

Polynomial approximations of symplectic dynamics and richness of chaos in non-hyperbolic area-preserving maps

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Abstract

It is shown that every symplectic diffeomorphism of R^{2n} can be approximated, in the C^∞ -topology, on any compact set, by some iteration of some map of the form $(x, y) \mapsto (y + \eta, -x + \nabla V(y))$ where $x \in R^n$, $y \in R^n$, and V is a polynomial $R^n \rightarrow R$ and $\eta \in R^n$ is a constant vector. For the case of area-preserving maps (i.e. $n = 1$), it is shown how this result can be applied to prove that C^r -universal maps (a map is universal if its iterations approximate dynamics of all C^r -smooth area-preserving maps altogether) are dense in the C^r -topology in the Newhouse regions.

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1. Polynomial approximations by Hénon-like maps

It is shown here that almost every symplectic dynamics in R^{2n} can be realized by iterations of Hénon-like maps, i.e. symplectic maps of the following special form

$$\bar{x} = y, \quad \bar{y} = -x + \nabla V(y), \quad (1)$$

where $x \in R^n$, $y \in R^n$ and V is a smooth function $R^n \rightarrow R$. The precise formulation is given in theorem 1, an interesting application is discussed in section 2. The main step of the construction is based on the fact that an arbitrary symplectic diffeomorphism of R^{2n} admits, on any compact set, arbitrarily good approximations by polynomial symplectic maps of a special form – by compositions of polynomial Hénon-like maps, in fact. The possibility of a symplectic polynomial approximation to any symplectic map is not immediately obvious, though it is possible that somebody could have proved this before. However, I failed to find a corresponding reference, and my discussions with many experts in symplectic dynamics

showed that the possibility of such an approximation is not widely known. Therefore, a detailed proof of the symplectic polynomial approximation theorem is also included.

Hénon-like maps (1) are quite special. They are reversible, so the inverse of any of them is again a Hénon-like map. The compositions of polynomial Hénon-like maps form a group which is known [1] to coincide, in the two-dimensional case, with the so-called Cremona group of the symplectic polynomial maps with a polynomial inverse. The general interest in Hénon-like maps is explained simply by the fact that (1) gives, probably, the easiest way to define a polynomial map which preserves the standard symplectic form $dx \wedge dy$ (see also [2, 3]). Maps of type (1) also appear in symplectic discretizations of natural Lagrangian systems (see e.g. [4, 5, 30] and formulae (7), (8) below). My interest in Hénon-like maps is due to the fact that they appear as rescaled first-return maps at homoclinic bifurcations, near a homoclinic tangency in particular (see [6, 7, 11, 8, 9] and section 2).

The main result of the paper is the following theorem.

Theorem 1. *Let U be a ball in R^{2n} , and let F be a C^{r+1} -smooth symplectic diffeomorphism $U \rightarrow R^{2n}$. Then, for any compact set $C \subset U$ and for any $\varepsilon > 0$ there exists a polynomial $V : R^n \rightarrow R$, a constant vector $\eta \in R^n$ and an integer $N > 0$ such that the $4N$ th iteration of the symplectic map $f : (x, y) \mapsto (\bar{x}, \bar{y})$, where*

$$\bar{x} = y + \eta, \quad \bar{y} = -x + \nabla V(y), \quad (x \in R^n, y \in R^n), \quad (2)$$

approximates F with the accuracy ε in the C^r -topology:

$$\sup_{(x,y) \in C} \|F(x, y) - f^{4N}(x, y)\| + \|\nabla F(x, y) - \nabla f^{4N}(x, y)\| + \dots$$

$$\dots + \left\| \frac{\partial^r}{\partial(x, y)^r} F(x, y) - \frac{\partial^r}{\partial(x, y)^r} f^{4N}(x, y) \right\| < \varepsilon,$$

where we denote $f^{4N} = \underbrace{f \circ \dots \circ f}_{4N}$.

Before we start to prove the theorem, note that the map (2) is not a Hénon-like map at $\eta \neq 0$, but it coincides with a Hénon-like map (namely, with the map $(\bar{x} = y, \bar{y} = -x + \nabla V(y - \eta) + \eta)$) after the shift of coordinates $y \mapsto y + \eta$. It will also be clear from the proof that if the map F depends continuously, or smoothly, on some parameters μ , then the approximation by the map (2) with η constant and Φ depending, respectively continuously or smoothly, on μ can be done uniformly on any compact set of parameters μ , and along with the derivatives with respect to μ (in the case of smooth parameter dependence). With this remark, theorem 1 thus tells us that every dynamical phenomenon which is robustly present in any symplectic map or in any finite-parameter family of symplectic maps can indeed be found in iterations of polynomial Hénon-like maps (up to a shift of coordinates).

Theorem 1 is, in fact, easily derived from the following formally weaker statement.

Theorem 2 (Symplectic polynomial approximation). *Given any C^{r+1} -smooth symplectic diffeomorphism $F : U \rightarrow R^{2n}$, for any compact set $C \subset U$ and for any $\varepsilon > 0$ there exists a sequence of polynomial Hénon-like maps f_1, \dots, f_{4N} such that the map F is approximated on C by the composition $f_{4N} \circ \dots \circ f_1$ with the accuracy ε in the C^r -topology.*

Remark 1. We will prove, in fact, that when F depends continuously on some parameter $\mu \in W$, where W is a finite-dimensional ball, the functions f_i , $i = 1, \dots, 4N$, depend continuously on μ as well, and the approximation is uniform on any compact subset of W . When F is C^{r+1} -smooth with respect to (x, y, μ) , the functions f_i are C^r -smooth with respect to (x, y, μ) as well, and the composition $f_{4N} \circ \dots \circ f_1$ is ε -close to F on C along with all the derivatives with respect to (x, y, μ) , uniformly on any compact set of the parameter values.

The rest of this section is occupied by the proof of these two theorems.

Proof of theorem 1. Assume for a moment that theorem 2 is proven, then theorem 1 is obtained as follows. Fix any $\varepsilon > 0$ and let f_1, \dots, f_{4N} be the corresponding sequence of the Hénon-like maps from theorem 2:

$$f_i : (x, y) \mapsto (y, -x + \nabla V_i(y)). \quad (3)$$

Let (x_0, y_0) be an arbitrary point in C and let $(x_i, y_i) = f_i \circ \dots \circ f_1(x_0, y_0)$. Note that $x_i = y_{i-1}$ at $i \geq 1$, according to (3). Let L be such that the image $f_i \circ \dots \circ f_1(C)$ lies inside the ball of radius L around the point (x_i, y_i) , for every $i = 1, \dots, 4N$ (if F depends on parameters, we assume that this holds true for all parameter values under consideration). Choose some η and take a sequence of points (x_i^*, y_i^*) ($i = 0, \dots, 4N$) such that $(x_0^*, y_0^*) = (x_0, y_0)$, $(x_{4N}^*, y_{4N}^*) = (x_{4N}, y_{4N})$, and $x_i^* = y_{i-1}^* + \eta$ for $i \geq 1$. If we take $\eta > 2L + \|x_{4N} - y_0\|$, then we can always choose y_i^* such that $\|y_i^* - y_j^*\| > 2L$ for all $0 \leq i < j \leq 4N - 1$ (our assumptions fix $y_{4N-1}^* = x_{4N} - \eta$, so we must have η sufficiently large in order to push y_{4N-1}^* on a distance larger than L from $y_0^* = y_0$). Now, define the function V such that

$$V(y) = V_i(y - y_{i-1}^* + y_{i-1}) + (x_{i-1}^* - x_{i-1} + y_i^* - y_i)y$$

on the cylinder $\|y - y_{i-1}^*\| \leq L$, $i = 1, \dots, 4N$ (the functions V_i are defined by (3)). Due to our choice of y_i^* , the cylinders corresponding to different i do not intersect, therefore, such C^{r+1} -smooth functions V can indeed be defined. By construction (recall that $x_i = y_{i-1}$ and $x_i^* = y_{i-1}^* + \eta$), the map $f : (x, y) \mapsto (y + \eta, -x + \nabla V(y))$ acts on the cylinder $\|y - y_{i-1}^*\| \leq L$ ($i = 1, \dots, 4N$) as a composition of the parallel translation $(x, y) \mapsto (x - x_{i-1}^* + x_{i-1}, y - y_{i-1}^* + y_{i-1})$ (that takes the cylinder $\|y - y_{i-1}^*\| \leq L$ onto the cylinder $\|y - y_{i-1}\| \leq L$; at $i = 1$ this is just the identity map since $(x_0^*, y_0^*) = (x_0, y_0)$), the map f_i , and the parallel translation $(x, y) \mapsto (x - x_i + x_i^*, y - y_i + y_i^*)$. Since every image $f_i \circ \dots \circ f_1(C)$ lies in the cylinder $\|y - y_{i-1}\| \leq L$, it follows that $f^i|_C$ is a composition of $f_i \circ \dots \circ f_1$ and the parallel translation $(x, y) \mapsto (x - x_i + x_i^*, y - y_i + y_i^*)$. Since $(x_{4N}^*, y_{4N}^*) = (x_{4N}, y_{4N})$, we have finally that $f^{4N}|_C \equiv f_{4N} \circ \dots \circ f_1|_C$. Hence, f^{4N} indeed gives the required approximation to the original map F on C (as the composition $f_{4N} \circ \dots \circ f_1$ does according to theorem 2). By construction, the map f has the required form (2) (the function V can be made polynomial by an arbitrarily small perturbation—this will make approximation only slightly worse). \square

Remark 2. Note that one cannot assume $\eta = 0$ in theorem 1, because any Hénon-like map f of the form (1) (i.e. with $\eta = 0$) is reversible with respect to the involution $R : (x, y) \leftrightarrow (y, x)$, i.e. $f^{-1} = R \circ f \circ R$, and the same holds true for any iteration of such a map. So, if the map F which is to be approximated is not reversible with respect to R and if it, for example, has a non-parabolic fixed point $Q(x, y)$ while the point $RQ(y, x)$ is not a fixed point of F , any sufficiently close C^1 -approximation of F must have a fixed point Q^* close to Q while RQ^* cannot be a fixed point. Hence, any sufficiently close C^1 -approximation of such map F cannot be reversible with respect to R , i.e. any iteration of any Hénon-like map must be far from F in C^1 -topology on a neighbourhood of the points Q and RQ .

The map f_η given by (2) with $\eta \neq 0$ is also reversible: $f_\eta^{-1} = R_\eta \circ f_\eta \circ R_\eta$ where R_η is the involution $(x, y) \mapsto (y + \eta, x - \eta)$. However, by taking $\|\eta\|$ large enough one can always achieve that $R_\eta C \cap FC = \emptyset$ where C is the compact set on which we want to approximate the map F . This means that the reversibility of the approximating map f_η^{4N} does not create obstacles anymore (by approximating the map F on the set C one automatically obtains an approximation to F^{-1} on the set FC ; on the other hand, if the approximation is reversible, its inverse is uniquely defined on the set $R_\eta C$ —this would obviously create a problem if FC and $R_\eta C$ had a non-empty intersection, so we cancel this problem by taking $\|\eta\|$ large enough).

Proof of theorem 2. It is well-known that for any C^{r+1} -smooth symplectic diffeomorphism F taking an open $2n$ -dimensional ball U into R^{2n} , for any compact set $C \subset U$ there exists a time-dependent Hamiltonian $H(x, y, t)$ defined at all $x \in R^n, y \in R^n$, continuous in t and C^{r+1} -smooth in (x, y) (and its gradient is C^r -smooth with respect to (x, y, μ) if the map F is C^r -smooth with respect to some parameter μ as well), such that F coincides on C with the time-1 shift by the flow defined via

$$\dot{x} = \frac{\partial H}{\partial y}(x, y, t), \quad \dot{y} = -\frac{\partial H}{\partial x}(x, y, t). \quad (4)$$

Thus, every symplectic diffeomorphism can be understood as a shift by a non-autonomous Hamiltonian flow, so we will work with the flow given by (4) from the very beginning. Since polynomials are dense among smooth functions, it is enough to prove the theorem for the case when H is polynomial in (x, y) (with the coefficients continuously depending on time and continuously or smoothly depending on parameters μ), so let this be our standing assumption. Let $F_{t,\tau}$ denote the map defined by the flow (4) from the moment time t to the moment $t+\tau$. It is obvious that we will prove the theorem if we can show that any map $F_{t,\tau}$ can be approximated, as $\tau \rightarrow 0$, by the composition of Hénon-like maps with the accuracy $O(\tau^2)$, uniformly for any compact interval of values of t and on any compact set in the (x, y, μ) -space. Indeed, since

$$F \equiv F_{0,1} = F_{1-1/m,1/m} \circ \cdots \circ F_{0,1/m}$$

for any integer m , it follows that if all the maps $F_{l/m,1/m}$ admit, uniformly for all $l = 0, \dots, m-1$, an $O(1/m^2)$ -approximation by the compositions of Hénon-like maps, then the composition of these compositions will provide an $O(1/m)$ -approximation to $F_{0,1}$: if we denote the approximation to $F_{l/m,1/m}$ as G_l , then

$$\begin{aligned} \|F_{0,1} - G_{m-1} \circ G_{m-2} \circ \cdots \circ G_0\|_{C^r} &\leq \sum_{l=1}^m \|F'_{l/m,1-1/m}\|_{C^r} \cdot \|F_{(l-1)/m,1/m} - G_{l-1}\|_{C^r} \\ &= O(m) \cdot O\left(\frac{1}{m^2}\right) = O\left(\frac{1}{m}\right) \end{aligned}$$

(we use the fact that all the derivatives of the map $F_{t,1-t}$ are uniformly bounded for all $t \in [0, 1]$).

It follows then immediately that the theorem holds true for the Hamiltonians of the type

$$H = \frac{1}{2}y^2 + V(x, t), \quad (5)$$

i.e. for systems of the type

$$\dot{x} = y, \quad \dot{y} = -\Psi(x, t), \quad (6)$$

where $\Psi \equiv \nabla_x V$. Indeed, the time- τ shift by the flow of (6) has the form

$$x_{t+\tau} = x_t + \tau y + O(\tau^2), \quad y_{t+\tau} = y_t - \tau \Psi(x_t, t) + O(\tau^2),$$

i.e. it is uniformly $O(\tau^2)$ -close to the symplectic map $(x, y) \mapsto (\bar{x}, \bar{y})$:

$$\bar{x} = x + \tau y, \quad \bar{y} = y - \tau \Psi(x + \tau y, t). \quad (7)$$

Now note that the latter map is the composition of four Hénon-like maps: $(x, y) \mapsto (x_1, y_1)$, $(x_1, y_1) \mapsto (x_2, y_2)$, $(x_2, y_2) \mapsto (x_3, y_3)$, $(x_3, y_3) \mapsto (\bar{x}, \bar{y})$, where

$$\begin{aligned} (\bar{x} = y_3, \bar{y} = -x_3), \quad (x_3 = y_2, y_3 = -x_2), \\ (x_2 = y_1, y_2 = -x_1 + \tau \Psi(-y_1, t)), \quad (x_1 = y, y_1 = -x - \tau y). \end{aligned} \quad (8)$$

Below we will prove the following lemma.

Lemma 1. *Given any $H(x, y, \mu, t)$, polynomial in (x, y) , there exists a function $V(x, \mu, s)$, polynomial in x , such that the time- τ map $F_{t,\tau}$ of system (4) is uniformly $O(\tau^2)$ -close (on any compact domain in the (x, y, μ) -space and any compact interval of the values of t) to the time- 2π map of the Hamiltonian system:*

$$\dot{x}_j = y_j, \quad \dot{y}_j = -\Omega_j^2 x_j - \tau \frac{\partial}{\partial x_j} V(x_1, x_2, \dots, x_n, \mu, s), \quad (j = 1, \dots, n) \quad (9)$$

for some appropriately chosen integers $\Omega_1, \dots, \Omega_n$.

Note that the theorem follows from this lemma immediately: since (9) is a Hamiltonian system of type (6), its time- 2π map can be arbitrarily closely approximated by a composition of Hénon-like maps; hence, $F_{t,\tau}$ can be $O(\tau^2)$ -approximated by such composition as well, which gives the theorem as was explained above. \square

Proof of lemma 1. Let us, first, consider the case $n = 1$, when x and y are scalars. Take $\Omega_1 = 1$. Equations (9) take the form

$$\dot{x} = y, \quad \dot{y} = -x - \tau V'_x(x, s) \quad (10)$$

(we suppress, notationally, the dependence on the parameters μ from now on). Solutions of (10) at small τ can be written as

$$\begin{aligned} x(s) &= x(0) \cos s + y(0) \sin s - \tau \int_0^s V'_x(x(\sigma), \sigma) \sin(s - \sigma) d\sigma + O(\tau^2), \\ y(s) &= -x(0) \sin s + y(0) \cos s - \tau \int_0^s V'_x(x(\sigma), \sigma) \cos(s - \sigma) d\sigma + O(\tau^2), \end{aligned} \quad (11)$$

where $x(\sigma) = x(0) \cos \sigma + y(0) \sin \sigma$.

Thus, the time- 2π map is written as

$$\begin{aligned} \bar{x} &= x + \tau \frac{\partial}{\partial y} \int_0^{2\pi} V(x \cos s + y \sin s, s) ds + O(\tau^2), \\ \bar{y} &= y - \tau \frac{\partial}{\partial x} \int_0^{2\pi} V(x \cos s + y \sin s, s) ds + O(\tau^2). \end{aligned} \quad (12)$$

The time- τ map $F_{t,\tau}$ of the Hamiltonian system (4) has the form

$$\bar{x} = x + \tau \frac{\partial}{\partial y} H(x, y, t) + O(\tau^2), \quad \bar{y} = y - \tau \frac{\partial}{\partial x} H(x, y, t) + O(\tau^2). \quad (13)$$

By comparing (12) and (13) we see that to prove the lemma we must show that for every polynomial function $H(x, y)$ there exists a polynomial in z function $V(z, s)$ such that

$$H(x, y) \equiv \int_0^{2\pi} V(x \cos s + y \sin s, s) ds. \quad (14)$$

Let H be a polynomial of degree M :
$$H(x, y) = \sum_{0 \leq p+q \leq M} h_{pq} x^p y^q.$$

We will look for $V(z, s)$ in the form

$$V(z, s) = \sum_{0 \leq k \leq M} v_k(s) z^k,$$

where the coefficients $v_k(s)$ have to be defined in such a way that (14) would be satisfied.

It is easy to see that relation (14) is fulfilled if and only if, for every $k = 0, \dots, M$,

$$h_{pq} = C_k^q \int_0^{2\pi} v_k(s) \cos^p s \sin^q s ds \quad (15)$$

for all $p \geq 0, q \geq 0$ such that $p + q = k$. We will look for the function $v_k(s)$ in the form

$$v_k(s) = \sum_{0 \leq q \leq k} \alpha_q g_{qk}(s),$$

where we denote $g_{qk}(s) = \cos^{k-q} s \sin^q s$, and α_q are coefficients to be defined. So, the problem reduces to finding such coefficients α_q , which satisfy the following system of linear equations:

$$\frac{h_{pq}}{C_k^q} = \sum_{0 \leq q' \leq k} \alpha_{q'} \langle g_{q'k}, g_{qk} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of functions: $\langle g_{q'k}, g_{qk} \rangle = \int_0^{2\pi} g_{q'k}(s) g_{qk}(s) ds$. This system has a solution if and only if the system of the functions $g_{qk}(s)$ (here k is fixed and q runs integer values from 0 to k) is linearly independent. Moreover, the coefficients α_q are determined uniquely and depend linearly on the coefficients h_{pq} involved. Now note that the linear independence of the system of functions $\cos^k s, \cos^{k-1} s \sin s, \dots, \cos s \sin^{k-1} s, \sin^k s$ is equivalent to the linear independence of the system of functions $1, \tan s, \dots, \tan^k s$, and the latter is obvious, of course. This gives the lemma in the case $n = 1$.

Consider now the case $n > 1$. It is more involved: we should be careful with the choice of the frequencies $\Omega_1, \dots, \Omega_n$. At small τ the solution of system (9) has the form

$$x_j(s) = x_j(0) \cos \Omega_j s + \frac{1}{\Omega_j} y_j(0) \sin \Omega_j s - \frac{\tau}{\Omega_j} \int_0^s \Psi_j(x^\sigma(\sigma), \sigma) \sin \Omega_j(s - \sigma) d\sigma + O(\tau^2),$$

$$y_j(s) = -\Omega_j x_j(0) \sin \Omega_j s + y_j(0) \cos \Omega_j s - \tau \int_0^s \Psi_j(x^\sigma(\sigma), \sigma) \cos \Omega_j(s - \sigma) d\sigma + O(\tau^2),$$

where $x_j^\sigma(\sigma) = x_j(0) \cos \Omega_j \sigma + \frac{1}{\Omega_j} y_j(0) \sin \Omega_j \sigma$,

and we denote $\Psi_j \equiv (\partial/\partial x_j) V$. Since Ω_j are integers, the time- 2π map is written as

$$\begin{aligned} \bar{x}_j &= x_j + \tau \frac{\partial}{\partial y_j} \int_0^{2\pi} W(x, y, s) ds + O(\tau^2), \\ \bar{y}_j &= y_j - \tau \frac{\partial}{\partial x_j} \int_0^{2\pi} W(x, y, s) ds + O(\tau^2), \end{aligned} \quad (17)$$

where we denote

$$W(x, y, s) = V \left(x_1 \cos \Omega_1 s + \frac{1}{\Omega_1} y_1 \sin \Omega_1 s, \dots, x_n \cos \Omega_n s + \frac{1}{\Omega_n} y_n \sin \Omega_n s, s \right). \quad (18)$$

The time- τ map $F_{t,\tau}$ of the Hamiltonian system (4) is still given by the formula (13) (but x and y are not scalars now). By comparing (17), (18) with (13) we see that to prove the lemma we must find for every polynomial function $H(x, y)$ a polynomial in z function $V(z, s)$ such that

$$H(x, y) \equiv \int_0^{2\pi} V(x_1 \cos \Omega_1 s + \frac{1}{\Omega_1} y_1 \sin \Omega_1 s, \dots, x_n \cos \Omega_n s + \frac{1}{\Omega_n} y_n \sin \Omega_n s, s) ds, \quad (19)$$

for some set of integers $\Omega_1, \dots, \Omega_n$.

Let H be a polynomial of degree M :

$$H(x, y) = \sum_{0 \leq |p|+|q| \leq M} h_{p_1, \dots, p_n, q_1, \dots, q_n} x_1^{p_1} x_2^{p_2} \dots x_n^{p_n} y_1^{q_1} \dots y_n^{q_n}.$$

Take $\Omega_1 = 1, \Omega_j = (M^j - 1)/(M - 1), j = 2, \dots$. What is important for us in this choice is that for every $n > 1$

$$\Omega_n > M \max(\Omega_1, \dots, \Omega_{n-1}). \quad (20)$$

We will look for $V(z, s)$ in the form

$$V(z, s) = \sum_{0 \leq |k| \leq M} v_{k_1, \dots, k_n}(s) z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n},$$

where every function $v_k(s)$ is a linear combination of the functions

$$g_{qk}(s) = \prod_{j=1}^n \cos^{k_j - q_j} \Omega_j s \cdot \sin^{q_j} \Omega_j s \quad (21)$$

where $q = (q_1, \dots, q_n)$ runs over all possible multi-indices such that $0 \leq q_j \leq k_j$, $j = 1, \dots, n$.

Relation (19) is fulfilled if and only if the following holds for every $k = (k_1, \dots, k_n)$ and every $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ such that $p_j + q_j = k_j$ for all $j = 1, \dots, n$:

$$\frac{h_{pq} \prod_{j=1}^n \Omega_j^{q_j}}{\prod_{j=1}^n C_{k_j}^{q_j}} = \int_0^{2\pi} v_k(s) g_{qk}(s) ds. \quad (22)$$

In other words, given any multi-index k , the problem reduces to finding coefficients α_q such that the standard inner product of the function $g_{qk}(s)$ with the function

$$v_k(s) = \sum_q \alpha_q g_{qk}(s) \quad (23)$$

acquires the value $h_{pq} \prod_{j=1}^n \Omega_j^{q_j} / \prod_{j=1}^n C_{k_j}^{q_j}$. This problem is equivalent to the following system of linear equations for the coefficients α :

$$\frac{h_{pq} \prod_{j=1}^n \Omega_j^{q_j}}{\prod_{j=1}^n C_{k_j}^{q_j}} = \sum_{q'} \alpha_{q'} \langle g_{q'k}, g_{qk} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes, as before, the inner product of functions. If the system of functions $g_{qk}(s)$ (here k is fixed and q runs over all possible multi-indices such that $0 \leq q_j \leq k_j$, $j = 1, \dots, n$) is linearly independent, the coefficients α_q are determined uniquely and depend linearly on the corresponding coefficients h_{pq} . Therefore, if we prove the linear independence of the functions $g_{qk}(s)$ for any fixed k , it will follow that the required function V exists and its coefficients depend on coefficients of H smoothly. This will provide that the maps (17) and (13) are $O(\tau^2)$ -close to each other, uniformly on compact intervals of values of t . Hence, to finish the lemma and the theorem, it remains to show that if

$$\sum_q \beta_q g_{qk}(s) \equiv 0 \quad (24)$$

for some choice of coefficients β_q , then all β_q must be zero.

We will do it by induction in n (we have already considered the case $n = 1$). Before we go further, note that $g_{qk}(s) = e^{-i(k, \Omega)s} P_{qk}(e^{2is})$ where P_{qk} is a polynomial of degree $(k, \Omega) = k_1 \Omega_1 + k_2 \Omega_2 + \cdots + k_n \Omega_n$. Any linear combination of the functions g_{qk} (with fixed k) has, obviously, the same structure, so any such linear combination has no more than (k, Ω) zeros at $0 \leq s < \pi$.

If the functions g_{qk} were linearly dependent, then at least one of the coefficients β_q in (24) could be non-zero. Let Q be the maximal value of q_n for which there exists a non-zero β_{q_1, \dots, q_n} . Then, (24) can be rewritten as

$$\sin^Q \Omega_n s \cos^{k_n - Q} \Omega_n s \cdot \sum_{q'} \beta_{q', Q} g_{q'k'}(s) \equiv - \sum_{q_n=0}^{Q-1} \sin^{q_n} \Omega_n s \cos^{k_n - q_n} \Omega_n s \cdot \sum_{q'} \beta_{q', q_n} g_{q'k'}(s),$$

or, after cancelling the common multiplier $\cos^{k_n-Q} \Omega_n s$, in the form

$$\sin^Q \Omega_n s \cdot \sum_{q'} \beta_{q',Q} g_{q'k'}(s) \equiv -\cos \Omega_n s \sum_{q_n=0}^{Q-1} \sin^{q_n} \Omega_n s \cos^{Q-q_n-1} \Omega_n s \cdot \sum_{q'} \beta_{q',q_n} g_{q'k'}(s), \quad (25)$$

where we denote $k' = (k_1, \dots, k_{n-1})$, $q' = (q_1, \dots, q_{n-1})$ runs over multi-indices of length $(n-1)$ such that $0 \leq q'_j \leq k_j$, $j = 1, \dots, n-1$, and at least one of the coefficients $\beta_{q',Q}$ is non-zero—hence, by the induction hypothesis $\sum_{q'} \beta_{q',Q} g_{q'k'}(s)$ is not identically zero.

If $Q = 0$, the right-hand side of (25) is zero, but the left-hand side is non-zero, as we just mentioned, so we have $Q > 0$. The linear combination $\sum_{q'} \beta_{q',Q} \tilde{g}_{q'k'}(s)$ has no more than $(k', \Omega) = k_1 \Omega_1 + \dots + k_{n-1} \Omega_{n-1}$ zeros at $0 \leq s < \pi$, as was explained above. Hence, the left-hand side of (25) has no more than (k', Ω) zeros different from the zeros of $\sin \Omega_n s$. On the other hand, the right-hand side is a multiple of $\cos \Omega_n s$, so it has at least Ω_n such zeros. Since $|k| \leq M$, we have that $(k', \Omega) \leq M \max(\Omega_1, \dots, \Omega_{n-1})$, so $(k', \Omega) < \Omega_n$ by virtue of (20). Hence, identity (25) cannot hold, which means linear independence of the functions g_{qk} . \square

2. Concluding remarks: universal maps in Newhouse regions

The results above show that the dynamics of Hénon-like symplectic maps is as rich as the dynamics of all symplectic maps. This statement can be used in proving a much more discouraging result, as I will demonstrate now for the example of area-preserving maps of a plane.

Let f be a C^r -smooth area-preserving map of R^2 . Define the *dynamical conjugacy class* of f as the set of all maps $f_{n,\psi}$ of a unit disc U_1 into R^2 obtained by the rule $f_{n,\psi} = \psi^{-1} \circ f^n \circ \psi$, where n is an integer, f^n is the n th iteration of f and ψ is an arbitrary C^r -smooth map of U_1 into R^2 with a constant Jacobian (so, by construction, all the maps $f_{n,\psi}$ in the class are area-preserving).

When we speak about dynamics of the map, we somehow describe its iterations, and the description should be independent of smooth coordinate transformations. Therefore, the class of the map f , as we have just introduced it, indeed gives some representation of the dynamics of f . Note that the coordinate transformations ψ are not area-preserving (they preserve the standard symplectic form up to a constant factor), i.e. the image $\psi(U_1)$ can be a disc of an arbitrarily small radius, with the centre situated anywhere. Thus, the class of f contains information about the behaviour of arbitrarily long iterations of f on arbitrarily fine spatial scales.

The general intuition here is that if the class of the map is large, then the dynamics is rich, while if the dynamics is sufficiently simple, then the class is somehow restricted. For example, if the topological entropy of f is zero, then every map in the class of f has zero entropy as well. Or if f possesses a uniformly hyperbolic structure (like, say the linear map $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$), then every map in the class is uniformly hyperbolic. It is interesting in these examples that the uniform hyperbolicity is a robust (or rough) property—it cannot be destroyed by a small smooth perturbation of f , whereas examples of maps of zero entropy can be provided which can be perturbed to produce a maximally rich dynamical class (see below).

Definition. A C^r -smooth symplectic map f is called *universal* (or C^r -universal) if its dynamical class is dense in the C^r -topology among all C^r -smooth symplectic diffeomorphisms $U_1 \rightarrow R^2$.

By the definition, the detailed understanding of the dynamics of any single universal map is not simpler than understanding of all other area-preserving maps *altogether*, i.e. it is beyond human abilities. What is surprising, is that such universal maps are, in fact, quite common. Namely, one can expect that the following statement is valid.

Proposition A. *C^r -universal maps exist in any neighbourhood (in the C^r -topology) of any area-preserving map with a homoclinic tangency.*

Let us discuss this claim in more detail. A homoclinic tangency is a point where the stable and unstable manifolds of some saddle fixed point or a periodic orbit are tangent to each other. Typically, the tangency is quadratic, although higher order tangencies are also possible. For example, a time-1 map of a conservative flow with a homoclinic loop (an orbit which is asymptotic to a saddle equilibrium state both as $t \rightarrow \pm\infty$) has a homoclinic tangency of infinite order (the stable and unstable curves of the saddle fixed point which corresponds to the saddle equilibrium of the flow coincide). Such a map has zero entropy, but proposition A says that after an arbitrarily small perturbation the dynamical class of the map can become extremely rich.

A quadratic homoclinic tangency is a codimension-1 bifurcation: in a generic one-parameter unfolding the tangency between the stable and unstable invariant curves is removed near a given point. However, the stable and unstable curves are not compact, so one cannot immediately reject the possibility that while the original tangency is removed some new homoclinic tangencies appear. Indeed, it was proved by Newhouse [10] that maps with homoclinic tangencies are dense in some open regions in the space of C^r -maps (with $r \geq 2$). In [11, 12] Duarte proved that in the space of area-preserving maps the Newhouse regions exist arbitrarily close (in the C^r -topology) to any area-preserving map with a homoclinic tangency. In fact, a combination of the results of [11, 12] and [13] shows that the Newhouse regions exist in any generic one-parameter unfolding of any area-preserving map with a homoclinic tangency [14], e.g. in the quadratic Hénon map $(x, y) \mapsto (y, M - y^2 - x)$ (M is a parameter), in the standard map $(x, y) \mapsto (y, -x + M \sin y)$ (see also [15]), etc.

The Newhouse regions do not just exist in popular examples, they also seem to be quite large. In [16] Newhouse proved that maps with homoclinic tangencies are dense in the C^1 -topology among all area-preserving maps which are not uniformly hyperbolic. Of course, the C^1 -topology is inadequate for symplectic dynamics. However, this result suggests a conjecture on the C^r -denseness of the maps with homoclinic tangencies among all non-hyperbolic area-preserving maps for any r . This seems to be a very difficult conjecture to prove. However, regardless of it, the fact that homoclinic tangencies appear in so many models, for so many parameter values, and practically near any point in the phase plane, can be taken as an experimental observation (see [17], or take any area-preserving map with chaotic behaviour and follow, numerically, its stable and unstable curves; the usual picture is that, after a number of iterations, folds in the unstable curve come sufficiently close to the stable curve, so the tangencies can be created by fine parameter tuning).

Proposition A says that the universal maps are as common as homoclinic tangencies are. So, if we have any explicitly given area-preserving map with chaotic behaviour, then by changing parameters slightly we can, probably, encounter a homoclinic tangency. Then, if we have enough parameters to tune, we can make some iteration of our map become as close as we want to any other area-preserving map in some carefully chosen coordinates. In other words, it looks like any two-dimensional area-preserving dynamics can be modelled by any non-hyperbolic map after a proper transformation of coordinates and a small variation of parameters.

This statement sounds strange, but proposition A strongly supports it. Something like that can be true in the higher-dimensional case as well (here, obviously, the absence of uniform *partial* hyperbolicity should be assumed to create universal maps). One can fantasize about an infinite-dimensional case as well. Can it be true that any nonlinear wave equation with chaotic dynamics describes any possible dynamical process after an appropriate transformation of the variables and time and an appropriate tuning of the nonlinearity?

Proposition A is deduced from the following statement.

Proposition B. *Area-preserving maps, each having, for every $n \geq 1$, infinitely many isolated homoclinic tangencies of order n , exist in any neighbourhood (in the C^r -topology) of any area-preserving map with a quadratic homoclinic tangency.*

An analogous statement about non-symplectic maps was proved in [20] by Gonchenko, Shilnikov and myself (a very detailed version of the proof is published in [21]). The proof of proposition B for the symplectic case is in preparation now and we will publish it in a forthcoming paper.

The fact that, say, cubic tangencies can be obtained by a small C^r -smooth perturbation of a quadratic tangency seems to contradict the usual scheme of the singularity theory where the order of degeneracy decreases in the unfolding. Here, of course, the order of the original tangency does not increase, but some new tangencies appear after iterations of the map, and the order of these tangencies can be made arbitrarily high indeed. For an illustration one may again take, say, a quadratic Hénon map and start to iterate stable and unstable manifolds of a saddle fixed point. Then, after some number of iterations, folds on the unstable manifold (which appear because the nonlinearity is quadratic) are folded once again, and one can see how ‘inflection points’ on the unstable manifold are created. Tangencies of order higher than cubic are harder to see, but the technique of proving their existence is available from the non-symplectic case (see [22] for more explanations and illustrations).

A homoclinic tangency of order n is a bifurcation phenomenon of codimension n , i.e. it may appear in general position only in n' -parameter families of maps where $n' \geq n$. A quadratic homoclinic tangency is a codimension-1 bifurcation phenomenon. The main reason why bifurcations of higher codimensions can be created by small perturbations of bifurcations of low codimension (in proposition B: homoclinic tangencies of any order n by a small perturbation of a map with a quadratic homoclinic tangency) is the existence of some hidden parameters, the so-called local moduli, in the unperturbed system. The existence of moduli (continuous invariants) of the local Ω -conjugacy, and even the existence of infinitely many independent moduli, is a typical feature of systems with homoclinic tangency [23–25, 20]. Arbitrarily small variations in the value of any of these invariants change, by definition, the structure of the set of orbits lying in a small neighbourhood of the orbit of the point of the homoclinic tangency, without destroying the tangency. Thus, by varying the values of moduli one can obtain new bifurcating orbits without destroying the original one; hence, the degeneracy of the bifurcation can indeed be increased by an arbitrarily small perturbation (see [26] for more discussions). The moduli which were used in [20, 21] in the proof of proposition B in the non-conservative case do not exist (they degenerate into constants) in the area-preserving case. Moduli for the conservative case were found in [9], and proposition B can indeed be proved with the use of them.

Let us now show how proposition B implies proposition A. Let f be a map with infinitely many isolated homoclinic tangencies of any order. Let P be a point of homoclinic tangency of order n . Since P belongs to the stable manifold of some saddle periodic orbit, its forward iterations tend to this periodic orbit (we assume, for simplicity, that this is a fixed point of f ; otherwise, one should consider some power of f for which this periodic orbit is a fixed

point—then, the rest of the construction will be the same), i.e. $P_m \equiv f^m P \rightarrow Q$ as $m \rightarrow +\infty$ where $fQ = Q$. Analogously, since P is a homoclinic point, it belongs to the unstable manifold of Q as well, so its backward iterations also tend to Q : $P_m \rightarrow Q$ as $m \rightarrow -\infty$.

Let P^+ and P^- be a pair of points of the orbit of the homoclinic point P , lying in the local stable manifold W_{loc}^s and, respectively, in the local unstable manifold W_{loc}^u of the fixed point Q , sufficiently close to Q . Since P^- and P^+ are the points of the same orbit, it follows that $f^{\bar{m}} P^- = P^+$ for some $\bar{m} \geq 1$.

The point P^+ lies in the stable manifold of Q , so its forward iterations never leave a small neighbourhood of Q . However, the points arbitrarily close to P^+ which are not lying in W_{loc}^s will leave the small neighbourhood of Q after sufficiently many iterations of f . Moreover, one can show (see [27] or [21]) that for any small neighbourhoods Π^+ and Π^- of the points P^+ and P^- , respectively, and for any sufficiently large k there exist points in Π^+ whose k th forward iteration belongs to Π^- . We denote the set of these points as σ_k (for an illustration: if f is locally linear, i.e. if it is written as $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$ near $Q(0, 0)$, where $0 < \lambda < 1$, then $W_{\text{loc}}^s = \{y = 0\}$, $W_{\text{loc}}^u = \{x = 0\}$ and $P^+ = (x^+, 0)$, $P^- = (0, y^-)$, for some small x^+ , y^- ; if Π^\pm are the ε -neighbourhoods of P^\pm for some small $\varepsilon > 0$, then $\sigma_k = \{|x - x^+| < \varepsilon, \|y - \lambda^k y^-\| < \lambda^k \varepsilon\}$). Recall that the map $f^{\bar{m}}$ takes a small neighbourhood of P^- into the small neighbourhood of P^+ , and we also have that f^k takes σ_k into a small neighbourhood of P^- . Hence, the map $T_k \equiv f^{\bar{m}+k} : \sigma_k \rightarrow \Pi^+$, called a *first-return map*, is defined for all sufficiently large k .

We will also consider n -parameter perturbations f_μ of the map f . We assume that the perturbations are localized in a small neighbourhood of one homoclinic point (say, the point $f^{-1}P^+$), i.e. f_μ coincides with f outside this small neighbourhood. Thus, our perturbations will not affect other homoclinic tangencies which f has (because the homoclinic tangency we consider is isolated by assumption). We choose our perturbations in such a way that the tangency between the stable and unstable invariant curves of Q at the point P^+ unfolds generically. It is a tangency of order n , so we need n -parameters (μ_1, \dots, μ_n) for the unfolding. In the local coordinates (ξ, η) near point P^+ in which the stable manifold is a curve $\eta = \phi(\xi)$ and the unstable manifold is a curve $\eta = \psi(\xi)$, we have $\phi(\xi) - \psi(\xi) = C\xi^{n+1} + O(\xi^{n+1})$ for the map f itself (C is a non-zero constant) and $\phi(\xi) - \psi(\xi) = \sum_{i=0}^{n-1} \mu_{i+1} \xi^i + C\xi^{n+1} + O(\xi^{n+1})$ for the map f_μ .

At small μ , the first-return maps T_k are still defined for all sufficiently large k . Since the domain σ_k where the map T_k is defined is very small (of size $O(\lambda^k)$ in the direction transverse to the stable manifold), it makes sense to rescale the coordinates in order to make the size of the domain bounded away from zero (this has been proved to be quite useful in the study of homoclinic bifurcations). Doing this by formulae given in [22], lemma 2 (see also [9] for the case of quadratic tangency), one can see that there exists a rescaling (i.e. a smooth coordinate transformation) which brings the map T_k to the following Hénon-like form

$$\bar{X} = Y + O(1), \quad \bar{Y} = -X + \sum_{i=0}^{n-1} M_{i+1} Y^i + C Y^{n+1} + O(1)$$

where the $O(1)$ -terms tend to zero along with all derivatives as $k \rightarrow +\infty$, uniformly on any compact set of values of (X, Y) . The coefficients (M_1, \dots, M_n) are functions of (μ_1, \dots, μ_n) and k , they can be considered as free parameters, no longer small: when μ run an arbitrarily small ball around zero, M run a ball of an arbitrarily large fixed size, provided k is taken sufficiently large (roughly, we have $M_i \sim \mu_i \lambda^{-k(n+2-i)/n}$). The domain of T_k in the coordinates (X, Y) becomes large as well and it covers all finite values in the limit $k \rightarrow +\infty$ (essentially, the rescaling blows the $O(\lambda^k)$ -thin strip σ_k up to a rectangle of size $O(\lambda^{-k/n})$).

Hence, by taking k sufficiently large, we can find μ arbitrarily close to zero, such that the map f_μ will have in its dynamical conjugacy class a map which is as close as we want to any given polynomial Hénon-like map of degree n . Our perturbations are localized, so they neither destroy other homoclinic tangencies, nor influence the dynamics near these tangencies. We can apply the same procedure near the rest of the homoclinic tangencies as well, and since we have infinitely many tangencies of arbitrarily high orders, this will give us, at the end, a map f^* , arbitrarily close to the original map f , the closure of whose dynamical class contains all Hénon-like maps. Now, we may apply theorem 1, which says that such a map f^* is universal. Since, by proposition B, our map f with infinitely many homoclinic tangencies could be taken arbitrarily close to any map with a quadratic homoclinic tangency, proposition A follows.

I would like to note that in other (non-conservative) situations universal maps were discussed in [18] and, in the C^1 -topology, in [19], more or less with the same implications; striking results on the complexity of the closure of the invariant manifolds of a hyperbolic fixed point of a dissipative map with a homoclinic tangency see e.g. in [28, 29].

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