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Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps

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Abstract
We show that maps with homoclinic tangencies of arbitrarily high orders and, as a consequence, with arbitrarily degenerate periodic orbits are dense in the Newhouse regions in spaces of real-analytic area-preserving two-dimensional maps and general real-analytic two-dimensional maps (the result was earlier known only for the space of smooth non-conservative maps). Based on this, we show that a generic area-preserving map from the Newhouse region is 'universal' in the sense that its iterations approximate the dynamics of any other area-preserving map with arbitrarily good accuracy. In fact, we show that every dynamical phenomenon which occurs generically in any open set of symplectic diffeomorphisms of a two-dimensional disc, or in any open set of finite-parameter families of such diffeomorphisms, can be encountered at a perturbation of any area-preserving two-dimensional map with a homoclinic tangency.

Mathematics Subject Classification: 37C20, 37J10, 37G25, 37C15, 37E20

1. Statement of results

In [1, 2, 3, 9] we established that an arbitrarily small smooth perturbation of a two-dimensional map with a quadratic homoclinic tangency can produce homoclinic tangencies of arbitrarily high orders and, as a consequence, arbitrarily degenerate periodic orbits. These results show that global bifurcations of codimension 1 can be accompanied by bifurcations of arbitrarily high codimension, i.e. the unfolding of global bifurcations can lead to an increase in the level of degeneracy, contrary to the usual logic coming from the singularity theory.

Based on this, we reached the conclusion that a complete description of dynamics and bifurcations of systems with homoclinic tangencies is impossible in principle (see further discussions in [3, 6, 9] and higher-dimensional results in [5, 10]). We recall that systems with homoclinic tangencies are dense in open regions (the so-called Newhouse regions) in space of smooth dynamical systems [11, 12, 14]. Moreover, these regions exist near any system with a homoclinic tangency [14, 17, 18]. In fact, homoclinic tangencies and, hence, Newhouse regions in the parameter space have been found in a huge variety of different models with chaotic dynamics. Thus, they exist in the Hénon map (see discussion in [19]), in the standard map [16] and in 'soft billiards' [20], they appear in the process of the development of a Smale horseshoe (after period-doubling), they play a central role in the transition from quasiperiodicity to chaos (the destruction of invariant tori) [21–23, 55], they are present in Lorenz-like models beyond the boundary of the region of existence of the Lorenz attractor [24, 25], in systems with 'spiral chaos', such as the Chua circuit or the Rössler model (see [26, 27]), and with wild spiral attractor [28], in periodically forced Lorenz attractors [29, 30], etc. According to [1–10], in all these models one should expect an incomprehensibly complex behaviour.

Recently, it has been realized that the density of systems with homoclinic tangencies of arbitrarily high orders in the Newhouse regions is a useful working tool for proving that many seemingly exotic dynamical phenomena are, in fact, generic. Thus, it was shown in [31] that a certain interpretation of the results of [3] disproves Smale's conjecture on the genericity of the exponential growth of the number of periodic orbits with period (also see [32, 34] for higher-dimensional generalizations). In [15], our results were used to show that generic two-dimensional C^r-diffeomorphisms from the Newhouse regions, with r finite, cannot be topologically conjugate to any C^∞-diffeomorphism and that they have transitive sets of full Hausdorff dimension; in [36] the ultimate topological complexity of such sets was established. In the same manner, it was shown in [37] that the measure of the Newhouse set, i.e. the set of parameter values that correspond to the coexistence of infinitely many stable periodic orbits, is positive for a dense set of finite-parameter families (interestingly, the set of finite-parameter families for which the measure of the Newhouse set is zero is, quite probably, also dense among the finite-parameter families that intersect the Newhouse regions, as results of [33, 35] suggest). In the present paper, we continue this line by showing, in particular, that a generic area-preserving map from the Newhouse region is 'universal' [38] in the sense that its iterations approximate the dynamics of any other area-preserving map with arbitrarily good accuracy (in appropriately chosen coordinates).

The fact that systems with homoclinic tangencies of arbitrarily high orders are dense in the Newhouse regions was proven in [3, 9] for the space of general smooth maps, and one of our genericity conditions excluded area-preserving maps. Therefore, the validity of the result (and the above cited results based on it) in the area-preserving case can be questioned. In the present paper we close the problem and provide a unified proof which works in the area-preserving case as well. Moreover, we enhance our perturbation technique so that the new proof covers the real-analytic case too.

Let f be a map (locally a diffeomorphism) of a two-dimensional manifold without a boundary. Assume f to be C^r ($r = 2, \ldots, \infty$, or $r = \omega$—that corresponds to the real-analytic case). Let f have a saddle periodic orbit $L(f)$ whose stable and unstable manifolds have a quadratic tangency at some point M (see figure 1). This is a tangency of invariant manifolds; therefore, they are tangent at each point of the orbit of M. By construction, this orbit Γ is homoclinic to L, i.e. it closes on L both at forward and backward iterations of f.

Fix a compact region K in phase space (in the theorem below we assume that K contains a neighbourhood of $\Gamma \cup L$). We will say that two C^r-smooth maps are δ-close if a C^r-distance between them on K does not exceed δ (the C^∞-distance may be defined as $\rho_\infty(f_1, f_2) = \sum_0^\infty (1/(r+1)^2)(\rho_r(f_1, f_2)/1 + \rho_r(f_1, f_2))$ where ρ_r is a C^r-distance). In the real-analytic case we fix some small complex neighbourhood Q of K and say that two

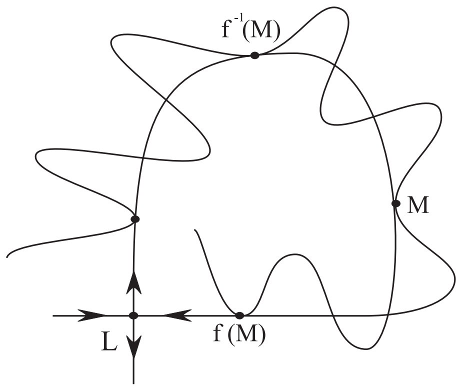

Figure 1. A two-dimensional diffeomorphism with a homoclinic tangency.

C^ω-maps are δ-close if they differ not more than δ at every point of Q. Obviously, any map g close to f will have a saddle periodic orbit $L(g)$ close to $L(f)$.

Theorem 1. *Arbitrarily close to f there exists a map f^* (an area-preserving one if f itself is area-preserving) which has infinitely many orbits of homoclinic tangency of every order between the stable and unstable manifolds of $L(f^*)$.*

The proof occupies sections 2–4. In section 2 we give necessary formulae for the Poincaré return maps near the periodic and homoclinic orbits and describe the form of the perturbations that we use (lemma 1). In section 3 we prove certain key lemmas and in section 4 we make the actual construction of the sequence of perturbations which lead from f to f^*.

The main reason why the homoclinic tangency can be perturbed in such a way that a tangency of a higher order is created is the presence of a 'hidden degeneracy' in the system. Thus, it was established in [42–44] that non-conservative systems with a homoclinic tangency of the 'third class' in the terminology of [39] have a modulus (i.e. a continuous invariant) of local Ω-conjugacy (i.e. the topological conjugacy on the set of non-wandering orbits which lie entirely in a small neighbourhood of the orbit of homoclinic tangency). Such a modulus is, for example, the ratio $\theta = -\ln|\gamma|/\ln|\lambda|$ of the logarithms of the multipliers λ and γ of the saddle periodic orbit L to which the given homoclinic orbit converges. Thus, two such systems cannot be locally Ω-conjugate if the corresponding values of θ are different. As a result, θ can be taken as an additional bifurcational parameter, by changing which homoclinic tangencies of a higher degree of degeneracy can be obtained (see further discussion in [9,45]).

Two-dimensional area-preserving maps with homoclinic tangencies have no local moduli [41,49] (for instance, $\theta \equiv 1$ for such systems). Therefore, in order to prove theorem 1 in the area-preserving case, we first prove (lemmas 2, 3) that a small perturbation of a map with a homoclinic tangency can produce a heteroclinic cycle with two different saddle periodic orbits, one transverse heteroclinic orbit and one orbit of heteroclinic tangency; moreover, such a heteroclinic cycle belongs to the third class of [46] (see figure 8(*a*)). Since the heteroclinic cycles of the third class have local Ω-moduli in both dissipative [46] and conservative [47,48] cases, we can prove that systems with homoclinic tangencies of arbitrarily high orders are dense among the systems with such heteroclinic cycles, by applying a refined version (lemmas 4, 5) of the machinery developed in [3,9].

In fact, we prove more than theorem 1 in section 4. Namely, we show that the map f^* constructed in theorem 1 has a non-trivial uniformly hyperbolic set (a horseshoe) which includes the original saddle periodic orbit L, and there exist infinitely many orbits of tangency of every order between stable and unstable manifolds of every periodic orbit in this hyperbolic set.

We might as well assume the existence of a horseshoe from the very beginning. Namely, our proof in section 4 provides the following generalization of theorem 1.

Theorem 2. *Let f be a C^r-map ($r = 2, \ldots, \infty, \omega$) of a two-dimensional manifold. Assume that f has a locally maximal, compact, transitive, uniformly hyperbolic invariant set $\Lambda(f)$ whose stable and unstable manifolds have a tangency. Then, arbitrarily close to f there exists a map f^* (an area-preserving one if f itself is area-preserving) which has infinitely many orbits of tangency of every order between the stable and unstable manifolds of every pair of periodic orbits of $\Lambda(f^*)$.*

As we mentioned, the C^r-closure ($r = 2, \ldots, \infty, \omega$) of the set of C^r-maps with homoclinic tangencies contains open (Newhouse) regions. For the space of all two-dimensional C^r-maps this statement was proved in [14] (see a multidimensional version in [4, 57]). Extending this result onto the space of two-dimensional area-preserving C^r-maps was a long-standing open problem, until the proof was obtained in [17, 18] (it also follows from [13] that the C^1-closure of the Newhouse regions in space of two-dimensional area-preserving diffeomorphisms coincides with the set of all non-Anosov area-preserving diffeomorphisms—whether the same remains true in the C^r-topology with $r \geqslant 2$ is so far an intractable question). Theorem 1 immediately implies

Theorem 3. *Maps with infinitely many homoclinic tangencies of all orders are dense in the Newhouse regions, both in space of all two-dimensional C^r-maps and in space of area-preserving C^r-maps ($r = 2, \ldots, \infty, \omega$).*

According to [1–9], in the case of maps which are not area-preserving these results imply that maps with arbitrarily degenerate periodic orbits are dense in the Newhouse regions. In this paper we further develop the corresponding theory for the area-preserving case.

1.1. Area-preserving case

Recall the definitions. Let a two-dimensional area-preserving C^r-map ($r = 3, \ldots, \infty, \omega$) have an elliptic periodic point, i.e. a periodic point with multipliers on the unit circle: $\nu_{1,2} = \mathrm{e}^{\pm i\varphi}, 0 < \varphi < \pi$. When φ/π is irrational, for any m such that $2m + 1 \leqslant r$ there exists an analytic area-preserving change in variables which brings the first-return map near such a point locally to the Birkhoff normal form

$$\bar{z} = \mathrm{e}^{i\varphi} z(1 + B_1 zz^* + B_2(zz^*)^2 + \cdots + B_m(zz^*)^m) + o(|z|^{2m+1}). \tag{1.1}$$

Here z is a scalar complex variable; the coefficients B_j are called Birkhoff coefficients; the first non-zero Birkhoff coefficient has zero real part always. The elliptic point is non-degenerate if $B_1 \neq 0$. For degenerate elliptic points it is natural to introduce *the order of degeneracy*: it is equal to k where B_{k+1} is the first non-zero Birkhoff coefficient. In section 5.3 we prove the following theorem.

Theorem 4. *Maps with infinitely many elliptic periodic orbits of every order of degeneracy are dense in the Newhouse regions in space of two-dimensional area-preserving analytic maps*[3].

[3] We say that a set A is dense in a certain subset B of space of real-analytic maps, if given any diffeomorphism f from B and any compact subset K of phase space there exists a complex neighbourhood Q of K such that A is dense in an open neighbourhood of f in space of maps holomorphic on Q.

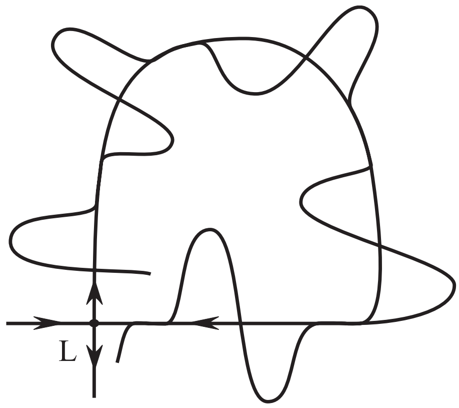

Figure 2. An example of a homoclinic band.

We can make a stronger statement in the smooth case. If the stable and unstable manifolds of some periodic orbit coincide along some curve, we will call this a homoclinic band (see figure 2); a curve filled by periodic orbits of the same period is a periodic band. If there is a non-empty open region whose all points are periodic of the same period, we will call this region a periodic spot. It was shown in [9, 31] that two-dimensional maps with homoclinic and periodic bands are dense in the Newhouse regions in space of smooth non-conservative maps. In this paper we show that an analogous (though stronger) statement holds true for the conservative case.

Theorem 5. *Maps with homoclinic bands and periodic spots are dense in the Newhouse regions in space of two-dimensional area-preserving C^r-diffeomorphisms ($r = 2, \ldots, \infty$) (proof in section 5.3).*

As we see, bifurcations of homoclinic tangencies and elliptic orbits can lead to phenomena of arbitrarily high complexity. In order to investigate this issue further, we use the following scheme from [38]. Let f be an area-preserving C^r-map ($r = 1, \ldots, \infty, \omega$, i.e. the real-analytic case is included as well) of a two-dimensional manifold $\mathcal{M}$. We want to give a unified description for arbitrarily long iterations of f on arbitrarily small spatial scales. To this aim, we will take small discs in $\mathcal{M}$ and consider corresponding return maps. Namely, let $\mathcal{B}$ be any such disc, i.e. let $\mathcal{B} = \psi(U)$ where U is the closed unit disc in R^2 and ψ is some C^r-diffeomorphism of U into $\mathcal{M}$ with a constant Jacobian (thus, the map ψ defines some symplectic coordinates on $\mathcal{B}$). Given positive n, the map $f^n|_{\mathcal{B}}$ is a return map if $f^n(\mathcal{B}) \cap \mathcal{B} \neq \emptyset$. By construction, the return map $f^n|_{\mathcal{B}}$ is smoothly conjugate with the map $f_{n,\psi} = \psi^{-1} \circ f^n \circ \psi$ (for the map $f_{n,\psi}$ to be properly defined, we assume that ψ admits an extension, as a C^r-diffeomorphism with a constant Jacobian, onto some larger disc $V \supseteq U$ such that $f^n(\mathcal{B}) \subseteq \psi(V)$). Evidently, the map $f_{n,\psi}$ is an area-preserving C^r-map $U \to R^2$, and it is solely defined by the choice of the coordinate transformation ψ and the number of iterations n (the choice of the map $\psi : U \to \mathcal{M}$ fixes the disc $\mathcal{B} = \psi(U)$ as well). We will call the maps $f_{n,\psi}$ obtained by such a procedure *renormalized iterations of* f. The set $\cup_{n,\psi} f_{n,\psi}$ of all possible renormalized iterations of f will be called *the dynamical conjugacy class* of f.

When we speak about dynamics of the map, we somehow describe its iterations, and the description should be insensitive to coordinate transformations. Therefore, the class of the map f, as we just have introduced it, gives some representation of the dynamics of f indeed. Note that the coordinate transformations ψ are not area-preserving (they preserve the standard symplectic form up to a constant factor), i.e. the image $\psi(U)$ can be a disc of arbitrarily small radius, with the centre situated anywhere. Thus, the class of f captures all arbitrarily fine details of the long-time behaviour of f.

Definition. An area-preserving C^r-map f $(r = 1, \dots, \infty, \omega)$ is called *universal* (or C^r-universal) if the C^r-closure of its dynamical conjugacy class contains all area- and orientation-preserving C^r-diffeomorphisms of the unit disc U into R^2.

By the definition, the dynamics of any single universal map is ultimately complicated and rich, and the detailed understanding of it is not simpler than the understanding of all symplectic diffeomorphisms *altogether*. Still, the property of a map to be universal occurs to be generic. Namely, in section 5.4 we prove the following theorem.

Theorem 6. *In the Newhouse regions of space of two-dimensional area-preserving C^r-maps ($r = 2, \dots, \infty, \omega$) there is a residual subset consisting of C^r-universal maps*[4].

It follows that every symplectic diffeomorphism of a two-dimensional disc can be approximated by a diffeomorphism, analytically conjugate to a perturbation of any given area-preserving map with a homoclinic tangency (the latter map has to be restricted to an appropriately chosen domain in phase space). This statement can be extended onto finite-parameter families of symplectic diffeomorphisms.

Consider space $D_{k,r}$ $(r = 2, \dots, \infty, \omega)$ of all k-parameter families of area- and orientation- preserving diffeomorphisms g_ε of the unit disc U into R^2, of class C^r with respect to the phase variables and ε (the parameter $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ runs the closed unit ball in R^k). We will call a family $g_\varepsilon \in D_{k,r}$ *minimal* if for every area-preserving two-dimensional C^r-map f with a homoclinic tangency, for any arbitrarily small $\delta > 0$ there exists a k-parameter family $\tilde{f}_\varepsilon$ of area-preserving two-dimensional C^r-maps δ-close to f such that a certain renormalized iteration of $\tilde{f}_\varepsilon$ coincides with g_ε for all ε. In other words, every bifurcation happening in a minimal family occurs near every map with a homoclinic tangency.

Theorem 7. *Minimal families form a residual set in $D_{k,r}$ (proof in section 5.4).*

Thus, *every* dynamical phenomenon which is generic for some open set of symplectic diffeomorphisms of a two-dimensional disc, or which occurs in generic finite-parameter families of such diffeomorphisms, can be encountered arbitrarily close to *any* area-preserving two-dimensional map from the Newhouse regions.

2. Preliminary constructions

2.1. Local map

Consider a C^r-map f of a two-dimensional manifold, $r = 2, ..., \infty, \omega$. As it is our standing assumption, we assume that f is locally a diffeomorphism. Let f have a saddle periodic orbit L. This means that there is a point O such that $f^m O = O$ (where m is the period of $L = \{O, fO, \dots, f^{m-1}O\}$) and that one can introduce coordinates (x, y) with the origin at O so that the map $f^m : (x, y) \mapsto (\bar{x}, \bar{y})$ will have the following form near O:

$$\bar{x} = \lambda x + o(x, y), \qquad \bar{y} = \gamma y + o(x, y), \tag{2.1}$$

[4] If $r = \omega$, we define residual sets as follows: a set A is residual in a certain subset B of space of real-analytic maps, if given any diffeomorphism f from B and any compact subset K of phase space, there exists a complex neighbourhood Q of K such that the intersection of A with some open neighbourhood X of f in space of maps holomorphic on Q is the intersection of a countable collection of open and dense subsets of X.

where $|\lambda| < 1$ and $|\gamma| > 1$. In the case of area-preserving map f, we have

$$|\lambda\gamma| = 1. \tag{2.2}$$

We denote f^m near O as T_0 and call it the local map. The stable and unstable invariant manifolds of the saddle fixed point $O(0,0)$ of T_0 have, locally, the form $y = \psi(x)$ and $x = \varphi(y)$, respectively, with $\varphi(0) = \varphi'(0) = 0$, $\psi(0) = \psi'(0) = 0$.

Let $x \mapsto \phi(x)$ be the inverse to $x \mapsto x - \varphi(\psi(x))$, i.e. $\phi(x - \varphi(\psi(x))) \equiv x$ at small x. The area-preserving coordinate transformation

$$(x, y)^{\text{new}} = (x - \varphi(y),\ y - \psi(\phi(x^{\text{new}}))) \tag{2.3}$$

straightens the local invariant manifolds, i.e. they take the form $x^{\text{new}} = 0$ and $y^{\text{new}} = 0$. Hence, the local map (2.1) takes the following form in the new coordinates

$$\bar{x} = \lambda x + p(x, y)x, \qquad \bar{y} = \gamma y + q(x, y)y, \tag{2.4}$$

where p and q vanish at the origin. Note that if we consider a family f_ε of maps, C^r with respect to both (x, y) and the parameters ε, then the stable and unstable invariant manifolds of O will enjoy the same smoothness, so the transformation which brings the local map near a saddle periodic point to the form (2.4) will be C^r with respect to (x, y, ε).

When proving the results of sections 3 and 5 we will use the fact (see [40,41,44]) that by an additional, close to identity, coordinate transformation, one may achieve the identical vanishing of p and q both at $x = 0$ and $y = 0$:

$$\begin{aligned} &p(x, 0) \equiv 0, \qquad q(0, y) \equiv 0,\\ &p(0, y) \equiv 0, \qquad q(x, 0) \equiv 0. \end{aligned} \tag{2.5}$$

If f is area-preserving, then such coordinate transformation can be chosen area-preserving too [48]. Indeed, as p is at least C^1, one can locally linearize the restriction $x \mapsto \lambda x + p(x, 0)x$ of map (2.4) onto the stable manifold, i.e. there exists a C^r-function $\zeta(x) = o(x)$ such that $p(x, 0)$ will vanish after the transformation $x^{\text{new}} = x + \zeta(x)$ (see, e.g. [50]). We make this coordinate transformation area-preserving by choosing $y^{\text{new}} = y/(1+\zeta'(x))$. Next, we locally linearize the restriction of (2.4) onto the unstable manifold in the same way. Thus, map (2.4) in the new coordinates will satisfy the first line of (2.5), and it will remain area-preserving. The latter means that

$$\det \begin{pmatrix} \lambda + p(x, y) + p'_x(x, y)x & p'_y(x, y)x \\ q'_x(x, y)y & \gamma + q(x, y) + q'_y(x, y)y \end{pmatrix} = \lambda\gamma$$

for all small (x, y). By plugging $y = 0$ or $x = 0$ in this identity, we immediately see that the first line of (2.5) implies the second line of (2.5) as well.

When the smoothness r of f is finite, the given coordinate transformation is C^{r-1} and it is, in general, only C^{r-2} with respect to ε if f depends on parameters ε. If $r = \infty$ or $r = \omega$, the coordinate transformation is also C^∞ or, respectively, C^ω with respect to (x, y), and we may ensure any finite smoothness with respect to ε (see [48,51]).

According to [52,53], for any small $x^{(0)}$ and $y^{(k)}$ and for any $k \geqslant 0$ there exist uniquely defined small $x^{(k)}$ and $y^{(0)}$ such that $(x^{(k)}, y^{(k)}) = T_0^k(x^{(0)}, y^{(0)})$ and all the points in the orbit $\{(x^{(0)}, y^{(0)}), T_0(x^{(0)}, y^{(0)}), \dots, T_0^k(x^{(0)}, y^{(0)})\}$ lie in a small neighbourhood of zero. We denote

$$x^{(k)} = \lambda^k x^{(0)} + \lambda^k \xi_k(x^{(0)}, y^{(k)}), \qquad y^{(0)} = \gamma^{-k} y^{(k)} + \gamma^{-k}\eta_k(x^{(0)}, y^{(k)}) \tag{2.6}$$

(if the map depends on parameters, then ξ_k and η_k are functions of ε as well). By [40,41,44,51], when identities (2.5) are satisfied, the functions ξ_k and η_k are uniformly small along with all

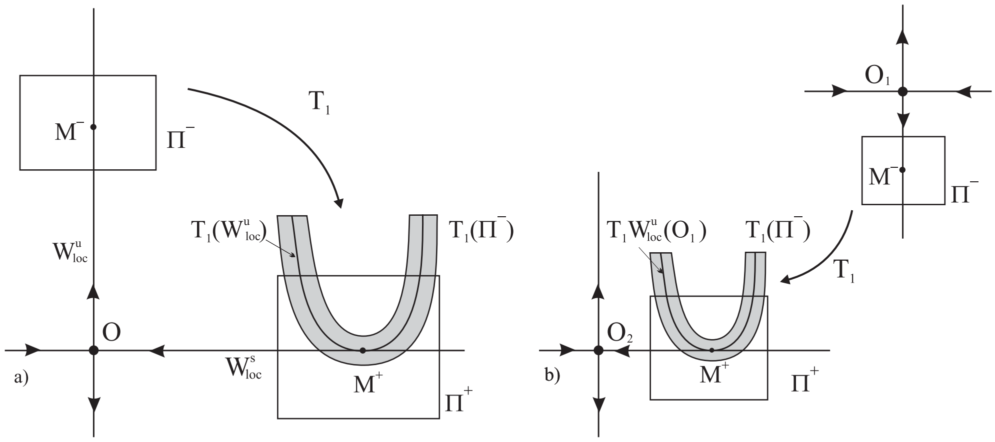

Figure 3. The global map T_1 in the case of homoclinic (*a*) and heteroclinic (*b*) tangency. The points $M^+ \in W^s_{\rm loc}$ and $M^- \in W^u_{\rm loc}$ belong to the same homoclinic or heteroclinic orbit.

the derivatives up to the order $(r-1)$ with respect to (x_0, y_k) and up to the order $(r-2)$ with respect to parameters

$$\|\xi_k, \eta_k\| = o(1)_{k\to+\infty} \tag{2.7}$$

(in the case of infinite r we have uniform smallness for all the derivatives up to any given finite order).

2.2. *Global map*

Let the map f have an orbit of homoclinic tangency. It means that in the local unstable manifold $W^u_{\rm loc}$ of O there is a point $M^-(0, y^-)$ such that its image $M^+ = f^l M^-$ for some positive integer l lies in the local stable manifold $W^s_{\rm loc}(O)$, and the curve $f^l W^u_{\rm loc}$ is tangent to $W^s_{\rm loc}$ at M^+. The corresponding orbit is homoclinic, because it tends to O both at forward and backward iterations of f.

We call the map f^l in a small neighbourhood of M^- the global map and denote it by T_1 (see figure 3). It can be written as

$$\begin{aligned} \bar{x} - x^+ &= ax + b(y-y^-) + g_1(x, y), \\ \bar{y} &= cx + \Phi(y) + g_2(x, y), \end{aligned} \tag{2.8}$$

where the functions g_1 and g_2 do not contain linear terms and g_2 vanishes identically at $x = 0$. By (2.8), the equation of the curve $T_1 W^u_{\rm loc}$ is

$$\bar{x} = x^+ + b(y-y^-) + g_1(0, y), \qquad \bar{y} = \Phi(y). \tag{2.9}$$

The condition of the tangency of $T_1 W^u_{\rm loc}$ and $W^s_{\rm loc}$ at $y = y^-$ reads as

$$\Phi(y^-) = 0, \qquad \Phi'(y^-) = 0. \tag{2.10}$$

Note that f is locally a diffeomorphism, hence det $T_1'(M^-) \neq 0$; i.e.

$$bc \neq 0. \tag{2.11}$$

Similarly, when we consider a heteroclinic tangency (figure 3(*b*)) involving two saddle periodic points, O_1 and O_2, the global map T_1 acting from a small neighbourhood of $M^- \in W^u_{\rm loc}(O_1)$ into a small neighbourhood of $M^+ \in W^s_{\rm loc}(O_2)$ is defined (where M^- and

M^+ are points of the heteroclinic orbit under consideration). If the local invariant manifolds are straightened, i.e. the local maps are brought to form (2.4) near O_1 and O_2, then formulae (2.8)–(2.11) hold for the map T_1.

The homoclinic or heteroclinic tangency has the order n if $\Phi^{(n+1)}(y^-) \neq 0$ while $\Phi^{(j)}(y^-) = 0$ for all $j \leqslant n$ (so the quadratic tangency is of order 1). Of course, to define the tangency of order n, we should require from our map f at least the smoothness $r \geqslant n+1$. When f depends on parameters ε, the global map T_1 can still be written in the form (2.8), but the functions g_1, g_2, Φ and the coefficients a, b, c, x^+ and y^- may now depend on ε. If we have a homoclinic (or heteroclinic) tangency of order n at $\varepsilon = 0$, we may choose $y^-(\varepsilon)$ such that $\Phi^{(n)}(y^-) = 0$ for all small ε. We will always fix this choice of $y^-(\varepsilon)$, and we denote, under this assumption,

$$\mu_j(\varepsilon) = \Phi^{(j)}(y^-)/j! \qquad (j = 0, \ldots, n-1), \tag{2.12}$$

so that

$$\Phi(y, \varepsilon) = \mu_0 + \cdots + \mu_{n-1}(y - y^-)^{n-1} + d(y - y^-)^{n+1} + o((y - y^-)^{n+1}) \tag{2.13}$$

with $d \neq 0$. The tangency is said to be split generically if at ε small

$$\operatorname{rank} \partial(\mu_0, \ldots, \mu_{n-1})/\partial\varepsilon = n.$$

Let us rewrite the parametric equation (2.9) for the curve $T_1 W^u_{\mathrm{loc}}$ in the explicit form: $\bar{y} = \Psi_\varepsilon(\bar{x})$. When there is a tangency of order n, we have $\Psi^{(n+1)}(x^+) \neq 0$, while $\Psi^{(j)}(x^+) = 0$ for all $j \leqslant n$ at $\varepsilon = 0$. It follows that we may choose $x^+(\varepsilon)$ in such a way that $\Psi^{(n)}(x^+) = 0$ for all small ε. If we denote, under this assumption,

$$\tilde{\mu}_j(\varepsilon) = \Psi^{(j)}(x^+)/j! \qquad (j = 0, \ldots, n-1), \tag{2.14}$$

then, obviously, the vector $(\tilde{\mu}_0, \ldots, \tilde{\mu}_{n-1})$ and the vector of the functionals μ_j defined by (2.12) are related by a diffeomorphism. Hence, the equivalent condition for the tangency to be split generically is

$$\operatorname{rank} \partial(\tilde{\mu}_0, \ldots, \tilde{\mu}_{n-1})/\partial\varepsilon = n.$$

Analogously, one can easily see from (2.8) that the curve $T_1^{-1} W^s_{\mathrm{loc}}$ near M^- is given by

$$x = \hat{\mu}_0 + \cdots + \hat{\mu}_{n-1}(y - y^-)^{n-1} + \hat{d}(y - y^-)^{n+1} + o((y - y^-)^{n+1}), \tag{2.15}$$

with $\hat{d} \neq 0$, and with $(\hat{\mu}_0, \ldots, \hat{\mu}_{n-1})$ related to the vector of μ_j by a diffeomorphism. This gives us one more equivalent condition for the tangency to be split generically:

$$\operatorname{rank} \partial(\hat{\mu}_0, \ldots, \hat{\mu}_{n-1})/\partial\varepsilon = n.$$

2.3. Splitting of homoclinic and heteroclinic tangencies

Here we describe (lemma 1) the perturbations which we use in order to ensure a generic splitting of a given finite number of homoclinic or heteroclinic tangencies. Namely, given a map f with homoclinic or heteroclinic tangencies, we embed f into a family $f_\varepsilon \equiv F_\varepsilon \circ f$ where $F_0 = id$ and F_ε depends on the parameters $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{\bar{n}})$ which run a small ball centred at $\varepsilon = 0$. The map F_ε is real-analytic and area-preserving, so the maps f_ε are C^r-close to the unperturbed f, provided f itself is of class C^r ($r = 2, \ldots, \infty, \omega$), and the perturbation does not lead out of the class of area-preserving maps if f is area-preserving itself. The goal is to construct the family F_ε in such a way that certain (a finite number) homoclinic or heteroclinic tangencies of the map f chosen in advance are split generically and independently as ε varies.

The independence of the splitting means that, for each of the tangencies $\Gamma_1, \ldots, \Gamma_k$ under consideration, there exists a smooth manifold $S(\Gamma_i)$ in space of parameters ε such that when

ε varies within S_i the tangency Γ_i is split generically, while the other tangencies are not split at all, and the manifolds S_i and S_j corresponding to different tangencies intersect at $\varepsilon = 0$ transversely. In fact, we can achieve more. We can perturb f in such a way that a given tangency will be split generically while the other tangencies are not split, and neither are the values of the multipliers of the saddle periodic orbits involved changed. As well, we can change a multiplier of any of the saddles, without changing the multipliers of the other ones or destroying the tangencies.

The proof is done in two steps. First (see (2.21)), we construct a family of finitely smooth maps F_ε with the use of smooth cut-off functions. Here, $F_\varepsilon = id$ everywhere except for small neighbourhoods of a finite number of the homoclinic, heteroclinic or periodic points involved. For the corresponding family f_ε, the manifolds S_i simply coincide with coordinate hyperplanes in space of parameters ε. In the second step, we approximate F_ε by an appropriate family of real-analytic maps. While the manifolds S_i will no longer be hyperplanes, still they will exist and remain transverse as required.

Further, we also show that if, in addition, the map f has a number of elliptic periodic orbits, then the values of any given finite number of their Birkhoff coefficients can be changed independently as well.

Let us start with the case of a single homoclinic or heteroclinic tangency. Let f be a C^r-map with a tangency of order $n < r$. Let $\bar{y} = \Psi(\bar{x})$ be the equation of the curve $T_1 W^u_{\mathrm{loc}}$. Take any finite integer r_0 such that $r_0 \leqslant r$. Include the function Ψ into any C^{r_0}-smooth n-parameter family of functions Ψ_ε, such that $\Psi_0 = \Psi$. Fix a small $\delta > 0$ and denote

$$H_\varepsilon(x, y) = -\chi_\delta(x - x^+, y) \int_{x^+}^{x} (\Psi_\varepsilon(s) - \Psi_0(s))\,\mathrm{d}s, \tag{2.16}$$

where $\chi_\delta(u, v)$ is a C^{r_0+1}-smooth cut-off function which vanishes identically at $\|u, v\| \geqslant 2\delta$ and is equal to 1 at $\|u, v\| \leqslant \delta$. Let F_ε be the time-1 map by the orbits of the Hamiltonian system

$$\dot{x} = \frac{\partial H_\varepsilon}{\partial y}, \qquad \dot{y} = -\frac{\partial H_\varepsilon}{\partial x}.$$

By construction, F_ε is a C^{r_0}-smooth area-preserving map, which is equal to the identity outside a small neighbourhood of the point M^+ for all ε; at $\varepsilon = 0$ it is equal to the identity everywhere. Near M^+ the map F_ε acts as

$$(x, y) \mapsto (x, y + \Psi_\varepsilon(x) - \Psi_0(x)). \tag{2.17}$$

Consider the family $F_\varepsilon \circ f$. Since every map of the family coincides with f outside a small neighbourhood of $f^{-1}M^+$, it follows that the global map $(F_\varepsilon \circ f)^l$ from a small neighbourhood of M^- to a neighbourhood of M^+ equals $F_\varepsilon \circ T_1$ (where T_1 is the global map for the map f). By (2.17), the equation of the curve $F_\varepsilon \circ T_1 W^u_{\mathrm{loc}}$ near this point is $\bar{y} = \Psi_\varepsilon(\bar{x})$.

Now, take any Ψ_ε such that

$$\det \frac{\partial(\tilde{\mu}_0, \ldots, \tilde{\mu}_{n-1})}{\partial \varepsilon} \neq 0, \tag{2.18}$$

(where the functionals $\tilde{\mu}_j$ are given by (2.14)). In particular, we may take $\Psi_\varepsilon(\bar{x}) = \sum_{j=0}^{n-1} \varepsilon_j (\bar{x} - x^+)^j$, which would correspond to

$$H_\varepsilon(x, y) = -\chi_\delta(x - x^+, y) \sum_{j=0}^{n-1} \varepsilon_j \frac{(\bar{x} - x^+)^{j+1}}{j + 1} \tag{2.19}$$

and $\tilde{\mu}_j = \varepsilon_j$ in this case. We now recall that inequality (2.18) means exactly that the tangency between $T_1 W^u_{\mathrm{loc}}$ and W^s_{loc} is split generically in the family $F_\varepsilon \circ f$.

Note that this construction allows us to transform locally the piece of the unstable manifold $T_1 W^u_{loc}$ near the point M^+ into any sufficiently close curve by a small (in the C^{r_0}-norm, given any finite r_0) perturbation which does not destroy the area-preservation property of the map f. In particular, a homoclinic tangency of order n can be transformed to a small homoclinic band by a C^n-small perturbation. Recall also that the perturbation is localized in a small neighbourhood of one homoclinic (heteroclinic) point, so it does not affect any other homoclinic or heteroclinic tangencies away from this point.

In the same way one can show that the multiplier λ of the saddle periodic orbit O can be changed by a small smooth localized perturbation of the map f (the perturbation is area-preserving, if f is area-preserving), without destroying any finite number of given homoclinic or heteroclinic tangencies. Indeed, consider a one-parameter family $f_\varepsilon = F_\varepsilon \circ f$, where the area-preserving diffeomorphism F_ε is the time-1 shift by the flow defined by the Hamiltonian

$$H_\varepsilon(x, y) = -\varepsilon \chi_\delta(x, y) x y, \tag{2.20}$$

where (x, y) are the coordinates near O for which the local invariant manifolds are straightened and χ_δ is the cut-off function (such as in (2.16)) with some sufficiently small and fixed $\delta > 0$. By construction, $F_0 = id$, hence $f_0 = f$. At non-zero ε, the map F_ε can differ from the identity only in the δ-neighbourhood of O, so if δ is small enough, then the new local map is $T_0 \circ F_\varepsilon$. Direct computation of the multiplier gives then $\lambda_\varepsilon = \mathrm{e}^{-\varepsilon}\lambda$, so

$$\frac{\partial \lambda_\varepsilon}{\partial \varepsilon} = -\lambda_\varepsilon \neq 0.$$

The lines $x = 0$ and $y = 0$ are invariant with respect to the map F_ε; hence, they remain local unstable and, respectively, stable invariant manifolds of O for all small ε. Since the position of the local invariant manifolds is not changed and since the perturbation is localized in a sufficiently small neighbourhood of O, any given number of homoclinic or heteroclinic tangencies is not split by such perturbation.

Now, let a two-dimensional C^r-map f have a number of saddle periodic orbits $L_1, \ldots, L_s$ and a number of homoclinic or heteroclinic orbits to them $\Gamma_1, \ldots, \Gamma_m$, corresponding to the tangency of the stable and unstable manifolds of orders $n_1, \ldots, n_m$; we assume $r \geqslant r_0 \equiv 1 + \max(n_1, \ldots, n_m)$. For all maps C^{r_0}-close to f we can define $s + n_1 + \ldots n_m$ smooth functionals: the multipliers λ_i $(i = 1, \ldots, s)$ of the periodic orbits L_i and the functionals μ^i_j $(j = 0, \ldots, n_i - 1; i = 1, \ldots, m)$ which determine the splitting of the homoclinic and heteroclinic tangencies (see (2.13)). Although the functionals μ^i_j depend on the choice of the coordinate transformation that straightens the local invariant manifolds, we can define them unambiguously by assuming that this transformation is canonically given by (2.3).

Consider the C^{r_0+1}-smooth Hamiltonian

$$H_\varepsilon(x, y) = \varepsilon_1 \zeta_1(x, y) + \cdots + \varepsilon_{\bar{n}} \zeta_{\bar{n}}(x, y), \tag{2.21}$$

where ζ_l $(l = 1, \ldots, \bar{n})$ are the functions given by the right-hand sides of (2.20) and (2.19), localized in sufficiently small neighbourhoods of the appropriately chosen periodic and homo/heteroclinic points, respectively (the functions χ_δ are C^{r_0+1}-smooth cut-off functions). As it follows from our considerations above, the family $f_\varepsilon = F_\varepsilon \circ f$, where F_ε is the time-1 map by the flow defined by Hamiltonian (2.21), satisfies

$$\det \frac{\partial(\lambda_1(f_\varepsilon), \ldots, \lambda_s(f_\varepsilon), \mu^1_0(f_\varepsilon), \ldots, \mu^m_{n_m-1}(f_\varepsilon))}{\partial(\varepsilon_1, \ldots, \varepsilon_{\bar{n}})} \neq 0. \tag{2.22}$$

This inequality, obviously, means that in the constructed family f_ε all the tangencies are split generically and independently, and the multipliers of the saddles are changing independently too.

Now note that inequality (2.22) preserves under any sufficiently small, in the C^{r_0}-metric, perturbation of the family f_ε. Let us construct such a perturbation in the following way. If the phase manifold $\mathcal{M}$ is orientable, we replace the functions ζ_l in (2.21) by their sufficiently close (in the C^{r_0+1}-metric) analytic approximations $\bar{\zeta}_l$. Due to the orientability, the area density defines a symplectic form on the manifold, so the Hamiltonian

$$\bar{H}_\varepsilon = \varepsilon_1\bar{\zeta}_1 + \cdots + \varepsilon_{\bar{n}}\bar{\zeta}_{\bar{n}} \tag{2.23}$$

defines an analytic Hamiltonian flow on $\mathcal{M}$. The corresponding time-1 map $\bar{F}_\varepsilon$ is analytic, area-preserving and it is close to F_ε; hence, the family $\bar{F}_\varepsilon \circ f$ satisfies inequality (2.22).

In the non-orientable case, we lift the vector field X defined by Hamiltonian (2.21) onto an orientable double-covering manifold $\widetilde{\mathcal{M}}$. This will give us an area-preserving vector field $\widetilde{X}$ on $\widetilde{\mathcal{M}}$. As $\widetilde{\mathcal{M}}$ is orientable, the area density defines a symplectic form on $\widetilde{\mathcal{M}}$ and the field $\widetilde{X}$ is defined by some Hamiltonian of form (2.21) where the functions ζ_l are now localized each in a pair of small discs (projected into one disc on $\mathcal{M}$ by the covering map). Replacing the functions ζ_l by their sufficiently close (in the C^{r_0+1}-metric) analytic approximations $\bar{\zeta}_l$ will provide us with a real-analytic Hamiltonian vector field $\bar{X}$ on $\widetilde{\mathcal{M}}$, close to the field $\widetilde{X}$. Back on the non-orientable manifold $\mathcal{M}$, the projection of the field $\bar{X}$ by the covering map is a field of vector pairs, close to X, real-analytic and divergence-free. By taking a half-sum of the vectors in each pair, we finally obtain a divergence-free real-analytic vector field on $\mathcal{M}$, close to the original field X. So, the corresponding (real-analytic and area-preserving) time-1 map $\bar{F}_\varepsilon$ is close to F_ε, and the family $\bar{F}_\varepsilon \circ f$ satisfies inequality (2.22).

Thus, we have proved the following lemma.

Lemma 1. *There exists an analytic $\bar{n}$-parameter family of real-analytic area-preserving diffeomorphisms $\bar{F}_\varepsilon$ such that in the family $f_\varepsilon = \bar{F}_\varepsilon \circ f$ the tangencies $\Gamma_1, \ldots, \Gamma_m$ split generically and independently, and the multipliers of the periodic orbits $L_1, \ldots, L_s$ change independently as well.*

This result can be further extended in order to include perturbations of elliptic periodic orbits. Let the map f be area-preserving and let it have an elliptic periodic point O with the multipliers $\mathrm{e}^{\pm i\varphi}$. We focus only on the case where $\varphi/2\pi$ is irrational. Then one can fix any m and choose local coordinates near O such that the first-return map f^q (where q is the period of O) takes the following normal form

$$(\bar{x}, \bar{y}) = \mathcal{F}(x, y) + o(\|x, y\|^{2m+1}), \tag{2.24}$$

where $\mathcal{F}$ is the time-1 map of the flow

$$\begin{aligned} \dot{x} &= (\omega + b_1(x^2+y^2) + \cdots + b_m(x^2+y^2)^m)y, \\ \dot{y} &= -(\omega + b_1(x^2+y^2) + \cdots + b_m(x^2+y^2)^m)x. \end{aligned} \tag{2.25}$$

The same normal form is obtained for every C^{2m+1}-close map. We can fix any normalizing procedure, then the coefficients $\omega, b_1, \ldots, b_m$ become functions of the map. It is easy to check that $b_1, \ldots, b_m$ are related to the Birkhoff coefficients $B_1, \ldots, B_m$ of (1.1) by a diffeomorphism and that first k coefficients $b_1, \ldots, b_k$ vanish if and only if the first k Birkhoff coefficients $B_1, \ldots, B_k$ vanish.

Flow (2.25) is defined by the Hamiltonian

$$H(x, y) = \frac{\varphi}{2}(x^2+y^2) + \sum_{s=1}^{m} \frac{b_s}{2(s+1)}(x^2+y^2)^{s+1}. \tag{2.26}$$

Define $F_\varepsilon = \mathcal{F}_\varepsilon \circ \mathcal{F}^{-1}$ where $\mathcal{F}$ is the time-1 map of the flow of (2.26) and $\mathcal{F}_\varepsilon$ is the time-1 map of the flow defined by the Hamiltonian

$$H_\varepsilon(x, y) = H(x, y) + \chi_\delta(x, y)\left(\frac{\varepsilon_0}{2}(x^2 + y^2) + \sum_{s=1}^{m} \frac{\varepsilon_s}{2(s+1)}(x^2 + y^2)^{s+1}\right), \tag{2.27}$$

where $H(x, y)$ is given by (2.26) and χ_δ is the cut-off function such as in (2.20), with a sufficiently small fixed $\delta > 0$. Consider an $(m+1)$-parameter perturbation $f_\varepsilon = F_\varepsilon \circ f$ of the original map f. As F_ε can differ from the identity only in the δ-neighbourhood of O, it follows that $f_\varepsilon^q = f_0^q \circ F_\varepsilon$, i.e. the first-return map near O keeps its form (2.24) with $\mathcal{F}$ replaced by $\mathcal{F}_\varepsilon$. As we see, one can make arbitrary and independent small changes in the values of the multiplier of the elliptic point and of any fixed number m of its Birkhoff coefficients by a localized area-preserving perturbation.

Now, arguing as in the proof of lemma 1, we find that given an area-preserving C^r-map ($r = 2, \ldots, \infty, \omega$), by applying a C^r-small perturbation which does not lead out of the class of area-preserving maps, we can independently change the multipliers of any finite number of saddle periodic orbits and split generically any finite number of homoclinic or heteroclinic tangencies without changing the order of degeneracy of any finite number of elliptic periodic orbits. We will use this fact in the proof of theorem 4 in section 5.3.

3. Lemmas on secondary homoclinic and heteroclinic tangencies

Here we show how perturbations of homoclinic and heteroclinic cycles can produce secondary homoclinic and heteroclinic tangencies of various types. Statements analogous to the following lemma were given in [4,14,56] for different situations. We give a unified proof which includes the area-preserving case.

Lemma 2. *Let f_μ be a one-parameter family of two-dimensional C^r-diffeomorphisms ($r \geqslant 3$). Let f_0 have a saddle periodic point O with an orbit of quadratic homoclinic tangency which splits generically as μ varies. Then, arbitrarily close to $\mu = 0$ there exists a value of μ for which the map f_μ has an orbit of quadratic homoclinic tangency (which splits generically as μ varies) and a homoclinic orbit corresponding to a transverse intersection of the invariant manifolds of O.*

Proof. Note that orbits of transverse homoclinic intersection may exist already at the moment of the original homoclinic tangency; in this case the sought value of μ is 0. Otherwise, we have to find a converging to zero sequence of non-zero values of μ for which the map has new, secondary homoclinic tangencies accompanied by transverse homoclinic orbits. We will explore both possibilities in the proof.

As the given homoclinic tangency is quadratic, the global map T_1 is written in the following form (see (2.8),(2.13)):

$$\begin{aligned} \bar{x} - x^+ &= ax + b(y - y^-) + g_1(x, y), \\ \bar{y} &= cx + \mu + d(y - y^-)^2 + g_2(x, y). \end{aligned} \tag{3.1}$$

It follows (see (2.9)) that the equation of $T_1(W^u_{\mathrm{loc}})$ is

$$y = \mu + \frac{d}{b^2}(x - x^+)^2 + o((x - x^+)^2). \tag{3.2}$$

At $\mu d < 0$ this curve intersects $W^s_{\mathrm{loc}} \cap \Pi^+ : \{y = 0\}$ transversely at two points.

Assume that the coordinates near O are chosen such that (2.4), (2.5) hold. Then, by (2.6), (3.2), the image $(\hat{x}, \hat{y})$ of a point $(x, y) \in T_1(W^u_{\rm loc})$ by the map T_0^k gets in a small neighbourhood of the homoclinic point M^- (the domain of the global map T_1, see figure 3), if and only if

$$\hat{x} = \lambda^k x + \lambda^k \xi_k(x, \hat{y}), \qquad y = \gamma^{-k}\hat{y} + \gamma^{-k}\eta_k(x, \hat{y}). \tag{3.3}$$

The point (x, y) will be homoclinic if $T_1(\hat{x}, \hat{y}) \in W^s_{\rm loc}$, which means (see (3.1))

$$0 = c\hat{x} + \mu + d(\hat{y} - y^-)^2 + o(|\hat{x}| + (\hat{y} - y^-)^2).$$

Thus, a homoclinic point (x, y): $T_1T_0^k(x, y) \in W^s_{\rm loc}$, $T_1^{-1}(x, y) \in W^u_{\rm loc}$ corresponds to a solution of the system

$$\begin{aligned} 0 &= \mu + c\lambda^k x^+ + c\lambda^k X + dY^2 + \phi_1(X, Y, \mu) + \phi_2(Y, \mu), \\ 0 &= \mu - \gamma^{-k} y^- - \gamma^{-k} Y + \frac{d}{b^2}X^2 + \phi_3(X, Y, \mu) + \phi_4(X, \mu), \end{aligned} \tag{3.4}$$

where we denote $X = x - x^+$, $Y = \hat{y} - y^-$. The functions ϕ satisfy

$$\phi_1 = o(\lambda^k), \qquad \phi_2 = o(Y^2), \qquad \phi_3 = o(\gamma^{-k}), \qquad \phi_4 = o(X^2). \tag{3.5}$$

Note that the right-hand sides of system (3.4) are at least C^2 with respect to X and Y: the map f is at least C^3 by assumption, but we lose one smoothness when we introduce the coordinates bringing the local map to the form (2.4), (2.5). Thus, the first derivative of the right-hand side with respect to (X, Y) is at least C^1 with respect to X, Y and μ. So, the coefficients λ, γ, x^+, y^-, b and c are C^1-functions of μ (while d is a constant).

Note that $d \neq 0$ implies that we may shift the origin of the coordinates: $(X, Y) \mapsto (X + o(\gamma^{-k}), Y + o(\lambda^k))$ so that the first derivative of the right-hand side of the first equation in (3.4) with respect to Y and of the right-hand side of the second equation with respect to X will vanish at $(X, Y) = 0$. After that, the system will take the form

$$\begin{aligned} 0 &= \mu + \nu_k^1 + c\lambda^k X + dY^2 + \tilde{\phi}_1(X, Y, \mu) + \tilde{\phi}_2(Y, \mu), \\ 0 &= \mu + \nu_k^2 - \gamma^{-k} Y + \frac{d}{b^2}X^2 + \tilde{\phi}_3(X, Y, \mu) + \tilde{\phi}_4(X, \mu), \end{aligned} \tag{3.6}$$

where we denote by $\nu_k^{1,2}$ the terms independent of X and Y:

$$\nu_k^1 = c\lambda^k x^+ + o(\lambda^k), \qquad \nu_k^2 = -\gamma^{-k} y^- + o(\gamma^{-k}), \tag{3.7}$$

and the functions $\tilde{\phi}$ satisfy

$$\tilde{\phi}_1 = o(\lambda^k X), \qquad \tilde{\phi}_2 = o(Y^2), \qquad \tilde{\phi}_3 = o(\gamma^{-k} Y), \qquad \tilde{\phi}_4 = o(X^2). \tag{3.8}$$

The non-degenerate solutions of (3.6) correspond to transverse homoclinics, and the degenerate ones correspond to homoclinic tangencies. We will consider system (3.6) for even k only, so $\lambda^k > 0$ and $\gamma^{-k} > 0$, no matter what are the signs of λ and γ. It is easy to see then that when $cx^+d < 0$ and $y^-d > 0$ the system has non-degenerate solutions

$$Y = \pm|\lambda|^{k/2}\sqrt{\frac{cx^+}{d} + o(1)}, \qquad X = \pm b|\gamma|^{-k/2}\sqrt{\frac{y^-}{d} + o(1)}$$

at $\mu = 0$ for all sufficiently large k. This means that at $\mu = 0$, in addition to the original orbit of homoclinic tangency, we have also transverse homoclinic orbits (see figure 4). That gives us the lemma in this case.

If, on the contrary, $cx^+d > 0$ or $y^-d < 0$, we will search for secondary homoclinic tangencies at small $\mu \neq 0$. They correspond to solutions of (3.4) for which the Jacobian of the right-hand side vanishes. Thus, they solve the system

$$\begin{aligned} 0 &= c\lambda^k\gamma^{-k} + 4\frac{d^2}{b^2}(X + \varphi_1(X, Y))(Y + \varphi_2(X, Y)) + o(\lambda^k\gamma^{-k}), \\ 0 &= \nu_k + c\lambda^k X + \gamma^{-k} Y + dY^2 - \frac{d}{b^2}X^2 + o(\lambda^k|X| + Y^2 + X^2 + \gamma^{-k}|Y|), \end{aligned} \tag{3.9}$$

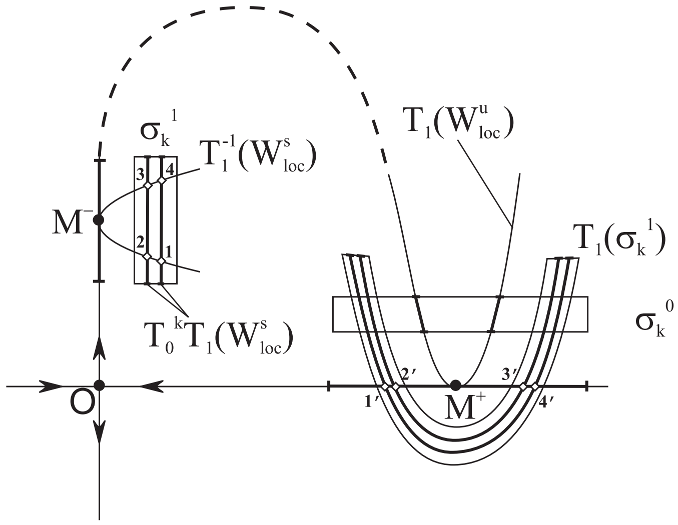

Figure 4. Secondary transverse homoclinic orbits in the case $cx^+d < 0$, $y^-d > 0$.

where the constant term ν_k is given by

$$\nu_k = c\lambda^k x^+ + \gamma^{-k} y^- + o(\lambda^k + \gamma^{-k}), \tag{3.10}$$

and

$$\varphi_1 = o(|X| + \gamma^{-k}|Y|), \qquad \varphi_2 = o(|Y| + \lambda^k|X|). \tag{3.11}$$

We obtained the second equation in (3.9) as follows: from the first equation of (3.6) we expressed μ as a function of (X, Y) and plugged the result into the second equation. Thus, in (3.9) there is no dependence on μ, so by $\lambda, \gamma, c, x^+, y^-$ in (3.9), (3.10) we mean the values of these coefficients at $\mu = 0$.

The non-degenerate solutions of (3.9) correspond to quadratic homoclinic tangencies for μ found from either one of the two equations in (3.6). Note that the value of parameter μ is found uniquely for any given small X and Y, and this remains true for an arbitrary small perturbation of the map under consideration, which means that the corresponding tangency splits indeed generically as μ varies.

Consider, first, the case $|\lambda\gamma| < 1$ (hence $\lambda^k\gamma^k \to 0$ as $k \to +\infty$). If $y^-d > 0$ and $cx^+d > 0$ (the case $y^-d > 0$ and $cx^+d < 0$ has already been considered), we scale the variables as follows:

$$(X, Y) \mapsto b|\gamma|^{-k/2}\sqrt{\frac{y^-}{d}}\left(X^{\text{new}}, -\frac{c}{4y^-d}\lambda^k\, Y^{\text{new}}\right).$$

In the new variables, system (3.9) recasts as

$$1 = XY + o(1)_{k\to+\infty}, \qquad 1 = X^2 + o(1)_{k\to+\infty}. \tag{3.12}$$

For all k large enough this system has non-degenerate solutions $X = Y = \pm 1 + o(1)$. In the non-rescaled variables it corresponds to $X = O(|\gamma|^{-k/2})$, $Y = O(\lambda^k|\gamma|^{-k/2})$. The first equation of (3.6) gives us then that the corresponding homoclinic tangencies happen at $\mu = \mu_k = -cx^+\lambda^k(1 + o(1))$, see figure 5. By our current assumptions, we have $cx^+d > 0$, hence $\mu_k d < 0$, i.e. the found secondary homoclinic tangencies coexist with primary transverse homoclinic orbits, as required.

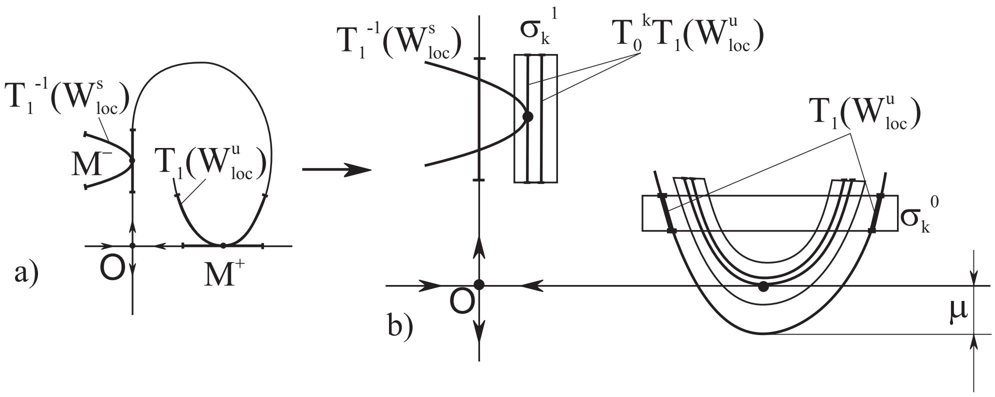

Figure 5. Case $cx^+d > 0$, $y^-d > 0$—creation of secondary homoclinic tangencies.

In the case $y^-d < 0$, we use the following scaling:

$$(X, Y) \mapsto |\gamma|^{-k/2}\sqrt{|y^-/d|}\left(-\frac{cb^2}{4|y^-d|}\lambda^k(X^{\text{new}} - \varphi_1(0, Y)), Y^{\text{new}}\right).$$

In the new variables, system (3.9) recasts as

$$1 = XY + o(1)_{k\to+\infty}, \qquad 1 = Y^2 + o(1)_{k\to+\infty}. \tag{3.13}$$

For all k large enough, this system has non-degenerate solutions $X = Y = \pm1 + o(1)$. In the non-rescaled variables it corresponds to $X = o(|\gamma|^{-3k/2}), Y = O(|\gamma|^{-k/2})$. The second equation of (3.6) gives us then that the corresponding homoclinic tangencies happen at $\mu = \mu_k = y^-\gamma^{-k}(1 + o(1))$, see figure 6. Since we assume here $y^-d < 0$, it follows that $\mu_k d < 0$, i.e. the found secondary homoclinic tangencies coexist with primary transverse homoclinic orbits in this case too. This finishes the proof of the lemma in the case $|\lambda\gamma| < 1$.

The case $|\lambda\gamma| > 1$ is reduced to the previous one if we consider the map f^{-1} instead of f. Thus, it remains to prove the lemma for the case $|\lambda\gamma| = 1$, which includes the area-preserving maps. Let us choose the sequence of (even) values of k such that there exists the limit (finite or infinite)

$$\lim_{k\to+\infty} \nu_k/\lambda^{2k} = M. \tag{3.14}$$

If $M = \pm\infty$, we scale

$$(X, Y) \mapsto b\sqrt{\frac{\nu_k}{d}}(X^{\text{new}}, -\frac{c}{4d}\frac{\lambda^{2k}}{\nu_k}Y^{\text{new}}) \qquad \text{if } \nu_k d > 0,$$

$$(X, Y) \mapsto \sqrt{|\nu_k/d|}\,(-\frac{cb^2}{4|d|}\frac{\lambda^{2k}}{|\nu_k|}X^{\text{new}}, Y^{\text{new}}) \qquad \text{if } \nu_k d < 0.$$

In the first case, system (3.9) takes the form (3.12). The non-degenerate solutions correspond to $(X = O(|\lambda|^{k/2}), Y = O(|\lambda|^{3k/2}))$, and the first equation of (3.6) gives $\mu = \mu_k = -cx^+\lambda^k(1+o(1))$ for the moments of homoclinic tangencies, see figure 7(*a*). The assumption $\nu_k d > 0$ implies (see (3.10)) that either $cx^+d > 0$ or $y^-d > 0$. The latter implies $cx^+d > 0$ as well (because the case $cx^+d < 0$, $y^-d > 0$ has already been considered). Thus $\mu_k d < 0$, i.e. the found secondary homoclinic tangencies coexist with primary transverse homoclinic orbits.

In the second case, system (3.9) reduces to (3.13). The non-degenerate solutions correspond to $(X = O(|\lambda|^{3k/2}), Y = O(|\lambda|^{k/2}))$, and the second equation of (3.6) gives

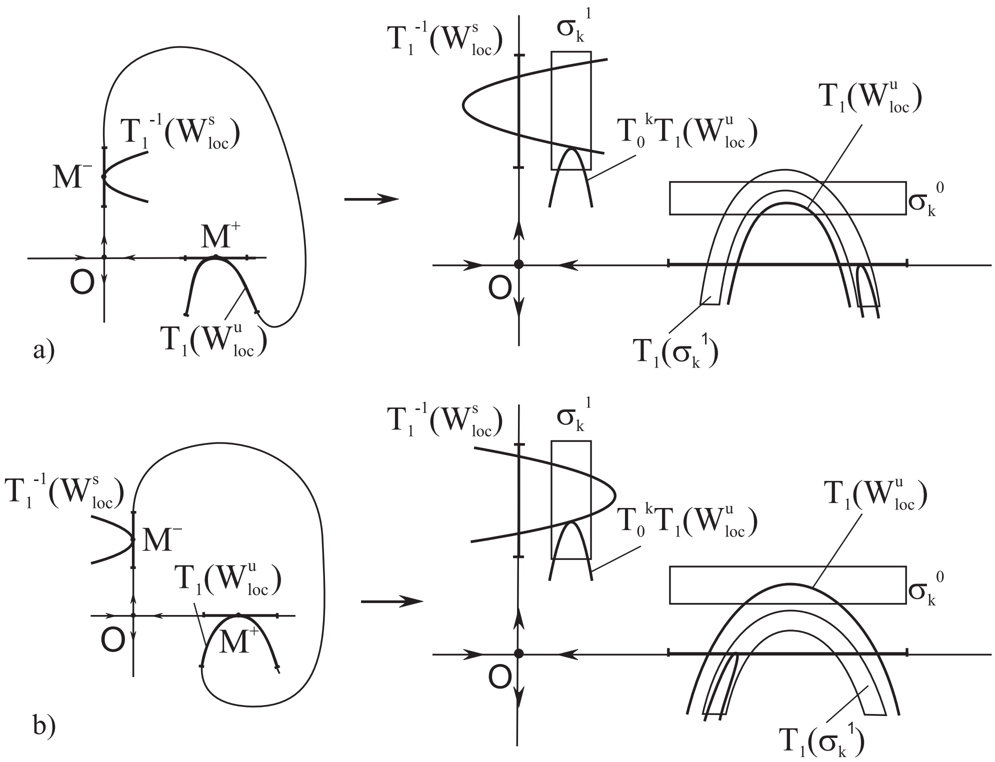

Figure 6. Case $y^-d < 0$. Secondary homoclinic tangencies in cases (*a*) $cx^+d > 0$ and (*b*) $cx^+d < 0$.

$\mu = \mu_k = y^-\lambda^k(1 + o(1))$ for the moments of homoclinic tangencies, see figures 7(*b*) and (*c*). The assumption $\nu_k d < 0$ implies either $cx^+d < 0$ or $y^-d < 0$, but the former inequality implies $y^-d < 0$ anyway. Thus $\mu_k d < 0$ in this case too.

It remains to consider the case where M is finite in (3.14). It follows, in particular, that $cx^+ + y^- = 0$, which implies $cx^+d > 0$. After the rescaling $(X, Y) \mapsto (1/d)\lambda^k(b^2cX^{\text{new}}, -Y^{\text{new}})$, system (3.9) takes the form

$$0 = 1 - 4XY + o(1)_{k\to+\infty}, \qquad 0 = Md - (bc)^2(X^2 - X) + Y^2 - Y + o(1)_{k\to+\infty}. \qquad (3.15)$$

It is easy to see that this system has a non-degenerate solution in the region $\{X < 0, Y < 0\}$ for any Md and $bc \neq 0$. In the non-rescaled variables this solution gives $(X, Y) = O(\lambda^k)$, and from (3.6) we have $\mu = \mu_k = -cx^+\lambda^k(1+o(1))$ for the moments of homoclinic tangencies, see figure 7(*d*). Again we have $\mu_k d < 0$, so the coexistence of the secondary homoclinic tangencies with primary transverse homoclinics is established in this last remaining case as well. □

Remark. In exactly the same way as is done in the proof of the lemma for other cases, one can check that a sequence of values of $\mu = \mu_k$ corresponding to secondary homoclinic tangencies exists in the case $y^-d > 0$ and $cx^+d < 0$ too. Here, however, $\mu_k d > 0$ for even k, which means that we cannot guarantee the existence of primary transverse homoclinic orbits at these values of μ. On the other hand, as we have shown in the proof of the lemma, secondary transverse homoclinics exist already at $\mu = 0$. Any finite number of them survives, obviously,

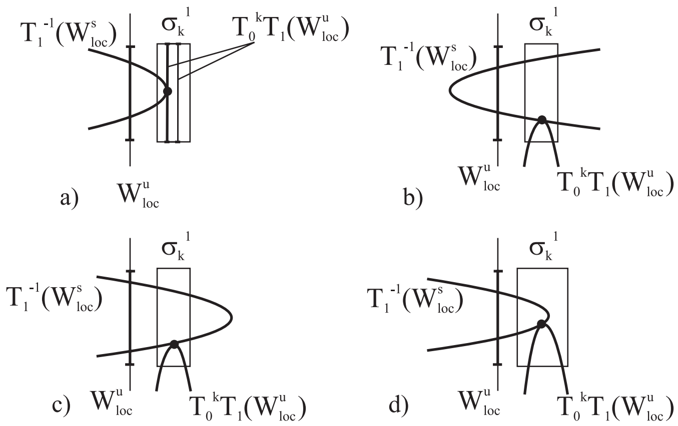

Figure 7. Secondary tangencies in the case $|\lambda\gamma| = 1$—(*a*) $M = \infty$, $\nu_k d > 0$; (*b*) $M = \infty$, $\nu_k d < 0$, $cx^+d > 0$; (*c*) $M = \infty$, $\nu_k d < 0$, $cx^+d < 0$; (*d*) $M \neq \infty$.

for all μ sufficiently small. Thus, in the case $y^-d > 0$ and $cx^+d < 0$ we obtain secondary homoclinic tangencies which coexist with secondary transverse homoclinics.

The existence of a transverse homoclinic to O implies [52] the existence of a non-trivial compact, locally maximal, transitive uniformly-hyperbolic invariant set Λ which contains O. Thus, lemma 2 can be reformulated as the existence of a non-trivial basic hyperbolic set whose stable and unstable sets $W^s(\Lambda)$ and $W^u(\Lambda)$ have a quadratic tangency at values of μ arbitrarily close to zero. Since the stable and unstable manifolds of any periodic orbit in Λ approximate, in the C^r-topology, any leaf in, respectively, $W^s(\Lambda)$ and $W^u(\Lambda)$, and since the found homoclinic tangency splits generically as μ changes, it follows that for any two periodic points in Λ a quadratic heteroclinic tangency of their invariant manifolds can be obtained by an arbitrarily small variation of μ, and this tangency also splits generically.

Let O_1, O_2 be two different saddle periodic points in Λ, different from O and with all multipliers positive (such periodic points always exist in any horseshoe). Let us fix some small μ for which $W^u(O_2)$ and $W^s(O_1)$ have a quadratic heteroclinic tangency (figure 8(*a*)). A heteroclinic orbit corresponding to a transverse intersection of $W^u(O_1)$ and $W^s(O_2)$ also exists at the same μ, because O_1 and O_2 belong to the same transitive hyperbolic set Λ.

We introduce coordinates (x_i, y_i) near the points O_i $(i = 1, 2)$ such that the local invariant manifolds are straightened, i.e. the local maps T_{0i} take the form (see (2.4))

$$\bar{x}_i = \lambda_i x_i + p_i(x_i, y_i)x_i, \qquad \bar{y}_i = \gamma_i y_i + q_i(x_i, y_i)y_i,$$

where $0 < \lambda_i < 1 < \gamma_i$. Note that the point O_i divides its stable and unstable manifolds into two invariant components each; we will denote these components as $W^{u+}(O_i)$, $W^{u-}(O_i)$ and $W^{s+}(O_i)$, $W^{s-}(O_i)$.

Choose a pair of heteroclinic points from the same orbit of heteroclinic tangency: $M_1^+(x_1^+, 0)$ in a small neighbourhood of O_1 and $M_2^-(0, y_2^-)$ in a small neighbourhood of O_2 (see figure 8(*a*)). The global map T_{21} from a small neighbourhood of M_2^- into a small neighbourhood of M_1^+ is written as follows (see (2.8),(2.13)):

$$x_1 - x_1^+ = ax_2 + b(y_2 - y_2^-) + \cdots, \qquad y_1 = cx_2 + d(y_2 - y_2^-)^2 + \cdots. \tag{3.16}$$

Choose also a pair of heteroclinic points that belong to a transverse heteroclinic orbit: $M_1^-(0, y_1^-)$ in a neighbourhood of O_1 and $M_2^+(x_2^+, 0)$ in a neighbourhood of O_2. In the terms

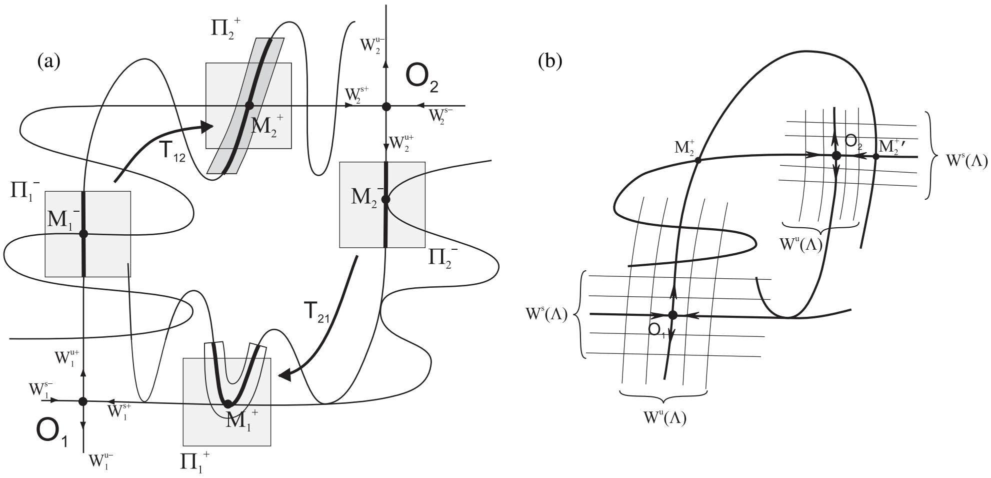

Figure 8. (*a*) Non-transverse heteroclinic cycle. (*b*) When O_1 and O_2 are 'inner' points of the horseshoe Λ and the manifolds $W^u(O_2)$ and $W^{s-}(O_1)$ are tangent, one can always choose an orbit of transverse intersection of $W^u(O_1)$ and $W^s(O_2)$ such that the corresponding heteroclinic cycle will be of the third class. Indeed, $W^u(O_1)$ has intersection points with both $W^{s+}(O_2)$ and $W^{s-}(O_2)$. Since $x_2^+ > 0$ at one of these points (M_2^+) and $x_2^+ < 0$ at the other point ($M_2^{+\prime}$), we can always choose the transverse heteroclinic in such a way that (3.17) will hold.

of [7, 46], the heteroclinic cycle belongs to the 'third class' when

$$cy_1^- x_2^+ > 0. \tag{3.17}$$

We can always choose the points O_1 and O_2 among the 'inner' points of the horseshoe Λ, i.e. the leaves of $W^s(\Lambda)$ and $W^u(\Lambda)$ converge, respectively, to $W^s(O_1)$ and $W^u(O_1)$ from both sides, and the same holds true for O_2 (see figure 8(*b*)). In other words, the points O_1 and O_2 can be chosen such that each of the manifolds $W^{u+}(O_1)$, $W^{u-}(O_1)$, has, in Λ, an orbit of transverse intersection with each of the manifolds $W^{s+}(O_2)$, $W^{s-}(O_2)$. These four orbits correspond to different signs of y_1^- and x_2^+. Hence, no matter what is the sign of c, we can always choose a transverse heteroclinic in such a way that (3.17) holds. Thus, we have the following lemma.

Lemma 3. *Under the conditions of lemma 2, arbitrarily close to $\mu = 0$, there exist values of μ for which the map f_μ has a non-trivial transitive hyperbolic set Λ which includes the point O and two saddle periodic points O_1 and O_2 such that $W^u(O_2)$ and $W^s(O_1)$ have a quadratic heteroclinic tangency which splits generically as μ varies. In Λ there exists also an orbit of transverse intersection of $W^u(O_1)$ and $W^s(O_2)$ such that the corresponding heteroclinic cycle belongs to the third class.*

The peculiarity of the maps with heteroclinic cycles of the third class is that they have moduli of local Ω-conjugacy [46]. In particular, the value[5]

$$\alpha = -\frac{\ln \gamma_1}{\ln \lambda_2} \tag{3.18}$$

is such a modulus: if two maps with a heteroclinic cycle of the third class have different values of α, they are not locally Ω-conjugate. It follows that any change in α must lead to bifurcations

[5] It is well known that α is an invariant of topological conjugacy for maps with a quadratic heteroclinic tangency [54–56]; however, the fact that it is also an invariant of local Ω-conjugacy holds true only for maps with heteroclinic cycles of the third class [46].

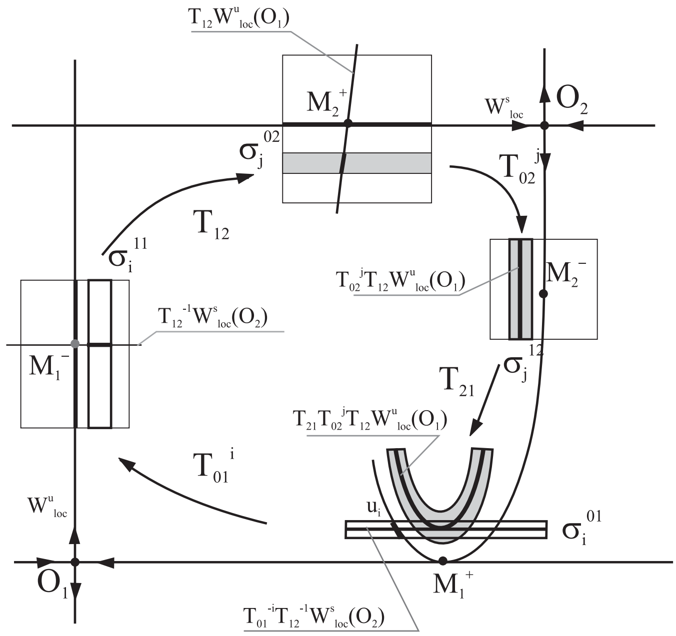

Figure 9. Heteroclinic tangency between the manifolds $W^u(O_1)$ and $W^s(O_2)$. The heteroclinic tangency between $W^u(O_2)$ and $W^s(O_1)$ is not split.

in the set of orbits lying in a small neighbourhood of the heteroclinic cycle. In particular, we have the following lemma.

Lemma 4. *Let f_ε be any smooth family of C^r-diffeomorphisms ($r \geqslant 3$) such that all diffeomorphisms in the family have a heteroclinic cycle of the third class, i.e. there are two periodic points O_1 and O_2, an orbit of transverse intersection of $W^u(O_1)$ and $W^s(O_2)$, and an orbit of quadratic heteroclinic tangency between $W^u(O_2)$ and $W^s(O_1)$ which does not split as ε varies, plus condition (3.17) holds. If α changes monotonically with ε, i.e.*

$$\frac{\partial}{\partial\varepsilon}\,\alpha(f_\varepsilon) \neq 0,$$

then there is a dense set of values of ε for which the map f_ε has a quadratic heteroclinic tangency (which splits generically as ε varies) between $W^u(O_1)$ and $W^s(O_2)$ (see figure 9 for an illustration).

A complete proof is given in [7], theorem 7. Here we recall the scheme of it. Note that by the lambda-lemma, since $W^u(O_1)$ and $W^s(O_2)$ intersect transversely, there are pieces of $W^s(O_2)$ which converge in the C^r-topology to $W^s_{\text{loc}}(O_1)$ and pieces of $W^u(O_1)$ which converge to $W^u_{\text{loc}}(O_2)$. By (2.6), these pieces of $W^s(O_2)$ near the point M_1^+ form an infinite sequence of curves

$$W_i^s : y_1 \sim \gamma_1^{-i} y_1^-, \tag{3.19}$$

and the pieces of $W^u(O_1)$ near M_2^- form an infinite sequence of curves

$$W_j^u : x_2 \sim \lambda_2^j x_2^+, \tag{3.20}$$

where y_1^- and x_2^+ are the coordinates of, respectively, M_1^- and M_2^+. By (3.16), the curves $T_1 W_j^u$ form a sequence of parabola-like curves, extended towards positive y_1 if $d > 0$ and towards

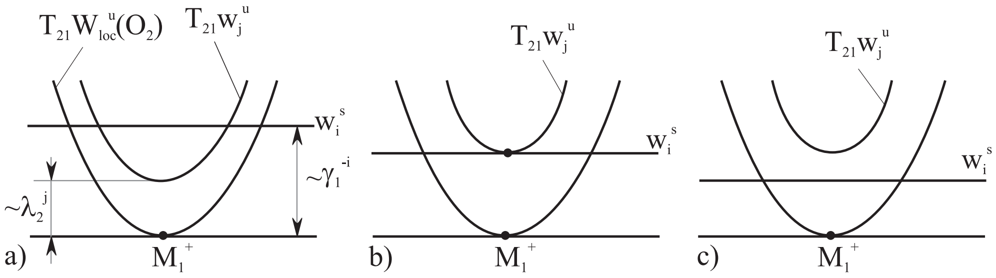

Figure 10. As the Ω-modulus $\alpha = -\ln\gamma_1/\ln\lambda_2$ decreases, we have that in (*a*) $i \ll j\alpha$ and $T_{21}(w^u_j) = T_{21}T^j_{02}(T_{12}W^u_{\rm loc}(O_1) \cap \sigma^{02}_j)$ intersects with $w^s_i = T^{-i}_{01}(T^{-1}_{12}W^s_{\rm loc}(O_2) \cap \sigma^{11}_i)$, while in (*c*) $i \gg j\alpha$ and $T_{21}(w^u_j)$ does not intersect w^s_i. On the way, at $i \sim j\alpha$, a tangency must occur between $T_{21}(w^u_j)$ and w^s_i, as in (*b*).

negative y_1 if $d < 0$, with the tops at $y_1 \sim cx_2^+\lambda_2^j$. Thus, W^s_i and $T_1W^u_j$ intersect transversely when

$$d\left(\frac{\gamma_1^{-i}y_1^-}{cx_2^+\lambda_2^j} - 1\right) \gg 0 \tag{3.21}$$

and have no intersection when

$$d\left(\frac{\gamma_1^{-i}y_1^-}{cx_2^+\lambda_2^j} - 1\right) \ll 0. \tag{3.22}$$

Now take any two arbitrarily close values $\varepsilon_1 \neq \varepsilon_2$. By assumption, $\alpha(\varepsilon_1) \neq \alpha(\varepsilon_2)$, and we may assume $\alpha(\varepsilon_1) > \alpha(\varepsilon_2)$. Hence, we can find two sufficiently large integers i and j such that $i\ln\gamma_1 - j|\ln\lambda_2| \gg 0$ at $\varepsilon = \varepsilon_1$ and $i\ln\gamma_1 - j|\ln\lambda_2| \ll 0$ at $\varepsilon = \varepsilon_2$. Now, by (3.17), it follows that one of the inequalities (3.21), (3.22) is fulfilled at $\varepsilon = \varepsilon_1$ and another at $\varepsilon = \varepsilon_2$. We see that for such chosen i and j the intersection of the curves $T_1W^u_i$ and W^s_j disappears when ε runs the interval between ε_1 and ε_2 (figure 10). Hence, the two curves must have a tangency at some ε from this interval, which is the required heteroclinic tangency between $W^u(O_1)$ and $W^s(O_2)$ (figure 10(*b*)). □

The following lemma deals with secondary heteroclinic or homoclinic tangencies of high orders. It is a version of lemma 2 from [9]. Since our formulation here is slightly different, we give a complete proof.

Lemma 5. *Let f_ε (where $\varepsilon = (\mu_0, \ldots, \mu_{n-1}, \nu)$) be a smooth $(n+1)$-parameter family of two-dimensional C^r-maps ($r > n+2$) which have saddle periodic points O_1, O_2, O_3 (not necessarily different) such that at $\mu = 0$ the manifolds $W^u(O_1)$ and $W^s(O_2)$ have a tangency of order n, and at $\nu = 0$ the manifolds $W^u(O_2)$ and $W^s(O_3)$ have a quadratic tangency (see figure 11(a)). Suppose that the tangency between $W^u(O_1)$ and $W^s(O_2)$ splits generically as μ varies, and the tangency between $W^u(O_2)$ and $W^s(O_3)$ splits generically as ν varies. Then there exists a sequence $\varepsilon_k \to 0$ such that the map f_ε has an orbit of tangency of order $(n+1)$ between $W^u(O_1)$ and $W^s(O_3)$ at $\varepsilon = \varepsilon_k$* (see figure 11 for an illustration).

Proof. Let (x, y) be the C^{r-1}-coordinates near O_2 for which identities (2.4), (2.5) hold for the local map T_0; hence formulae (2.6) hold for its iterations T_0^k. Let $M^+(x^+, 0)$ be a point at which $W^u(O_1)$ has the tangency of order n with $W^s_{\rm loc}(O_2)$ at $\mu = 0$ and $M^-(0, y^-)$ be a point at which $W^s(O_3)$ has the quadratic tangency with $W^u_{\rm loc}(O_2)$ at $\nu = 0$. As was explained in

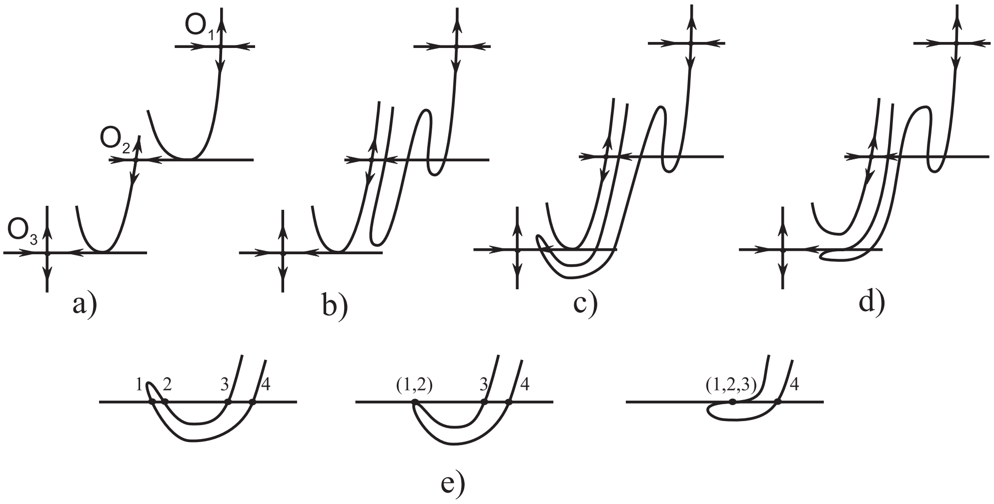

Figure 11. An even order tangency (d) obtained from two tangencies of odd orders (a). The way from (c) to (d), where three homoclinic intersections collide forming a second order (cubic) tangency, is shown in (e).

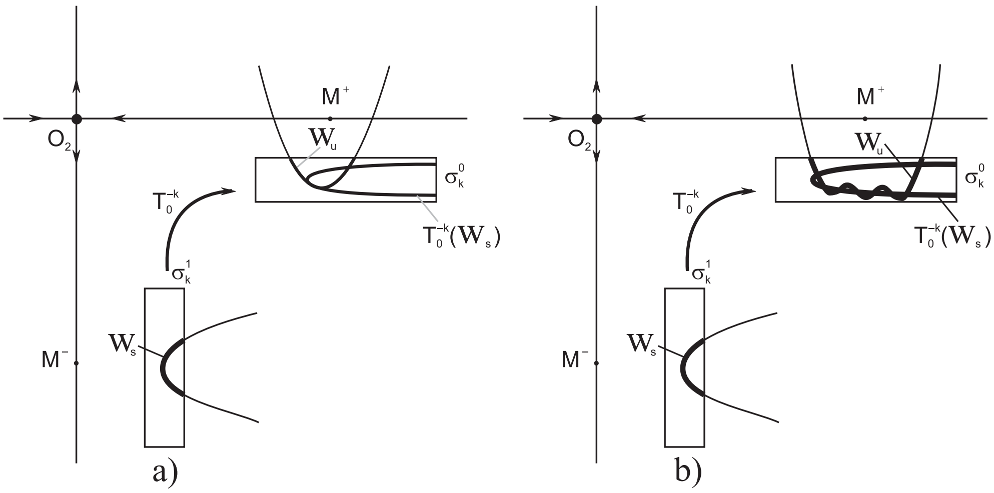

Figure 12. (a) A cubic tangency between w_u and $T_0^{-k}w_s$, where w_u is the piece of curve (3.23) in σ_k^0 and w_s is the piece of curve (3.24) in σ_k^1; (b) creating a tangency of order $(n+1)$ at the unfolding of the tangency of order n.

section 2.2, we may choose parameters μ in such a way (see (2.14)) that the equation of the piece of $W^u(O_1)$ near M^+ will be

$$y = \mu_0 + \cdots + \mu_{n-1}(x - x^+)^{n-1} + d(x - x^+)^{n+1} + o((x - x^+)^{n+1}). \tag{3.23}$$

The parameter ν can be chosen in such a way that the equation of the piece of $W^s(O_3)$ near M^- will be (see (2.15))

$$x = \nu + \hat{d}(y - y^-)^2 + o((y - y^-)^2). \tag{3.24}$$

By (2.6), the image of curve (3.24) by the map T_0^{-k} (see figure 12) has the sought tangency of order $(n + 1)$ with curve (3.23) if and only if the following curves have a tangency of order $(n+1)$:

$$\gamma^{-k}(Y+y^- + \eta_k(X+x^+, Y+y^-)) = \mu_0 + \cdots + \mu_{n-1}X^{n-1} + dX^{n+1} + o(X^{n+1}) \tag{3.25}$$

and

$$\lambda^k(X+x^+ + \xi_k(X+x^+, Y+y^-)) = \nu + \hat{d}Y^2 + o(Y^2) \tag{3.26}$$

(here $(x^+ + X)$ is the x-coordinate of the point of tangency and $(y^- + Y)$ is the y-coordinate of the image of the same point by the map T_0^k).

One can rewrite equations (3.25), (3.26) in the explicit form:

$$\gamma^{-k}Y = \bar{\mu}_0 + \cdots + \bar{\mu}_{n-1}X^{n-1} + \bar{\mu}_n X^n + dX^{n+1} + o(X^{n+1}) \tag{3.27}$$

and

$$\lambda^k X = \bar{\nu} + \bar{\nu}_1 Y + \hat{d}Y^2 + o(Y^2), \tag{3.28}$$

where

$$\bar{\mu}_0 = \mu_0 - \gamma^{-k}y^- + o(\gamma^{-k}), \qquad \bar{\mu}_n = o(\gamma^{-k}),$$
$$\bar{\mu}_j = \mu_j + o(\gamma^{-k}) \qquad (j = 1, \ldots, n-1),$$

and

$$\bar{\nu} = \nu - \lambda^k x^+ + o(\lambda^k), \qquad \bar{\nu}_1 = o(\lambda^k).$$

After the change in variables $X^{\text{new}} = X - (\bar{\nu}_1/2\hat{d})$, $Y^{\text{new}} = Y - (\bar{\mu}_n/(n+1)d)$, equations (3.27), (3.28) are recast as

$$\gamma^{-k}Y = \mu'_0 + \cdots + \mu'_{n-1}X^{n-1} + dX^{n+1} + o(X^{n+1}) \tag{3.29}$$

and

$$\lambda^k X = \nu' + \hat{d}Y^2 + o(Y^2), \tag{3.30}$$

where

$$\begin{aligned} &\mu'_0 = \mu_0 - \gamma^{-k}y^- + o(\gamma^{-k}), \qquad \nu' = \nu - \lambda^k x^+ + o(\lambda^k), \\ &\mu'_j = \mu_j + o(\gamma^{-k}) \qquad (j = 1, \ldots, n-1). \end{aligned} \tag{3.31}$$

Let us rescale the variables:

$$X = -\frac{1}{(\hat{d}d^2)^{1/(2n+1)}}\lambda^{k/(2n+1)}\gamma^{-(2k/2n+1)}X^{\text{new}},$$

$$Y = (-1)^{n+1}\frac{1}{(\hat{d}^{n+1}d)^{1/(2n+1)}}\lambda^{k((n+1)/(2n+1))}\gamma^{-k/(2n+1)}Y^{\text{new}}.$$

Equations (3.29), (3.30) change to

$$Y = M_0 + \cdots + M_{n-1}X^{n-1} + X^{n+1} + o(1)_{k\to+\infty} \tag{3.32}$$

and

$$X = N - Y^2 + o(1)_{k\to+\infty} \tag{3.33}$$

where

$$\begin{aligned} M_j &= \mu'_j \frac{(-1)^{n+1-j}}{d}(\hat{d}d^2)^{(n+1-j)/(2n+1)}\lambda^{-(k(n+1-j)/(2n+1)}\gamma^{(2k(n+1-j))/(2n+1)} \\ &\qquad (j = 0, \ldots, n-1), \\ N &= \nu'(\hat{d}d^2)^{1/(2n+1)}\lambda^{-k((2n+2)/(2n+1))}\gamma^{2k/(2n+1)}. \end{aligned} \tag{3.34}$$

It is proven in [9] (also see similar results in [8,31]) that for all sufficiently large k there are uniquely defined, uniformly (with respect to k) bounded values of $N, M_0, \ldots, M_{n-1}$ for which curves (3.32) and (3.33) have a tangency of order $(n+1)$. The proof is quite straightforward. Indeed, equation (3.33) can be rewritten as

$$Y = \pm\sqrt{N - X + o(1)}. \tag{3.35}$$

We obtain the sought tangency of order $(n+1)$ if and only if the first $(n+2)$ terms of the Taylor expansion of (3.32) coincide at some X^* with the corresponding terms of the Taylor expansion of one of the two branches of (3.35). Given any N and X^*, we can always make the initial segments of length n of the two expansions coincide by an appropriate choice of $M_0, \ldots, M_{n-1}$. The coincidence conditions for the two remaining terms read as

$$\begin{aligned} (n+1)X^* &= \pm\frac{\sigma_n}{n!}(N - X^*)^{\frac{1}{2}-n} + o(1)_{k\to+\infty}, \\ 1 &= \pm\frac{\sigma_{n+1}}{(n+1)!}(N - X^*)^{-\frac{1}{2}-n} + o(1)_{k\to+\infty}, \end{aligned} \tag{3.36}$$

where $\sigma_n = \frac{-1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-3}{2} < 0$. Equalities (3.36) show that there can be no tangency of order $(n+1)$ with the 'plus' branch of (3.35), while for the 'minus' branch there exist uniquely defined values of N and X^* corresponding to the sought tangency:

$$N = X^*\left(n + \frac{1}{2}\right) + o(1)_{k\to+\infty}, \qquad X^* = \frac{2}{2n-1}\left(\frac{|\sigma_{n+1}|}{(n+1)!}\right)^{2/(2n+1)} + o(1)_{k\to+\infty}.$$

The found tangency of order $(n+1)$ between curves (3.32) and (3.33) corresponds to the tangency of order $(n+1)$ between curves (3.25) and (3.26). Since the corresponding values of $N, M_0, \ldots, M_{n-1}$ are uniformly bounded for all large k, the respective values of $\varepsilon = (\nu, \mu_0, \ldots, \mu_{n-1})$ tend to zero as $k \to +\infty$ (see (3.31), (3.34)). $\square$

4. Homoclinic tangencies of arbitrarily high orders. Proof of theorems 1, 2

In this section we prove theorems 1 and 2. As we have already mentioned, theorem 3 follows from theorem 1 immediately. Let f be a two-dimensional C^r-map ($r = 2, \ldots, \infty, \omega$) having a saddle periodic point O and an orbit of a quadratic homoclinic tangency of $W^u(O)$ and $W^s(O)$. By lemma 1, we may include f into a one-parameter family f_μ of C^r-maps (all area-preserving if f itself is area-preserving) for which the homoclinic tangency splits generically. Then, by lemma 2, arbitrarily close to f in this family we find a map $\tilde{f} = f_{\tilde{\mu}}$ with a non-trivial transitive hyperbolic set Λ which includes the point O, and some leaf of the unstable set of Λ has a quadratic tangency (which splits generically as μ varies) with some leaf of the stable set of Λ.

In another setting (as in theorem 2) we may assume the existence of such a set Λ from the very beginning. In any case, as lemma 3 gives it to us, we may achieve by an additional arbitrarily small perturbation that the map $\tilde{f}$ will have a quadratic heteroclinic tangency between stable and unstable manifolds of two periodic points O_1 and O_2 in Λ, and the orbit of this tangency will be a part of a heteroclinic cycle of third class.

This tangency is split generically as the parameter μ varies. It follows that for any family of maps which approximates f_μ sufficiently closely (at least in the C^2-topology) there will exist a value of the parameter corresponding to a quadratic heteroclinic tangency between $W^s(O_1)$ and $W^u(O_2)$. Hence, even if our original map f is not analytic, we may choose a sufficiently close approximation to f_μ which will be real-analytic in a small neighbourhood U of the heteroclinic cycle (this neighbourhood is a finite collection of disjoint discs; recall that if f_μ is a family of area-preserving maps of a disc into R^2, it can always be approximated by a family of real-analytic and even polynomial [38] area-preserving maps). Then we find near f a map $\tilde{f}$ (area-preserving if f is area-preserving) which has an orbit of a quadratic tangency between $W^s(O_1)$ and $W^u(O_2)$ and which is real-analytic in U. Thus, from now on, all the maps we obtain as small perturbations of f will be real-analytic in a neighbourhood of the heteroclinic cycle under consideration. Moreover, all the maps will be area-preserving if f is (all the perturbations below will be of the type given by lemma 1).

Let $n_1, n_2, \ldots$ be an arbitrary infinite sequence of positive integers, and (M_{11}, M_{12}), (M_{21}, M_{22}), (M_{31}, M_{32}), $\ldots$ be an arbitrary sequence of pairs of periodic points from the hyperbolic set Λ. We will prove that $\tilde{f}$ can be perturbed in such a way that the perturbed map will have an infinite sequence of homoclinic/heteroclinic tangencies, and these will be exactly the tangencies of orders n_k between $W^u(M_{k1})$ and $W^s(M_{k2})$, $k = 1, 2, \ldots$. The perturbation which we construct can be as small as we need, and it does not lead out of the class of area-preserving maps if the original map f is area-preserving. This will, obviously, give us a proof of both theorems 2 and 1 (by letting the numbers n_k take all natural values infinitely many times, and by choosing all M_{kj} equal to the same periodic point O we will have infinitely many homoclinic tangencies of every order).

Take an arbitrarily small $\delta > 0$, let $\delta_k > 0$ be such that $\delta_1 + \delta_2 + \cdots = \delta$. We will construct a sequence of maps f_k, $f_0 \equiv \tilde{f}$, such that each of them retains the tangency between $W^s(O_1)$ and $W^u(O_2)$, and f_k has k additional orbits of tangency: between $W^u(M_{11})$ and $W^s(M_{12})$ of order n_1, between $W^u(M_{21})$ and $W^s(M_{22})$ of order n_2, etc. We will construct maps f_k in such a way that the distance between f_{k+1} and f_k will be less than δ_k. The sequence f_k will have a limit f^* which lies at a distance less than δ from $\tilde{f}$ and has the required infinite sequence of tangencies.

Thus, in order to prove both theorems 1 and 2, we need to prove that given a map f_k which has the heteroclinic cycle of the third class and $k \geqslant 0$ additional orbits of homoclinic/heteroclinic tangencies, one can perturb it, destroying neither the heteroclinic tangency between $W^u(O_2)$ and $W^s(O_1)$ nor the k additional tangencies, nor changing the order of these tangencies, such that the new map f_{k+1} will have one more orbit of tangency, between $W^u(M_{k+1,1})$ and $W^s(M_{k+1,2})$, of the given order n_{k+1}. We will construct such a perturbation from f_k to f_{k+1} as a finite sequence of perturbations each of which can be made arbitrarily small, so the total size of the resulting perturbation will be less than δ_{k+1}, as required.

By lemma 1, we can include f_k into a one-parameter family $f_k(\varepsilon)$ such that neither of the given heteroclinic and homoclinic tangencies splits as ε varies nor the multipliers of the point O_1 change, while the multiplier λ_2 of O_2 changes with non-zero velocity. It follows that the Ω-modulus α is changing with non-zero velocity as well. Hence, by lemma 4, arbitrarily close to f_k we find in this family a map which has an additional quadratic heteroclinic tangency between $W^u(O_1)$ and $W^s(O_2)$. When ε changes, this tangency splits generically. Recall that the points O_1, O_2, $M_{k+1,1}$ and $M_{k+1,2}$ belong to the same transitive hyperbolic set Λ; therefore $W^u(M_{k+1,1})$ accumulates onto $W^u(O_1)$ and $W^s(M_{k+1,1})$ accumulates onto $W^s(O_2)$. Thus, by an additional arbitrarily small change in ε, when splitting the tangency between $W^u(O_1)$ and $W^s(O_2)$ we can obtain a new quadratic homoclinic tangency between $W^u(M_{k+1,1})$ and $W^s(M_{k+1,1})$.

If $n_{k+1} > 1$, we repeat the procedure n_{k+1} times, obtaining each time a new quadratic homoclinic tangency between $W^u(M_{k+1,1})$ and $W^s(M_{k+1,1})$, without perturbing the other tangencies. Next, we include the map into a two-parameter family of maps for which two of the newly obtained quadratic homoclinic tangencies split generically and independently, while all the other tangencies are kept in place. By lemma 5, a cubic homoclinic tangency can be obtained by an arbitrarily small variation of parameters within such a family. If $n_{k+1} > 2$, we include the map into a three-parameter family for which the cubic tangency and one of the remaining quadratic homoclinic tangencies split generically and independently, while all the other tangencies remain unperturbed. By lemma 5 again, by an arbitrarily small variation of the parameters, a quartic tangency is obtained, etc. Thus, repeating the procedure, after a finite number of arbitrarily small consecutive perturbations we obtain a new homoclinic tangency of order n_{k+1} between $W^u(M_{k+1,1})$ and $W^s(M_{k+1,1})$, in addition to the heteroclinic tangency between $W^u(O_2)$ and $W^s(O_1)$ and to the k tangencies between $W^u(M_{11})$ and $W^s(M_{12})$, $W^u(M_{21})$ and $W^s(M_{22})$,..., $W^u(M_{k1})$ and $W^s(M_{k2})$ which the map f_k already had. If $M_{k+1,2} = M_{k+1,1}$, it means that we have found the tangency we sought, and the map f_{k+1} is constructed. If $M_{k+1,2} \neq M_{k+1,1}$, we note that $M_{k+1,1}$ and $M_{k+1,2}$ belong to the same transitive hyperbolic set Λ; therefore $W^s(M_{k+1,2})$ accumulates onto $W^s(M_{k+1,1})$. Hence, by an arbitrarily small perturbation which splits the homoclinic tangency between $W^u(M_{k+1,1})$ and $W^s(M_{k+1,1})$ generically (and does not split the other tangencies) we obtain the sought tangency between $W^u(M_{k+1,1})$ and $W^s(M_{k+1,2})$. □

5. Richness of dynamics of area-preserving maps from the Newhouse regions

5.1. A scaling lemma

Let f be a two-dimensional C^r-map ($r = 3, \dots, \infty, \omega$) that has a saddle periodic point O whose invariant manifolds have a homoclinic tangency of order $n \leqslant r - 1$. Let $M^-(0, y^-)$ be a corresponding homoclinic point in the local unstable manifold $W^u_{\mathrm{loc}}(O)$ and $M^+(x^+, 0) = T_1 M^-$ be a homoclinic point in the local stable manifold $W^s_{\mathrm{loc}}(O)$. Consider two sufficiently small rectangular neighbourhoods $\Pi^+ : \{|x - x^+| < \delta, |y| < \delta\}$ and $\Pi^- : \{|y - y^-| < \delta, |x| < \delta\}$ of M^+ and M^-. By (2.6), (2.7), the piece of $W^s_{\mathrm{loc}}(O)$ near M^+ is the limit of the countable sequence of strips $\sigma_k = \Pi^+ \cap T_0^{-k} \Pi^- = \{|\gamma^k y - y^-| < \delta + o(1)_{k\to+\infty}\}$ on which the first-return maps $\tilde{T}_k \equiv T_1 T_0^k$ are defined, where T_0 and T_1 are the local and global maps.

The map $\tilde{T}_k$ takes the strip $\sigma_k \subset \Pi^+$ back into Π^+. The same holds true for every map close to f. Let us include f into an n-parameter family f_ε of C^r-maps for which the homoclinic tangency splits generically. Let the parameters $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ run a ball of a fixed, arbitrarily small size. The following lemma is a version of lemma 2 from [6] (see similar statements in [35, 56, 57]).

Lemma 6. *For $|\lambda\gamma| \leqslant 1$ there exist a C^{r-1}-smooth coordinate transformation on σ_k: $(x, y) \mapsto (X, Y)$ with a constant Jacobian and a C^{r-n-1}-smooth transformation of the parameters $\varepsilon \mapsto (E_0, \dots, E_{n-1})$ which bring the first-return map $\tilde{T}_k$ to the form*

$$\begin{aligned} \bar{X} &= Y + o(1)_{k\to+\infty}, \\ \bar{Y} &= -J(\lambda\gamma)^k X + E_0 + E_1 Y + \cdots + E_{n-1} Y^{n-1} + d\ Y^{n+1} + o(1)_{k\to+\infty}, \end{aligned} \tag{5.1}$$

where the range of values of the new variables (X, Y) and parameters $E_0, \dots, E_{n-1}$ covers a centred-at-zero ball whose radius tends to infinity as k increases. The constant d is the coefficient d of the global map T_1 in (2.13), the constant J is the Jacobian of T_1 at the point M^+ at $\varepsilon = 0$ ($J = bc$ in terms of (2.8)) and λ and γ are the multipliers of O. The $o(1)$-terms tend to zero uniformly on any compact, along with the derivatives with respect to (X, Y) up to the order $(r - 1)$ and with respect to E up to the order $(r - n - 1)$ (if $r = \infty$—along with any finite number of derivatives, if $r = \omega$—uniformly on some complex neighbourhood of any compact in the (X, Y) plane, along with any finite number of derivatives with respect to E).

Note that in the area-preserving case $|J| = 1$ and $|\lambda\gamma| = 1$, so the coefficient of X in the second equation of (5.1) is ± 1 in this case. If, in addition, f is an orientation-preserving map (or, at least, the local and global maps T_0 and T_1 are), then $J = 1$ and $\lambda\gamma = 1$, so (5.1) takes the form

$$\begin{aligned}\bar{X} &= Y + o(1)_{k\to+\infty},\\ \bar{Y} &= -X + E_0 + E_1 Y + \cdots + E_{n-1}Y^{n-1} + d\ Y^{n+1} + o(1)_{k\to+\infty}.\end{aligned} \tag{5.2}$$

The same form is taken by the rescaled first-return map (5.1) when the local map T_0 is orientation-reversing, provided the parity of k is chosen appropriately.

Proof of the lemma. First, we introduce a new coordinate y on σ_k such that

$$y = \gamma^{-k} y_{\text{new}} + \gamma^{-k}\eta_k(x, y_{\text{new}}) \tag{5.3}$$

(i.e. y_{new} is equal to $y^{(k)}$ from (2.6)). Then, by virtue of (2.6), (2.8), (2.13), the map $\tilde{T}_k : (x, y) \mapsto (\bar{x}, \bar{y})$ will satisfy the following relations:

$$\begin{aligned}&\bar{x} - x^+ = a\lambda^k x + b(y - y^-) + o(y - y^-) + o(\lambda^k),\\ &\gamma^{-k}\bar{y} + \gamma^{-k}\eta_k(\bar{x}, \bar{y}, \varepsilon) = c\lambda^k x + \mu_0 + \cdots + \mu_{n-1}(y - y^-)^{n-1} + d(y - y^-)^{n+1}\\ &\qquad + o((y - y^-)^{n+1})_{y\to y^-, \varepsilon\to 0} + O(\lambda^k x(y - y^-)) + o(\lambda^k),\end{aligned} \tag{5.4}$$

where $d \neq 0$ is a constant the coefficients $\lambda, \gamma, a, b, c, x^+, y^-, \mu$ are functions of ε, at least C^{r-n-1}-smooth[6].

It is easy to see that one can shift the origin:

$$x_{\text{new}} = x - x^+ + \alpha_k, \qquad y_{\text{new}} = y - y^- + \beta_k, \tag{5.5}$$

where α_k and β_k are certain constants $O(\lambda^k)$, in such a way that $\bar{x}_{\text{new}}$ will be zero at $(x_{\text{new}}, y_{\text{new}}) = 0$, and on the right-hand side of the second equation there will be no y^n_{new}-term. Thus, after transformation (5.5) map (5.4) will take the form

$$\begin{aligned}&\bar{x} = by + o(\|x, y\|),\\ &\gamma^{-k}\bar{y} + o(\gamma^{-k}\|\bar{x}, \bar{y}\|) = c\lambda^k x + \tilde{\mu}_0 + \cdots + \tilde{\mu}_{n-1}y^{n-1} + dy^{n+1}\\ &\qquad + o(y^{n+1}) + o(\lambda^k\|x, y\|) + O(\lambda^k xy),\end{aligned} \tag{5.6}$$

where the modified values of the splitting parameters are $\tilde{\mu}_j = \mu_j + O(\lambda^k)$ $(j = 1, \ldots, n-1)$ and $\tilde{\mu}_0 = \mu_0 + c\lambda^k x^+ - \gamma^{-k}y^- + o(\gamma^{-k})$ (we collected all constant terms in the second equation of (5.6) into $\tilde{\mu}_0$, i.e. the left-hand side vanishes identically at $(\bar{x}, \bar{y}) = 0$, and the value of the right-hand side at $(x, y) = 0$ is equal to $\tilde{\mu}_0$).

Now, if we normalize the coordinates and the parameters in the following way:

$$\begin{aligned}&x = b\gamma^{-k/n}X, \qquad y = \gamma^{-k/n}Y,\\ &\tilde{\mu}_j = \gamma^{-k(1-(j-1)/n)}E_j \qquad (j = 0, \ldots, n-1),\end{aligned} \tag{5.7}$$

then map (5.6) is reduced to the desired form (5.1).

The coordinate transformation which we used to transform the map $\tilde{T}_k$ to form (5.1) is the composition of transformations (5.3), (5.5), (5.7). Its Jacobian equals to

$$b\gamma^{2k/n}\frac{\gamma^k}{1 + \eta'_y(x^+ + \alpha_k + b\gamma^{-k/n}X, y^- + \beta_k + \gamma^{-k/n}Y)}.$$

It is clear that if we perform the following additional coordinate transformation

$$Y_{\text{new}} = Y + \gamma^{k/n}(\eta(x^+ + \alpha_k + b\gamma^{-k/n}X, y^- + \beta_k + \gamma^{-k/n}Y) - \eta(x^+ + \alpha_k, y^- + \beta_k)),$$

[6] We lose one smoothness when we bring the map near O to the form (2.4), (2.5), ensuring estimates (2.7) for the map T_0^k; we further lose n derivatives with respect to ε when defining $y^-(\varepsilon)$ and the splitting parameters $\mu(\varepsilon)$ (see (2.12)).

then the Jacobian of the resulting transformation will become a constant $(b\gamma^{k(1+2/n)})$, while the map (5.1) will retain its form in the new coordinates. $\square$

Remark. Note that if we slightly 'overscale' the variables (x, y), i.e. if we scale (x, y) in (5.7) to a factor which, as $k \to +\infty$, tends to zero a bit faster than $\gamma^{-k/n}$, then the term $d\ y^{n+1}$ in (5.1) will disappear, i.e. the first-return map will take the form

$$\begin{aligned}\bar{X} &= Y + o(1)_{k\to+\infty},\\ \bar{Y} &= -J(\lambda\gamma)^k X + \hat{E}_0 + \hat{E}_1 Y + \cdots + \hat{E}_{n-1}Y^{n-1} + o(1)_{k\to+\infty},\end{aligned} \tag{5.8}$$

where the range of values of the new rescaled variables (X, Y) and parameters $\hat{E}_0, \ldots, \hat{E}_{n-1}$ will still cover all finite values with the increase in k. In the area-preserving case the coefficient of X in the second equation of (5.1) is ± 1. If, in addition, the local and global maps T_0 and T_1 are orientation-preserving or the local map T_0 is orientation-reversing and the parity of k is chosen appropriately, then the coefficient of X is equal to (-1).

5.2. Degenerate elliptic orbits near homoclinic tangencies

From now on, we will focus on area-preserving maps only. According to lemma 6, any bifurcation occurring generically in symplectic maps (5.2) in a finite region of the values of (X, Y) must happen at the unfolding of every homoclinic tangency of order n in the class of area-preserving maps (provided the corresponding local and global maps are both orientation-preserving or the local map is orientation-reversing).

For example, a symplectic map of type (5.2) has an elliptic fixed point with the multipliers $e^{\pm i\varphi}$ at the point $(X, Y) = 0$ when

$$E_0 = 0 + \cdots, \qquad E_1 = 2\cos\varphi + \cdots, \tag{5.9}$$

where the dots stand for terms which tend to zero as $k \to +\infty$. If $\varphi/2\pi$ is irrational, then, given any $m \leqslant (r-1)/2$, the map in a neighbourhood of the fixed point can be written in the following complex normal form (z^* is complex conjugate to z)

$$\bar{z} = e^{i\varphi} z(1 + B_1 zz^* + \cdots + B_m(zz^*)^m) + o(|z|^{2m+1}). \tag{5.10}$$

This is done in the following way. First, make the linear change in coordinates

$$X = \xi, \qquad Y = \cos\varphi\cdot\xi - \sin\varphi\cdot\eta.$$

Map (5.2) is reduced to the form

$$\begin{aligned}\bar{\xi} &= \cos\varphi\cdot\xi - \sin\varphi\cdot\eta + o(1)_{k\to+\infty},\\ \bar{\eta} &= \sin\varphi\cdot\xi + \cos\varphi\cdot\eta - \frac{1}{\sin\varphi}R(Y) + o(1)_{k\to+\infty},\end{aligned} \tag{5.11}$$

where $R(Y) = E_2Y^2 + \cdots + E_{n-1}Y^{n-1} + d\ Y^{n+1}$. In the complex variables $z = \xi + i\eta$, $z^* = \xi - i\eta$ map (5.11) takes the following form

$$\bar{z} = ze^{i\varphi} - \frac{i}{\sin\varphi}R(\mathrm{Re}(ze^{i\varphi})) + o(1)_{k\to+\infty}. \tag{5.12}$$

Next, one can kill all quadratic monomials by means of a quadratic coordinate transformation whose coefficients depend only on φ and on the coefficients of the quadratic terms in (5.12), i.e. the coefficients of the transformation are equal to E_2 times some rational function of $\sin\varphi$ and $\cos\varphi$, plus certain $o(1)_{k\to+\infty}$ terms. After this transformation, the new values of the coefficients of the cubic terms will be equal to the old ones plus E_2 times rational functions of

$\sin\varphi$, $\cos\varphi$ plus $o(1)_{k\to+\infty}$ terms. Then, a cubic coordinate transformation kills all the cubic terms, except for the resonant one, $e^{i\varphi}B_1z^2z^*$. Note that its coefficient does not change at this transformation nor does it change at further normalizing transformations. Thus, the coefficient B_1 in the normal form (5.10) is given by the formula

$$B_1 = -\frac{3i}{8\sin\varphi}E_3 + b_1(E_2, \sin\varphi, \cos\varphi) + o(1)_{k\to+\infty},$$

where the function b_1 depends only on E_2 and φ. Similarly, one consecutively kills all the terms of even orders and all the non-resonant terms of odd orders. The coefficient B_j in the normal form (5.10) is equal to the coefficient of the term $z^{j+1}(z^*)^j$, as it appears after the map is brought to the normal form up to the order $2j$. The corresponding normalizing transformation is a polynomial of order $2j$, whose coefficients depend only on φ and on the coefficients of the terms of orders not greater than $2j$ in the original map (5.12). It follows that B_j may differ from the coefficient of $z^{j+1}(z^*)^j$ in (5.12) only on a function of φ and of the coefficients of the terms of orders lower than $(2j+1)$. Hence, if $j < (n-1)/2$, then

$$B_j = -i\frac{C^j_{2j+1}}{2^{2j+1}\sin\varphi}E_{2j+1} + b_j(E_1, E_2, E_3, \ldots, E_{2j}) + o(1)_{k\to+\infty}, \tag{5.13}$$

where b_j are certain smooth functions, irrelevant for our purposes. It follows that we can get all the Birkhoff coefficients B_j with $j < (n-1)/2$ vanished in (5.10) by an appropriate choice of $E_2, \ldots, E_{n-1}$. In other words, we have established that map (5.2) has an elliptic fixed point of the degeneracy order $[(n-2)/2]$ at certain values of the coefficients $E_0, \ldots, E_{n-1}$. Since map (5.2) is the first-return map near a homoclinic tangency of order n (lemma 6), we obtain the following result.

Lemma 7. *For every family f_ε of area-preserving C^r-maps for which a homoclinic tangency of order $n < r$ is split generically and either both the maps T_0 and T_1 for the given tangency are orientation-preserving or the map T_0 is orientation-reversing, arbitrarily close to the moment of tangency there are values of parameters that correspond to the existence of an elliptic periodic orbit of the degeneracy order* $[(n-2)/2]$.

In fact, the requirement on T_1 to be orientation-preserving is not crucial. Indeed, we have the following lemma.

Lemma 8. *Let f_ε be a family of area-preserving C^r-maps for which a homoclinic tangency of order $n < r$ is split generically. Then arbitrarily close to the moment of tangency there are values of parameters that correspond to the existence of an elliptic periodic orbit of the degeneracy order* $[(n-3)/2]$.

Proof. If T_0 is orientation-reversing, or both T_0 and T_1 are orientation-preserving, the lemma is contained in lemma 7. In the remaining case, where T_0 is orientation-preserving and T_1 is orientation-reversing, we note that since the tangency of order $n > 1$ is split generically, one can always find a parameter value for which there exist two homoclinic points, M_1 and M_2, which correspond, respectively, to a homoclinic tangency of order $(n-1)$ and to a transverse homoclinic intersection (figure 13). Take a small piece w of the unstable manifold $T_1W^u_{loc}$ through the point M_2. It is transverse to W^s_{loc}; therefore its images $(T_0)^kw$ converge, in the C^r-topology, to W^u_{loc} by virtue of the lambda-lemma. It follows that the piece of the unstable manifold $T_1W^u_{loc}$ through the point M_1 is accumulated by the curves $T_1(T_0)^kw$ as $k\to+\infty$. Hence, as the tangency of order $(n-1)$ between $T_1W^u_{loc}$ and W^s_{loc} at the point M_1 unfolds, a tangency of order $(n-1)$ between $T_1(T_0)^kw$ and W^s_{loc} appears for some sufficiently large k, see figure 13. It is a tangency between the curves $T_1(T_0)^kT_1W^u_{loc}$ and W^s_{loc}, so the global map for

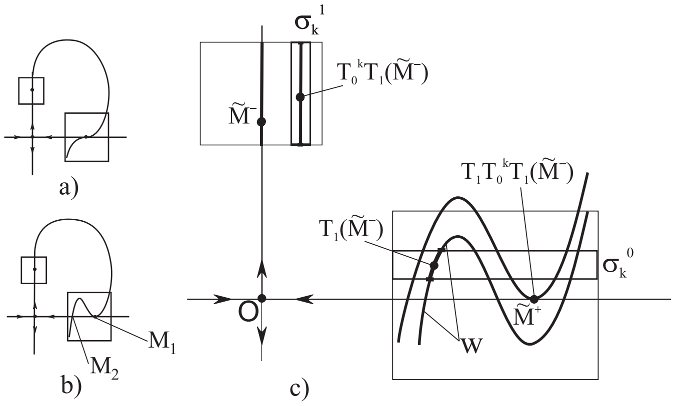

Figure 13. Creation of a homoclinic tangency of order $(n-1)$ whose global map is orientation-preserving: (*a*) the initial tangency of order $n > 1$; (*b*) the initial tangency splits into two homoclinic points, M_1 and M_2, that correspond, respectively, to a tangency of order $(n-1)$ and to a transverse intersection; (*c*) as the tangency of order $(n-1)$ unfolds, secondary homoclinic points $\tilde{M}^- \in W^u_{loc}$ and $\tilde{M}^+ \in W^s_{loc}$ are created such that $\tilde{M}^+ = T_1 T_0^k T_1(\tilde{M}^-)$ and the piece $T_1 T_0^k w$ of $W^u(O)$ has a tangency of order $(n-1)$ with W^s_{loc} at the point $\tilde{M}^+$.

the newly obtained homoclinic tangency is $T_1(T_0)^k T_1$. It is, obviously, orientation-preserving, so, according to lemma 7, an elliptic periodic orbit of the degeneracy order $[((n-1)-2)/2]$ is born as this tangency unfolds. □

5.3. *Proof of theorems 4 and 5*

We can now prove theorems 4 and 5.

Proof of theorem 4. We use the same construction as in the proof of theorem 3 (see section 4). By definition, maps with quadratic homoclinic tangencies are dense in the Newhouse regions. We pick any such map and perturb it in order to obtain some special heteroclinic cycle. Then, by a perturbation which does not destroy this cycle, we are able to obtain a homoclinic tangency of any arbitrarily high given order. Now, by lemma 8, an elliptic orbit of a high order of degeneracy is born when this tangency is split generically. By lemma 1, this can be done without destroying our heteroclinic cycle.

Repeating the arguments, we can obtain one more homoclinic tangency of high order near the heteroclinic cycle. When we do this (see section 4), we apply a sequence of perturbations, each belonging to a certain finite-parameter family the only requirement on which is that a certain homoclinic or heteroclinic tangency is split generically, or a multiplier of a certain saddle periodic orbit changes with a non-zero velocity as a parameter varies. As we showed in section 2.3 (see comments after lemma 1), these requirements can be satisfied by perturbations which do not change the values of the Birkhoff coefficients of a given elliptic orbit. Thus, we can obtain the new homoclinic tangency of a high order and, moreover, split it in such a way that a new elliptic orbit of a high order of degeneracy will be born, without destroying the degenerate elliptic orbit obtained before. This procedure can be repeated again and again, each time adding a new elliptic orbit of a given arbitrarily high order of degeneracy, without affecting the degenerate elliptic orbits obtained in the previous steps. At the end we have the required infinite sequence of coexisting degenerate elliptic orbits. □

Proof of theorem 5. In the proof of theorem 4 we described the procedure which, starting from any map from the Newhouse regions, produces, by arbitrarily small perturbations, arbitrarily degenerate homoclinic tangencies and elliptic periodic orbits step by step, without destroying the homoclinic tangencies or elliptic orbits obtained in the previous steps. Thus, given any finite r, arbitrarily close to any map from the Newhouse regions in space of C^r-smooth area-preserving maps, one can find a map having both an infinite sequence of homoclinic tangencies of order r and an infinite sequence of elliptic periodic orbits of degeneracy order $[(r-1)/2]$. Each of these homoclinic tangencies can be transformed to a small homoclinic band by a perturbation which is localized in an arbitrarily small given neighbourhood of one point of the homoclinic orbit under consideration and which is arbitrarily small in the C^r-topology (see section 2.3). The first-return map near each of the degenerate elliptic periodic orbits can be perturbed to become, locally, a linear rotation in appropriate coordinates. Indeed, when the first-return map near the degenerate elliptic orbit is brought to the normal form (1.1), it reads

$$\bar{z} = e^{i\varphi}z + o(|z|^r),$$

which means that this map is close in the C^r-topology to the linear rotation

$$\bar{z} = e^{i\varphi}z$$

in a small neighbourhood of $z = 0$. By an additional, arbitrarily small change φ, one can make $\varphi/(2\pi)$ rational, thus turning a neighbourhood of $z = 0$ into a periodic spot. Moreover, such perturbations can be localized in an arbitrary small neighbourhood of the degenerate periodic orbit under consideration. Since each of the perturbations which make homoclinic bands from degenerate homoclinic tangencies and periodic spots from degenerate elliptic periodic orbits can be localized in such a way that they do not affect the other homoclinic and periodic orbits from our sequence the sum of these perturbations will provide an infinite set of coexisting homoclinic bands and periodic spots. □

5.4. Proof of theorems 6 and 7

The proof of theorems 6 and 7 is based on the following result.

Lemma 9. *Given any integer $k \geqslant 0$, and given any k-parameter family f_ε of area-preserving, orientation-preserving C^r-diffeomorphisms of a unit disc into R^2, and given any $\delta > 0$ there exists a real η and a C^r-family of polynomials g_ε such that a certain iteration of the Hénon-like map*

$$\bar{x} = y + \eta, \qquad \bar{y} = -x + g_\varepsilon(y) \tag{5.14}$$

is δ-close to f_ε on the unit disc.

The statement is proven in [38] (see theorem 1 there) for the smooth case ($1 \leqslant r \leqslant \infty$). To include the real-analytic case ($r = \omega$), we need a minor modification of the proof in [38]; therefore, we repeat the corresponding argument.

Proof of the lemma. It is shown in [38] (see theorem 2 there) that there exists a finite sequence of polynomials $g_{\varepsilon 1}, \ldots, g_{\varepsilon n}$ such that the given family f_ε is approximated on the unit disc by the composition of the maps

$$G_i : (x, y) \mapsto (y, -x + g_{\varepsilon i}(y)), \qquad i = 1, \ldots, n, \tag{5.15}$$

with the accuracy $\frac{1}{2}\delta$ in the C^r-topology (this is proven for any r, including $r = \omega$).

Let (x_0, y_0) be an arbitrary point in the closed unit disc U and let $(x_i, y_i) = G_i \circ \cdots \circ G_1(x_0, y_0)$. Note that $x_i = y_{i-1}$ at $i \geqslant 1$, according to (5.15). Let R be a real such that the image $G_i \circ \cdots \circ G_1(U)$ lies, for all ε, strictly inside the disc of radius R around the point (x_i, y_i), for every $i = 1, \dots, n$. Choose some η and take a sequence of points (x_i^*, y_i^*) $(i = 0, \dots, n)$ such that $(x_0^*, y_0^*) = (x_0, y_0)$, $(x_n^*, y_n^*) = (x_n, y_n)$ and $x_i^* = y_{i-1}^* + \eta$ for $i \geqslant 1$. If we take $\eta > 2R + |x_n - y_0|$, then we can always choose y_i^* such that

$$\|y_i^* - y_j^*\| > 2R \tag{5.16}$$

for all $0 \leqslant i < j \leqslant n - 1$ (our assumptions fix $y_{n-1}^* = x_n - \eta$, so we must have η sufficiently large in order to push y_{n-1}^* on a distance larger than $2R$ from $y_0^* = y_0$). Now, choose a polynomial g_ε such that $g_\varepsilon(y)$ will be sufficiently close to

$$g_{\varepsilon i}(y - y_{i-1}^* + y_{i-1}) + x_{i-1}^* - x_{i-1} + y_i^* - y_i \tag{5.17}$$

uniformly on the square $Y_i : \{|\mathrm{Re}\ y - y_{i-1}^*| \leqslant R,\ |\mathrm{Im}\ y| \leqslant 1\}$ in the complex plane, for each $i = 1, \dots, n$ (the functions $g_{\varepsilon i}$ are defined by (5.15)). Due to our choice of y_i^* (see (5.16)), the squares corresponding to different i do not intersect, so the sought polynomial g_ε that gives a good approximation to (5.17) exists indeed, due to the classical Runge theorem.

With the chosen g_ε, define the map G_* by formula (5.14). By construction (recall that $x_i = y_{i-1}$ and $x_i^* = y_{i-1}^* + \eta$), the map G_* is close at $y \in Y_i$ $(i = 1, \dots, n)$ to the map $I_i^{-1} \circ G_i \circ I_{i-1}$, where I_i is the parallel translation $(x, y) \mapsto (x - x_i^* + x_i, y - y_i^* + y_i)$ that takes the region $|\mathrm{Re}\ y - y_i^*| \leqslant R$ onto the region $|\mathrm{Re}\ y - y_i| \leqslant R$; at $i = 0$ this is just the identity map since $(x_0^*, y_0^*) = (x_0, y_0)$. Since every image $G_i \circ \cdots \circ G_1(U)$ lies in the set $|\mathrm{Re}\ y - y_{i-1}| \leqslant R$, it follows that $G_*^i|_U$ is close to the composition $I_i^{-1} \circ G_i \circ \cdots \circ G_1$. Since $(x_n^*, y_n^*) = (x_n, y_n)$, it follows that $I_n = id$, and we have that $G_*^n|_U$ is close to $G_n \circ \cdots \circ G_1|_U$. Since g_ε can be chosen as close to (5.17) on Y_i as we need, the distance between $G_*^n|_U$ and $G_n \circ \cdots \circ G_1|_U$ can be made less than $\frac{1}{2}\delta$. Thus, the distance between $G_*^n|_U$ and the original map f_ε is less than δ. □

Proof of theorem 6. Take any area- and orientation-preserving C^r-diffeomorphism g of the unit disc into R^2 and take any $\delta > 0$. Let $\mathcal{V}(g, \delta)$ be the set of all two-dimensional area-preserving C^r-maps whose dynamical conjugacy class intersects the open δ-neighbourhood of g (i.e. whose certain renormalized iteration is at a distance smaller than δ from g). This set is open by definition. It is also dense in the Newhouse regions. Indeed, given any arbitrarily large n, the maps with homoclinic tangencies of order $(n + 1)$ are dense in the Newhouse regions according to theorem 3. As was shown in the proof of lemma 8, arbitrarily close to any such map there is a map with an order n homoclinic tangency for which either both the local and global maps are orientation-preserving or the local map is orientation-reversing. According to the remark for lemma 6, by an arbitrarily small perturbation of any such homoclinic, a rescaled first-return map (see (5.8)) can be made arbitrarily close to any map from the family

$$\bar{x} = y, \qquad \bar{y} = -x + E_0 + E_1 Y + \cdots + E_{n-1} Y^{n-1}. \tag{5.18}$$

In other words, by an arbitrarily small perturbation of any map from the Newhouse regions one can obtain a map whose certain renormalized iteration gives as good approximation to any given map from family (5.18) as we want. It remains to note that an iteration of a map which is affine equivalent to some of the maps (5.18) with n sufficiently large provides, in turn, an approximation of any aforehand given degree of accuracy to the given map g (see lemma 9). Hence, the maps whose certain renormalized iteration is δ-close to f are dense in the Newhouse regions indeed.

Let us now take a countable sequence of maps g_i which is dense in space of area- and orientation-preserving C^r-diffeomorphisms g of the unit disc into R^2 and a sequence δ_j of positive reals converging to zero. Then, by construction, the intersection $\cap_{i,j}\mathcal{V}(g_i, \delta_j)$ is the sought residual set composed of universal maps. □

Proof of theorem 7. Take any two-dimensional area-preserving C^r-diffeomorphism f with a homoclinic tangency and take any $\delta > 0$. Let $\mathcal{W}(f, \delta)$ be the set of all k-parameter families of orientation-preserving C^r-diffeomorphisms of the unit disc into R^2 such that each of these families can be obtained as a certain renormalized iteration of a family of area-preserving C^r-maps from the open δ-neighbourhood of f (the parameter is assumed to run the closed unit ball in R^k). By definition, the set $\mathcal{W}(f, \delta)$ is open in space $D_{k,r}$ of the k-parameter families of area- and orientation-preserving diffeomorphisms of the unit disc into R^2, of class C^r with respect to both the phase variables and the parameters. It is also dense in this space. Indeed, by lemma 9, every family from $D_{k,r}$ can be arbitrarily well approximated (up to a shift of coordinates) by an iteration of a certain k-parameter subfamily of the polynomial family (5.18) with some sufficiently large n. By the remark for lemma 6, every such subfamily can be obtained as a renormalized iteration of a family of maps lying in an arbitrarily small neighbourhood of any area-preserving map with a homoclinic tangency of order n with both local and global map orientation-preserving, or with the local map orientation-reversing. As we argued in the proof of theorem 6, maps with such homoclinics exist arbitrarily close to the given map f. Thus, for every $\delta > 0$ there exists indeed a set of k-parameter families of maps from the open δ-neighbourhood of the given map f, such that the renormalized iterations of these families are dense in $D_{k,r}$. Let us now take a countable sequence of maps f_i which is dense in the Newhouse regions and a sequence δ_j of positive reals converging to zero. The intersection $\cap_{i,j}\mathcal{W}(f_i, \delta_j)$ is the sought residual set composed of minimal families. □

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