove these lemmas under the assumption that A_1 is an $(m_1 \times m_1)$-matrix whose eigenvalues, at $\varepsilon = 0$, are all equal to some $\lambda \in (0, 1)$ in absolute value and B_1 is an $(n_1 \times n_1)$-matrix the absolute values of the eigenvalues of which are, at $\varepsilon = 0$, all equal to some $\gamma > 1$. The absolute values of the eigenvalues of the $(m - m_1) \times (m - m_1)$-matrix A_2 and the $(n - n_1) \times (n - n_1)$-matrix B_2 are assumed to be strictly less than λ and, respectively, strictly greater than γ. Here $1 \leqslant m_1 \leqslant m$, $1 \leqslant n_1 \leqslant n$, and $x \in R^{m_1}$, $u \in R^{m-m_1}$, $y \in R^{n_1}$, $v \in R^{n-n_1}$ in (1.1).

By a C^r-transformation of coordinates, one can straighten the local stable and unstable manifolds of O_ε, so that they acquire equations W^s_{loc} : $\{y = 0, v = 0\}$ and W^u_{loc} : $\{x = 0, u = 0\}$. After that, the map T_0 takes, locally, the form

$$\begin{aligned} \bar{x} &= A_1(\varepsilon)x + p_1(x, u, y, v, \varepsilon), \qquad & \bar{u} &= A_2(\varepsilon)u + p_2(x, u, y, v, \varepsilon), \\ \bar{y} &= B_1(\varepsilon)y + q_1(x, u, y, v, \varepsilon), \qquad & \bar{v} &= B_2(\varepsilon)v + q_2(x, u, y, v, \varepsilon), \end{aligned} \tag{2.1}$$

where the nonlinearities $p_{1,2}$ and $q_{1,2}$ vanish at the origin along with their first derivatives, and for all small $(x, u, y, v, \varepsilon)$ we have

$$p(0, 0, y, v, \varepsilon) \equiv 0, \qquad q(x, u, 0, 0, \varepsilon) \equiv 0. \tag{2.2}$$

Bringing the local map to the form (2.1) is not enough for our purposes because the iterations T_0^k can deviate too much from those of the linearized map. Essentially, this means that the right-hand side of (2.1) contains 'too many' non-resonant terms. Fortunately, infinitely many of the non-resonant terms can be eliminated by means of some additional smooth transformation of coordinates, as the following lemma shows.

Lemma 6. *At all sufficiently small ε, there exists a local C^r-transformation of coordinates after which the map $T_0(\varepsilon)$ keeps its form (2.1) and (2.2) while the functions p and q now satisfy additional identities*

$$p_1(x, u, 0, 0, \varepsilon) \equiv 0, \qquad q_1(0, 0, y, v, \varepsilon) \equiv 0, \tag{2.3}$$

$$\frac{\partial p}{\partial x}(0, 0, y, v, \varepsilon) \equiv 0, \qquad \frac{\partial q}{\partial y}(x, u, 0, 0, \varepsilon) \equiv 0. \tag{2.4}$$

Proof. See the appendix. □

Remarks.

(1) At finite r, the coordinate transformation is only C^{r-2} with respect to ε; more precisely, its second derivative with respect to (x, y, u, v) is C^{r-2} with respect to $(x, y, u, v, \varepsilon)$. When $r = +\infty$ or $r = \omega$, we can guarantee, in general, only arbitrarily large finite smoothness with respect to ε. However, if the absolute values of the eigenvalues of the matrix $A_1(\varepsilon)$ stay, at all small ε, equal to each other and the same holds true for the matrix $B_1(\varepsilon)$ (this is always true when condition A is fulfilled), then the transformation is C^∞ with respect to the parameters.

(2) When identities (2.3) are fulfilled the non-leading manifolds are straightened: W^{ss}_{loc} : $\{(y, v) = 0, x = 0\}$, W^{uu}_{loc} : $\{(x, u) = 0, y = 0\}$. The invariant foliations F^{ss} on W^s_{loc} and F^{uu} on W^u_{loc} are also straightened and have the form $\{x = \text{const}, (y, v) = 0\}$ and $\{y = \text{const}, (x, u) = 0\}$, respectively. Concerning the extended stable and unstable manifolds W^{se} and W^{ue} (see condition E), note that the tangents to W^{se}_{loc} at the points of W^s_{loc} form an invariant (with respect to the derivative of the map T_0) continuous family of $(n - n_2)$-dimensional spaces which is a unique such family transverse to W^{uu}_{loc} at O. One can easily see that the second of the identities (2.4) implies that the space $\{v = 0\}$ is invariant with respect to the derivative of the map (2.1) at any point of W^s_{loc}. By uniqueness, it follows that the spaces $\{v = 0\}$ form the family of tangents to W^{se}_{loc}. In

particular, $H_s = \mathcal{T}_{M^+} W^{se} = \{v = 0\}$. Analogously, $H_u = \mathcal{T}_{M^-} W^{ue} = \{u = 0\}$ due to the first of the identities (2.4) (see [19, 42]).

When the map $T_0(\varepsilon)$ is written in the form (2.1) with p, q satisfying identities (2.2), (2.3) and (2.4), we will say that it is in *the main normal form*. It occurs that when T_0 is brought to this normal form, the iterations $T_0^k : U_0 \to U_0$ of the local map do not differ essentially from the iterations of the linearized map at all large k. Namely, let $(x_k, u_k, y_k, v_k) = T_0^k(x_0, u_0, y_0, v_0)$. It has been known since [57] (see also [58, 59]) that (x_k, u_k, y_0, v_0) are uniquely defined functions of (x_0, u_0, y_k, v_k) for any $k \geqslant 0$.

Lemma 7. *When the local map T_0 is brought to the main normal form, the following relations hold for all small ε and all large k:*

$$\begin{aligned} x_k - A_1^k(\varepsilon)x_0 &= \hat{\lambda}^k \xi_k(x_0, u_0, y_k, v_k, \varepsilon), \qquad & u_k &= \hat{\lambda}^k \hat{\xi}_k(x_0, u_0, y_k, v_k, \varepsilon), \\ y_0 - B_1^{-k}(\varepsilon)y_k &= \hat{\gamma}^{-k} \eta_k(x_0, u_0, y_k, v_k, \varepsilon), \qquad & v_0 &= \hat{\gamma}^{-k} \hat{\eta}_k(x_0, u_0, y_k, v_k, \varepsilon), \end{aligned} \tag{2.5}$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are some constants such that $0 < \hat{\lambda} < \lambda$, $\hat{\gamma} > \gamma$, and the functions $\xi_k, \eta_k, \hat{\xi}_k, \hat{\eta}_k$ are uniformly bounded for all k, along with the derivatives up to the order $(r-2)$. For the derivatives of order $(r-1)$ the following estimates hold as $k \to +\infty$:

$$\begin{aligned} \left\| \frac{\partial^{r-1}(x_k - A_1(\varepsilon)^k x_0, u_k)}{\partial(x_0, u_0, y_k, v_k, \varepsilon)^{r-1}} \right\| &= o(\|A_1(\varepsilon)\|^k), \\ \left\| \frac{\partial^{r-1}(y_0 - B_1(\varepsilon)^{-k} y_k, v_0)}{\partial(x_0, u_0, y_k, v_k, \varepsilon)^{r-1}} \right\| &= o(\|B_1^{-1}(\varepsilon)\|^k), \end{aligned} \tag{2.6}$$

while the derivatives of order r are estimated as follows:

$$\|x_k, u_k, y_0, v_0\|_{C^r} = o(1)_{k\to\infty}; \tag{2.7}$$

these estimates do not include derivatives with more than $(r-2)$ differentiations with respect to ε (such may not exist, see remark 1 to lemma 6).

See the proof in appendix. Lemmas 6 and 7 strengthen similar results in [23, 42, 43, 46, 56, 60]; in particular, we achieve maximal possible smoothness here.

Concerning the global map $T_1(\varepsilon)$, in the local coordinates of lemma 6, we may also find a convenient form for it, using conditions A, C, D, E. Also recall that the transversality of the family f_ε to the bifurcational surface $\mathcal{H}$ means that among the parameters ε we can distinguish the parameter μ which measures the splitting of the invariant manifolds of O near the chosen homoclinic point M^+.

Lemma 8. *The homoclinic points M^+, M^- and the system of coordinates in U for which the local map T_0 is kept in the main normal form can be chosen in such a way that at all small ε, in cases (1,1) and (2,1) (i.e. at $y \in R^1$), the v-coordinates of any point M from a small neighbourhood of M^- are uniquely defined by the (x, u, y)-coordinates of M and by the v-coordinates of the point $\bar{M}(\bar{x}, \bar{u}, \bar{y}, \bar{v}) = T_1 M$. Moreover,*

$$\frac{\partial \bar{x}}{\partial y} \neq 0, \qquad \frac{\partial \bar{y}}{\partial x} \neq 0, \qquad \frac{\partial^2 \bar{y}}{\partial y^2} \neq 0, \tag{2.8}$$

and there exists such $y^-(\varepsilon)$, tending to the y-coordinate of the homoclinic point M^- as $\mu \to 0$, that at the point $(x = 0, u = 0, y = y^-(\varepsilon), \bar{v} = 0)$ we have

$$\frac{\partial \bar{y}}{\partial y} = 0, \quad \bar{y} = \mu, \qquad \text{and, in case } (2,1),\ \frac{\partial \bar{x}_2}{\partial y} = 0 \tag{2.9}$$

(recall that μ is the splitting parameter, and $x = (x_1, x_2) \in R^2$ in case (2,1)).

In cases (1,2) and (2,2) (here $y \in R^2$) the (y_2, v)-coordinates of M are uniquely defined by the (x, u, y_1)-coordinates and by the (y_2, v)-coordinates of $\bar{M}$. Moreover,

$$\frac{\partial \bar{x}}{\partial y_1} \neq 0, \qquad \frac{\partial \bar{y}_1}{\partial x} \neq 0, \qquad \frac{\partial^2 \bar{y}_1}{\partial y_1^2} \neq 0, \tag{2.10}$$

and, for an appropriately chosen $y_1^-(\varepsilon)$, at $(x, u, \bar{y}_2, \bar{v}) = 0$, $y_1 = y_1^-(\varepsilon)$, we have

$$\frac{\partial \bar{y}_1}{\partial y_1} = 0, \quad \frac{\partial \bar{y}_1}{\partial \bar{y}_2} = 0, \quad \bar{y}_1 = \mu, \qquad \text{and, in case } (2,2),\ \frac{\partial \bar{x}_2}{\partial y_1} = 0. \tag{2.11}$$

This lemma says that we can write the map $T_1 : (x, u, y, v) \mapsto (\bar{x}, \bar{y}, \bar{u}, \bar{v})$ in the *cross form*, i.e. as the correspondence $(x, u, y, \bar{v}) \mapsto (\bar{x}, \bar{u}, \bar{y}, v)$ in the case $y \in R^1$ and $(x, u, y_1, \bar{y}_2, \bar{v}) \mapsto (\bar{x}, \bar{u}, \bar{y}_1, y_2, v)$ in the case $y \in R^2$. Moreover, we have the following.

Corollary 1. *The Taylor expansion for the cross form of T_1 at $(x = 0, u = 0, y = y^-(\varepsilon), \bar{v} = 0)$ or, respectively, at $(x = 0, u = 0, y_1 = y_1^-(\varepsilon), \bar{y}_2, \bar{v} = 0)$ is as follows:*

Case (1,1)

$$\begin{aligned}
\bar{x} - x^+ &= ax + b_0(y - y^-) + \alpha_1 u + \beta_1 \bar{v} + \cdots,\\
\bar{y} &= \mu + cx + D_0(y - y^-)^2 + \nu_1 u + \rho_1 \bar{v} + \cdots,\\
v - v^- &= \tilde{c}x + \tilde{d}(y - y^-) + \nu_2 u + \rho_2 \bar{v} + \cdots,\\
\bar{u} - u^+ &= \tilde{a}x + \tilde{b}(y - y^-) + \alpha_2 u + \beta_2 \bar{v} + \cdots;
\end{aligned} \tag{2.12}$$

Case (2,1)

$$\begin{aligned}
\bar{x} - x^+ &= ax + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} (y - y^-) + \alpha_1 u + \beta_1 \bar{v} + \cdots,\\
\bar{y} &= \mu + c_1 x_1 + c_2 x_2 + D_0(y - y^-)^2 + \nu_1 u + \rho_1 \bar{v} + \cdots,\\
v - v^- &= \tilde{c}x + \tilde{d}(y - y^-) + \nu_2 u + \rho_2 \bar{v} + \cdots,\\
\bar{u} - u^+ &= \tilde{a}x + \tilde{b}(y - y^-) + \alpha_2 u + \beta_2 \bar{v} + \cdots
\end{aligned} \tag{2.13}$$

(here $x = (x_1, x_2) \in R^2$);
Case (1,2)

$$\begin{aligned}
\bar{x} - x^+ &= ax + b_0(y_1 - y_1^-) + b_1 \bar{y}_2 + \alpha_1 u + \beta_1 \bar{v} + \cdots,\\
\bar{y}_1 &= \mu + c_0 x + D_0(y_1 - y_1^-)^2 + \nu_{11} u + \rho_{11} \bar{v} + \cdots\\
y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2 \bar{y}_2 + c_1 x + \nu_{12} u + \rho_{12} \bar{v} + \cdots,\\
v - v^- &= \tilde{c}x + \tilde{d}_1(y_1 - y_1^-) + \tilde{d}_2 \bar{y}_2 + \nu_2 u + \rho_2 \bar{v} + \cdots,\\
\bar{u} - u^+ &= \tilde{a}x + \tilde{b}_1(y_1 - y_1^-) + \tilde{b}_2 \bar{y}_2 + \alpha_2 u + \beta_2 \bar{v} + \cdots;
\end{aligned} \tag{2.14}$$

Case (2,2)

$$\begin{aligned}
\bar{x} - x^+ &= ax + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} (y_1 - y_1^-) + b_1 \bar{y}_2 + \alpha_1 u + \beta_1 \bar{v} + \cdots,\\
\bar{y}_1 &= \mu + c_{01} x_1 + c_{02} x_2 + D_0(y_1 - y_1^-)^2 + \nu_{11} u + \rho_{11} \bar{v} + \cdots,\\
y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2 \bar{y}_2 + c_1 x + \nu_{12} u + \rho_{12} \bar{v} + \cdots,\\
v - v^- &= \tilde{c}x + \tilde{d}_1(y_1 - y_1^-) + \tilde{d}_2 \bar{y}_2 + \nu_2 u + \rho_2 \bar{v} + \cdots,\\
\bar{u} - u^+ &= \tilde{a}x + \tilde{b}_1(y_1 - y_1^-) + \tilde{b}_2 \bar{y}_2 + \alpha_2 u + \beta_2 \bar{v} + \cdots;
\end{aligned} \tag{2.15}$$

where $b_0 \neq 0$, $c_0 \neq 0$, $D_0 \neq 0$, $d_2 \neq 0$ *and* $\det \rho_2 \neq 0$. *All the coefficients in (2.12)–(2.15) depend on* ε *(at least* C^{r-2}*-smoothly).*

Proof of lemma 8. Let $M^+ \in W^s_{\text{loc}}$ and $M^- \in W^u_{\text{loc}}$ be a pair of homoclinic points at $\mu = 0$. In the coordinates of lemma 6 the manifolds W^s_{loc} and W^u_{loc} are straightened, so the (y, v)-coordinates of M^+ and the (x, u)-coordinates of M^- are zero. Let $M^+ = M^+(x^+, u^+, 0, 0)$ and $M^- = M^-(0, 0, y^-, v^-)$. Since $T_1 M^- = M^+$ at $\varepsilon = 0$, the map $T_1(\varepsilon)$ may be written in the following form at small ε:

$$\begin{aligned}
\bar{x} - x^+(\varepsilon) &= \hat{a}_1 x + \hat{\alpha}_1 u + \hat{b}_1(y - y^-) + \hat{\beta}_1(v - v^-) + \cdots, \\
\bar{u} - u^+(\varepsilon) &= \hat{a}_2 x + \hat{\alpha}_2 u + \hat{b}_2(y - y^-) + \hat{\beta}_2(v - v^-) + \cdots, \\
\bar{y} &= y^+(\varepsilon) + \hat{c}_1 x + \hat{\nu}_1 u + \hat{d}_1(y - y^-) + \hat{\rho}_1(v - v^-) + \cdots, \\
\bar{v} &= v^+(\varepsilon) + \hat{c}_2 x + \hat{\nu}_2 u + \hat{d}_2(y - y^-) + \hat{\rho}_2(v - v^-) + \cdots,
\end{aligned} \tag{2.16}$$

where the dots stand for nonlinear terms, all the coefficients depend on ε and $y^+(0) = 0$, $v^+(0) = 0$.

By condition E2, the space $T_1^{-1}(H_s)$ is transverse to l^{uu} at the point M^-. The leaf l^{uu} of the foliation F^{uu} through the point M^- is given by equations $\{x = 0, u = 0, y = y^-\}$, and the tangent plane H_s to the extended stable manifold W^{se} at M^+ is $\bar{v} = 0$ (see remark 2 to lemma 6). Hence, the condition of transversality of $T_1^{-1}(H_s)$ and F^{uu} reads as $\det \hat{\rho}_2 \neq 0$. It follows that the last equation of (2.16) can be resolved with respect to $(v - v^-)$, i.e. $(v - v^-)$ can be understood as a function of $(x, u, y - y^-, \bar{v})$. Thus, (2.16) may be recast in the cross form:

$$\begin{aligned}
\bar{x} - x^+ &= ax + \alpha_1 u + b(y - y^-) + \beta_1 \bar{v} + \cdots, \\
\bar{y} &= y^+ + cx + \nu_1 u + d(y - y^-) + \rho_1 \bar{v} + \cdots, \\
v - v^- &= \tilde{c}x + \nu_2 u + \tilde{d}(y - y^-) + \rho_2 \bar{v} + \cdots, \\
\bar{u} - u^+ &= \tilde{a}x + \alpha_2 u + \tilde{b}(y - y^-) + \beta_2 \bar{v} + \cdots
\end{aligned} \tag{2.17}$$

with some new ε-dependent coefficients $x^+, u^+, y^+, v^-, a, b, \ldots$.

By condition E.1 (see remark 2 to lemma 6), the following manifolds are transverse at $\varepsilon = 0$: the leaf $l^{ss} = \{\bar{x} = x^+, \bar{y} = 0, \bar{v} = 0\}$ of the strong-stable foliation F^{ss} through the point M^+ and the image $T_1(H_u)$ of the space $u = 0$ (the tangent space to W^{ue} at the point M^-). This transversality condition means that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0. \tag{2.18}$$

Now, according to condition C, the manifold $T_1 W^u_{\text{loc}}$ has, at the point M^+, exactly one common tangent vector with W^s_{loc} at $\varepsilon = 0$. Since W^u_{loc} and W^s_{loc} are, respectively, given by the equations $(x = 0, u = 0)$ and $(\bar{y} = 0, \bar{v} = 0)$, it follows from (2.17) that the intersection of the tangent spaces to $T_1 W^u_{\text{loc}}$ and W^s_{loc} at M^+ is one-dimensional if and only if the equation $d(y - y^-) = 0$ has a one-parameter family of solutions at $\varepsilon = 0$. Hence, in cases (1,1) and (2,1) where $y \in R^1$ and d is a scalar, we have

$$d = 0 \qquad \text{at} \qquad \varepsilon = 0. \tag{2.19}$$

Note that (2.18) and (2.19) imply $b \neq 0$ and $c \neq 0$.

In cases (1,2) and (2,2), $y \in R^2$ and d is a (2×2)-matrix, so condition C reads as

$$\det d = 0 \qquad \text{and} \qquad \operatorname{rank} d = 1 \quad \text{at } \varepsilon = 0. \tag{2.20}$$

Let us now focus on the case $y \in R^1$. By (2.19), the second equation in (2.17) may be written in the following form at $\varepsilon = 0$:

$$\bar{y} = cx + \nu_1 u + \rho_1 \bar{v} + D_0(y - y^-)^2 + \cdots . \tag{2.21}$$

Condition D of the quadraticity of the homoclinic tangency simply means that

$$D_0 \equiv \left.\frac{\partial^2 \bar{y}}{\partial y^2}\right|_{(x=0,u=0,y=y^-,\bar{v}=0,\varepsilon=0)} \neq 0. \tag{2.22}$$

Indeed, we must show (see comments after condition D) that one can introduce such coordinates (z_1, z_2, w_1, w_2) near the point M^+ that, at $\varepsilon = 0$, the manifold W^s_{loc} would have the equation $(z_1 = 0, w_1 = 0)$ and the piece of the unstable manifold $T_1 W^u_{\mathrm{loc}}$ would acquire the equation $(z_1 = \Psi(z_2), w_2 = 0)$ with a function Ψ such that $\Psi(0) = 0$, $\Psi'(0) = 0$ and that $\Psi''(0) \neq 0$ if and only if $D_0 \neq 0$. Let us check this. By (2.17) and (2.21) the equation of $T_1 W^u_{\mathrm{loc}}$ is given, at $\varepsilon = 0$, by

$$\begin{aligned}
\bar{x} - x^+ &= b(y - y^-) + \beta_1 \bar{v} + \cdots , \\
\bar{y} &= \rho_1 \bar{v} + D_0(y - y^-)^2 + \cdots , \\
\bar{u} - u^+ &= \tilde{b}(y - y^-) + \beta_2 \bar{v} + \cdots .
\end{aligned} \tag{2.23}$$

It follows from (2.18) and (2.19) that $b \neq 0$ here. If $x = (x_1, x_2)$ is a vector, then at least one component of the vector b must be non-zero, so we assume that it is the first component, b_0. Then, if we denote $z_2 = \bar{x} - x^+$ in the case $x \in R^1$ and $z_2 = \bar{x}_1 - x_1^+$ in the case $x \in R^2$, $(y - y^-)$ can be expressed by the first equation of (2.23), as a function of z_2 and $\bar{v}$: $y - y^- = z_2/b_0 - \beta_1/b_0 \bar{v} + \ldots$. Now, the second equation of (2.23) takes the form $\bar{y} = \rho_1 \bar{v} + D_0 z_1^2/b_0^2 + \cdots$, so one can see that we may indeed introduce a variable $z_1 = \bar{y} - \rho_1 \bar{v} + \cdots$ such that $T_1 W^u_{\mathrm{loc}}$ near M^+ would take the form $z_1 = \Psi(z_2)$ with $\Psi(0) = 0$, $\Psi'(0) = 0$. Here, $\Psi''(0) = D_0/b_0^2$, so the quadraticity of the tangency of $T_1 W^u_{\mathrm{loc}}$ with W^s_{loc} is given by (2.22) indeed.

By (2.22), we may choose $y^- = y^-(\varepsilon)$ in such a way that $\partial \bar{y}/\partial y = 0$ at $(x = 0, u = 0, y = y^-(\varepsilon), \bar{v} = 0)$ at all small ε. Then, at small ε the third equation of (2.17) may be written in the form $\bar{y} = y^+(\varepsilon) + cx + \nu_1 u + \rho_1 \bar{v} + D_0(y - y^-(\varepsilon))^2 + \cdots$. Thus, $y^+(\varepsilon)$ indeed is equal to the splitting parameter μ.

We have almost proved the lemma in the case $y \in R^1$. The only remaining claim is that in case (2,1) the coordinate system can be chosen such that the second component of the vector b in (2.17) will be zero. It is obvious that this can indeed be achieved by a linear rotation of the coordinates (x_1, x_2) and that such rotations keep the local map T_0 in the main normal form and do not change the form of the second equation in (2.13).

It remains to prove the lemma in the case $y \in R^2$ (i.e. cases (1, 2) and (2, 2)). Equations for $\bar{y}$ in (2.17) are written as follows:

$$\begin{aligned}
\bar{y}_1 &= y_1^+(\varepsilon) + c_0 x + d_{11}(y_1 - y_1^-) + d_{12}(y_2 - y_2^-) + \nu_{11} u + \rho_{11} \bar{v} + \cdots , \\
\bar{y}_2 &= y_2^+(\varepsilon) + c_1 x + d_{21}(y_1 - y_1^-) + d_{22}(y_2 - y_2^-) + \nu_{12} u + \rho_{12} \bar{v} + \cdots .
\end{aligned} \tag{2.24}$$

Note that rotations of the (y_1, y_2)-coordinates do not change the form of equations (2.24). At $\varepsilon = 0$, since $\det d = 0$, one can make a rotation of the y-coordinates so that

$$d_{11} = 0, \qquad d_{12} = 0. \tag{2.25}$$

Since $\operatorname{rank} d = 1$, it follows then that at least one of the coefficients d_{21} or d_{22} must be non-zero. If $d_{22} = 0$, we consider another homoclinic point, $T_0^{-1}(M^-)$, as a new point M^-. For the new global map ($T_{1\mathrm{new}} = T_1 T_0$) the matrix d will have the following form (see lemma 6): $d_{\mathrm{new}} = d \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. By (2.25), $d_{\mathrm{new}} = \begin{pmatrix} 0 & 0 \\ d_{21}\cos\varphi + d_{22}\sin\varphi & -d_{21}\sin\varphi + d_{22}\cos\varphi \end{pmatrix}$.

Thus, if $d_{22} = 0$, then $d_{22new} \neq 0$ (since $d_{21} \neq 0$ and $\sin\varphi > 0$). Hence, we may always assume $d_{22} \neq 0$.

Taking into account quadratic terms, we write the first equation of (2.24) as

$$\bar{y}_1 = c_1 x + \nu_{11} u + \rho_{11}\bar{v} + D_1(y_1 - y_1^-)^2 + D_2(y_1 - y_1^-)(y_2 - y_2^-) + D_3(y_2 - y_2^-)^2 + \cdots \tag{2.26}$$

at $\varepsilon = 0$. Since $d_{22} \neq 0$, the second equation in (2.24) can be written as

$$y_2 - y_2^- = \frac{1}{d_{22}}\bar{y}_2 - \frac{d_{21}}{d_{22}}(y_1 - y_1^-) + c_2 x + \nu_{12} u + \rho_{12}\bar{v} + \cdots, \tag{2.27}$$

with some new coefficients $y_2^-, c_2, \nu_{12}, \rho_{12}$. Plugging (2.27) into (2.26), we obtain

$$\bar{y}_1 = c_1 x + \nu_{11} u + \rho_{11}\bar{v} + D_0(y_1 - y_1^-)^2 + \tilde{D}_2(y_1 - y_1^-)\bar{y}_2 + \tilde{D}_3\bar{y}_2^2 + \cdots \tag{2.28}$$

at $\varepsilon = 0$, where $D_0 \equiv D_1 - D_2\frac{d_{21}}{d_{22}} + D_3\frac{d_{21}^2}{d_{22}^2}$. Absolutely analogously to the case $y \in R^1$, one can now check that condition D reads as $D_0 \neq 0$. It follows from these conditions that one may always choose $y_1^-(\varepsilon)$ and make an additional rotation of the y-coordinates on the angle of order ε, so that the coefficients $d_{11}(\varepsilon)$ and $d_{12}(\varepsilon)$ in the equation for $\bar{y}_1$ will vanish identically at all small ε. Then, again as we did in the case $y \in R^1$, one can check that the value $y^+(\varepsilon)$ of $\bar{y}_1$ at $(x = 0, u = 0, y_1 = y_1^-(\varepsilon), \bar{y}_2 = 0, \bar{v} = 0)$ is the splitting parameter μ.

Plugging (2.27) into (2.17), we arrive, after an obvious adaptation of notation, at the desired equations (2.14) and (2.15); to obtain formulae (2.15) one also needs an additional rotation of the x-coordinates as was done above for case (2,1). □

3. Rescaled first-return maps. Proof of lemmas 1–5

In this section we study the first-return maps $T^{(k)}(\varepsilon) \equiv T_1 T_0^k$ at sufficiently large k, $k = \bar{k}, \bar{k}+1, \ldots$, and small ε. The domain of the map $T^{(k)}$ is the strip $\sigma_k^0 = T_0^{-k}(\Pi^-) \cap \Pi^+$. Denote $(x_k, u_k, y_k, v_k) = T_0^k(x_0, u_0, y_0, v_0)$ and take (x_0, u_0, y_k, v_k) as the new coordinates on σ_k^0—this can indeed be done, according to lemma 7. The relation between these coordinates and the standard coordinates (x_0, u_0, y_0, v_0) is given by (2.5). If the neighborhoods Π^- and Π^+ are chosen as the boxes $\|x - x^+, u - u^+, y, v\| \leqslant \epsilon_0$ and $\|x, u, y - y^-, v - v^-\| \leqslant \epsilon_1$ with sufficiently small positive ϵ_0 and ϵ_1, then in the coordinates (x_0, u_0, y_k, v_k) the strip σ_k is the box $\|x_0 - x^+, u_0 - u^+\| \leqslant \epsilon_0$, $\|y_k - y^-, v_k - v^-\| \leqslant \epsilon_1$.

3.1. First-return maps in case (1,1)

Here the leading coordinates x and y are one-dimensional and, respectively, $A_1 = \lambda_1$, $B_1 = \gamma_1$. By plugging (2.5) into (2.12), we obtain the following formula for the first-return map $T^{(k)} \equiv T_1 T_0^k : (x_0, u_0, y_k, v_k) \mapsto (\bar{x}_0, \bar{u}_0, \bar{y}_k, \bar{v}_k)$:

$$\begin{aligned}
\bar{x}_0 - x^+ = {} & a(\lambda_1^k x_0 + \hat{\lambda}^k \xi_k(x_0, u_0, y_k, v_k, \varepsilon)) + b_0(y_k - y^-) + O((y_k - y^-)^2) \\
& + \alpha_1 \hat{\lambda}^k \hat{\xi}_k(x_0, u_0, y_k, v_k, \varepsilon) + \beta_1 \hat{\gamma}^{-k} \hat{\eta}_k(\bar{x}_0, \bar{u}_0, \bar{y}_k, \bar{v}_k, \varepsilon) + O((\lambda^k |x_0| \\
& + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\hat{\eta}_k\|)|y_k - y^-| + \lambda^{2k} x_0^2 + \hat{\lambda}^{2k}\|\xi_k, \hat{\xi}_k\|^2 + \hat{\gamma}^{-2k}\|\hat{\eta}_k\|^2), \\
\gamma_1^{-k}\bar{y}_k = {} & -\hat{\gamma}^{-k}\eta_k(\bar{x}_0, \bar{u}_0, \bar{y}_k, \bar{v}_k, \varepsilon) + \mu + c(x_0\lambda_1^k + \hat{\lambda}^k \xi_k(x_0, u_0, y_k, v_k, \varepsilon)) \\
& + D_0(y_k - y^-)^2 + o((y_k - y^-)^2) + \nu_1 \hat{\lambda}^k \hat{\xi}_k(x_0, u_0, y_k, v_k, \varepsilon) \\
& + \rho_1 \hat{\gamma}^{-k}\hat{\eta}_k(\bar{x}_0, \bar{u}_0, \bar{y}_k, \bar{v}_k, \varepsilon) + O((\lambda^k |x_0| + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\hat{\eta}_k\|)|y_k - y^-| \\
& + \lambda^{2k} x_0^2 + \hat{\lambda}^{2k}\|\xi_k, \hat{\xi}_k\|^2 + \hat{\gamma}^{-2k}\|\hat{\eta}_k\|^2),
\end{aligned}$$

$$v_k - v^- = \lambda_1^k \tilde{c} x_0 + \tilde{d}(y_k - y^-) + O((y_k - y^-)^2 + \lambda^k |x_0|\,|y_k - y^-| + \lambda^{2k} x_0^2 + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k} \|\hat{\eta}_k\|),$$

$$\bar{u}_0 - u^+ = \lambda_1^k \tilde{a} x_0 + \tilde{b}(y_k - y^-) + O((y_k - y^-)^2 + \lambda^k |x_0||y_k - y^-| + \lambda^{2k} x_0^2 + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k} \|\hat{\eta}_k\|). \tag{3.1}$$

Note that the right-hand sides of (3.1) are also functions of (x_0, u_0, y_k, v_k) and $(\bar{x}_0, \bar{u}_0, \bar{y}_k, \bar{v}_k)$. They are of class C^r with respect to the variables and C^{r-2} with respect to the parameters (see remark 1 to lemma 6). Below, in order to simplify notation, we denote as $\mathcal{O}_1$ unspecified, uniformly bounded terms, linear with respect to $(x, u, y, v, \bar{x}, \bar{u}, \bar{y}, \bar{v})$, and $\mathcal{O}_{2+}$ will denote terms of the second order and higher with respect to $(x, u, y, v, \bar{x}, \bar{u}, \bar{y}, \bar{v})$, uniformly bounded along with all derivatives.

Recall that $\lambda\gamma < 1$ according to condition B (recall that $\lambda = |\lambda_1|$, $\gamma = |\gamma_1|$). Therefore, we may assume $\lambda\hat{\gamma} < 1$ also; hence

$$\hat{\lambda}^k \ll \lambda^k \ll \hat{\gamma}^{-k} \ll \gamma^{-k} \tag{3.2}$$

at large k. Now, one can rewrite (3.1) as

$$\begin{aligned} \bar{x}_0 - x^+ &= b_0(y_k - y^-) + O((y_k - y^-)^2) + O(\hat{\gamma}^{-k}),\\ \gamma_1^{-k} \bar{y}_k &= \mu + D_0(y_k - y^-)^2 + o((y_k - y^-)^2) + O(\hat{\gamma}^{-k}),\\ v_k - v^- &= \tilde{d}(y_k - y^-) + O((y_k - y^-)^2) + O(\hat{\gamma}^{-k}),\\ \bar{u}_0 - u^+ &= \tilde{b}(y_k - y^-) + O((y_k - y^-)^2) + O(\hat{\gamma}^{-k}). \end{aligned} \tag{3.3}$$

The terms denoted here as $O(\hat{\gamma}^{-k})$ include functions $\xi_k, \eta_k, \hat{\xi}_k, \hat{\eta}_k$. Therefore, while their C^{r-2}-norms are of order $\hat{\gamma}^{-k}$, the norms of the derivatives of the order $(r-1)$ and r are estimated as $o(\gamma^{-k})$ and $o(1)$, respectively, see lemma 7.

Shift the origin:

$$\begin{aligned} x &= x_0 - x^+(\varepsilon) + \cdots, & y &= y_k - y^-(\varepsilon) + \cdots,\\ u &= u_0 - u^+(\varepsilon) + \cdots, & v &= v_k - v^-(\varepsilon) + \cdots, \end{aligned} \tag{3.4}$$

in order to nullify the constant terms (i.e. terms depending only on parameters) in the first, third and fourth equations of (3.3) and the linear in y term in the second equation. It is easy to see that this indeed can be done, with small corrections to (x^+, u^+, y^-, v^-) (that are denoted by the dots in (3.4)) being of order $O(\hat{\gamma}^{-k})$. As a result, system (3.3) is recast as

$$\begin{aligned} (\bar{x}, \bar{u}, v) &= O(y) + o(\gamma^{-k})\mathcal{O}_1 + o_{k\to\infty}(1)\mathcal{O}_{2+},\\ \bar{y} &= \gamma_1^k M_1 + D_0 \gamma_1^k y^2 + \gamma^k o(y^2) + \gamma^k \cdot (o(\gamma^{-k})\mathcal{O}_1 + o_{k\to\infty}(1)\mathcal{O}_{2+}), \end{aligned} \tag{3.5}$$

where

$$M_1 = \mu - \gamma_1^{-k} y^-(1 + \cdots) + c\lambda_1^k x^+(1 + \cdots). \tag{3.6}$$

Let us now introduce rescaled coordinates (X, U, Y, V) by the rule

$$x = \gamma^{-k} X/\delta_k, \qquad y = -\frac{1}{D_0}\gamma_1^{-k} Y, \qquad u = \gamma^{-k} U/\delta_k, \qquad v = \gamma^{-k} V/\delta_k, \tag{3.7}$$

where δ_k tends sufficiently slowly to zero as $k \to \infty$. System (3.5) takes the form

$$(\bar{X}, \bar{U}, V) = o(1), \qquad \bar{Y} = M - Y^2 + o(1), \tag{3.8}$$

where $M = -D_0\gamma_1^{2k} M_1 = -D_0\gamma_1^{2k}[\mu - \gamma_1^{-k} y^-(1 + \cdots) + c\lambda_1^k x^+(1 + \cdots)]$.

Recall that, by our construction, the $o_{k\to\infty}(1)$-terms in (3.8) are functions of $(X, U, Y, V, \bar{X}, \bar{Y}, \bar{U}, \bar{V}, \varepsilon)$. It is obvious, however, that for any bounded region of values of M one can resolve (3.8) so that the resulting expressions for $(\bar{X}, \bar{Y}, \bar{U}, V)$ as functions of $(X, Y, U, \bar{V}, M)$ will still be given by equations of exactly the form (3.8).

Note that for any bounded region of values of M and on any finite size ball in the (X, U, Y, V)-space, the first-return map $T^{(k)}$ which we have brought to the form (3.8) is strongly expanding in V for all sufficiently large k. Obviously, one can continue the map onto the whole R^{n+m} without losing this property. More precisely, given any R, one can modify the right-hand sides of (3.8) in the region $\|X, U, Y, \bar{V}\| \geqslant R$ in such a way that formula (3.8) would, for every $(X, U, Y, \bar{V})$, define a smooth correspondence $(X, U, Y, \bar{V}) \mapsto (\bar{X}, \bar{U}, \bar{Y}, V)$ such that $\frac{\partial V}{\partial(X,U,Y,\bar{V})}$ tends uniformly to zero as $k \to +\infty$ and $\frac{\partial(\bar{X},\bar{U},\bar{Y})}{\partial(X,U,Y,\bar{V})}$ is uniformly bounded. This means that at all sufficiently large k, conditions of theorem 4.4 of [42] hold for the inverse of the map $T^{(k)}$. Thus, applying that theorem, we obtain the existence of a normally hyperbolic repelling invariant smooth manifold $\mathcal{M}_k^u$ of the form $V = \phi_k(X, U, Y)$ where $\phi_k = o(1)$ in the C^r-norm (and in the C^{r-2}-norm with respect to the parameters). By (3.8), the map $T^{(k)}$ on this manifold has the form (1.3), which proves lemmas 1 and 2 in case (1,1).

3.2. The first-return maps in case (2,1)

Here, the leading coordinates are $x = (x_1, x_2) \in R^2$ and $y \in R^1$ and, respectively, $A_1 \equiv \lambda\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$, $B_1 \equiv \gamma_1$. By virtue of (2.5) and (2.13) the first-return map $T^{(k)} \equiv T_1 T_0^k$ can be written in the following form for all sufficiently large k and all small ε:

$$
\begin{aligned}
\bar{x}_{01} - x_1^+ &= \lambda^k A_{11}(k\varphi)x_{01} + \lambda^k A_{12}(k\varphi)x_{02} + b_0(y_k - y^-)\\
&\qquad + O((y_k - y^-)^2 + \lambda^k\|x_0\||y_k - y^-| + \hat{\lambda}^k(\|\xi_k, \hat{\xi}_k\| + \|x_0\|^2) + \hat{\gamma}^{-k}\|\hat{\eta}_k\|),\\
\bar{x}_{02} - x_2^+ &= \lambda^k A_{21}(k\varphi)x_{01} + \lambda^k A_{22}(k\varphi)x_{02}\\
&\qquad + O((y_k - y^-)^2 + \lambda^k\|x_0\||y_k - y^-| + \hat{\lambda}^k(\|\xi_k, \hat{\xi}_k\| + \|x_0\|^2) + \hat{\gamma}^{-k}\|\hat{\eta}_k\|),\\
\gamma^{-k}\bar{y}_k &= \mu + \lambda^k[(c_1\cos k\varphi + c_2\sin k\varphi)x_{01} + (c_2\cos k\varphi - c_1\sin k\varphi)x_{02}] + D_0(y_k - y^-)^2\\
&\qquad + o((y_k - y^-)^2) + O(\lambda^k\|x_0\||y_k - y^-| + \hat{\lambda}^k(\|\xi_k, \hat{\xi}_k\| + \|x_0\|^2) + \hat{\gamma}^{-k}\|\eta_k, \hat{\eta}_k\|),\\
v_k - v^- &= \tilde{d}(y_k - y^-) + O((y_k - y^-)^2 + \lambda^k|x_0| + \hat{\lambda}^k\|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\hat{\eta}_k\|),\\
\bar{u}_0 - u^+ &= \tilde{b}(y_k - y^-) + O((y_k - y^-)^2 + \lambda^k\|x_0\| + \hat{\lambda}^k\|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\hat{\eta}_k\|),
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
A_{11}(k\varphi) &= a_{11}\cos k\varphi - a_{12}\sin k\varphi, \qquad & A_{12}(k\varphi) &= a_{12}\cos k\varphi + a_{11}\sin k\varphi,\\
A_{21}(k\varphi) &= a_{21}\cos k\varphi + a_{22}\sin k\varphi, \qquad & A_{22}(k\varphi) &= a_{22}\cos k\varphi - a_{21}\sin k\varphi.
\end{aligned}
\tag{3.10}
$$

If $\lambda\gamma < 1$, then $\hat{\lambda}^k \ll \lambda^k \ll \hat{\gamma}^{-k} \ll \gamma^{-k}$ at large k. Then, (3.9) is recast as

$$
\begin{aligned}
\bar{x}_0 - x^+ &= \begin{pmatrix} b_0\\ 0\end{pmatrix} \cdot (y_k - y^-) + O((y_k - y^-)^2) + O(\hat{\gamma}^{-k}),\\
\gamma_1^{-k}\bar{y}_k &= \mu + D_0(y_k - y^-)^2 + o((y_k - y^-)^2) + O(\hat{\gamma}^{-k}),\\
v_k - v^- &= \tilde{d}(y_k - y^-) + O((y_k - y^-)^2) + O(\hat{\gamma}^{-k}),\\
\bar{u}_0 - u^+ &= \tilde{b}(y_k - y^-) + O((y_k - y^-)^2) + O(\hat{\gamma}^{-k}).
\end{aligned}
\tag{3.11}
$$

This system is completely analogous to (3.3), so bringing it to the form (1.3) is done in this case in the same way as in case (1,1). This completes the proof of lemmas 1 and 2 for the case (2,1) with $\lambda\gamma < 1$.

Consider now the case $\lambda\gamma > 1$. We have here $\hat{\gamma}^{-k} \ll \gamma^{-k} \ll \hat{\lambda}^k \ll \lambda^k$ at large k. In the third equation of (3.9), replace the arguments $\bar{x}$, $\bar{u}$ of the functions η_k, $\hat{\eta}_k$ and the argument v of ξ_k, $\hat{\xi}_k$ by their expressions in terms of $x, u, y, \bar{y}, \bar{v}$ which are obtained from the first, second and fourth equations. After that, we shift the origin:

$$x_1 = x_{01} - x_1^+(\varepsilon) + O(\lambda^k), \qquad x_2 = x_{02} - x_2^+(\varepsilon) + O(\lambda^k), \qquad y = y_1 - y^-(\varepsilon) + O(\lambda^k),$$
$$u = u_0 - u^+(\varepsilon) + O(\lambda^k), \qquad v = v_1 - v^-(\varepsilon) + O(\lambda^k),$$

in order to nullify the linear in y term in the third equation of (3.9) and all constant (independent of the coordinates) terms in the other equations.

System (3.9) takes the following form:

$$\begin{aligned}
\bar{x}_1 &= \lambda^k A_{11}(k\varphi)x_1 + \lambda^k A_{12}(k\varphi)x_2 + b_0 y + O(\lambda^k y) + o(\lambda^k)\mathcal{O}_1 + \mathcal{O}_{2+},\\
\bar{x}_2 &= \lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + O(\lambda^k y) + o(\lambda^k)\mathcal{O}_1 + \mathcal{O}_{2+},\\
\bar{y} + s_{k1}\bar{y} + s_{k2}\bar{v} &= \gamma_1^k M_1 + D_0\gamma_1^k y^2 + \gamma^k o(y^2) + \lambda^k\gamma_1^k[x_1(c_1\cos k\varphi + c_2\sin k\varphi)\\
&\qquad + x_2(c_2\cos k\varphi - c_1\cos k\varphi) + s_{k3}x + s_{k4}u)] + \lambda^k\gamma^k\mathcal{O}_{2+},\\
v - \tilde{d}y &= \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+}, \qquad \bar{u} = \tilde{b}y + \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+},
\end{aligned} \tag{3.12}$$

where

$$M_1 = \mu - \gamma_1^{-k}y^-(1+\cdots) + \lambda^k(\cos(k\varphi)(c_1x_1^+ + c_2x_2^+) + \sin(k\varphi)(c_2x_1^+ - c_1x_2^+) + \cdots). \tag{3.13}$$

Here, the coefficients s_{ki} in (3.12) tend to zero as $k \to \infty$ (the corresponding terms came from linearization of the functions $\eta_k, \hat{\eta}_k, \xi_k, \hat{\xi}_k$).

Introduce new coordinates $u_{\text{new}} = u - \dfrac{\tilde{b}}{b_0}x_1$, $v_{\text{new}} = v - \tilde{d}y$. Map (3.12) takes the following form:

$$\begin{aligned}
\bar{x}_1 &= \lambda^k A_{11}(k\varphi)x_1 + \lambda^k A_{12}(k\varphi)x_2 + b_0 y + O(\lambda^k y) + o(\lambda^k)\mathcal{O}_1 + \mathcal{O}_{2+},\\
\bar{x}_2 &= \lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + O(\lambda^k y) + o(\lambda^k)\mathcal{O}_1 + \mathcal{O}_{2+},\\
\bar{y} + \hat{s}_{k1}\bar{y} + s_{k2}\bar{v} &= \gamma_1^k M_1 + D_0\gamma_1^k y^2 + \gamma^k o(y^2) + \lambda^k\gamma_1^k[x_1(c_1\cos k\varphi + c_2\sin k\varphi + \hat{s}_{k31})\\
&\qquad + x_2(c_2\cos k\varphi - c_1\cos k\varphi + \hat{s}_{k32}) + s_{k4}u)] + \lambda^k\gamma^k\mathcal{O}_{2+},\\
v &= \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+}, \qquad \bar{u} = \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+}.
\end{aligned} \tag{3.14}$$

As $\lambda^2\gamma < 1$, we may choose a positive ρ such that $\lambda^2\gamma(1+\rho) < 1$. Recall that $\lambda\gamma > 1$ here; hence $\gamma^{-k} \ll \lambda^k \ll (1+\rho)^{-k}$ at large k. Rescale coordinates as follows:

$$x_1 = -b_0D_0^{-1}\gamma_1^{-k}X_1, \qquad x_2 = -b_0D_0^{-1}\gamma_1^{-k}\lambda^k(1+\rho)^kX_2, \qquad y = -D_0^{-1}\gamma_1^{-k}Y,$$
$$u = \gamma^{-k}\lambda^k(1+\rho)^kU, \qquad v = \gamma^{-k}V. \tag{3.15}$$

In the rescaled coordinates, map (3.14) is written as

$$\begin{aligned}
&\bar{X}_1 = Y + O(\lambda^k), \qquad \bar{X}_2 = (1+\rho)^{-k}A_{21}(k\varphi)X_1 + o((1+\rho)^{-k}),\\
&\bar{Y} + o_{k\to\infty}(1)\bar{V} = M - Y^2 + \vartheta_k(Y) + b_0\lambda^k\gamma_1^k[X_1(c_1\cos k\varphi + c_2\sin k\varphi + o_{k\to\infty}(1))\\
&\qquad + \lambda^k(1+\rho)^kX_2(c_2\cos k\varphi - c_1\cos k\varphi + o_{k\to\infty}(1)) + o(\lambda^k(1+\rho)^k)U)] + O(\lambda^k),\\
&V = O(\lambda^k), \qquad \bar{U} = O((1+\rho)^{-k}),
\end{aligned} \tag{3.16}$$

where $\vartheta_k(y) = o(y^2)$ and tends to zero as $k \to \infty$; and $M = -D_0\gamma^{2k} M_1$. By (3.13), the rescaled parameter M can take arbitrary finite values when μ varies near $\mu_k = \gamma_1^{-k} y^- - \lambda^k(\cos(k\varphi)(c_1 x_1^+ + c_2 x_2^+) + \sin(k\varphi)(c_2 x_1^+ - c_1 x_2^+))$. Denote

$$B = -b_0 \lambda^k \gamma_1^k (c_1 \cos k\varphi + c_2 \sin k\varphi + o_{k\to\infty}(1)) \tag{3.17}$$

(the coefficient of X_1 in (3.16)). It is not small (since $\lambda\gamma > 1$) and may assume arbitrary finite values (for large k) when φ varies near those values at which $c_1 \cos k\varphi + c_2 \sin k\varphi = 0$, i.e. near $\varphi = -\frac{1}{k}\arctan\frac{c_1}{c_2} + \pi\frac{j}{k}$, $j \in Z$ (recall that $c_1^2 + c_2^2 \neq 0$ by lemma 8). Obviously, these values of the angle φ fill densely in the interval $(0, \pi)$.

For any bounded region of values of M and B and on any finite size ball in the (X, U, Y, V)-space, map (3.16) is strongly expanding in V for all sufficiently large k: $\frac{\partial V}{\partial(X,U,Y,\bar V)} = O(\lambda^k)$ tends uniformly to zero as $k \to +\infty$ and $\frac{\partial(\bar X, \bar U, \bar Y)}{\partial(X,U,Y,\bar V)}$ is uniformly bounded. Like in case (1,1), by virtue of theorem 4.4 of [42], this implies, for all sufficiently large k, the existence of a normally hyperbolic repelling invariant smooth manifold $\mathcal{M}_k^u$ of the form $V = \lambda^k \phi_k(X, U, Y)$ where ϕ_k is uniformly bounded with all the derivatives. On this manifold, map (3.16) takes the form

$$\begin{aligned}
&\bar X_1 = Y + O(\lambda^k), \qquad \bar X_2 = (1+\rho)^{-k} A_{21}(k\varphi) X_1 + o((1+\rho)^{-k}), \qquad \bar U = O((1+\rho)^{-k}),\\
&\bar Y = M - Y^2 + \vartheta_k(Y) - BX_1 + b_0\lambda^{2k}\gamma_1^k(1+\rho)^k[X_2(c_2\cos k\varphi - c_1 \cos k\varphi + o_{k\to\infty}(1))\\
&\qquad + o_{k\to\infty}(1)U] + O(\lambda^k).
\end{aligned} \tag{3.18}$$

This map has form (1.4), i.e. we have proved lemmas 1 and 2 in case (2,1) with $\lambda\gamma > 1$.

3.3. *Proof of lemma 3*

Let us suppose that $B \neq 0$ in (3.18). Since $\lambda^{2k}\gamma_1^k(1+\rho)^k \to 0$ as $k \to \infty$, one can introduce a new variable

$$X_{1\text{new}} = X_1 - \frac{1}{B}\{b_0\lambda^{2k}\gamma_1^k(1+\rho)^k[X_2(c_2\cos k\varphi - c_1\cos k\varphi + o_{k\to\infty}(1)) + o_{k\to\infty}(1)U]\}.$$

Then, system (3.18) is rewritten as

$$\begin{aligned}
&\bar X_1 = Y - \frac{b_0 A_{21}(k\varphi)}{B}(c_2\cos k\varphi - c_1 \sin k\varphi)\lambda^{2k}\gamma_1^k X_1 + o(\lambda^{2k}\gamma^k),\\
&\bar Y = M - Y^2 + \vartheta_k(Y) - BX_1 + O(\lambda^k), \qquad (\bar X_2, \bar U) = O((1+\rho)^{-k}).
\end{aligned} \tag{3.19}$$

This map is strongly contracting with respect to (X_2, U), with the contraction coefficient tending to zero as $k \to \infty$. At the same time, $\left\| \frac{\partial(\bar X_1, \bar Y)}{\partial(X_1, Y)} \right\|^{-1}$ remains bounded, since B is bounded away from zero. Thus, by virtue of theorem 4.4 of [42], map (3.19) possesses, at every sufficiently large k, an invariant manifold $\mathcal{M}_k^c$ of the form $(X_2, U) = (1+\rho)^{-k}\phi_k(X_1, Y)$, where φ_k is uniformly bounded along with all the derivatives. On $\mathcal{M}_k^c$, the map has the following form:

$$\begin{aligned}
&\bar X_1 = Y - \frac{b_0 A_{21}(k\varphi)}{B}(c_2\cos k\varphi - c_1 \sin k\varphi)\lambda^{2k}\gamma_1^k X_1 + o(\lambda^{2k}\gamma^k),\\
&\bar Y = M - Y^2 + \vartheta_k(Y) - BX_1 + O(\lambda^k).
\end{aligned} \tag{3.20}$$

For bounded values of B, we have $c_1\cos k\varphi + c_2 \sin k\varphi = O(\lambda^{-k}\gamma^{-k})$ (see (3.17)). As $\lambda\gamma > 1$, at such φ we have $c_2\cos k\varphi - c_1\sin k\varphi = \pm\sqrt{c_1^2 + c_2^2} + O(\lambda^{-k}\gamma^{-k})$ and $A_{21}(k\varphi) \equiv a_{21}\cos k\varphi + a_{22}\sin k\varphi = \pm\frac{a_{21}c_2 - a_{22}c_1}{\sqrt{c_1^2+c_2^2}} + O(\lambda^{-k}\gamma^{-k})$. Thus, we have $b_0 A_{21}(k\varphi)(c_2\cos k\varphi - c_1 \sin k\varphi) = b_0(a_{21}c_2 - a_{22}c_1) + O(\lambda^{-k}\gamma^{-k})$. As easily seen from (2.13), the value

$$J_1 = b_0(a_{21}c_2 - a_{22}c_1) \tag{3.21}$$

is the Jacobian of the global map T_1 restricted to the leading coordinates (x_1, x_2, y), i.e. restricted to the manifold $W^{ue}_{\rm loc} \cap W^{se}_{\rm loc}$ at the homoclinic point $(x,u) = 0, (y,v) = (y^-, v^-)$ at $\varepsilon = 0$. By (2.18), $J_1 \neq 0$.

Denote $J_k = J_1 \lambda^{2k} \gamma_1^k$. Map (3.20) is written as follows:

$$\bar{X}_1 = Y - \frac{J_k}{B} X_1 + o(J_k), \qquad \bar{Y} = M - Y^2 - BX_1 + \vartheta_k(Y) + O(\lambda^k). \tag{3.22}$$

Introduce $Y_{\rm new} = Y - \frac{J_k}{B} X_1 + o(J_k) \equiv \bar{X}_1$. Map (3.22) becomes

$$\bar{X}_1 = Y, \qquad \bar{Y} = M - Y^2 - BX_1 - \frac{J_k}{B} Y - \frac{2J_k}{B} X_1 Y + \vartheta_k(Y) + o(J_k). \tag{3.23}$$

By the additional shift of the coordinate Y and the parameter M: $Y_{\rm new} = Y + \frac{J_k}{2B} + o(J_k)$, $M_{\rm new} = M + o(J_k)$, we bring map (3.23) to the form (1.9). This proves lemma 3 in case $r = 2$. If $r \geqslant 3$, the term $\gamma^k o(y^2)$ in the third equation of (3.14) is, in fact, $\gamma^k O(y^3)$. Therefore, after the rescaling (3.15), this term becomes $O(\gamma^{-k})$ and is absorbed by the $O(\lambda^k)$-term, i.e. the function $\vartheta_k(Y)$ does not appear in (3.16) and further. Thus, map (1.9) transforms into (1.8). This proves lemma 3 in the case $r \geqslant 3$.

3.4. *The first-return maps in the case (1,2)*

Here the leading coordinates are $x \in R^1$ and $y = (y_1, y_2) \in R^2$ and, respectively, $A_1 \equiv \lambda_1$, $B_1 \equiv \gamma\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$. By virtue of (2.5) and (2.14), the first-return map $T^{(k)} \equiv T_1 T_0^k$, for every sufficiently large k and all small ε, can be written as

$$\begin{aligned}
\bar{x}_0 - x^+ &= a\lambda_1^k x_0 + b_0(y_{k1} - y_1^-) + b_1\gamma^{-k}(\bar{y}_{k2}\cos k\psi - \bar{y}_{k1}\sin k\psi) \\
&\quad + O((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-| \cdot (\lambda^k |x_0| + \gamma^{-k}\|\bar{y}_k\|)) + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\eta_k, \hat{\eta}_k\|), \\
\gamma^{-k}(\bar{y}_{k1}\cos k\psi + \bar{y}_{k2}\sin k\psi) &= \mu + c_0\lambda_1^k x_0 + D_0(y_{k1} - y_1^-)^2 \\
&\quad + o((y_{k1} - y_1^-)^2) + O(|y_{k1} - y_1^-| \cdot (\lambda^k |x_0| + \gamma^{-k}\|\bar{y}_k\|)) + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\eta_k, \hat{\eta}_k\|), \\
y_{k2} - y_2^- &= c_1\lambda_1^k x_0 + d_1(y_{k1} - y_1^-) + d_2\gamma^{-k}(\bar{y}_{k2}\cos k\psi - \bar{y}_{k1}\sin k\psi) \\
&\quad + O((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-| \cdot (\lambda^k |x_0| + \gamma^{-k}\|\bar{y}_k\|)) + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\eta_k, \hat{\eta}_k\|), \\
v_k - v^- &= \tilde{c}\lambda_1^k x_0 + \tilde{d}_1(y_{k1} - y_1^-) + \tilde{d}_2\gamma^{-k}(\bar{y}_{k2}\cos k\psi - \bar{y}_{k1}\sin k\psi) \\
&\quad + O((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-| \cdot (\lambda^k |x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k \|\xi_k, \hat{\xi}_k\|), \\
\bar{u}_0 - u^+ &= \tilde{a}\lambda_1^k x_0 + \tilde{b}_1(y_{k1} - y_1^-) + \tilde{b}_2\gamma^{-k}(\bar{y}_{k2}\cos k\psi - \bar{y}_{k1}\sin k\psi) \\
&\quad + O((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-| \cdot (\lambda^k |x_0| + \gamma^{-k}\|\bar{y}_k\|)) + (\hat{\lambda}^k \|\xi_k, \hat{\xi}_k\| + \hat{\gamma}^{-k}\|\eta_k, \hat{\eta}_k\|).
\end{aligned} \tag{3.24}$$

We replace the arguments $\bar{x}$ and $\bar{u}$ of the functions $\eta_k, \hat{\eta}_k$ by their expressions given by the first and the fifth equations of (3.24). After that, we shift the origin:

$$x = x_0 - x^+(\varepsilon) + O(\gamma^{-k}), \qquad y_1 = y_{k1} - y_1^-(\varepsilon) + O(\gamma^{-k}), \qquad y_2 = y_{k2} - y_2^-(\varepsilon) + O(\gamma^{-k}),$$
$$u = u_k - u^+(\varepsilon) + O(\gamma^{-k}), \qquad v = v_k - v^-(\varepsilon) + O(\gamma^{-k}),$$

in order to nullify the constant terms in the first, third, fourth and fifth equations of (3.24) and the linear in y_1 term in the second equation. System (3.24) is rewritten as

$$\begin{aligned}
&(\bar{x},\bar{u}) = (b_0,\tilde{b}_1)y_1 + \gamma^{-k}\mathcal{O}_1 + \mathcal{O}_{2+},\\
&\gamma^{-k}\{(\cos k\psi + s_{k1})\bar{y}_1 + (\sin k\psi + s_{k2})\bar{y}_2 + s_{k3}\bar{v}\} = M_1 + c_0\lambda_1^k x + D_0 y_1^2 + o(y_1^2) + \gamma^{-k}\mathcal{O}_{2+},\\
&y_2 - (d_1+p_k)y_1 = d_2\gamma^{-k}\{(\cos k\psi + s_{k4})\bar{y}_2 - (\sin k\psi + s_{k5})\bar{y}_1 + s_{k6}\bar{v}\} + \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+},\\
&v - \tilde{d}y_1 = \lambda_k O(|x| + \|u\|) + \gamma^{-k}O(\|y,\bar{y},\bar{v}\|) + \mathcal{O}_{2+},
\end{aligned} \tag{3.25}$$

where $s_{kj} = o_{k\to\infty}(1)$ and $p_k = O(\gamma^{-k})$ are certain coefficients, and

$$M_1 = \mu - \gamma^{-k}(y_1^- \cos k\psi + y_2^- \sin k\psi + \cdots) + c\lambda_1^k(x^+ + \cdots). \tag{3.26}$$

Introduce new coordinates y and v as follows:

$$\begin{aligned}
&y_{1\text{new}} = (\cos k\psi + s_{k4})y_2 - (\sin k\psi + s_{k5})y_1 + s_{k6}v,\\
&y_{2\text{new}} = y_2 - (d_1+p_k)y_1, \qquad v_{\text{new}} = v - \tilde{d}y_1.
\end{aligned} \tag{3.27}$$

This coordinate transformation is non-degenerate for all large k such that $s_0 \equiv s_0(k\psi) = d_1\cos k\psi - \sin k\psi$ is bounded away from zero.

After the transformation, system (3.25) takes the following form:

$$\begin{aligned}
&(\bar{x},\bar{u}) = O(\|y\| + \|v\|) + \gamma^{-k}\mathcal{O}_1 + \mathcal{O}_{2+},\\
&\frac{1}{s_0}\{(\cos k\psi + d_1\sin k\psi + \hat{s}_{k1})\bar{y}_1 - (1+\hat{s}_{k2})\bar{y}_2 + \hat{s}_{k3}\bar{v}\} = \gamma^k M_1 + c_0\lambda_1^k\gamma^k x\\
&\qquad + \frac{D_0}{s_0^2}\gamma^k y_1^2 + \gamma^k O(|y_1|(|y_2| + \|v\|) + y_2^2 + \|v\|^2) + o(\gamma^k y_1^2) + \mathcal{O}_{2+},\\
&y_2 = d_2\gamma^{-k}\bar{y}_1 + \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+}, \qquad v = \lambda^k O(|x| + \|u\|) + \gamma^{-k}O(\|y,\bar{y},\bar{v}\|) + \mathcal{O}_{2+}.
\end{aligned} \tag{3.28}$$

Now, we rescale the coordinates by the rule

$$(x,u) = \gamma^{-2k}(1+\rho)^k(X,U), \qquad y_1 = \frac{d_2 s_0}{D_0}\gamma^{-2k}Y_1, \qquad y_2 = \frac{d_2^2 s_0}{D_0}\gamma^{-3k}Y_2,$$

$$v = \frac{1}{\delta_k}\gamma^{-3k}V,$$

where ρ is a small positive constant such that $\lambda\gamma^2(1+\rho) < 1$, and δ_k tends sufficiently slowly to zero as $k \to \infty$. In the rescaled coordinates, system (3.28) is recast as

$$\begin{aligned}
&(\bar{X},\bar{U},V) = o_{k\to\infty}(1), \qquad Y_2 = \bar{Y}_1 + o_{k\to\infty}(1),\\
&\frac{\gamma^k}{d_2 s_0}(\cos k\psi + d_1\sin k\psi + \hat{s}_{k1})\bar{Y}_1 - \bar{Y}_2 = -M + O(\lambda_1^k\gamma^{2k}(1+\rho)^k)X + Y_1^2 + o_{k\to\infty}(1),
\end{aligned} \tag{3.29}$$

where $M = -\gamma^{4k}\frac{D_0}{s_0 d_2^2}M_1$ (see (3.26)).

As $\gamma^k \to \infty$, the coefficient $C(k\psi) \equiv \frac{\gamma^k}{d_2 s_0}(\cos k\psi + d_1\sin k\psi + \hat{s}_{k1})$ can take arbitrary finite values (negative and positive) when $\cos k\psi + d_1\sin k\psi$ is close to zero, i.e. at ψ close to $\frac{1}{k}\arctan(-\frac{1}{d_1}) + \pi\frac{j}{k}$, $j \in Z$. Note, that at such ψ the coefficient s_0 is non-zero: $s_0^2 = 1 + d_1^2$. This ensures that the coordinate transformation (3.27) is non-degenerate, as required. Note also, that these values of ψ fill densely in $(0,\pi)$. Therefore, for any finite $Q > 0$, for every sufficiently large k, in any neighbourhood of any point $\psi_0 \in (0,\pi)$ there exist intervals of size $\sim Q\gamma^{-k}$ such that the coefficient $C(k\psi)$ runs over all values from $[-Q,Q]$ when ψ varies in any of these intervals.

When C and M are finite, for finite values of the rescaled coordinates, possible expansion along (X,Y,U)-directions is finite, while expansion along the V-directions is strong ($\|\frac{\partial V}{\partial \bar V}\| \to 0$ as $k \to \infty$). Thus, exactly like in cases (1,1) and (2,1), map (3.29) has an invariant manifold $\mathcal{M}_k^u$ of the form $V = \phi_k(X,Y,U)$ where $\phi_k \to 0$ (along with all derivatives) as $k \to \infty$. On this manifold, map (3.29) takes the desired form (1.5).

3.5. The first-return maps in the case (2,2)

Here, the leading coordinates $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are both two-dimensional and, respectively, $A_1 \equiv \lambda\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, $B_1 \equiv \gamma\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$. By virtue of (2.5) and (2.15), the first-return map $T^{(k)} \equiv T_1 T_0^k$, at every sufficiently large k and all small ε, can be written in the following form:

$$
\begin{aligned}
\bar x_0 - x^+ &= \lambda^k[(a_1\cos k\varphi + a_2 \sin k\varphi)x_{01} + (a_2\cos k\varphi - a_1\sin k\varphi)x_{02}] \\
&\quad + \begin{pmatrix} b_0 \\ 0 \end{pmatrix}(y_{k1} - y_1^-) + O((y_{k1}-y_1^-)^2) + b_1\gamma^{-k}(\bar y_{k2}\cos k\psi - \bar y_{k1}\sin k\psi) \\
&\quad + O(|y_{k1}-y_1^-|\cdot(\lambda^k\|x_0\| + \gamma^{-k}\|\bar y_k\|)) + \hat\lambda^k\|\xi_k,\hat\xi_k\| + \hat\gamma^{-k}\|\eta_k,\hat\eta_k\|), \\
\gamma^{-k}(\bar y_{k1}\cos k\psi + \bar y_{k2}\sin k\psi) &= \mu + \lambda^k[(c_{01}\cos k\varphi + c_{02}\sin k\varphi)x_{01} + (c_{02}\cos k\varphi - c_{01}\sin k\varphi)x_{02}] \\
&\quad + D_0(y_{k1}-y_1^-)^2 + o((y_{k1}-y_1^-)^2) + O(|y_{k1}-y_1^-|\cdot(\lambda^k\|x_0\| + \gamma^{-k}\|\bar y_k\|)) \\
&\quad + \hat\lambda^k\|\xi_k,\hat\xi_k\| + \hat\gamma^{-k}\|\eta_k,\hat\eta_k\|), \\
y_{k2} - y_2^- &= \lambda^k[(c_{11}\cos k\varphi + c_{12}\sin k\varphi)x_{01} + (c_{12}\cos k\varphi - c_{11}\sin k\varphi)x_{02}] \\
&\quad + d_1(y_{k1}-y_1^-) + O((y_{k1}-y_1^-)^2) + d_2\gamma^{-k}(\bar y_{k2}\cos k\psi - \bar y_{k1}\sin k\psi) \\
&\quad + O(|y_{k1}-y_1^-|\cdot(\lambda^k\|x_0\| + \gamma^{-k}\|\bar y_k\|)) + \hat\lambda^k\|\xi_k,\hat\xi_k\| + \hat\gamma^{-k}\|\eta_k,\hat\eta_k\|), \\
v_k - v^- &= \lambda^k[(\tilde c_1\cos k\varphi + \tilde c_2\sin k\varphi)x_{01} + (\tilde c_2\cos k\varphi - \tilde c_1\sin k\varphi)x_{02}] \\
&\quad + \tilde d_1(y_{k1}-y_1^-) + O((y_{k1}-y_1^-)^2) + \tilde d_2\gamma^{-k}(\bar y_{k2}\cos k\psi - \bar y_{k1}\sin k\psi) \\
&\quad + O(|y_{k1}-y_1^-|\cdot(\lambda^k\|x_0\| + \gamma^{-k}\|\bar y_k\|) + \hat\lambda^k\|\xi_k,\hat\xi_k\|), \\
\bar u_0 - u^+ &= \lambda^k[(\tilde a_1\cos k\varphi + \tilde a_2\sin k\varphi)x_{01} + (\tilde a_2\cos k\varphi - \tilde a_1\sin k\varphi)x_{02}] \\
&\quad + \tilde b_1(y_{k1}-y_1^-) + O((y_{k1}-y_1^-)^2) + \tilde b_2\gamma^{-k}(\bar y_{k2}\cos k\psi - \bar y_{k1}\sin k\psi) \\
&\quad + O(|y_{k1}-y_1^-|\cdot(\lambda^k\|x_0\| + \gamma^{-k}\|\bar y_k\|)) + (\hat\lambda^k\|\xi_k,\hat\xi_k\| + \hat\gamma^{-k}\|\eta_k,\hat\eta_k\|).
\end{aligned}
\tag{3.30}
$$

Recall that $\lambda\gamma < 1$ by assumption, therefore $\lambda^k = o(\gamma^{-k})$.

Replace the arguments $\bar x_{1,2}$ and $\bar u$ of the functions $\eta_k, \hat\eta_k$ by their expressions given by the first and the fifth equations of (3.30). After that, shift the origin:

$$
\begin{aligned}
x &= x_0 - x^+(\varepsilon) + O(\gamma^{-k}), \qquad & y_1 &= y_{k1} - y_1^-(\varepsilon) + O(\gamma^{-k}), \qquad & y_2 &= y_{k2} - y_2^-(\varepsilon) + O(\gamma^{-k}), \\
u &= u_k - u^+(\varepsilon) + O(\gamma^{-k}), & v &= v_k - v^-(\varepsilon) + O(\gamma^{-k}),
\end{aligned}
$$

in order to nullify the constant terms in the first, third, fourth and fifth equations of (3.24) and the linear in y_1 term in the second equation. System (3.24) is recast as

$$
\begin{aligned}
&(\bar{x}_1,\bar{u})=(b_0,\tilde{b}_1)y_1+\gamma^{-k}\mathcal{O}_1+\mathcal{O}_{2+}, \qquad \bar{x}_2=\gamma^{-k}\mathcal{O}_1+\mathcal{O}_{2+},\\
&\gamma^{-k}\{(\cos k\psi+s_{k1})\bar{y}_1+(\sin k\psi+s_{k2})\bar{y}_2+s_{k3}\bar{v}\}=M_1+D_0y_1^2+o(y_1^2)\\
&\qquad +\lambda^k[(c_{01}\cos k\varphi+c_{02}\sin k\varphi+q_{k1})x_1+(c_{02}\cos k\varphi-c_{01}\sin k\varphi+q_{k2})x_2]+\gamma^{-k}\mathcal{O}_{2+},\\
&y_2-(d_1+p_k)y_1=d_2\gamma^{-k}\{(\cos k\psi+s_{k4})\bar{y}_2-(\sin k\psi+s_{k5})\bar{y}_1+s_{k6}\bar{v}\}+\lambda^k\mathcal{O}_1+\mathcal{O}_{2+},\\
&v-\tilde{d}y_1=\lambda_k O(|x|+\|u\|)+\gamma^{-k}O(\|y,\bar{y},\bar{v}\|)+\mathcal{O}_{2+},
\end{aligned}
\tag{3.31}
$$

where $s_{kj},q_{kj}=o_{k\to\infty}(1)$ and $p_k=O(\gamma^{-k})$ are certain coefficients, and

$$
\begin{aligned}
M_1=\mu&-\gamma^{-k}(y_1^-\cos k\psi+y_2^-\sin k\psi+\cdots)\\
&+\lambda^k[(c_{01}x_1^++c_{02}x_2^+)\cos k\varphi+(c_{02}x_1^+-c_{01}x_2^+)\sin k\varphi+\cdots].
\end{aligned}
\tag{3.32}
$$

Introduce new coordinates y, u and v as follows:

$$
\begin{aligned}
&u_{\mathrm{new}}=u-\frac{\tilde{b}_1}{b_0}x_1, \qquad v_{\mathrm{new}}=v-\tilde{d}y_1,\\
&y_{1\mathrm{new}}=(\cos k\psi+s_{k4})y_2-(\sin k\psi+s_{k5})y_1+s_{k6}v, \qquad y_{2\mathrm{new}}=y_2-(d_1+p_k)y_1.
\end{aligned}
\tag{3.33}
$$

This coordinate transformation is non-degenerate for all large k such that $s_0\equiv s_0(k\psi)=d_1\cos k\psi-\sin k\psi$ is bounded away from zero. In the new coordinates, map (3.31) takes the form

$$
\begin{aligned}
&\bar{x}_1=b_0y_1+O(|y_2|+\|v\|)+\gamma^{-k}\mathcal{O}_1+\mathcal{O}_{2+}, \qquad (\bar{x}_2,\bar{u})=\gamma^{-k}\mathcal{O}_1+\mathcal{O}_{2+},\\
&\frac{1}{s_0}\{(\cos k\psi+d_1\sin k\psi+\hat{s}_{k1})\bar{y}_1-(1+\hat{s}_{k2})\bar{y}_2+\hat{s}_{k3}\bar{v}\}=\gamma^kM_1+\frac{D_0}{s_0^2}\gamma^ky_1^2\\
&\qquad +(\lambda\gamma)^k[(c_{01}\cos k\varphi+c_{02}\sin k\varphi+\hat{q}_{k1})x_1+(c_{02}\cos k\varphi-c_{01}\sin k\varphi+q_{k2})x_2]\\
&\qquad +\gamma^kO(|y_1|(|y_2|+\|v\|)+y_2^2+\|v\|^2)+o(\gamma^ky_1^2)+\mathcal{O}_{2+},\\
&y_2=d_2\gamma^{-k}\bar{y}_1+\lambda^k\mathcal{O}_1+\mathcal{O}_{2+}, \qquad v=\lambda^kO(|x|+\|u\|)+\gamma^{-k}O(\|y,\bar{y},\bar{v}\|)+\mathcal{O}_{2+}.
\end{aligned}
\tag{3.34}
$$

Rescale the coordinates

$$
(x_2,u,v)=\frac{1}{\delta_k}\gamma^{-3k}(X_2,U,V), \qquad (x_1,y_1)=\frac{d_2s_0}{D_0}\gamma^{-2k}(b_0X_1,Y_1), \qquad y_2=\frac{d_2^2s_0}{D_0}\gamma^{-3k}Y_2,
\tag{3.35}
$$

where δ_k tends sufficiently slowly to zero as $k\to\infty$. Map (3.34) takes the form

$$
\begin{aligned}
&\bar{X}_1=Y_1+o_{k\to\infty}(1), \qquad Y_2=\bar{Y}_1+o_{k\to\infty}(1),\\
&\frac{\gamma^k}{d_2s_0}(\cos k\psi+d_1\sin k\psi+\hat{s}_{k1})\bar{Y}_1-\bar{Y}_2=-M+Y_1^2+b_0s_0\lambda^k\gamma^{2k}(c_{11}\cos k\varphi+c_{12}\sin\varphi+\hat{q}_{k1})X_1\\
&\qquad +O\left(\frac{1}{\delta_k}(\lambda\gamma)^k\right)X_2+o_{k\to\infty}(1),\\
&(\bar{X}_2,\bar{U},V)=o_{k\to\infty}(1),
\end{aligned}
\tag{3.36}
$$

where $M=-\gamma^{4k}\frac{D_0}{d_2^2}M_1$ (see (3.32)).

Since $\lambda\gamma<1$, the coefficient $O(\frac{1}{\delta_k}(\lambda\gamma)^k)$ of X_2 in the third equation tends to zero as $k\to\infty$. Note also that if $\lambda\gamma^2<1$, then the coefficient of X_1 in the same equation also tends to zero. Moreover, after an additional rescaling $X_1 \to X_1(1+\rho)^k$ with some small $\rho > 0$ we obtain $\bar{X}_1 = o(1)$ and system (3.36) will be recast as

$$
\begin{aligned}
&Y_2 = \bar{Y}_1 + o_{k\to\infty}(1), \qquad (\bar{X}_1, \bar{X}_2, \bar{U}, V) = o_{k\to\infty}(1),\\
&\frac{\gamma^k}{d_2 s_0}(\cos k\psi + d_1 \sin k\psi + \hat{s}_{k1})\bar{Y}_1 - \bar{Y}_2 = -M + Y_1^2\\
&\qquad\qquad + b_0 s_0 \lambda^k \gamma^{2k}(1+\rho)^k (c_{11}\cos k\varphi + c_{12}\sin\varphi + \hat{q}_{k1})X_1 + o_{k\to\infty}(1).
\end{aligned}
\tag{3.37}
$$

Thus, when $\lambda\gamma^2 < 1$, the map takes a form completely analogous to (3.29), and the rest of the arguments go in this case exactly in the same way as in case (1,2).

It remains to consider the case $\lambda\gamma^2 > 1$ (still $\lambda\gamma < 1$). Denote $C = \frac{\gamma^k}{d_2 s_0}(\cos k\psi + d_1 \sin k\psi + \hat{s}_{k1})$, $B = -b_0 s_0 \lambda^k \gamma^{2k}(c_{11}\cos k\varphi + c_{12}\sin\varphi + \hat{q}_{k1})$ in (3.36). As $\gamma^k \to \infty$ and $\lambda^k\gamma^{2k} \to \infty$, the coefficients B and C may, for sufficiently large k, take arbitrary finite values when φ and, respectively, ψ vary. As in the case (1,2), bounded values of C correspond to non-zero s_0 (where $s_0^2 = 1 + d_1^2 + \cdots$), i.e. transformation (3.33) is non-degenerate, as required.

As in the other cases, for bounded values of the rescaled parameters M, B, C and the rescaled variables (X, U, Y, V) the first-return map $T^{(k)}$ in the form (3.36) has an invariant manifold $\mathcal{M}_k^u$ of the form $V = \phi_k(X, Y, U)$, where ϕ_k tends to zero along with all derivatives as $k \to \infty$. Obviously, on such a manifold, map (3.36) takes the form (1.6). This completes the proof of lemmas 1 and 2.

3.6. *Proof of lemmas 4 and 5*

We prove lemmas 4 and 5 simultaneously. In the case $\lambda\gamma < 1$, we obviously have $(\lambda\gamma)^k \to 0$ as $k \to +\infty$. If $\lambda\gamma = 1$ at $\mu = 0$, then $(\lambda\gamma)^k = (1 + O(\mu))^k$ at all small μ. We will consider only the values of μ of order $O(\lambda^k + \gamma^{-k})$ (that corresponds to a finite range of values of M in (1.12), see (1.11)), hence, in this case,

$$
(\lambda\gamma)^k = 1 + O(k\gamma^{-k}). \tag{3.38}
$$

Case (1,1). Let us write formula (3.1) for the first-return map $T^{(k)}$ in the form

$$
\begin{aligned}
&\bar{x}_0 - x^+ = b_0(y_k - y^-) + O((y_k - y^-)^2) + O(\gamma^{-k}),\\
&\gamma_1^{-k}\bar{y}_k = \mu + D_0(y_k - y^-)^2 + c x_0 \lambda_1^k + o((y_k - y^-)^2) + o(\gamma^{-k}),\\
&v_k - v^- = \tilde{d}(y_k - y^-) + O((y_k - y^-)^2) + O(\gamma^{-k}),\\
&\bar{u}_0 - u^+ = \tilde{b}(y_k - y^-) + O((y_k - y^-)^2) + \lambda_1^k O((\gamma^{-k}).
\end{aligned}
\tag{3.39}
$$

This is analogous to (3.3); we, however, take into account that $\lambda\gamma$ now is not necessarily strictly less than 1; hence we have to replace $O(\hat{\gamma}^{-k})$ by $O(\gamma^{-k})$ (by (3.38) $\lambda^k = O(\gamma^{-k})$ in any case). Now, shifting the origin as in (3.4), we obtain

$$
\begin{aligned}
&\bar{x} = b_0 y + O(y^2) + O(\gamma^{-k})\mathcal{O}_1 + o_{k\to\infty}(1)\mathcal{O}_{2+},\\
&\bar{y} = \gamma_1^k M_1 + D_0\gamma_1^k y^2 + \gamma_1^k o(y^2) + c\lambda_1^k\gamma_1^k x + \gamma_1^k \cdot [o(\gamma^{-k})\mathcal{O}_1 + o_{k\to\infty}(1)\mathcal{O}_{2+}],\\
&\bar{u} = O(y) + O(\gamma^{-k})\mathcal{O}_1 + o_{k\to\infty}(1)\mathcal{O}_{2+}, \qquad v = O(y) + O(\gamma^{-k})\mathcal{O}_1 + o_{k\to\infty}(1)\mathcal{O}_{2+},
\end{aligned}
\tag{3.40}
$$

with the constant term M_1 given by (3.6).

The rescaled coordinates (X, U, Y, V) are introduced as follows:

$$
x = -\frac{b_0}{D_0}\gamma_1^{-k}X, \qquad y = -\frac{1}{D_0}\gamma_1^{-k}Y, \qquad u = \gamma^{-k}U/\delta_k, \qquad v = \gamma^{-k}V/\delta_k, \tag{3.41}
$$

where δ_k tends sufficiently slowly to zero as $k\to\infty$. It is quite similar to (3.7), but we have a weaker scaling factor for x now. After the rescaling, map (3.40) is recast as

$$\begin{aligned}\bar{X}&=Y+o_{k\to\infty}(1),\\ \bar{Y}&=M-Y^2-b_0c\lambda_1^k\gamma_1^kX+o_{k\to\infty}(1), \qquad (\bar{U},V)=o_{k\to\infty}(1),\end{aligned} \tag{3.42}$$

where M is given by (1.11). At finite values of M, the expansion in the V variables is much stronger than a possible expansion in (X,Y,U) (by (3.38) the coefficient $\lambda_1^k\gamma_1^k$ is bounded at bounded M even if $\lambda\gamma=1$ at $\mu=0$). Therefore, as in the proof of lemma 2, by theorem 4.4 of [42], we obtain the existence of the invariant manifold $\mathcal{W}_k^u$ on which map $T^{(k)}$ has the form

$$\begin{aligned}\bar{X}&=Y+o_{k\to\infty}(1), \qquad \bar{U}=o_{k\to\infty}(1),\\ \bar{Y}&=M-Y^2-b_0c\lambda_1^k\gamma_1^kX+o_{k\to\infty}(1).\end{aligned} \tag{3.43}$$

If $\lambda\gamma<1$, this gives lemma 4 immediately. If $\lambda\gamma=1$ at $\mu=0$, then, by (3.38), $\lambda_1(\mu)^k\gamma_1(\mu)^k=(\text{sign}\,\lambda_1\text{sign}\,\gamma_1)^k+o(1)_{k\to+\infty}$ at finite M. Since $b_0c\neq0$ (by lemma 8), the coefficient $b_0c\lambda_1^k\gamma_1^k$ is bounded away from zero; hence the contraction in (X,Y) variables is much weaker than the contraction in U. Then the existence of an attracting invariant two-dimensional C^r-manifold for map (3.43) follows immediately, and the map takes the required form (1.12) on this manifold, with

$$B_k=b_0c(\text{sign}\lambda_1\text{sign}\gamma_1)^k. \tag{3.44}$$

Case (2,1) Like in the previous case, after a shift of the coordinate origin, we may rewrite formula (3.9) for the first-return map $T^{(k)}$ in the following form:

$$\begin{aligned}\bar{x}&=\begin{pmatrix}b_0\\0\end{pmatrix}\cdot y+O(y^2)+O(\gamma^{-k})\mathcal{O}_1+o_{k\to\infty}(1)\mathcal{O}_{2+},\\
\gamma_1^{-k}\bar{y}&=M_1+D_0y^2+\lambda^k[(c_1\cos k\varphi+c_2\sin k\varphi)x_1+(c_2\cos k\varphi-c_1\sin k\varphi)x_2]\\
&\quad+o(y^2)+[o(\gamma^{-k})\mathcal{O}_1+o_{k\to\infty}(1)\mathcal{O}_{2+}],\\
v&=O(y)+O(\gamma^{-k})\mathcal{O}_1+o_{k\to\infty}(1)\mathcal{O}_{2+}, \qquad \bar{u}=O(y)+O(\gamma^{-k})\mathcal{O}_1+o_{k\to\infty}(1)\mathcal{O}_{2+},\end{aligned} \tag{3.45}$$

where $x=x_0-x^++O(\gamma^{-k})$, $y=y_k-y^-+O(\gamma^{-k})$, $u=u_0-u^++O(\gamma^{-k})$, $v=v_k-v^-+O(\gamma^{-k})$ and

$$M_1=\mu-\gamma_1^{-k}y^-(1+\cdots)+\lambda^k(\cos(k\varphi)(c_1x_1^++c_2x_2^+)+\sin(k\varphi)(c_2x_1^+-c_1x_2^+)+\cdots).$$

The rescaling $(x,u,y,v)\to(X,U,Y,V)$ is analogous to (3.41):

$$x_1=-\frac{b_0}{D_0}\gamma_1^{-k}X_1, \qquad x_2=\delta_k\gamma^{-k}X_2, \qquad y=-\frac{1}{D_0}\gamma_1^{-k}Y, \qquad u=\gamma^{-k}U/\delta_k,$$
$$v=\gamma^{-k}V/\delta_k, \tag{3.46}$$

where δ_k tends sufficiently slowly to zero as $k\to\infty$. In the rescaled coordinates, system (3.45) takes the following form:

$$\begin{aligned}\bar{X}_1&=Y+o_{k\to\infty}(1), \qquad \left(\bar{X}_2,\bar{U},V\right)=o_{k\to\infty}(1),\\ \bar{Y}&=M-Y^2+b_0(c_1\cos k\varphi+c_2\sin k\varphi)\lambda^k\gamma_1^kX_1+o_{k\to\infty}(1).\end{aligned} \tag{3.47}$$

This is absolutely analogous to (3.42), so lemma 4 now follows immediately, and we obtain lemma 5 provided the sequence of values of k is chosen such that $c_1\cos k\varphi+c_2\sin k\varphi$ stays bounded away from zero as $k\to+\infty$; we may always do it because $\varphi\in(0,\pi)$ and $c_1^2+c_2^2\neq0$ by lemma 8. The coefficient B_k in (1.12) is

$$B_k=b_0(c_1\cos k\varphi+c_2\sin k\varphi)(\text{sign}\gamma_1)^k. \tag{3.48}$$

Case (1, 2). Consider the first-return map $T^{(k)}$ in the form (3.25) (using (3.38), one can check that (3.25) remains valid if $\lambda\gamma \leqslant 1$ at $\mu = 0$). Introduce new u-coordinates by the rule $u_{\text{new}} = u - (\tilde{b}_1/b_0)x$. Then, the first equation of (3.25) is recast as $(\bar{x}, \bar{u}) = (b_0, 0)y_1 + \gamma^{-k}\mathcal{O}_1 + \mathcal{O}_{2+}$. The form of the other equations in (3.25) does not change. As in the proof of lemma 2, we make the transformation (3.27) and bring the map to the following form (cf (3.28)):

$$\bar{x} = \frac{b_0}{s_0}y_1 - \frac{b_0}{s_0}\cos k\psi\, y_2 + O(v) + \gamma^{-k}\mathcal{O}_1 + \mathcal{O}_{2+}, \qquad \bar{u} = \gamma^{-k}\mathcal{O}_1 + \mathcal{O}_{2+},$$

$$\frac{1}{s_0}\{(\cos k\psi + d_1 \sin k\psi + \hat{s}_{k1})\bar{y}_1 - (1 + \hat{s}_{k2})\bar{y}_2 + \hat{s}_{k3}\bar{v}\} = \gamma^k M_1 + c_0\lambda_1^k\gamma^k x$$
$$+ \frac{D_0}{s_0^2}\gamma^k y_1^2 + \gamma^k O(|y_1|(|y_2| + \|v\|) + y_2^2 + \|v\|^2) + o(\gamma^k y_1^2) + \mathcal{O}_{2+},$$

$$y_2 = d_2\gamma^{-k}\bar{y}_1 + \lambda^k\mathcal{O}_1 + \mathcal{O}_{2+}, \qquad v = \lambda^k O(|x| + \|u\|) + \gamma^{-k}O(\|y, \bar{y}, \bar{v}\|) + \mathcal{O}_{2+}. \tag{3.49}$$

Now, we rescale the coordinates in (3.49) as follows:

$$x = \gamma^{-k}X, \qquad y_1 = \gamma^{-k}Y_1, \qquad u = \gamma^{-k}\delta_k U, \qquad y_2 = \gamma^{-k}\delta_k Y_2, \qquad v = \gamma^{-k}\delta_k V, \tag{3.50}$$

where δ_k tends to zero sufficiently slowly. Map (3.49) takes the form

$$\begin{aligned} &\bar{X} = \frac{b_0}{s_0}Y_1 + o_{k\to\infty}(1), \qquad (\bar{U}, Y_2, V) = o_{k\to\infty}(1), \\ &\frac{1}{s_0}(\cos k\psi + d_1\sin k\psi)\bar{Y}_1 = \gamma^{2k}M_1 + c_0\lambda_1^k\gamma^k X + \frac{D_0}{s_0^2}Y_1^2 + o_{k\to\infty}(1), \end{aligned} \tag{3.51}$$

where, recall, $s_0 = d_1\cos k\psi - \sin k\psi$ (see (3.27)). Denote $q_0 = \cos k\psi + d_1 \sin k\psi$. Choose a sequence of values of $k \to \infty$ such that both q_0 and s_0 stay bounded away from zero (we can always do it as $d_1 \neq 0$, see lemma 8). For such k, we introduce

$$X_{\text{new}} = -\frac{b_0}{q_0 D_0}X, \qquad Y_{1\text{new}} = -\frac{s_0}{q_0 D_0}, \qquad (U, Y_2, V)_{\text{new}} = (U, Y_2, V). \tag{3.52}$$

Then, map (3.51) is recast as

$$\begin{aligned} &\bar{X} = Y_1 + o_{k\to\infty}(1), \qquad (\bar{U}, Y_2, V) = o_{k\to\infty}(1), \\ &\bar{Y}_1 = M + \frac{b_0 c_0}{q_0}\lambda_1^k\gamma^k X - Y_1^2 + o_{k\to\infty}(1), \end{aligned}$$

where $M = -D_0\gamma^{2k}M_1$ and M_1 is given by (3.26). This is absolutely analogous to (3.42), so lemmas 4 and 5 follow immediately, with B_k given by

$$B_k = \frac{b_0 c_0}{\cos k\psi + d_1 \sin k\psi}(\operatorname{sign}\lambda_1)^k. \tag{3.53}$$

Case (2, 2). We start with formula (3.34) for the first-return map $T^{(k)}$, where we assume, again, that values of $k \to \infty$ are such that both $q_0 \equiv \cos k\psi + d_1 \sin k\psi$ and $s_0 \equiv d_1 \cos k\psi - \sin k\psi$ stay bounded away from zero. We introduce rescaled coordinates (X, U, Y, V) by the following formulae (analogous to (3.50) and (3.52)):

$$x_1 = -\gamma^{-k}\frac{b_0}{q_0 D_0}X_1, \qquad y_1 = -\gamma^{-k}\frac{s_0}{q_0 D_0}Y_1, \qquad (x_2, u, y_2, v) = \gamma^{-k}\delta_k(X_2, U, Y_2, V),$$

where δ_k tends to zero sufficiently slowly. After that, map (3.34) takes the form

$$\begin{aligned} &\bar{X}_1 = Y_1 + o_{k\to\infty}(1), \qquad (\bar{X}_2, \bar{U}, Y_2, V) = o_{k\to\infty}(1), \\ &\bar{Y}_1 = M + \frac{b_0}{q_0}(c_{11}\cos k\varphi + c_{12}\sin\varphi)\lambda^k\gamma^k X_1 - Y_1^2 + o_{k\to\infty}(1), \end{aligned} \tag{3.54}$$

where $M = -D_0\gamma^{2k} M_1$ with M_1 given by (3.32). This map is absolutely analogous to (3.42), so lemmas 4 and 5 follow, with B_k given by

$$B_k = -\frac{b_0(c_{11}\cos k\varphi + c_{12}\sin\varphi)}{\cos k\psi + d_1 \sin k\psi}. \tag{3.55}$$

3.7. Bifurcations in the first-return maps

According to lemmas 1–5 the first-return maps $T^{(k)}$ (when restricted to an invariant manifold, if necessary) are close to one of the five 'truncated' maps: parabola map (3.56), Hénon map (3.57), generalized Hénon map (3.60), Mira map (3.58) or three-dimensional Hénon map (3.59). Let us consider bifurcations in these maps.

Parabola map. Consider the following one-dimensional map

$$\bar{y} = M - y^2. \tag{3.56}$$

Let ν_1 be the multiplier of some fixed point. The coordinate y of this fixed point satisfies equations $M = y + y^2$ and $2y = -\nu_1$. Thus, the parabola map has a fixed point with the multiplier ν_1 at $M = \frac{\nu_1^2}{4} - \frac{\nu_1}{2}$. In particular, $M = -1/4$ corresponds to the saddle-node bifurcation and $M = 3/4$ corresponds to period-doubling. It is well known that these bifurcations are non-degenerate in this map (the first Lyapunov values are non-zero). Accordingly, at all k large enough, the saddle-node bifurcation in the corresponding first-return map $T^{(k)}$ is non-degenerate at $r \geqslant 2$ and the period-doubling is non-degenerate at $r \geqslant 3$ (see case (i) of lemmas 1, 2 and 4).

Hénon map. Consider a Hénon map (the limit map for (1.4)):

$$\bar{x} = y, \qquad \bar{y} = M - Bx - y^2. \tag{3.57}$$

Let ν_1 and ν_2 be the multipliers of some fixed point (they are either both real or they comprise a complex-conjugate pair). The coordinates $x = y$ of the fixed point satisfy the equation $M = y(1+B) + y^2$. The characteristic equation is $\nu^2 + 2y\nu + B = 0$. It is easy to find that $B(\nu_1, \nu_2) = \nu_1\nu_2$, $M(\nu_1, \nu_2) = \frac{\nu_1+\nu_2}{4}(\nu_1 + \nu_2 - 2\nu_1\nu_2 - 2)$, i.e. for every admissible ν_1 and ν_2 there exist B and M for which map (3.57) has a fixed point with the multipliers ν_1 and ν_2.

In particular, the curve $M = -\frac{1}{4}(1+B)^2$ corresponds to a saddle-node bifurcation (one multiplier equal to +1), the curve $M = \frac{3}{4}(B^2 - 1)$ corresponds to a period-doubling (one multiplier equal to (-1)) and the curve $\{B = 1, M \in (-1, 3)\}$ corresponds to a pair of multipliers $\nu_{1,2} = \mathrm{e}^{\pm i\omega}$. The codimension-2 points $(B = 1, M = -1)$, $(B = 1, M = 3)$ and $(B = -1, M = 0)$ correspond to $(\nu_1, \nu_2) = (1, 1)$, $(-1, -1)$ and $(1, -1)$, respectively (see figure 1(*a*)).

The saddle-node and period-doubling bifurcations in the Hénon map are non-degenerate. However, the bifurcations corresponding to a fixed point with two multipliers on the unit circle (i.e. at $|B| = 1$) are degenerate. Indeed, in the generic case, crossing the curve corresponding to $\nu_{1,2} = \mathrm{e}^{\pm i\omega}$ should lead to the birth of a closed invariant curve, but the Hénon map cannot have closed invariant curves at $|B| \neq 1$, as the Jacobian of the map is constant (it equals B). Similar reasoning gives us the degeneracy of the point $(B = -1, M = 0)$ that corresponds to $\nu_{1,2} = \pm 1$.

Mira map. Consider the map (limit for (1.5))

$$\bar{y}_1 = y_2, \qquad \bar{y}_2 = M - Cy_2 - y_1^2. \tag{3.58}$$

Let ν_1 and ν_2 be the multipliers of some fixed point (again, the multipliers are either both real or they comprise a complex-conjugate pair). The coordinates $y_1 = y_2 = y$ of the fixed point

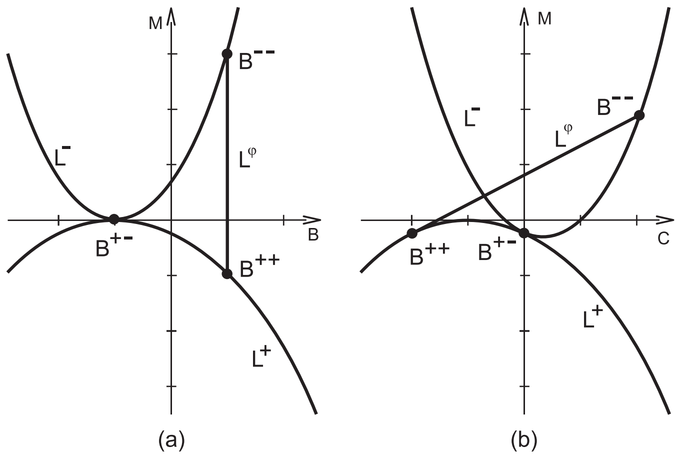

Figure 1. Bifurcation curves for fixed points of (*a*) the Hénon map (3.57) and (*b*) the Mirá map (3.58). Bifurcation curves L^+, L^- and L^φ correspond to multipliers +1, −1 and $e^{\pm i\omega}$, respectively; codimension-2 points B^{++}, B^{+-} and B^{--} correspond to pairs of multipliers (+1, +1), (+1, −1) and (−1, −1), respectively.

satisfy the equation $M=y(1+C)+y^2$; the characteristic equation is $\nu^2+C\nu+2y=0$. One can easily find $C=-(\nu_1+\nu_2)$, $M=\frac{\nu_1\nu_2}{2}(1+C)+\frac{(\nu_1\nu_2)^2}{4}$. We thus obtain three bifurcation curves (see figure 1(*b*)): $\{M=-\frac{1}{4}(C+1)^2, C\neq 0, C\neq -2\}$ corresponds to a non-degenerate saddle-node bifurcation, $\{M=\frac{1}{4}(C-1)(3C+1), C\neq 0, C\neq 2\}$ to a non-degenerate period-doubling and $\{M=\frac{1}{4}(3+2C), |C|<2\}$ to $\nu=e^{\pm i\omega}$. By [61], the latter bifurcation is non-degenerate and the first Lyapunov value is equal to $L_1=-\frac{1}{4}(C+3)/(2+C)$. As $|C|=2|\cos\omega|<2$, it follows that $L_1<0$, i.e. a stable closed invariant curve is born from the fixed point when the bifurcation curve is crossed.

Three-dimensional Hénon map. Consider the following map (limit for (1.6)):

$$\bar{x}=y_1, \qquad \bar{y}_1=y_2, \qquad \bar{y}_2=M-Bx-Cy_2-y_1^2. \tag{3.59}$$

Let ν_1,ν_2,ν_3 be the multipliers of some fixed point (either all three of them are real or one multiplier is real and the other two comprise a complex-conjugate pair). The coordinates $x=y_1=y_2$ of the fixed point satisfy the equation $M=x(1+B+C)+x^2$, and the characteristic equation is $-\nu^3-C\nu^2-2x\nu-B=0$. It follows that $B=-\nu_1\nu_2\nu_3$, $C=-(\nu_1+\nu_2+\nu_3)$, $M=\frac{1}{2}(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3)(1+B+C)+\frac{1}{4}(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3)^2$. These formulae give us the following equations for codimension-1 bifurcation surfaces (see figure 2): $M=-\frac{1}{4}(B+C+1)^2$ corresponds to a non-degenerate saddle-node, $M=\frac{1}{4}(B+C-1)(3B+3C+1)$ corresponds to a non-degenerate period-doubling, $M=\frac{1}{4}(1+B(C-B))(3+2(C+B)+B(C-B))$ corresponds to a pair of multipliers $e^{\pm i\omega}$ (the third multiplier is equal to $-B$ in this case). Note that map (3.59) degenerates into Mira map (3.58) at $B=0$, and one can check that the first Lyapunov value L_1 for the fixed point with the multipliers $e^{\pm i\omega}$ is close, at small B, to that for the Mira map. So, $L_1<0$ at small B. Hence, there exists an open region of parameter values for which map (3.59) has a stable closed invariant curve.

Generalized Hénon map. As we have seen, the analysis of the Hénon map is not sufficient for the study of bifurcations of the birth of closed invariant curves in the first-return maps $T^{(k)}$

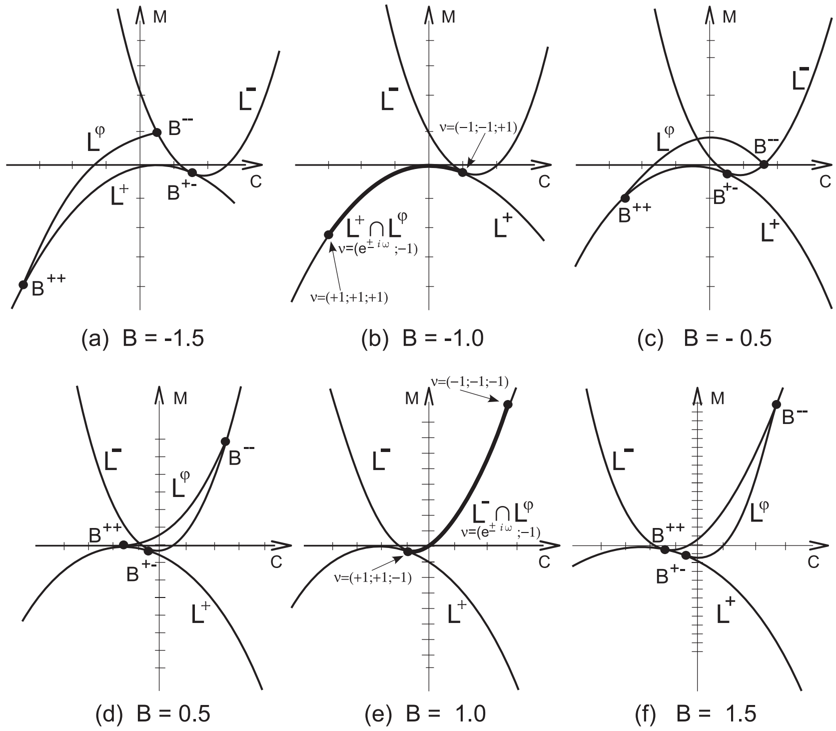

Figure 2. Bifurcation diagrams for different values of $B = \text{const}$ for the three-dimensional Hénon map (3.59). The diagram for $B = 0$ is shown in figure 1(*b*).

in case $(2, 1)$ with $\lambda\gamma > 1$ (case (ii) of lemmas 1 and 2). In this case, according to lemma 3, the map $T^{(k)}$ (restricted to an invariant manifold) is close to the generalized Hénon map:

$$\bar{x} = y, \qquad \bar{y} = M - y^2 - Bx - Q_k xy, \tag{3.60}$$

where $Q_k \equiv \frac{2J_1}{B}(\lambda^2\gamma)^k \to 0$ as $k \to +\infty$. This map undergoes a non-degenerate saddle-node bifurcation of the fixed point at $M = -\frac{(1+B)^2}{4(1+Q_k)}$, and a non-degenerate period-doubling bifurcation at $M = \frac{(1+B)^2(3+Q_k)}{4}$ (see [32]). The bifurcation curve that corresponds to $\nu_{1,2} = e^{\pm i\omega}$ is given by $(B - 1 - Q_k)^2 = Q_k^2(1 + M)$, $|B - 1| < \frac{|Q_k|}{1+Q_k/2}$. In contrast to the Hénon map, this bifurcation in the generalized Hénon map is non-degenerate (at $Q_k \neq 0$): the first Lyapunov coefficient of the fixed point with the multipliers $e^{\pm i\omega}$ equals

$$L_1 = -\frac{Q_k}{16(1 - \cos\omega)} + o(Q_k) \tag{3.61}$$

(see [32]). It means that a closed invariant curve is born from the fixed point when the bifurcation curve is crossed. The invariant curve is born stable if $Q_k > 0$ and unstable if $Q_k < 0$. The first Lyapunov coefficient is a function of the coefficients of the Taylor expansion of the map at the fixed point up to the terms of the third order. Therefore, for every map that is $o(Q_k)$-close to (3.60) in C^3-topology, the first Lyapunov value is still given by (3.61), i.e. it is negative at $Q_k < 0$ and positive at $Q_k > 0$. It means that the corresponding bifurcation remains non-degenerate. Thus, by lemma 3, if $r \geqslant 3$, the first-return map $T^{(k)}$ has, in some region of parameter values, a stable closed invariant curve, provided $J_1 > 0$.

In case $J_1 < 0$, the existence of a stable closed invariant curve in the truncated map (3.60) follows from the analysis of a fixed point with the pair of multipliers $(-1,-1)$ which was done in [32]. Applying the results of [32], one can show the existence of a stable closed invariant curve in any map which is $o(Q_k)$-close to (3.60) in C^3-topology. Since we want to consider the C^2-case also, we will consider below another mechanism of the birth of closed invariant curves.

Recall that if a C^2-map of a plane has a parabolic periodic point, then the map in a neighbourhood of this point can be brought to the following normal form:

$$\bar{x} = y, \qquad \bar{y} = -x + 2y + ay^2 + bxy + o(x^2 + y^2). \tag{3.62}$$

The point is non-degenerate if $ab \neq 0$. For a non-degenerate point, any generic two-parameter unfolding of class C^2 can be written in the form

$$\bar{x} = y, \qquad \bar{y} = \varepsilon_1 + (-1+\varepsilon_2)x + 2y + ay^2 + bxy + o(x^2 + y^2),$$

where a, b depend continuously on the governing parameters $\varepsilon_{1,2}$. If

$$ab > 0, \tag{3.63}$$

then there exists an open region in the plane of parameters $(\varepsilon_1, \varepsilon_2)$ which corresponds to the existence of a stable closed invariant curve. If $ab < 0$, then an unstable closed invariant curve is born.

By lemma 3, the first-return map $T^{(k)}$ in case $(2,1)$ with $\lambda\gamma > 1$ has the form

$$\bar{x} = y, \qquad \bar{y} = M - y^2 - Bx + \vartheta_k(y) - Q_k xy + o(Q_k), \tag{3.64}$$

where $Q_k = \frac{2J_1}{B}(\lambda^2\gamma)^k$ and $\vartheta_k = o(1)_{k\to+\infty}$. Map (3.64) has a non-degenerate parabolic fixed point at $B = -1 + o(1)_{k\to+\infty}$, $M = -1/4 + o(1)_{k\to+\infty}$: the coefficients a and b are given by $a = -1 + o(1)_{k\to+\infty}$, $b = -Q_k + o(Q_k)$. Thus, if $J_1 > 0$, we have the existence of a stable closed invariant curve in the case $r = 2$ also (see (3.63)).

In order to show the existence of stable closed invariant curves in the case $J_1 < 0$, we will study parabolic points of period three. In the Hénon map (3.57) such a point exists at $M = B = 1$; it is the point $(0,0)$. The third iteration of the Hénon map at $M = B = 1$ has the form

$$\begin{aligned} \bar{\bar{\bar{x}}} &= 1 - y - (1 - x - y^2)^2 = 2x - y - x^2 + 2y^2 + o(x^2 + y^2), \\ \bar{\bar{\bar{y}}} &= x + y^2 - (\bar{\bar{\bar{x}}})^2 = x - 4x^2 + 4xy + o(x^2 + y^2). \end{aligned} \tag{3.65}$$

In order to bring this map to the normal form (3.62), we introduce the new coordinates $u = y, v = x + y^2 - (\bar{\bar{\bar{x}}})^2$. Map (3.65) takes the form

$$\bar{u} = v, \qquad \bar{v} = -u + 2v - v^2 + 4uv - 2u^2 + o(u^2 + v^2).$$

After one more transformation $z = u - u^2, w = v - v^2$ the map takes the form

$$\bar{z} = w, \qquad \bar{w} = -z + 2w + w^2 + o(z^2 + w^2).$$

Since map (3.64) is at least C^2-close to the Hénon map as $k \to +\infty$, it also has a parabolic point of period three and the normal form is close to that of the Hénon map, i.e. the normal form is

$$\bar{z} = w, \qquad \bar{w} = -z + 2w + (1 + o(1)_{k\to+\infty})w^2 + R_k zw + o(z^2 + w^2), \tag{3.66}$$

where R_k is a certain coefficient that tends to zero as $k \to +\infty$. Let us show that $R_k = -Q_k + o(Q_k)$. Indeed, it is easy to see that the Jacobian of the third iteration of map (3.64) equals

$$\mathcal{J} = (B + Q_k y)(B + Q_k \bar{y})(B + Q_k \bar{\bar{y}}) + o(Q_k) = B^3 + Q_k(x + o(|x| + |y|)) + o(Q_k). \tag{3.67}$$

Let us move the origin of the coordinate system to that point of the period three orbit which is close to zero (recall that for the Hénon map the corresponding point is exactly $(0, 0)$). Obviously, the Jacobian will keep its form (3.67).

At large k, the linear part of the third iteration $(T^{(k)})^3$ of (3.64) at the period three point will be close to the linear part of the third iteration of the Hénon map (see (3.65)), i.e. $(T^{(k)})^3$ has the form

$$\begin{pmatrix} \overline{\overline{\overline{x}}} \\ \overline{\overline{\overline{y}}} \end{pmatrix} = D \begin{pmatrix} x \\ y \end{pmatrix} + o(|x|+|y|), \tag{3.68}$$

where $D = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} + o(1)_{k\to+\infty}$. Recall that the periodic point is parabolic, which means that $\operatorname{tr} D = 2$ and $\det D = 1$. Therefore, by an $o(1)_{k\to+\infty}$-close to identity linear transformation we may bring this matrix to the form

$$D = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.69}$$

Note that this transformation does not change the form (3.67) of the Jacobian.

Next we make nonlinear normalizing transformations that bring the map to the form (3.66). These are coordinate transformations with the linear part equal to identity. It is easy to see that the derivative $\frac{\partial \mathcal{J}}{\partial x} + \frac{\partial \mathcal{J}}{\partial y}$ at $(x, y) = 0$ is an invariant of such transformations (for maps whose linear part is given by (3.69)). Since this derivative for the normal form (3.66) is equal to $-R_k$, we immediately find from (3.67) that $R_k = -Q_k + o(Q_k)$. The closed invariant curve that is born at the perturbation of map (3.66) is stable when $R_k > 0$ (see (3.63)). Therefore, map (3.64) has a period-3 stable closed invariant curve at $Q_k < 0$, i.e. at $J_1 < 0$. As we see, the first-return map $T^{(k)}$ of the form (1.9) has a stable closed invariant curve (either of period one or of period three) for both cases of the sign of J_1.

Infinitely many coexisting stable closed invariant curves

Proof of theorem 6. As we have just shown, in all cases with $d_e \geqslant 2$ (case (2,1) with $\lambda\gamma > 1$, case (1,2) and case (2,2)) the rescaled first-return map $T^{(k)}$ has, in some region of parameters (M, B), (M, C) or (M, C, B), a stable closed invariant curve. Thus, in the Newhouse regions with $d_e \geqslant 2$, the required genericity of maps with infinitely many coexisting stable invariant curves follows exactly in the same way as in theorem 3.

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Appendix

Proof of lemma 6. First, we make coordinate transformations after which identities (2.3) will hold. Let

$$x^{\text{new}} = x + h_1(x, u, \varepsilon), \qquad y^{\text{new}} = y + h_2(y, v, \varepsilon) \tag{A.1}$$

(u and v remain unchanged). We require that $h_{1,2}(0, 0, \varepsilon) = 0$, $\frac{\partial h_1}{\partial(x,u)}(0, 0, \varepsilon) = 0$, $\frac{\partial h_2}{\partial(y,v)}(0, 0, \varepsilon) = 0$, so that identities (2.2) will persist in the new coordinates. In order to make the first of conditions (2.3) fulfilled in the new coordinates we have to achieve $\bar{x}^{\text{new}} = A_1 x^{\text{new}}$ at $(y,v)=0$. This gives us (see (2.1)) $0 = \bar{x}^{\text{new}} - A_1 x^{\text{new}} = \bar{x} + h_1(\bar{x},\bar{u},\varepsilon) - A_1 x - A_1 h_1(x,u,\varepsilon) = p_1(x,u,0,0,\varepsilon) + h_1(\bar{x},\bar{u},\varepsilon) - A_1 h_1(x,u,\varepsilon)$, which leads to the following equation for h_1:

$$h_1(\bar{x},\bar{u},\varepsilon) = A_1(\varepsilon) h_1(x,u,\varepsilon) - p_1(x,u,0,0,\varepsilon), \tag{A.2}$$

where

$$\bar{x} = A_1(\varepsilon)x + p_1(x,u,0,0,\varepsilon), \qquad \bar{u} = A_2(\varepsilon)u + p_2(x,u,0,0,\varepsilon) \tag{A.3}$$

(this is just the restriction of T_0 onto the stable manifold).

The fulfilment of the second of conditions (2.3) in the new coordinates means $\bar{y}^{\text{new}} = A_1 y^{\text{new}}$ at $(x,u)=0$. This gives us the following equation for h_2:

$$h_2(\bar{y},\bar{v},\varepsilon) = B_1(\varepsilon) h_2(y,v,\varepsilon) - q_1(0,0,y,v,\varepsilon), \tag{A.4}$$

where

$$\bar{y} = B_1(\varepsilon)y + q_1(0,0,y,v,\varepsilon), \qquad \bar{v} = B_2(\varepsilon)v + q_2(0,0,y,v,\varepsilon) \tag{A.5}$$

(the restriction of T_0 onto the unstable manifold).

It is easy to see that the following functions h_1, h_2 solve (A.2) and, respectively (A.4):

$$h_1(x,u,\varepsilon) = \sum_{j=0}^{\infty} A_1^{-j-1}(\varepsilon) p_1(x_j,u_j,0,0,\varepsilon), \qquad h_2(y,v,\varepsilon) = -\sum_{j=1}^{\infty} B_1^{j-1}(\varepsilon) q_1(0,0,y_j,v_j,\varepsilon). \tag{A.6}$$

Here the points $\{(x_j,u_j)\}$ are the forward orbit of $(x,u)\equiv(x_0,u_0)$ by map (A.3): $(x_{j+1},u_{j+1}) = (T_0|_{W^s_{\text{loc}}})(x_j,u_j)$, and the points $\{(y_j,v_j)\}$ are the backward orbit of $(y,v)\equiv(y_0,v_0)$ by map (A.5): $(y_j,v_j) = (T_0|_{W^u_{\text{loc}}})(y_{j+1},v_{j+1})$. As the maps $T_0|_{W^s_{\text{loc}}}$ and $T_0^{-1}|_{W^u_{\text{loc}}}$ are contractions, it follows that $(x_j,u_j)\to 0$, $(y_j,v_j)\to 0$.

Let us show that series (A.6) is convergent. Indeed, for any small $\delta>0$, we can choose a norm in the (x,u)-space such that $\|A_{1,2}\| \leqslant \lambda(1+\delta/2)$. Then $\|x_j,u_j\| + \|\frac{\partial(x_j,u_j)}{\partial(x,u)}\| \leqslant K\lambda^j(1+\delta)^j$ for all small (x,u,ε) and for some constant K. Recall that all the eigenvalues of $A_1(0)$ are equal to λ in absolute values, so $\|A_1^{-1}\| \leqslant \lambda^{-1}(1+\delta)$ for the same choice of the norm in the x-space. As $p_1=0$, $(p_1)'_x=0$ and $(p_1)'_u=0$ at $(x,u)=0$, we have $p_1(x,u,0,0,\varepsilon) = O(\|x\|^2+\|u\|^2)$, $\frac{\partial p_1}{\partial(x,u)}(x,u,0,0,\varepsilon) = O(\|x\|+\|u\|)$. Thus, $\|A_1^{-j-1}(\varepsilon)p_1(x_j,u_j,0,0,\varepsilon)\| = O(\lambda^j(1+\delta)^{3j})$, $\|A_1^{-j-1}(\varepsilon)\frac{\partial p_1}{\partial(x,u)}(x_j,u_j,0,0,\varepsilon)\frac{\partial(x_j,u_j)}{\partial(x,u)}\| = O(\lambda^j(1+\delta)^{3j})$; hence the first series in (A.6) converges uniformly, along with the first derivatives with respect to (x,u), i.e. it defines a C^1 function h_1. Similarly, the second series defines a C^1 function h_2. It follows directly from (A.6) that h and h' vanish at zero, as required.

Let us show that $h_{1,2}$ have continuous derivatives up to the order r, with a possible exception for the rth and $(r-1)$th derivatives with respect to ε. Indeed, note that the derivatives $\xi_1(x,u,\varepsilon) \equiv \frac{\partial h_1}{\partial(x,u)}(x,u,\varepsilon)$ and $\xi_2(y,v,\varepsilon) \equiv \frac{\partial h_2}{\partial(y,v)}(y,v,\varepsilon)$ satisfy the equations

$$\xi_1(\bar{x},\bar{u}) = \left[A_1\xi_1(x,u) - \frac{\partial p_1}{\partial(x,u)}(x,u,0,0)\right]\begin{pmatrix} A_1 + \dfrac{\partial p_1}{\partial x}(x,u,0,0) & \dfrac{\partial p_1}{\partial u}(x,u,0,0) \\ \dfrac{\partial p_2}{\partial x}(x,u,0,0) & A_2 + \dfrac{\partial p_2}{\partial u}(x,u,0,0)\end{pmatrix}^{-1},$$

$$\xi_2(\bar{y},\bar{v}) = \left[B_1\xi_2(y,v) - \frac{\partial q_1}{\partial(y,v)}(0,0,y,v)\right]\begin{pmatrix} B_1 + \dfrac{\partial q_1}{\partial y}(0,0,y,v) & \dfrac{\partial q_1}{\partial y}(0,0,y,v) \\ \dfrac{\partial q_2}{\partial y}(0,0,y,v) & B_2 + \dfrac{\partial q_2}{\partial u}(0,0,y,v)\end{pmatrix}^{-1} \tag{A.7}$$

(see (A.6), (A.3) and (A.5)). The first of these formulae can be read as the condition of the invariance of the manifold $\mathcal{W}_1:\ z=\xi_1(x,u,\varepsilon)$ with respect to the map

$$\bar{x}=A_1(\varepsilon)x+p_1(x,u,0,0,\varepsilon),\qquad \bar{u}=A_2(\varepsilon)u+p_2(x,u,0,0,\varepsilon),$$

$$\bar{z}=\left[A_1(\varepsilon)z-\frac{\partial p_1}{\partial(x,u)}(x,u,0,0,\varepsilon)\right]\begin{pmatrix} A_1(\varepsilon)+\dfrac{\partial p_1}{\partial x}(x,u,0,0,\varepsilon) & \dfrac{\partial p_1}{\partial x}(x,u,0,0,\varepsilon)\\ \dfrac{\partial p_2}{\partial x}(x,u,0,0,\varepsilon) & A_2(\varepsilon)+\dfrac{\partial p_2}{\partial u}(x,u,0,0,\varepsilon)\end{pmatrix}^{-1} \tag{A.8}$$

(z is an $m_1\times m$-matrix). The second of formulae (A.7) is the condition of the invariance of the manifold $\mathcal{W}_2:\ w=\xi_2(y,v,\varepsilon)$ with respect to the map

$$\bar{y}=B_1(\varepsilon)y+q_1(0,0,y,v,\varepsilon),\qquad \bar{v}=B_2(\varepsilon)v+q_2(0,0,y,v,\varepsilon),$$

$$\bar{w}=\left[B_1(\varepsilon)w-\frac{\partial q_1}{\partial(y,v)}(0,0,y,v,\varepsilon)\right]\begin{pmatrix} B_1(\varepsilon)+\dfrac{\partial q_1}{\partial y}(0,0,y,v,\varepsilon) & \dfrac{\partial q_1}{\partial y}(0,0,y,v,\varepsilon)\\ \dfrac{\partial q_2}{\partial y}(0,0,y,v,\varepsilon) & B_2(\varepsilon)+\dfrac{\partial q_2}{\partial u}(0,0,y,v,\varepsilon)\end{pmatrix}^{-1} \tag{A.9}$$

(w is an $n_1\times n$-matrix). Thus, we may estimate the smoothness of the functions $\xi_{1,2}$ (that is the smoothness of the manifolds $\mathcal{W}_{1,2}$) by using known facts from the theory of local invariant manifolds. Let us start with map (A.8). It has a fixed point at $(z=0, x=0, u=0)$. The spectrum of the linear part of the map is the union of the spectra of the following three operators: $z\mapsto A_1 z\begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}^{-1}$, $x\mapsto A_1 x$, $u\mapsto A_2 u$. It is well known from the theory of matrices that the spectrum of the operator $z\mapsto LzK$ consists of the products of the eigenvalues of L to the eigenvalues of K. Hence, the spectrum of the multipliers of the fixed point $(0,0,0)$ of map (A.8) consists of all possible ratios of the eigenvalues of A_1 to the eigenvalues of A_1 and A_2, and, besides, of the eigenvalues of A_1 and A_2. Recall that at $\varepsilon=0$ all the eigenvalues of A_1 have the same absolute value $\lambda<1$, while the absolute values of the eigenvalues of A_2 are strictly less than λ. Thus, the zero fixed point has, at $\varepsilon=0$, m_1^2 multipliers on the unit circle, $m_1 m_2$ multipliers strictly outside the unit circle and m eigenvalues strictly smaller than 1 in absolute value (the eigenvalues of A_1 and A_2).

It is known (see, e.g., [44] or [42]) that such a fixed point lies in a uniquely defined m-dimensional *strong-stable* invariant manifold which is tangent at the fixed point to the eigenspace of the linear part that corresponds to the multipliers strictly inside the unit circle: in our case it means that the strong-stable manifold is a graph of a smooth function $(x,y)\mapsto z$. By uniqueness, it is the invariant manifold $\mathcal{W}_1$. Analogously, the manifold $\mathcal{W}_2$ with ξ_2 satisfying the second equation of (A.7) is the uniquely defined *strong-unstable* manifold (the manifold tangent to the eigenspace that corresponds to the multipliers strictly outside the unit circle) of the zero fixed point of (A.9).

The strong-stable and strong-unstable manifolds have the same smoothness as the map itself. Therefore, as the right-hand sides of (A.8) and (A.9) are C^{r-1}, the functions $\xi_{1,2}$ are C^{r-1} with respect to (x,u) or, respectively, (y,v). As $\xi_1\equiv\frac{\partial h_1}{\partial(x,u)}$ and $\xi_2\equiv\frac{\partial h_2}{\partial(y,v)}$, this gives us the required C^r-smoothness of $h_{1,2}$.

When a map depends on parameters, the field of tangents to the strong-stable or strong-unstable manifold has the same smoothness with respect to variables and parameters as the first derivative of the map with respect to the phase variables [42]. Since the first derivatives of the right-hand sides of (A.8) and (A.9) with respect to (x,u,z) and, respectively, (y,v,w) are

C^{r-2}, it follows that the derivatives $\frac{\partial\xi_1}{\partial(x,u)}$ and $\frac{\partial\xi_2}{\partial(y,v)}$ are C^{r-2} with respect to all variables and ε. This gives us the required smoothness of $h_{1,2}$ with respect to ε (see remark 1 to the lemma).

In cases $r=\infty,\omega$, only finite smoothness with respect to ε can be expected for the strong-stable manifold in general [42]. More precisely, we know that given any ρ, the strong-stable manifold is C^ρ with respect to ε at $|\varepsilon|<\bar{\varepsilon}(\rho)$ where $\bar{\varepsilon}(\rho)$ may tend to zero as $\rho\to+\infty$. However, this implies that at $\varepsilon=0$ there exist infinitely many derivatives with respect to ε. In other words, there exist infinitely many derivatives at any value of ε for which all the multipliers of the fixed point which are not bounded away from the unit circle lie exactly on the unit circle or outside it. In our case, it follows that h_1 is C^∞ with respect to ε if at every ε the spectrum of the operator $z\mapsto A_1(\varepsilon)z\begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}^{-1}$ lies on the unit circle or outside it, i.e. for any ε for which all the eigenvalues of $A_1(\varepsilon)$ are equal to each other in absolute value. Analogously, $h_2\in C^\infty$ with respect to ε if all the eigenvalues of $B_1(\varepsilon)$ are equal to each other in absolute value for every small ε.

We see that the coordinate transformation (A.1) has exactly the same smoothness as described in remark 1. Also note that the new functions p, q enjoy the same smoothness as the functions h do.

After the map is brought to the form in which identities (2.3) hold, we make the coordinate transformation:

$$\begin{aligned} x^{\text{new}} &= x+g_{11}(x,y,v,\varepsilon), & u^{\text{new}} &= u+g_{12}(x,y,v,\varepsilon),\\ y^{\text{new}} &= y+g_{21}(x,u,y,\varepsilon), & v^{\text{new}} &= v+g_{22}(x,u,y,\varepsilon), \end{aligned} \tag{A.10}$$

where g_{ij} vanish identically both at $(x,u)=0$ and at $(y,v)=0$. Thus, this transformation is identical on the stable and unstable manifolds, e.g. it does not destroy identities (2.2) and (2.3). Denote

$$\eta_{1j}(y,v,\varepsilon)=\frac{\partial g_{1j}}{\partial x}(0,y,v,\varepsilon), \qquad \eta_{2j}(x,u,\varepsilon)=\frac{\partial g_{2j}}{\partial y}(x,u,0,\varepsilon). \tag{A.11}$$

It is easy to see that identities (2.4) will be fulfilled after transformation (A.10) if η_{ij} satisfy the following equations:

$$\begin{aligned} \eta_{1j}(\bar{y},\bar{v}) &= \left(A_j\eta_{1j}(y,v)-\frac{\partial p_j}{\partial x}(0,0,y,v)+\frac{\partial p_j}{\partial u}(0,0,y,v)\eta_{12}(y,v)\right)\\ &\quad\times\left(A_1+\frac{\partial p_1}{\partial x}(0,0,y,v)-\frac{\partial p_1}{\partial u}(0,0,y,v)\eta_{12}(y,v)\right)^{-1}, \end{aligned} \tag{A.12}$$

where $(y,v)\mapsto(\bar{y},\bar{v})$ is given by (A.5), and

$$\begin{aligned} \eta_{2j}(\bar{x},\bar{u}) &= \left(B_j\eta_{2j}(x,u)-\frac{\partial q_j}{\partial y}(x,u,0,0)+\frac{\partial q_j}{\partial v}(x,u,0,0)\eta_{22}(x,u)\right)\\ &\quad\times\left(B_1+\frac{\partial q_1}{\partial y}(x,u,0,0)-\frac{\partial q_1}{\partial v}(x,u,0,0)\eta_{22}(x,u)\right)^{-1}, \end{aligned} \tag{A.13}$$

where $(x,u)\mapsto(\bar{x},\bar{u})$ is given by (A.3).

Formula (A.12) can be viewed as the condition of the invariance of the manifold $\mathcal{W}_3$: $\{z_1=\eta_{11}(y,v,\varepsilon), z_2=\eta_{12}(y,v,\varepsilon)\}$ with respect to the map

$$\begin{aligned} \bar{z}_j &= \left(A_j(\varepsilon)z_j-\frac{\partial p_j}{\partial x}(0,0,y,v,\varepsilon)+\frac{\partial p_j}{\partial u}(0,0,y,v,\varepsilon)z_2\right)\\ &\quad\times\left(A_1(\varepsilon)+\frac{\partial p_1}{\partial x}(0,0,y,v,\varepsilon)-\frac{\partial p_1}{\partial u}(0,0,y,v,\varepsilon)z_2\right)^{-1} \qquad (j=1,2), \end{aligned} \tag{A.14}$$

$$\bar{y}=B_1(\varepsilon)y+q_1(0,0,y,v,\varepsilon), \qquad \bar{v}=B_2(\varepsilon)v+q_2(0,0,y,v,\varepsilon),$$

with $z_1 \in R^{m_1\times m_1}$, $z_2 \in R^{(m-m_1)\times m_1}$. Similarly, formula (A.13) represents the conditions of the invariance of the manifold $\mathcal{W}_4$: $\{w_1 = \eta_{21}(x,u,\varepsilon), w_2 = \eta_{22}(x,u,\varepsilon)\}$ with respect to the following map (here, $w_1 \in R^{n_1\times n_1}$, $w_2 \in R^{(n-n_1)\times n_1}$):

$$\bar{w}_j = \left(B_j(\varepsilon)w_j - \frac{\partial q_j}{\partial y}(x,u,0,0,\varepsilon) + \frac{\partial q_j}{\partial v}(x,u,0,0,\varepsilon)w_2\right)$$
$$\times\left(B_1(\varepsilon) + \frac{\partial q_1}{\partial y}(x,u,0,0,\varepsilon) - \frac{\partial q_1}{\partial v}(x,u,0,0,\varepsilon)w_2\right)^{-1} \qquad (j=1,2), \quad \text{(A.15)}$$
$$\bar{x} = A_1(\varepsilon)x + p_1(x,u,0,0,\varepsilon), \qquad \bar{u} = A_2(\varepsilon)u + p_2(x,u,0,0,\varepsilon).$$

Map (A.14) has a fixed point at $(z_1 = 0, z_2 = 0, y = 0, v = 0)$. The multipliers of this point are the eigenvalues of the linearized map

$$z_1 \mapsto A_1(\varepsilon)z_1A_1(\varepsilon)^{-1} - \frac{\partial^2 p_1}{\partial x\partial(y,v)}(0,0,0,0,\varepsilon)\cdot(y,v)A_1(\varepsilon)^{-1},$$
$$z_2 \mapsto A_2(\varepsilon)z_2A_1(\varepsilon)^{-1} - \frac{\partial^2 p_2}{\partial x\partial(y,v)}(0,0,0,0,\varepsilon)\cdot(y,v)A_1(\varepsilon)^{-1},$$
$$y \mapsto B_1(\varepsilon)y, \qquad v \mapsto B_2(\varepsilon)v.$$

The spectrum of this operator is the union of the spectra of the four operators $z_1 \mapsto A_1z_1A_1^{-1}$, $z_2 \mapsto A_2z_2A_1^{-1}$, $y \mapsto B_1y$, $v \mapsto B_2v$. Since at $\varepsilon = 0$ all the eigenvalues of A_1 have the same absolute value $\lambda < 1$, while the absolute values of the eigenvalues of A_2 are less than λ, and the eigenvalues of B_1 and B_2 lie outside the unit circle, it follows that the zero fixed point of the map (A.14) has, at $\varepsilon = 0$, m_1^2 multipliers on the unit circle, m_1m_2 multipliers strictly inside the unit circle and n eigenvalues outside the unit circle. Such a fixed point lies in a uniquely defined n-dimensional strong-unstable invariant manifold which is the manifold $\mathcal{W}_3$. This proves the existence of smooth functions η_{11}, η_{12} that satisfy equations (A.12) and vanish at $(y,v) = 0$.

Analogously, the zero fixed point of map (A.15) has a uniquely defined m-dimensional strong-stable invariant manifold $\mathcal{W}_4$. This gives us the existence of smooth functions η_{21}, η_{22} that satisfy (A.13) and vanish at $(x,u) = 0$.

The first derivatives of the right-hand sides of maps (A.14) and (A.15) with respect to (y,v) or (x,u), respectively, are C^{r-2} with respect to the variables and ε. Therefore, the field of tangents to the strong-unstable and, respectively, strong-stable manifolds of the zero fixed point of these maps enjoys the same smoothness [42]. Thus, we have established that the first derivatives of the functions η_{11}, η_{12} with respect to (y,v) and the first derivatives of the functions η_{21}, η_{22} with respect to (x,u) are C^{r-2} with respect to (y,v,ε) and (x,u,ε), respectively. In cases $r = \infty, \omega$, as we explained above, these functions may have any finite smoothness with respect to ε, and if for every ε all the eigenvalues of $A_1(\varepsilon)$ are equal to each other in absolute value and all the eigenvalues of $B_1(\varepsilon)$ are equal to each other in absolute value, then the functions η_{ij} are C^∞ with respect to ε.

One can now take any functions g_{ij} that vanish both at $(x,u) = 0$ and $(y,v) = 0$ and that satisfy (A.11). By construction, the coordinate transformation (A.10) with such defined functions g_{ij} will bring the map T_0 to the desired form (it will make identities (2.4) fulfilled, while identities (2.3) were achieved in the previous step). As η_{ij} are derivatives of the functions g_{ij}, we may choose the functions g_{ij} to be C^r with respect to (x,u,y,v); the smoothness with respect to ε remains C^{r-2}. □

Proof of lemma 7. Introduce positive constants $\lambda_0 < 1$, $\gamma_0 > 1$, $\hat{\lambda}$ and $\hat{\gamma}$ such that

$$\|A_1(\varepsilon)\| < \lambda_0, \qquad \|B_1(\varepsilon)^{-1}\| < \gamma_0^{-1}, \qquad \lambda_0 > \hat{\lambda} > \|A_2(\varepsilon)\|, \qquad \gamma_0^{-1} > \hat{\gamma}^{-1} > \|B_2(\varepsilon)^{-1}\| \quad \text{(A.16)}$$

for all small ε. Since the spectra of the matrices $A_1(0)$ and $A_2(0)$, as well as of $B_1(0)$ and $B_2(0)$, are separated, such constants always exist for an appropriate choice of bases in the x-, u-, y- and v-spaces. We will also assume that

$$\hat{\lambda} > \lambda_0 \max(\lambda_0, \gamma_0^{-1}), \qquad \hat{\gamma} < \gamma_0 \min(\gamma_0, \lambda_0^{-1}). \tag{A.17}$$

Let $M_j(x_j, u_j, y_i, v_j)$ $(j = 0, \ldots, k)$ be a sequence of points in U_0 such that $T_0 M_j = M_{j+1}$, i.e these are the points of the trajectory of $M_0(x_0, u_0, y_0, v_0)$. By (2.1), we have the following equations for $\{(x_j, u_j, y_j, v_j)\}(j = 0, \ldots, k)$ (see [59]):

$$x_j = A_1^j x_0 + \sum_{s=0}^{j-1} A_1^{j-s-1} p_1(x_s, u_s, y_s, v_s, \varepsilon), \quad u_j = A_2^j u_0 + \sum_{s=0}^{j-1} A_2^{j-s-1} p_2(x_s, u_s, y_s, v_s, \varepsilon),$$

$$y_j = B_1^{j-k} y_k - \sum_{s=j}^{k-1} B_1^{j-s-1} q_1(x_s, u_s, y_s, v_s, \varepsilon), \quad v_j = B_2^{j-k} v_k - \sum_{s=j}^{k-1} B_2^{j-s-1} q_2(x_s, u_s, y_s, v_s, \varepsilon). \tag{A.18}$$

By lemma 3.1 of [59], for all sufficiently small $\delta > 0$, if $\max\{\|x_0\|, \|u_0\|, \|y_k\|, \|v_k\|\} \leqslant \delta/2$, system (A.18) has a unique solution $[(x_j^*, u_j^*, y_j^*, v_j^*)]_{j=0}^k$, where $\|x_j^*\| \leqslant \delta$, $\|u_j^*\| \leqslant \delta$, $\|y_j^*\| \leqslant \delta$, $\|v_j^*\| \leqslant \delta$. This follows from the fact [57, 59] that at small δ the operator $\Phi\colon [(x_j, u_j, y_j, v_j)]_{j=0}^k \mapsto [(\bar{x}_j, \bar{u}_j, \bar{y}_j, \bar{v}_j)]_{j=0}^k$ defined by

$$\bar{x}_j = A_1^j x_0 + \sum_{s=0}^{j-1} A_1^{j-s-1} p_1(x_s, u_s, y_s, v_s, \varepsilon), \quad \bar{u}_j = A_2^j u_0 + \sum_{s=0}^{j-1} A_2^{j-s-1} p_2(x_s, u_s, y_s, v_s, \varepsilon),$$

$$\bar{y}_j = B_1^{j-k} y_k - \sum_{s=j}^{k-1} B_1^{j-s-1} q_1(x_s, u_s, y_s, v_s, \varepsilon), \quad \bar{v}_j = B_2^{j-k} v_k - \sum_{s=j}^{k-1} B_2^{j-s-1} q_2(x_s, u_s, y_s, v_s, \varepsilon), \tag{A.19}$$

is contracting. The fixed point $[(x_j^*, u_j^*, y_j^*, v_j^*)]_{j=0}^k$ is the sought solution of (A.18).

In order to prove the lemma, we must show, first, that the functions $\xi_k = \hat{\lambda}^{-k}(x_k^* - A_1^k x_0)$, $\hat{\xi}_k = \hat{\lambda}^{-k} u_k^*$, $\eta_k = \hat{\gamma}^k (y_0^* - B_1^{-k} y_k)$ and $\hat{\eta}_k = \hat{\gamma}^k v_0^*$ are uniformly bounded for all k. In fact, we show that $[(x_j^*, u_j^*, y_j^*, v_j^*)]_{j=0}^k$ belongs to the set $\hat{R}$ of the sequences $[(x_j, u_j, y_j, v_j)]_{j=0}^k$ that satisfy the inequalities

$$\|x_j - A_1^j x_0\| \leqslant \delta \hat{\lambda}^j, \qquad \|u_j\| \leqslant \delta \hat{\lambda}^j, \qquad \|y_j - B_1^{j-k} y_k\| \leqslant \delta \hat{\gamma}^{j-k}, \qquad \|v_j\| \leqslant \delta \hat{\gamma}^{j-k}. \tag{A.20}$$

Since $[(x_j^*, u_j^*, y_j^*, v_j^*)]_{j=0}^k$ is a unique fixed point of the contracting map Φ, it is sufficient to check that $\Phi(\hat{R}) \subseteq \hat{R}$. Obviously, (A.20) implies

$$\|x_j, u_j\| \leqslant \delta \lambda_0^j, \qquad \|y_j, v_j\| \leqslant \delta \gamma_0^{j-k}, \tag{A.21}$$

for every sequence $[(x_j, u_j, y_j, v_j)]_{j=0}^k \in \hat{R}$. Also note that it follows from (2.2) that

$$\|p_i(x, u, y, v)\| \leqslant \max_{s \in [0,1]} \left\| \frac{\partial p_i}{\partial x}(sx, su, sy, sv) \right\| \cdot \|x\| + \max_{s \in [0,1]} \left\| \frac{\partial p_i}{\partial u}(sx, su, sy, sv) \right\| \cdot \|u\|,$$

$$\|q_i(x, u, y, v)\| \leqslant \max_{s \in [0,1]} \left\| \frac{\partial q_i}{\partial y}(sx, su, sy, sv) \right\| \cdot \|y\| + \max_{s \in [0,1]} \left\| \frac{\partial q_i}{\partial v}(sx, su, sy, sv) \right\| \cdot \|v\|. \tag{A.22}$$

Since $\frac{\partial(p,q)}{\partial(x,u,y,v)}$ is a smooth function vanishing at the origin,

$$\frac{\partial(p,q)}{\partial(x,u,y,v)} = O(\|x\|+\|u\|+\|y\|+\|v\|). \tag{A.23}$$

By (2.3), we have

$$\frac{\partial p_1}{\partial(x,u)} = O(\|y\|+\|v\|), \qquad \frac{\partial q_1}{\partial(y,v)} = O(\|x\|+\|u\|), \tag{A.24}$$

while it follows from (2.4) that

$$\frac{\partial p}{\partial x} = O(\|x\|), \qquad \frac{\partial q}{\partial y} = O(\|y\|). \tag{A.25}$$

By comparing (A.24) and (A.25), we find that $\frac{\partial p_1}{\partial x} = O(\min\{\|x\|, \|y\|+\|v\|\})$, $\frac{\partial q_1}{\partial y} = O(\min\{\|y\|, \|x\|+\|u\|\})$; hence, for any $\alpha \in [0,1]$,

$$\frac{\partial p_1}{\partial x} = O(\|x\|^{\alpha}\cdot(\|y\|+\|v\|)^{1-\alpha}), \qquad \frac{\partial q_1}{\partial y} = O(\|y\|^{\alpha}\cdot(\|x\|+\|u\|)^{1-\alpha}). \tag{A.26}$$

Thus, by (A.22)–(A.26), there exists $L>0$ such that

$$\begin{aligned} &\|p_1\| \leqslant L(\|x\|^{1+\alpha}\cdot(\|y\|+\|v\|)^{1-\alpha} + \|u\|\cdot(\|y\|+\|v\|)), \qquad \|p_2\| \leqslant L\|x\|^2 + L\delta\|u\|, \\ &\|q_1\| \leqslant L(\|y\|^{1+\alpha}\cdot(\|x\|+\|u\|)^{1-\alpha} + \|v\|\cdot(\|x\|+\|u\|)), \qquad \|q_2\| \leqslant L\|y\|^2 + L\delta\|v\|. \end{aligned} \tag{A.27}$$

Hence, by (A.21), (A.16) and (A.17), we obtain

$$\|p_1(x_s,u_s,y_s,v_s,\varepsilon)\| \leqslant L\delta^2(\hat\lambda^s\gamma_0^{s-k} + \lambda_0^{(1+\alpha)s}\gamma_0^{(1-\alpha)(s-k)}), \qquad \|p_2(x_s,u_s,y_s,v_s,\varepsilon)\| \leqslant L\delta^2\hat\lambda^s. \tag{A.28}$$

Let us fix the choice of $\alpha \in (0,1)$ in the following way:

$$\gamma_0^{1-\alpha}\lambda_0^{\alpha} = 1. \tag{A.29}$$

We also assume further that

$$\hat\lambda > \lambda_0^{1+\alpha} \qquad \text{and} \qquad \hat\gamma < \gamma_0^{2-\alpha}. \tag{A.30}$$

Now, for all $j=0,\ldots,k$, we have from (A.19) and (A.28)

$$\begin{aligned} &\|\bar x_j - A_1^j x_0\| \leqslant L\delta^2\lambda_0^{j-1}\gamma_0^{-k}\sum_{s=0}^{j-1}\left[\left(\frac{\hat\lambda\gamma_0}{\lambda_0}\right)^s + \gamma_0^{\alpha k}\right] \leqslant L\delta^2\left[\frac{\hat\lambda^j\gamma_0^{j-k}}{\hat\lambda\gamma_0-\lambda_0} + j\lambda_0^{j-1}\gamma_0^{-k(1-\alpha)}\right], \\ &\|u_j\| \leqslant \delta\left(\frac{1}{2}\hat\lambda^j + L\|A_2\|^{j-1}\delta\sum_{s=0}^{j-1}\left(\frac{\hat\lambda}{\|A_2\|}\right)^s\right) \leqslant \delta\hat\lambda^j\left(\frac{1}{2} + \frac{L\delta}{\hat\lambda - \|A_2\|}\right). \end{aligned} \tag{A.31}$$

Thus, $\bar x_j, \bar u_j$ $(j=0,\ldots,k)$ satisfy (A.20). Analogously, one verifies that the same holds true for $\bar y_j, \bar v_j$. So, we have indeed that if $[(x_j,u_j,y_j,v_j)]_{j=0}^k$ belongs to $\hat R$, then its image by Φ belongs to $\hat R$ also, from which the required uniform boundedness of the functions $\xi_k, \eta_k, \hat\xi_k, \hat\eta_k$ follows, as explained above.

Since Φ is a smooth contracting operator, it follows that its fixed point $[(x_j^*,u_j^*,y_j^*,v_j^*)]_{j=0}^k$ is also smooth—C^r with respect to (x_0,u_0,y_k,v_k) and C^{r-2} with respect to ε. As there is a substantial difference in the estimates for the derivatives of the orders up to $(r-2)$ and of the orders $(r-1)$ and r, we will first estimate the derivatives of $(x_j^*,u_j^*,y_j^*,v_j^*)$ in case $r=2$, in order to make the situation clear. In this case, we need to estimate the derivatives with respect to (x_0,u_0,y_k,v_k) (we do not differentiate with respect to ε, as we may guarantee only a continuous dependence on the parameter in this case). By the contraction mapping principle, the iterations of the initial point $[(x_j,u_j,y_j,v_j)]_{j=0}^k=0$ by the operator Φ converge to the unique fixed point along with the derivatives. The derivatives are iterated according to the following rule (obtained just by differentiation of (A.19)):

$$\begin{aligned}
D_1(\bar{x}_j-A_1^jx_0)&=\sum_{s=0}^{j-1}A_1^{j-s-1}\left[\frac{\partial p_1}{\partial x_s}D_1x_s+\frac{\partial p_1}{\partial u_s}D_1u_s+\frac{\partial p_1}{\partial(y_s,v_s)}D_1(y_s,v_s)\right],\\
D_1\bar{u}_j&=A_2^jD_1u_0+\sum_{s=0}^{j-1}A_2^{j-s-1}\left[\frac{\partial p_2}{\partial x_s}D_1x_s+\frac{\partial p_2}{\partial u_s}D_1u_s+\frac{\partial p_2}{\partial(y_s,v_s)}D_1(y_s,v_s)\right],\\
D_1(\bar{y}_j-B_1^{j-k}y_k)&=-\sum_{s=j}^{k-1}B_1^{j-s-1}\left[\frac{\partial q_1}{\partial(x_s,u_s)}D_1(x_s,u_s)+\frac{\partial q_1}{\partial y_s}D_1y_s+\frac{\partial q_1}{\partial v_s}D_1v_s\right],\\
D_1\bar{v}_j&=B_2^{j-k}D_1v_k-\sum_{s=j}^{k-1}B_2^{j-s-1}\left[\frac{\partial q_2}{\partial(x_s,u_s)}D_1(x_s,u_s)+\frac{\partial q_2}{\partial y_s}D_1y_s+\frac{\partial q_2}{\partial v_s}D_1v_s\right],
\end{aligned}\tag{A.32}$$

where we denote $D_1=\frac{\partial}{\partial(x_0,u_0,y_k,v_k)}$. By (2.2) and (2.4), we have $\frac{\partial p}{\partial(y,v)}=o(x)+O(u)$. By plugging (A.20) into this estimate and into (A.23)–(A.26), we find

$$\begin{aligned}
&\left\|\frac{\partial p_1}{\partial x_s}\right\|\leqslant L\delta\lambda_0^{\alpha s}\gamma_0^{(1-\alpha)(s-k)}, &&\left\|\frac{\partial p_1}{\partial u_s}\right\|\leqslant L\delta\gamma_0^{s-k},\\
&\left\|\frac{\partial p_2}{\partial x_s}\right\|\leqslant L\delta\lambda_0^{s}, \qquad \left\|\frac{\partial p_2}{\partial u_s}\right\|\leqslant L\delta, &&\left\|\frac{\partial(p_1,p_2)}{\partial(y_s,v_s)}\right\|=L\delta\cdot o(\|A_1\|^s)
\end{aligned}\tag{A.33}$$

(similar estimates hold true for q_1, q_2).

Let us show that there exist functions $\beta_{1,2}(s)$, satisfying $0<\beta_{1,2}(s)\leqslant 2$ and $\beta_{1,2}(s)=o(1)_{s\to+\infty}$, such that if

$$\|D_1(x_s-A_1^sx_0,u_s)\|\leqslant\beta_1(s)\|A_1\|^s,\qquad \|D_1(y_s-B_1^{s-k}y_k,v_s)\|\leqslant\beta_2(k-s)\|(B_1)^{-1}\|^{k-s}\tag{A.34}$$

on the right-hand side of (A.32), then the same estimates are fulfilled for $(\bar{x}_j,\bar{u}_j,\bar{y}_j,\bar{v}_j)$ on the left-hand side, with the same functions β_1 and β_2. This will immediately give us the same estimates for the derivatives of the fixed point, thus proving estimates (2.6) in the case under consideration. Note that (A.34) implies

$$\|D_1x_s\|\leqslant 3\|A_1\|^s,\qquad \|D_1(y_s,v_s)\|\leqslant 3\gamma_0^{s-k}.$$

Now, by (A.32), (A.33) and (A.29) we obtain for k large enough

$$\begin{aligned}
\|D_1(\bar{x}_j-A_1^jx_0)\|&\leqslant\sum_{s=0}^{j-1}\|A_1\|^{j-s-1}[3L\delta\lambda_0^{\alpha s}\gamma_0^{(1-\alpha)(s-k)}\|A_1\|^s\\
&\qquad+L\delta\gamma_0^{s-k}\beta_1(s)\|A_1\|^s+L\delta\gamma_0^{s-k}o(\|A_1\|^s)]\\
&\leqslant L\delta\|A_1\|^{j-1}\left(3j\gamma_0^{-(1-\alpha)k}+\gamma_0^{-k}\sum_{s=0}^{j-1}\gamma_0^s[\beta_1(s)+\phi(s)]\right),\\
\|D_1\bar{u}_j\|&\leqslant\|A_2\|^j+\sum_{s=0}^{j-1}\|A_2\|^{j-s-1}\left[3L\delta\lambda_0^s\|A_1\|^s+L\delta\beta_1(s)\|A_1\|^s+L\delta\gamma_0^{s-k}o(\|A_1\|^s)\right]\\
&\leqslant\hat{\lambda}^j+L\delta\hat{\lambda}^{j-1}\sum_{s=0}^{j-1}\left(\frac{\|A_1\|}{\hat{\lambda}}\right)^s[\beta_1(s)+\phi(s)],
\end{aligned}\tag{A.35}$$

where the positive function $\phi(s)=o(1)_{s\to+\infty}$ is independent of the choice of $\beta_{1,2}(s)$. By (A.35), we have that $(\bar{x}_j,\bar{u}_j)$ satisfies the required estimates (A.34) if

$$2\geqslant\beta_1(j)\geqslant\left(\frac{\hat{\lambda}}{\|A_1\|}\right)^j+\frac{3L\delta}{\|A_1\|}j\gamma_0^{-(1-\alpha)j}+\frac{L\delta}{\hat{\lambda}}\sum_{s=0}^{j-1}\left(\min\left\{\gamma_0,\frac{\|A_1\|}{\hat{\lambda}}\right\}\right)^{s-j}[\beta_1(s)+\phi(s)]. \tag{A.36}$$

Since $\gamma_0>1$ and $\|A_1\|/\hat{\lambda}>1$, and $L\delta(\hat{\lambda})^{-1}<1$, there always exists tending to zero $\beta_1(j)$ which satisfies (A.36). Thus, estimates (2.6) are proven for $r=2$ (the computations for $(\bar{y}_j,\bar{v}_j)$ are completely analogous, and we omit them).

For the second derivatives of $(\bar{x}_j,\bar{u}_j,\bar{y}_j,\bar{v}_j)$ we have

$$\begin{aligned}D_2(\bar{x}_j,\bar{u}_j)=\sum_{s=0}^{j-1}A^{j-s-1}\Bigg[&\frac{\partial p}{\partial(x_s,u_s)}D_2(x_s,u_s)+\frac{\partial p}{\partial(y_s,v_s)}D_2(y_s,v_s)+\frac{\partial^2 p}{\partial(y_s,v_s)^2}D_1(y_s,v_s)^2\\&+\frac{\partial^2 p}{\partial(x_s,u_s)\partial(x_s,u_s,y_s,v_s)}D_1(x_s,u_s)D_1(x_s,u_s,y_s,v_s)\Bigg],\\D_2(\bar{y}_j,\bar{v}_j)=\sum_{s=j}^{k-1}B^{j-s-1}\Bigg[&\frac{\partial q}{\partial(y_s,v_s)}D_2(y_s,v_s)+\frac{\partial q}{\partial(x_s,u_s)}D_2(x_s,u_s)+\frac{\partial^2 q}{\partial(x_s,u_s)^2}D_1(x_s,u_s)^2\\&+\frac{\partial^2 q}{\partial(y_s,v_s)\partial(x_s,u_s,y_s,v_s)}D_1(y_s,v_s)D_1(x_s,u_s,y_s,v_s)\Bigg],\end{aligned} \tag{A.37}$$

where $D_2=\frac{\partial^2}{\partial(x_0,u_0,y_k,v_k)^2}$. As we did in the estimates for the first derivatives, let us show that there exist functions $\beta_{3,4}(s)=o(1)_{s\to+\infty}$ such that if

$$\|D_2(x_s,u_s)\|\leqslant\beta_3(s),\qquad \|D_2(y_s,v_s)\|\leqslant\beta_4(k-s) \tag{A.38}$$

on the right-hand side of (A.37), then the same estimates are fulfilled for $(\bar{x}_j,\bar{u}_j,\bar{y}_j,\bar{v}_j)$ on the left-hand side, with the same functions $\beta_{3,4}$. This will give us estimates (2.7) for the derivatives of the fixed point, thus proving the lemma in the C^2 case. Note that estimates (A.34) imply that $D_1(x_s,u_s,y_s,v_s)$ are bounded uniformly for all s, and $D_1(x_s,u_s)=o(1)_{s\to+\infty}$. Also note that according to (2.2), all the derivatives of $p(x_s,u_s,y_s,v_s)$ with respect to (y_s,v_s) tend to zero as $(x_s,u_s)\to 0$, i.e. as $s\to+\infty$. Now, from (A.37) we find

$$\begin{aligned}\|D_2(\bar{x}_j,\bar{u}_j)\|&\leqslant\sum_{s=0}^{j-1}\|A\|^{j-s-1}[L\delta\|D_2(x_s,u_s)\|+\phi_1(s)\|D_2(y_s,v_s)\|+\phi_2(s)]\\&\leqslant\sum_{s=0}^{j-1}\lambda_0^{j-s-1}[L\delta\beta_3(s)+\phi_1(s)\beta_4(k-s)+\phi_2(s)],\\\|D_2(\bar{y}_j,\bar{v}_j)\|&\leqslant\sum_{s=j}^{k-1}\|B^{-1}\|^{s+1-j}[L\delta\|D_2(y_s,v_s)\|+\phi_3(k-s)\|D_2(x_s,u_s)\|+\phi_4(k-s)]\\&\leqslant\sum_{s=j}^{k-1}\gamma_0^{j-s-1}[L\delta\beta_4(k-s)+\phi_3(k-s)\beta_3(s)+\phi_4(k-s)],\end{aligned} \tag{A.39}$$

where the functions $\phi(s)=o(1)_{s\to+\infty}$ are independent of the choice of $\beta_{3,4}$. Since $L\delta<1$, $\lambda_0<1$ and $\gamma_0^{-1}<1$, one can show that for all k large enough there exist functions $\beta_{3,4}$ that tend to zero as $j \to +\infty$ and

$$\beta_3(j) = \sum_{s=0}^{j-1} \lambda_0^{j-s-1} [L\delta\beta_3(s) + \phi_1(s)\beta_4(k-s) + \phi_2(s)],$$

$$\beta_4(k-j) = \sum_{s=j}^{k-1} \gamma_0^{j-s-1} [L\delta\beta_4(k-s) + \phi_3(k-s)\beta_3(s) + \phi_4(k-s)].$$

It is obvious that with such chosen $\beta_{3,4}$ the derivative $D_2(\bar{x}_j, \bar{u}_j, \bar{y}_j, \bar{v}_j)$ will satisfy (A.38) indeed. This finishes the proof of the lemma in the case $r=2$.

Let us proceed to the case $r \geqslant 3$. Denote $D_l = \frac{\partial^{|l|}}{\partial(x_0, u_0, y_k, v_k, \varepsilon)^l}$, where l is a non-negative integer multi-index. Let us prove that for all l such that $|l| \leqslant r-2$ the following estimates hold (with some constants $Q_l > 0$):

$$\|D_l x_s\| \leqslant 3\lambda_0^s, \qquad \|D_l y_s\| \leqslant 3\gamma_0^{s-k}, \qquad \|D_l u_s\| \leqslant Q_l \hat{\lambda}^s, \qquad \|D_l v_s\| \leqslant Q_l \hat{\gamma}^{s-k}, \tag{A.40}$$

for every iteration of $[(x_j, u_j, y_j, v_j)]_{j=0}^k = 0$ by the operator Φ given by (A.19). This will imply that the same estimates hold true for the fixed point of this map, i.e. for the solution of (A.18). It is enough to check that if $D_l(x_s, u_s, y_s, v_s)$ satisfies (A.40), then $D_l(\bar{x}_j, \bar{u}_j, \bar{y}_j, \bar{v}_j)$ satisfies (A.40) also with the same constants Q_l (the values of $(\bar{x}_j, \bar{u}_j, \bar{y}_j, \bar{v}_j)$ are defined by (A.19)). Note that at $l=0$ the fulfilment of (A.40) follows from (A.21). Thus, we may assume that (A.40) is fulfilled for all l such that $|l| < \ell$ for some $\ell \leqslant r-2$, and it remains to check that (A.40) will have to be fulfilled for $D_l(\bar{x}_j, \bar{u}_j, \bar{y}_j, \bar{v}_j)$ at $|l| = \ell$.

The differentiation of (A.19) gives

$$\begin{aligned}
&D_l(\bar{x}_j - A_1(\varepsilon)^j x_0) \\
&\quad = \sum_{s=0}^{j-1} \left\{ A_1(\varepsilon)^{j-s-1} \left[\frac{\partial p_1}{\partial x_s} D_l x_s + \frac{\partial p_1}{\partial u_s} D_l u_s + \frac{\partial p_1}{\partial(y_s, v_s)} D_l(y_s, v_s) \right] + \Psi_{[x]}(s, j) \right\}, \\
&D_l \bar{u}_j = D_l(A_2(\varepsilon)^j u_0) \\
&\quad + \sum_{s=0}^{j-1} \left\{ A_2(\varepsilon)^{j-s-1} \left[\frac{\partial p_2}{\partial x_s} D_l x_s + \frac{\partial p_2}{\partial u_s} D_l u_s + \frac{\partial p_2}{\partial(y_s, v_s)} D_l(y_s, v_s) \right] + \Psi_{[u]}(s, j) \right\}, \\
&D_l(\bar{y}_j - B_1(\varepsilon)^{j-k} y_k) \\
&\quad = \sum_{s=j}^{k-1} \left\{ B_1(\varepsilon)^{j-s-1} \left[\frac{\partial q_1}{\partial y_s} D_l y_s + \frac{\partial q_1}{\partial v_s} D_l v_s + \frac{\partial q_1}{\partial(x_s, u_s)} D_l(x_s, u_s) \right] + \Psi_{[y]}(s, j) \right\}, \\
&D_l \bar{v}_j = D_l(B_2(\varepsilon)^{j-k} v_k) \\
&\quad + \sum_{s=j}^{k-1} \left\{ B_2(\varepsilon)^{j-s-1} \left[\frac{\partial q_2}{\partial y_s} D_l y_s + \frac{\partial q_2}{\partial v_s} D_l v_s + \frac{\partial q_2}{\partial(x_s, u_s)} D_l(x_s, u_s) \right] + \Psi_{[v]}(s, j) \right\}.
\end{aligned} \tag{A.41}$$

Here, $\Psi_{[x,u,y,v]}$ are certain expressions involving the derivatives $D_{l'}$ with $|l'| < \ell = |l|$ only. We describe these terms in more detail below, see (A.46). In particular, we will show that the assumed fulfilment of (A.40) at $|l'| < \ell$ implies that

$$\Psi_{[x]}(s,j) = O(\hat{\lambda}^s)\gamma_0^{(1-\alpha)(s-k)}\|A_1\|^{j-s}(j-s)^{l_0}, \quad \Psi_{[u]}(s,j) = O(\hat{\lambda}^s)\|A_2\|^{j-s}(j-s)^{l_0},$$
$$\Psi_{[y]}(s,j) = O(\hat{\gamma}^{s-k})\lambda_0^{\alpha s}\|B_1^{-1}\|^{s-j}(s-j+1)^{l_0}, \quad \Psi_{[v]}(s,j) = O(\hat{\gamma}^{s-k})\|B_2^{-1}\|^{s-j}(s-j+1)^{l_0} \tag{A.42}$$

at $\ell \leqslant r-2$, while at $\ell = r-1$ we have

$$\begin{aligned}
\Psi_{[x]} &= (o(\|A_1\|^s) + O(\hat{\lambda}^s)(j-s)^{l_0})\|A_1\|^{j-s}\gamma_0^{(1-\alpha)(s-k)},\\
\Psi_{[u]} &= (o(\|A_1\|^s) + O(\hat{\lambda}^s)(j-s)^{l_0})\|A_2\|^{j-s},\\
\Psi_{[y]} &= (o(\|B_1^{-1}\|^{k-s}) + O(\hat{\gamma}^{s-k})(s-j+1)^{l_0})\|B_1^{-1}\|^{s-j}\lambda_0^{\alpha s},\\
\Psi_{[v]} &= (o(\|B_1^{-1}\|^{k-s}) + O(\hat{\gamma}^{s-k})(s-j+1)^{l_0})\|B_2^{-1}\|^{s-j}.
\end{aligned} \tag{A.43}$$

In these formulae l_0 is the number of differentiations with respect to ε in D_l.

By (A.41), (A.33) and (A.42), at $\ell = |l| \leqslant r-2$ we have for some $L > 0$

$$\|D_l(\bar{x}_j - A_1(\varepsilon)^j x_0)\| \leqslant L\sum_{s=0}^{j-1}\lambda_0^{j-s-1}\Big[\delta\lambda_0^{\alpha s}\gamma_0^{(1-\alpha)(s-k)}\|D_l x_s\|$$
$$+\delta\gamma_0^{s-k}\|D_l u_s\| + \delta\lambda_0^s\|D_l(y_s, v_s)\| + \hat{\lambda}^s\gamma_0^{(1-\alpha)(s-k)}\Big],$$
$$\|D_l\bar{u}_j\| \leqslant \hat{\lambda}^j + L\sum_{s=0}^{j-1}\|A_2\|^{j-s-1}[\delta\lambda_0^s\|D_l x_s\| + \delta\|D_l u_s\| + \delta\lambda_0^s\|D_l(y_s, v_s)\| + \hat{\lambda}^s].$$

Now it is easy to see that there exists a constant C such that when estimates (A.40) hold for $D_l(x_s, u_s, y_s, v_s)$, the derivative $D_l(\bar{x}_j, \bar{u}_j)$ satisfies

$$\|D_l(\bar{x}_j - A_1(\varepsilon)^j x_0)\| \leqslant C(\delta + Q_l\gamma^{(1-\alpha)(j-k)}(\hat{\lambda}/\lambda_0)^j)\lambda_0^j, \qquad \|D_l\bar{u}_j\| \leqslant C(1+Q_l)\hat{\lambda}^j.$$

Thus, for sufficiently small δ and large k, one may choose the constants Q_l such that $D_l(\bar{x}_j, \bar{u}_j)$ will satisfy (A.40). By the symmetry of the problem, the same holds true for $D_l(\bar{y}_j, \bar{v}_j)$. Thus, by induction, estimates (A.40) hold true indeed for all $|l| \leqslant r-2$.

As we mentioned, this fact implies the validity of estimates (A.42) and (A.43). By plugging (A.42), (A.43) into (A.41) we find, in the same way as we did in the case $r = 2$, that the derivatives $D_l(x_j, u_j, y_j, v_j)$ of the solution of (A.18) satisfy the estimates

$$\begin{aligned}
&D_l(x_j - A_1(\varepsilon)^j x_0) = O(\hat{\lambda}^j\gamma_0^{(1-\alpha)(j-k)}), && \|D_l u_j\| = O(\hat{\lambda}^j),\\
&D_l(y_j - B_1(\varepsilon)^{j-k}y_k) = O(\hat{\gamma}^{j-k}\lambda_0^{\alpha j}), && \|D_l v_j\| = O(\hat{\gamma}^{j-k})
\end{aligned} \tag{A.44}$$

at $|l| \leqslant r-2$, and the estimates

$$\begin{aligned}
&D_l(x_j - A_1(\varepsilon)^j x_0) = o(\|A_1\|^j\gamma_0^{(1-\alpha)(j-k)}), && \|D_l u_j\| = o(\|A_1\|^j),\\
&D_l(y_j - B_1(\varepsilon)^{j-k}y_k) = o(\|B_1^{-1}\|^{k-j}\lambda_0^{\alpha j}), && \|D_l v_j\| = o(\|B_1^{-1}\|^{k-j})
\end{aligned} \tag{A.45}$$

at $|l| \leqslant r-1$. This gives us the estimates of the lemma for all the derivatives up to the order $(r-1)$—once (A.42), (A.43) are proven.

Thus, to finish the lemma for the derivatives up to the order $(r-1)$, it remains to prove that (A.40) implies (A.42) and (A.43) indeed. As before, it suffices to make computations only for $\Psi_{[x]}$ and $\Psi_{[u]}$; the estimates for $\Psi_{[y]}$ and $\Psi_{[v]}$ are obtained analogously, due to the symmetry of the problem.

By the chain rule, $\Psi_{[x]}$ and $\Psi_{[u]}$ in (A.41), are estimated by a constant times the sum of various terms of the kind

$$\|A_i(\varepsilon)\|^{j-s}(j-s)^{l_0-|l_3|}\left\|\frac{\partial^{|l_1|+|l_2|+|l_3|}p_i}{\partial(x,u)^{l_1}\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}(x_s,u_s,y_s,v_s,\varepsilon)\right\|$$
$$\times\|D_{t_1}(x_s,u_s)\|\cdots\|D_{t_{|l_1|}}(x_s,u_s)\|\times\|D_{t_{|l_1|+1}}(y_s,v_s)\|\cdots\|D_{t_{|l_1|+|l_2|}}(y_s,v_s)\|, \tag{A.46}$$

where $l_{1,2,3}$ run non-negative integer values such that $|l_1|+|l_2|+|l_3|\leqslant|l|$, and ts can take all possible positive integer values such that $|t_1|+\cdots+|t_{|l_1|+|l_2|}|=|l|$; the derivative of p_1 corresponds to $\Psi_{[x]}$ and the derivative of p_2 corresponds to $\Psi_{[u]}$. The factor $(j-s)^{l_0-|l_3|}$ appears when we differentiate the matrix A_i^{j-s-1} in (A.19) with respect to ε (the total number of differentiations with respect to ε in D_l equals l_0).

By virtue of (A.40), we immediately arrive at the following estimate:

$$\|\Psi_{[x]}\|\leqslant\text{const}\cdot\|A_1(\varepsilon)\|^{j-s}(j-s)^{l_0-|l_3|}$$
$$\times\sum_{|l_1|+|l_2|+|l_3|\leqslant|l|}\left\|\frac{\partial^{|l_1|+|l_2|+|l_3|}p_1}{\partial(x,u)^{l_1}\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}(x_s,u_s,y_s,v_s,\varepsilon)\right\|\cdot\lambda_0^{|l_1|s}\gamma_0^{|l_2|(s-k)},$$
$$\|\Psi_{[u]}\|\leqslant\text{const}\cdot\|A_2(\varepsilon)\|^{j-s}(j-s)^{l_0-|l_3|}$$
$$\times\sum_{|l_1|+|l_2|+|l_3|\leqslant|l|}\left\|\frac{\partial^{|l_1|+|l_2|+|l_3|}p_2}{\partial(x,u)^{l_1}\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}(x_s,u_s,y_s,v_s,\varepsilon)\right\|\cdot\lambda_0^{|l_1|s}\gamma_0^{|l_2|(s-k)}. \tag{A.47}$$

As we see, all the terms in the first line with $|l_1|\geqslant2$ and $|l_2|\geqslant1$ and all the terms with $|l_1|\geqslant2$ in the second line fit (A.42) and (A.43). Let us examine the other terms. We will consider below only estimates for $\Psi_{[x]}$, as the estimates for $\Psi_{[u]}$ are obtained analogously (and simpler). Thus, we want to show that

$$\Psi_{[x]}=\begin{cases}O(\hat\lambda^s)\gamma_0^{(1-\alpha)(s-k)}\|A_1(\varepsilon)\|^{j-s}(j-s)^{l_0-|l_3|} & \text{at } l\leqslant r-2,\\ (o(\|A_1\|^s)+(j-s)^{l_0-|l_3|}O(\hat\lambda^s))\gamma_0^{(1-\alpha)(s-k)}\|A_1(\varepsilon)\|^{j-s} & \text{at } l\leqslant r-1\end{cases} \tag{A.48}$$

(compare with (A.42) and (A.43)).

We start with the terms for which $|l_1|\geqslant2$ and $l_2=0$, i.e. we do not differentiate p_1 with respect to (y,v). As p_1 vanishes identically at $(y,v)=0$ (see (2.3)), we have

$$\frac{\partial^{|l_1|+|l_3|}p_1}{\partial(x,u)^{l_1}\partial\varepsilon^{l_3}}=O(\|y_s\|+\|v_s\|)=O(\gamma_0^{s-k}) \tag{A.49}$$

at $|l_1|+|l_3|\leqslant r-1$. Hence, all terms with $|l_1|\geqslant2$ (both with $l_2=0$ and with $|l_2|\geqslant1$) in the estimate (A.47) for $\Psi_{[x]}$ fit (A.48).

It remains to consider the terms with $|l_1|\leqslant1$. By (A.46), the terms with $|l_1|=1$ are estimated (modulo a constant factor times $\|A_1(\varepsilon)\|^{j-s}(j-s)^{l_0}$) as

$$\left\|\frac{\partial}{\partial x}\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}\right\|\cdot\gamma_0^{l_2(s-k)}\lambda_0^s\qquad\text{and}\qquad\left\|\frac{\partial}{\partial u}\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}\right\|\cdot\gamma_0^{l_2(s-k)}\hat\lambda^s. \tag{A.50}$$

The second term obviously gives a right contribution into (A.48) at $|l_2|\geqslant1$, while at $l_2=0$ it also fits (A.48) by virtue of (A.49). Thus, we are left to estimate the first term in (A.50). Recall that $\partial p_1/\partial x$ vanishes at $(x,u)=0$ (see (2.4)). Hence, as it is a C^{r-1}-function, it is $O(\|x\|+\|u\|)=O(\lambda_0^s)$ along with all the derivatives with respect to (y,v,ε) up to the order $(r-2)$. Thus, the first term in (A.50) fits (A.48) at $|l_2|\geqslant1$. If $l_2=0$, it takes the form $\left\|\frac{\partial^{1+|l_3|}p_1}{\partial x\partial\varepsilon^{l_3}}\right\|\lambda_0^s$. Since $\partial p_1/\partial x$ vanishes at both $(x,u)=0$ and $(y,v)=0$, the same holds true for $\frac{\partial^{1+|l_3|}p_1}{\partial x\partial\varepsilon^{l_3}}$. As this function is at least C^1 (recall that $|l_3|\leqslant r-2$ always), it follows that $\frac{\partial^{1+|l_3|}p_1}{\partial x\partial\varepsilon^{l_3}}=O(\min\{\|x,u\|,\|y,v\|\})=O(\|x,u\|^{\alpha}\|y,v\|^{1-\alpha})=O(\lambda_0^{\alpha s}\gamma_0^{(1-\alpha)(s-k)})$, and we see that (A.48) is satisfied at $l_2=0$ too.

Let us proceed to the terms with $l_1=0$ in (A.46). These are

$$\|A_1(\varepsilon)\|^{j-s}(j-s)^{l_0-|l_3|}\left\|\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}\right\|\cdot\gamma_0^{|l_2|(s-k)}, \tag{A.51}$$

with $|l_2|=|l|-l_0, |l_3|\leqslant l_0$. Note that p_1 and $\partial p_1/\partial x$ vanish at $(x,u)=0$. The same remains true for all their derivatives with respect to (y,v,ε). When $|l_2|+|l_3|\leqslant r-2$, the derivative $\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}$ is at least C^2. Since it vanishes at $(x,u)=0$ along with its first derivative with respect to x, it follows that

$$\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}=O(\|x\|^2+\|u\|)=O(\hat{\lambda}^s), \tag{A.52}$$

i.e. the term (A.51) satisfies (A.48) in this case, provided $|l_2|\geqslant 1$.

Now assume that $|l_2|+|l_3|=r-1$. This is possible only if $|l|=r-1$ and $l_0=|l_3|$ (i.e. the factor $(j-s)^{l_0-|l_3|}$ in (A.51) disappears). Now $\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}$ is only C^1, so we have an estimate worse than (A.52), namely, $\frac{\partial^{|l_2|+|l_3|}p_1}{\partial(y,v)^{l_2}\partial\varepsilon^{l_3}}=o(x)+O(u)=o(\|A_1\|^s)$. It follows that the term (A.51) with $|l_1|\geqslant 1$ satisfies in this case the part of (A.48) that corresponds to $|l|=r-1$.

The last remaining case corresponds to $l_1=0, l_2=0$, i.e. to the differentiation with respect to parameters ε only. The corresponding terms in (A.46) are given by $\|A_1(\varepsilon)\|^{j-s}(j-s)^{l_0-|l_3|}\|\frac{\partial^{|l_3|}p_1}{\partial\varepsilon^{l_3}}\|$. As the number of differentiations with respect to ε cannot exceed $(r-2)$ (see remark 1 to lemma 6), the derivative $\partial^{|l_3|}p_1/\partial\varepsilon^{l_3}$ is at least C^2. As it vanishes at both $(y,v)=0$ and $(x,u)=0$, along with its first derivative with respect to x, we have $\frac{\partial^{|l_3|}p_1}{\partial\varepsilon^{l_3}}=O((\|x\|^{1+\alpha}+\|u\|)\|y,v\|^{1-\alpha})=O(\hat{\lambda}^s\gamma_0^{-(1-\alpha)(k-s)})$. Thus, the corresponding terms in (A.46) satisfy (A.48) also. This finally gives us the part of the lemma that is concerned with the derivatives up to the order $(r-1)$.

To finish the lemma, we note that the derivatives of order r are estimated in absolutely the same way as they are in the case $r=2$: one shows that relations (A.38) are satisfied by the derivatives D_r, and the rest follows without changes. □

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