Splitting of separatrices, scattering maps, and energy growth for a billiard inside a time-dependent symmetric domain close to an ellipse

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Abstract

We study billiard dynamics inside an ellipse for which the axes lengths are changed periodically in time and an $O(\delta)$-small quartic polynomial deformation is added to the boundary. In this situation the energy of the particle in the billiard is no longer conserved. We show a Fermi acceleration in such system: there exists a billiard trajectory on which the energy tends to infinity. The construction is based on the analysis of dynamics in the phase space near a homoclinic intersection of the stable and unstable manifolds of the normally hyperbolic invariant cylinder Λ, parameterised by the energy and time, that corresponds to the motion along the major axis of the ellipse. The proof depends on the reduction of the billiard map near the homoclinic channel to an iterated function system comprised by the shifts along two Hamiltonian flows defined on Λ. The two flows approximate the so-called inner and scattering maps, which are basic tools that arise in the studies of the Arnold diffusion; the scattering maps defined by the projection along the strong stable and strong unstable foliations $W^{ss,uu}$ of the stable and unstable invariant manifolds $W^{s,u}(\Lambda)$ at the homoclinic points. Melnikov type calculations imply that the behaviour of the scattering map in this problem is quite unusual: it is only defined on a small subset of Λ that shrinks, in the large energy limit, to a set of parallel lines $t = const$ as $\delta \to 0$.

(Some figures may appear in colour only in the online journal)

1. Introduction and main results

Billiards are Hamiltonian dynamical systems, representing the motion of a point particle inside a domain Q (the billiard table) in a straight line with constant speed and elastically bouncing off the boundary of the domain ∂Q. The study of billiard systems was initiated by Birkhoff [1]. Depending on the boundary shape, billiard's dynamical behaviour may range from completely integrable to chaotic. The billiard inside an ellipse is the only known integrable strictly convex billiard [2]. The integrability of elliptic billiard is closely connected to the existence of a continuous family of caustics. A caustic is a curve such that if a billiard trajectory segment is tangent to it, all other segments of the trajectory are also tangent to the same curve. For the trajectories that do not intersect the segment connecting the foci of the ellipse, the caustics are confocal ellipses while for the trajectories that intersect this segment the caustics are confocal hyperbolas. The period two trajectory along the major axis is hyperbolic, with stable and unstable manifolds that coincide. The corresponding motions repeatedly go through the foci and converge to the major axis.

Billiards with time-dependent boundaries have received much attention in recent years [3]. One of the fundamental issues here is determining whether the particle energy may grow without bound as a result of repeated elastic collisions with the moving boundary. This phenomenon is called Fermi acceleration, after Fermi who first proposed it in his studies of highly energetic cosmic rays [4]. The existence of Fermi acceleration has been investigated theoretically and numerically in various billiard geometries. The simplest one-dimensional case corresponding to a particle bouncing between two periodically moving walls (Fermi-Ulam model) and its variants is already very subtle and the existence of Fermi acceleration has been shown to depend on the class of smoothness of the motion of the wall [5–7].

For domains in two dimensions and higher, it has been observed [3, 8, 11–14] that the acceleration depends on the structure of the phase space of the static 'frozen' billiard. It has been conjectured by Losktutov *et al* [8] and consequently proved in [15] that a sufficient condition for Fermi acceleration is the presence of a Smale horseshoe in the phase space of the frozen billiard. On the other hand, it has been shown [12] that the energy of trajectories in the time-dependent circle billiard stays bounded due to the angular momentum conservation. Earlier Koiler *et al* [17] numerically studied time-dependent perturbations of elliptic billiards and did not observe sustained energy growth. However, more detailed numerical simulations by Lenz *et al* [18–20] showed slow growth of the particle speed when initial conditions belong to the separatrix region. An elliptic billiard with a slow boundary perturbation and a slow angular velocity was also studied by Itin and Neishtadt [21] who investigated the destruction of adiabatic invariants near a separatrix. In this paper we further push the study of Fermi acceleration in periodically perturbed ellipse.

Fermi acceleration question is a part of the general question of energy growth in *a priori* unstable Hamiltonian systems, that also includes Mather acceleration problem [9, 16, 22]. Since time-dependent billiards on a plane are given by a nonautonomous Hamiltonian systems with two and a half degrees of freedom, the billiard map is exact symplectic four-dimensional

diffeomorphism [3, 26]. In particular, invariant KAM-tori, if exist, do not divide the phase space into invariant regions and Arnold diffusion [28] may occur. Arnold diffusion refers to the instability of action variables in a Hamiltonian system with $n > 2$ degrees of freedom of the form $H = H_0(I) + \epsilon H_1(I, \varphi, \epsilon)$ where (I, φ) are action-angle variables, ϵ is small, and H_0 is integrable. Following terminology in [27], a Hamiltonian system is called a-priori unstable if the integrable part H_0 has a normally hyperbolic invariant manifold Λ with stable and unstable manifolds $W^{s,u}(\Lambda)$ that coincide in a homoclinic loop. Under small perturbations, Λ and $W^{s,u}(\Lambda)$ persist but $W^{s,u}(\Lambda)$ may intersect transversally along a homoclinic set Γ. In this case the diffusing orbit stays near Λ most of the time, occasionally making a trip near Γ. It was shown by Treschev [10] and Delshams *et al* [30] that such homoclinic excursions can lead to a systematic drift of the action variable in the *a priori* unstable case.

The technique for the analysis of such excursions, which is also used in this paper, goes back to the works of Delshams *et al* [22, 29] where notions of the inner and scattering maps have been introduced and studied in detail. The inner map is the restriction of the dynamics to Λ, and the scattering map relates two points on Λ that are connected asymptotically in the past and future if the intersection of $W^{s,u}(\Lambda)$ is *strongly transverse* [31] along Γ. The iteration function system obtained by successive application of these two maps in an arbitrary order gives the diffusing orbit if they do not have common invariant curves [23–25, 31]. It was shown e.g. in [31] that the finite-length diffusing orbits of the iterated function system on Λ correspond to Arnold diffusion in the original diffeomorphism near $\Lambda \cup \Gamma$, under the assumption of strong transversality of homoclinic intersections. This result was generalised by Gidea *et al* [32] to the orbits of semi-infinite length.

In this paper we study the time-dependent four-dimensional billiard map B (defined in section 3) describing the motion of a billiard inside the planar domain with the time-dependent boundary

$$\frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} = 1 + \frac{2\delta y^4}{b^4(t)}, \tag{1}$$

where $0 < |\delta| \ll 1$ is a constant parameter and $0 < b(t) < a(t)$ are periodic C^{r+1}-smooth functions ($r \geqslant 4$) of time (so $t \in \mathbb{S}^1$), and $(x, y) \in \mathbb{R}^2$. The boundary (1) may be viewed as an ellipse with time-periodically changing semi-axes $a(t)$ and $b(t)$ plus an $O(\delta)$ quartic polynomial perturbation superimposed at each fixed value of t. Let $\mathcal{E}(t)$ be the kinetic energy of the particle in the billiard bounded by (1). We prove here the following

Theorem 1.1. *Let $0 < b(t) < a(t)$ be time-periodic C^{r+1}-functions ($r \geqslant 4$) such that the function $\frac{a(t)}{b(t)}$ has a nondegenerate critical point. Then, there exists a constant $C > 0$, independent of δ, such that for any $\mathcal{E}_0 \geqslant \frac{C}{|\delta|}$, there exists a billiard trajectory for which the energy $\mathcal{E}(t)$ grows from $\mathcal{E}_0$ to infinity.*

We note that if one replaces the $O(\delta)$ quartic polynomial perturbation in the right-hand side of (1) by another $O(\delta)$ perturbation that also destroys integrability of the static frozen ellipse for every fixed t (for instance symmetric entire perturbations studied in [35]), then theorem 1.1 should still hold. We however restrict ourselves to a particular form of the perturbation, in order to keep the computations explicit. We do not know whether the measure of the set of orbits for which the energy grows to infinity is positive. However, the same construction we use in the proof can show the existence of 'diffusing' orbits which take every sufficiently large value of energy infinitely many times, following an arbitrary given itinerary, so the evolution of energy is sensitive to initial conditions and has to be described by some random process.

The proof of theorem 1.1 is based on the study of inner and scattering maps and an application of the theory developed in [31]. The scheme of the proof is as follows.

Each time the particle hits the boundary of (1), one records the collision time moment $t \in \mathbb{S}^1$, the particle kinetic energy $\mathcal{E}$, the angular variable φ that determines the position of the collision point on the billiard boundary, and the post-collision reflection angle θ. Then, the particle motion is described by the billiard map B in the four-dimensional space of variables $(\varphi, \theta, \mathcal{E}, t)$. We assume that the speed $w = \sqrt{2\mathcal{E}}$ of the particle is large compared to the speed with which the boundary moves. This invokes the presence of two time scales in the problem: the variables $\mathcal{E}$ and t vary slowly, while (φ, θ) change fast. To make the presence of different time scales more apparent, we scale variables like it was done in [9]: take large speed w^*, introduce a small parameter $\varepsilon = \frac{1}{w^*}$ and the rescaled energy $E = \varepsilon^2 \mathcal{E}$. Then, for any bounded interval of E, the map B becomes near-identity in terms of (E, t), i.e. it becomes ε-close to the two-dimensional billiard map B_s corresponding to static boundary (1) at fixed t, with augmented phase space to account for (E, t). Hence, the map B may be expanded in series of $\varepsilon = \frac{1}{w^*}$ and δ. We will give full details of this construction in section 3.

If the particle moves along the major axis, it will never leave the major axis. This motion corresponds to an invariant manifold Λ in the phase space of the billiard map B, a two-dimensional cylinder parametrised by $(\mathcal{E}, t)$. In the static billiard, the motion along the major axis is a saddle periodic orbit for each frozen value of $\mathcal{E}$ and t. Therefore, because the map B is close so the static billiard map in the rescaled variables, it follows that the cylinder Λ is normally hyperbolic. In particular, it has three-dimensional stable and unstable manifolds $W^{s,u}(\Lambda)$ foliated by the strong stable and unstable foliations $W^{ss,uu}(\Lambda)$. These geometric objects are inherited by B from the stable and unstable separatrices of the static billiard's motion along the major axis.

The restriction of B to Λ is close to identity (when written in the coordinates (E, t) where E is the rescaled energy). It is well-known [33, 34] that a near-identity C^l-smooth (or analytic) symplectic map $x_1 = x_0 + \nu f(x_0)$ with small ν is approximated by a time-ν shift along the orbits of an autonomous Hamiltonian system up to accuracy $O(\nu^{l+1})$ (or exponential accuracy in ν for analytic case). Therefore, the map $B \mid_\Lambda$ (which we call the inner map) is approximated to a high level of accuracy by the time-shift along level curves of a certain Hamiltonian H_{in}.

As we mentioned, the billiard in ellipse is integrable, so the stable and unstable manifolds of the periodic orbit that corresponds to the motion along the major axis coincide. It is well-known that the resulting separatrix surface is a graph of a function θ of φ, where $\varphi \in (0, \pi)$. Therefore, for any small $\beta > 0$, the (perturbed) stable and unstable manifolds $W^{s,u}(\Lambda)$ at sufficiently small ε and δ can be expressed as graphs $\theta = \theta_{\varepsilon,\delta}^{s,u}(\varphi, E, t)$ (see section 4.3) over the interval $\varphi \in (\beta, \pi - \beta)$. They are $O(\varepsilon, \delta)$-close to the unperturbed manifolds $\theta = \theta_{0,0}^{s,u}$, therefore they can be expanded in series of ε, δ. The zeroes of the difference (see section 4.3) $\theta_{\varepsilon,\delta}^s - \theta_{\varepsilon,\delta}^u$ correspond to the primary homoclinic intersections $(\varphi_0, \theta_0, \mathcal{E}_0, t_0) \in \Gamma$ where Γ is the homoclinic set. If the corresponding unstable leaf of the foliation $W^{uu}(\Lambda)$ intersects transversely the stable manifold $W^s(\Lambda)$ at the homoclinic point $(\varphi_0, \theta_0, \mathcal{E}_0, t_0)$, the intersection is called strongly transversal (see section 4). When strong transversality condition is satisfied, projecting from the homoclinic point to Λ along the corresponding unstable fiber of $W^u(\Lambda)$ and stable fiber of $W^s(\Lambda)$ produces a pair of points in Λ which are related by what is called the scattering map S_Γ. Its domain of definition $\bar{\Lambda} \subset \Lambda$ is the projection of the set of strong-transverse primary homoclinic points by the strong unstable fibers; the image $S_\Gamma(\bar{\Lambda}) \subset \Lambda$ is the projection of the set of primary homoclinic points by the strong stable fibers.

The scattering map is exact symplectic [22, 31]. For any bounded interval of the rescaled energy E, this map is close to identity, so it is well approximated by the time-ε shift along the
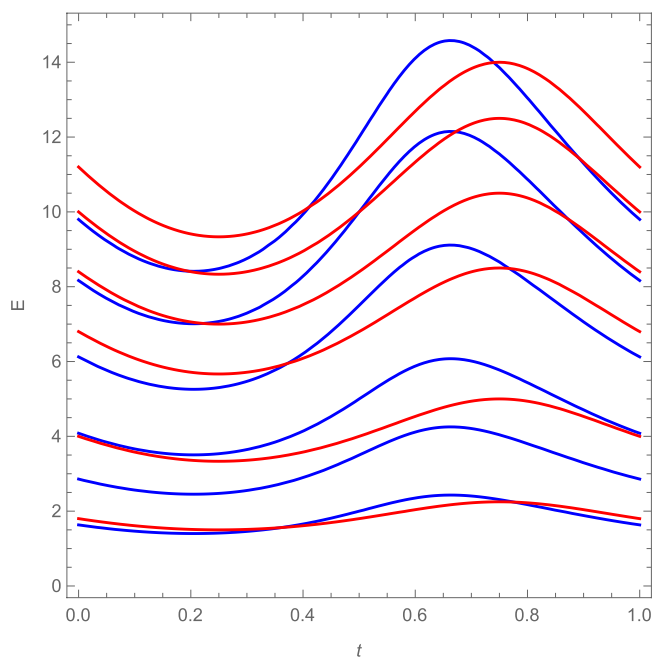
Figure 1. The level lines of H_{in} (in red) and H_{out} (in blue), in scaled variables (E, t), for $a(t) = 5 + \sin(2\pi t)$, $b(t) = 2 - \cos(2\pi t)$.

level curves of a Hamiltonian H_{out}. We build the trajectory whose energy grows to infinity by following level curves of H_{in} and H_{out} (see figure 1), and switching to the level curve which leads to the larger energy gain in the immediate future. This construction is similar to [16], however the application of inner and scattering maps is novel. Formally, we define H_{in} and H_{out} everywhere on Λ but the switch of the orbit of B to the level curve of H_{out} is only allowed at the domain of definition of the scattering map. We find that this domain has a non-trivial structure in our problem.

Namely, we find that the strong transversality is only satisfied in the limit of large energy if $\delta > 0$, and that at $\delta = 0$ the projection of the primary homoclinic set to the cylinder Λ by the strong unstable fibers shrinks to a set of parallel lines $t = const$ as $\varepsilon = 0$. To this aim, we put $\delta = 0$ and investigate the splitting of invariant manifolds $W^{s,u}(\Lambda)$. The first term of the power expansion in ε for the distance between $(\theta^s_{\varepsilon,0} - \theta^u_{\varepsilon,0})$ between perturbed $W^{s,u}(\Lambda)$ is given by the so-called Melnikov function $M_1(\varphi, \theta, E, t)$. Non-degenerate zeros of M_1 correspond to transverse primary intersections of W^s and W^u, if ε is small enough. We show

Theorem 1.2. *Consider the time-dependent elliptic billiard map B without the quartic perturbation (i.e. $\delta = 0$). The Melnikov function associated to the splitting of invariant manifolds $W^{s,u}(\Lambda)$ has zeroes only for the times t^* such that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a(t^*)}{b(t^*)}\right) = 0$, for all values of energy and reflection angle. If t^* is a nondegenerate critical point of $\frac{a(t)}{b(t)}$, there exists a corresponding transverse intersection of $W^{s,u}(\Lambda)$ along a 2-dimensional homoclinic surface where $(\theta, t) = (\theta^{s,u}_{0,0}, t^*) + O(\epsilon)$ are smooth functions of $(\mathcal{E}, \varphi)$.*

In the cylinder Λ, the image of the two-dimensional homoclinic intersection found in this theorem by the projection along the strong-unstable fibers is confined in a narrow strip around

the critical lines $t = t^*$. This means that the scattering map is not properly defined if $\delta = 0$ (to estimate the domain of definition of the scattering map we would need further expansion of the separatrix splitting function in powers of ε, but we suspect that it is small beyond all orders). We conjecture that the same structure is characteristic of a more general case of an integrable system with slowly varied parameters.

In order to have a scattering map defined, we add the δ-dependent term in (1). The noninte-grability of static elliptic billiards subject to polynomial perturbations was studied in [35–38]. While we use the Melnikov function calculations from these works, we also develop a novel Melnikov function technique for the computation of the scattering map for systems with normally-hyperbolic invariant manifolds (e.g. time-dependent billiards). In particular, we show that the domain $\bar{\Lambda}$ of definition of the scattering map S_Γ (see figure 2) in our situation has an unusual shape at small δ—it contains essential curves only at very large energies. Namely, to the first order in $\frac{1}{\sqrt{\mathcal{E}}}$ and δ the domain $\bar{\Lambda}$ is given by

$$\sqrt{\mathcal{E}} > \frac{|a\frac{db}{dt} - b\frac{da}{dt}|}{|\delta|} \, \phi(t),$$

where $\phi(t)$, defined by (50) is a strictly positive, continuous, periodic function of t. More precisely, we have the following

Theorem 1.3. *For any constant $k > 0$ there exists $C > 0$ such that all points $(\mathcal{E}, t)$ in the cylinder Λ, which satisfy*

$$\sqrt{\mathcal{E}} > \frac{|a\frac{db}{dt} - b\frac{da}{dt}|}{|\delta|}\phi(t) + k, \qquad \mathcal{E} \geqslant \frac{C(k)}{\delta}, \tag{2}$$
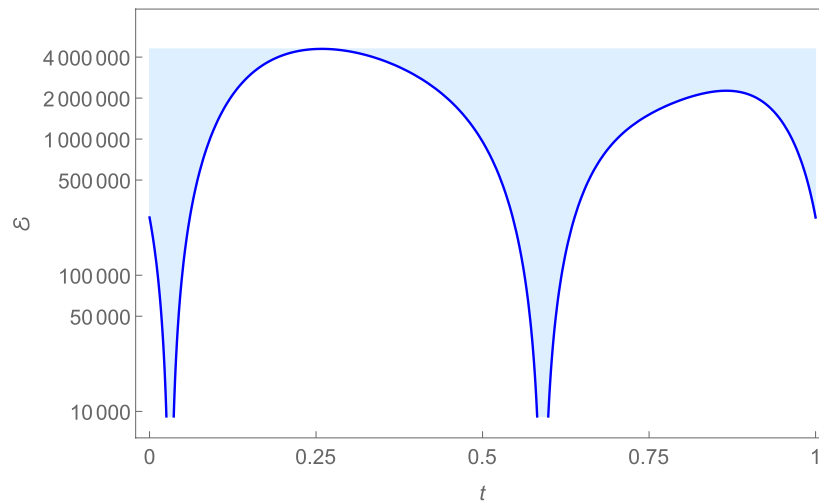
belong to the domain $\bar{\Lambda}$ of the definition of the scattering map S_Γ. Vice versa, the points which satisfy

$$\sqrt{\mathcal{E}} < \frac{|a\frac{db}{dt} - b\frac{da}{dt}|}{|\delta|}\phi(t) - k, \tag{3}$$

do not belong to $\bar{\Lambda} \cap \{\mathcal{E} \geqslant \frac{C(k)}{\delta}\}$.

It is seen from these formulas that $\bar{\Lambda}$ contains a circle $\mathcal{E} = const$ only if $\mathcal{E} > C_1\delta^{-2}$ where C_1 is some constant. In this region of energies techniques of [15] can be applied in order to prove the existence of orbits for which the energy tends to infinity starting from $\mathcal{E} \sim \delta^{-2}$. Our theorem 1.1 gives a stronger result by allowing to start at much lower energies $\mathcal{E} \sim \delta^{-1}$.

Our paper is organised as follows. In section 2 we review the known facts about the static elliptic billiard. Section 3 introduces the time-dependent, perturbed billiard map, where we show how the rescaling of billiard speed gives rise to a slow-fast billiard map. In section 4 we study the inner and scattering maps. We compute the splitting of stable and unstable invariant manifolds of Λ and use this result to derive a first order formula for the scattering map, and therefore give proofs of theorems 1.2 and 1.3. We also derive the Hamiltonians H_{in} and H_{out} that give the first order approximations to inner and scattering maps. We provide the estimates on energy growth via asymptotic study of inner and outer Hamiltonian vector fields and provide a proof of theorem 1.1 in section 5. The appendices A–E contain the derivation of the Melnikov function giving the first order distance between perturbed invariant manifolds $W^{s,u}(\Lambda)$ and its explicit computation with elliptic functions; they also provide computations for the scattering map.

Figure 2. The domain of definition $\bar{\Lambda}$ of the scattering map S_Γ, in physical variables $(\mathcal{E}, t)$. The dark blue curve is the graph of $\left(\frac{|a \frac{db}{dt} - b \frac{da}{dt}|}{|\delta|} \phi(t) \right)^2$, with $\phi(t)$ given by (50). The shaded region is the domain $\bar{\Lambda}$. Here $a(t) = 5 + \sin(2\pi t)$, $b(t) = 2 - \cos(2\pi t)$, $\delta = 0.05$.

2. Static elliptic billiard

The following facts are well-known, see for example [35, 37, 39]. Our exposition follows [37]. Let us consider a billiard inside a static ellipse. In Cartesian coordinates, we may define the analytic boundary Q of the ellipse by

$$Q = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}, \tag{4}$$

where $0 < b < a$. Here a is the half-length of the major axis and b is half-length of the minor axis. The foci are at $(\pm c, 0)$ where $c = \sqrt{a^2 - b^2}$. Let us parameterise the ellipse as $\gamma(\varphi) : [0, 2\pi) \to Q$, where

$$\gamma(\varphi) = \{(a \cos(\varphi), b \sin(\varphi)) : \varphi \in [0, 2\pi)\}. \tag{5}$$

Let us introduce the angle of reflection $\theta \in (0, \pi)$ of the particle velocity vector made with the positive tangent to $\gamma(\varphi)$ at the collision point. We define the static billiard map $B_s : (\varphi_n, \theta_n) \mapsto (\varphi_{n+1}, \theta_{n+1})$ (with subscript s for 'static'). Observe that Q is symmetric with regard to the origin. As in the work by Tabanov [37], we may exploit this symmetry for B_s, by identifying the points on the ellipse that are π across, hence defining $\varphi \pmod{\pi}$. The following formulas for B_s are known [37]:

$$\varphi_{n+1} = -\varphi_n + 2 \arctan \left(\frac{b(a \tan(\varphi_n) + b \tan(\theta_n))}{a(b - a \tan(\varphi_n) \tan(\theta_n))} \right) \pmod{\pi}, \tag{6}$$

$$\theta_{n+1} = -\theta_n + \arctan \left(\frac{b}{a \tan(\varphi_n)} \right) - \arctan \left(\frac{b}{a \tan(\varphi_{n+1})} \right) \pmod{\pi}. \tag{7}$$

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The map B_s is analytic and preserves the symplectic form $|\gamma'(\varphi)|\mathrm{d}\varphi \wedge \mathrm{d}\theta$, that becomes standard symplectic form $\mathrm{d}s \wedge d(\cos(\theta))$ in coordinates $(s, \cos(\theta))$, where s is the arc length associated to Q. The map B_s has a hyperbolic saddle fixed point $z = (0, \pi/2)$ with eigenvalues $\{\lambda, \frac{1}{\lambda}\}$, where

$$\lambda = \frac{a+c}{a-c} > 1. \tag{8}$$

The other fixed point $(\pi/2, \pi/2)$ is elliptic.

The elliptic billiard is integrable: the first integral I of B_s may be physically interpreted as the conservation of the inner product of angular momenta about the foci. The integral I may be written as

$$I(\varphi, \theta) = b^2 \cos^2(\theta) - c^2 \sin^2(\theta) \sin^2(\varphi). \tag{9}$$

Remark 2.1. Tabanov [37] gives the integral as $\tilde{I} = \cosh^2 \mu \cos^2(\theta) + \cos^2(\varphi) \sin^2(\theta)$ in the elliptical coordinates $x = h \cosh \mu \cos(\varphi)$, $\quad y = h \sinh \mu \sin(\varphi)$, where $h^2 = a^2 - b^2$. Upon changing from elliptical coordinates to the parameterisation $\gamma(\varphi)$ above and using $a^2 - b^2 = c^2$, we have

$$\tilde{I} = \frac{a^2}{c^2} \cos^2(\theta) + \cos^2(\varphi) \sin^2(\theta),$$

which gives us

$$c^2 \tilde{I}(\varphi, \theta) = a^2 \cos^2(\theta) + c^2 \cos^2(\varphi) \sin^2(\theta) = b^2 \cos^2(\theta) - c^2 \sin^2(\varphi) \sin^2(\theta) + c^2.$$

Rearranging, we have

$$I(\varphi, \theta) = c^2 \tilde{I} - c^2 = b^2 \cos^2(\theta) - c^2 \sin^2(\varphi) \sin^2(\theta),$$

which gives us (9).

The level set $I = -c^2$ corresponds to the elliptic fixed point; for $-c^2 < I < 0$ the billiard trajectories cross the major axis between the foci and have confocal hyperbolas as caustics, and for $0 < I < b^2$ trajectories cross the major axis outside the foci and have confocal ellipses as caustics. Zero level set, $I(\varphi, \theta) = 0$, corresponds to the union of homoclinic orbits that comprise two coincident branches $W_{1,2} = W_{1,2}^s(z) = W_{1,2}^u(z)$, the stable and unstable manifolds of z. From $I(\varphi, \theta) = 0$, the union $W(z) = W_1(z) \bigcup W_2(z)$ is given by the expression [37]:

$$\sin^2(\varphi) = \frac{b^2}{c^2 \tan^2(\theta)}. \tag{10}$$

Physically, $W_1(z)$ corresponds to the billiard trajectory segments repeatedly passing through the focus at $(c, 0)$ while $W_2(z)$ correspond to trajectories passing the focus at $(-c, 0)$. These trajectories asymptotically tend to the major axis of the ellipse, which corresponds to the saddle fixed point z of B_s (recall that we take φ by modulo π).

One can obtain explicit expressions for $B_s^n|_{W_{1,2}(z)}$ for $n \in \mathbb{Z}$. We have

$$B_s^n(\varphi_0, \theta_0)|_{W_1(z)} = \left(2 \arctan\left(\lambda^n \tan\left(\frac{\varphi_0}{2} \right) \right), \arctan\left(\frac{-b}{c \sin(\varphi_n)} \right) \right), \tag{11}$$

$$B_s^n(\varphi_0, \theta_0)|_{W_2(z)} = \left(2\arctan\left(\lambda^{-n}\tan\left(\frac{\varphi_0}{2}\right)\right), \arctan\left(\frac{b}{c\sin(\varphi_n)}\right)\right). \tag{12}$$

Let us introduce the variable $\xi \in (0, \infty)$ such that $\xi_n = \tan(\varphi_n/)2$, $n \in \mathbb{Z}$, so that (11) gives

$$\xi_n = \lambda^n \xi_0, \qquad \tan(\theta_n) = \frac{-b(1 + \xi_n^2)}{2c\xi_n}, \tag{13}$$

while (12) gives

$$\xi_n = \lambda^{-n}\xi_0, \qquad \tan(\theta_n) = \frac{b(1 + \xi_n^2)}{2c\xi_n}. \tag{14}$$

Remark 2.2. Upon making the change of coordinates $\nu = \|\gamma'(\varphi)\|\cos(\theta)$, as in [35], the expression (10) becomes $\nu = \pm c\sin(\varphi)$ and the phase portrait of B_s resembles the one of the pendulum Hamiltonian $H = p^2/2 + \cos(q) - 1$. In spite of the integrability, the existence of the hyperbolic fixed point with a homoclinic orbit implies that global action-angle variables cannot be introduced in an elliptic billiard: it is an example of an *a priori unstable* system.

3. Time-dependent perturbed elliptic billiard

3.1. Billiard map setup

Let us consider a billiard inside a time-dependent convex curve $Q(q, t, \delta)$ that is $O(\delta)$ quartic polynomial perturbation of the ellipse for each fixed time t:

$$Q(q, t; \delta) := \{q = (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{S}^1 : \frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} = 1 + \frac{2\delta y^4}{b^4(t)}\}, \tag{15}$$

where a and b are periodic C^{r+1} ($r \geqslant 4$) functions of time t, such that $0 < b(t) < a(t)$ for all t, and $0 < |\delta| \ll 1$. Let us parameterise $Q(q, t; \delta)$ as

$$Q(q, t; \delta) = \{(a(t)\cos(\varphi), b(t)\sin(\varphi)\left(1 + \delta\sin^2(\varphi)\right)) + O(\delta^2), \varphi \in [0, 2\pi), t \in \mathbb{S}^1\}. \tag{16}$$

The $O(\delta^2)$ terms do not play any role in our work.

Remark 3.1. Polynomial perturbations of billiards in ellipses were considered previously in a number of works [35–38], however these works considered static perturbations only, not time-dependent ones.

Assuming the billiard reflection at the moment of collision with the moving boundary is elastic, we define the time-dependent billiard map $B : (\varphi_n, \theta_n, \mathcal{E}_n, t_n) \mapsto (\varphi_{n+1}, \theta_{n+1}, \mathcal{E}_{n+1}, t_{n+1})$. Here φ_n is the collision point on the boundary at the nth collision, θ_n is the reflection angle of the post-collision particle velocity vector made with the positively oriented tangent to the boundary at the collision point, $\mathcal{E}_n$ is the particle post-collision energy, and t_n is the time of the nth collision. It is known that B is symplectic hence volume-preserving [3, 26]. Since the boundary curve (16) is analytic with respect to φ and C^{r+1} in t, it is known that B is a C^r diffeomorphism [3].

Denote the speed of the particle as w, its corresponding velocity as $\mathbf{w}$, its energy as $\mathcal{E} = \frac{w^2}{2}$; the speed of the boundary in the direction of outward normal is given by $u(q,t) = -\frac{1}{\nabla_q Q(q,t)} \frac{\partial Q(q,t;\delta)}{\partial t}$, and the unit outward normal is $\mathbf{n} = \frac{\nabla_q Q(q,t;\delta)}{\|\nabla_q Q(q,t;\delta)\|}$. We assume that positive u corresponds to outward motion of the boundary. The following formula [26] gives the change in velocity at the boundary collision:

$$\mathbf{w}_{n+1} = \mathbf{w}_n - 2\langle \mathbf{w}_n, \mathbf{n}_{n+1} \rangle \, \mathbf{n}_{n+1} + 2u(\varphi_{n+1}, t_{n+1})\mathbf{n}_{n+1}. \tag{17}$$

By analogy with [3], let us introduce the auxilliary variable θ^*, which denotes the angle of incidence at the $(n+1)$th impact with the tangent to the boundary, and let α denote the angle between the tangent to $Q(q,t;\delta)$ and the x-axis, given by $\tan(\alpha) = \frac{y'(\varphi)}{x'(\varphi)}$, with $' = \frac{d}{d\varphi}$ (at each fixed t) and $x(\varphi)$, $y(\varphi)$ defined by (16). Since (16) is symmetric with respect to the origin, let us define $\varphi \pmod{\pi}$ as in section 2, thus identifying points on the boundary that are π across. Define $u(\varphi_{n+1}, t_{n+1})$ to be the normal speed of the boundary at $(n+1)$th impact. In this way, we obtain an implicit form for the billiard map

$$B : (\varphi_n, \theta_n, \mathcal{E}_n, t_n) \mapsto (\varphi_{n+1}, \theta_{n+1}, \mathcal{E}_{n+1}, t_{n+1}),$$

given by the following formulas (more details can found in [3]):

$$a(t_{n+1})\cos(\varphi_{n+1}) = a(t_n)\cos(\varphi_n) + \sqrt{2\mathcal{E}_n}(t_{n+1} - t_n)\cos(\alpha_n + \theta_n),$$
$$b(t_{n+1})\sin(\varphi_{n+1})\left(1 + \delta\sin^2(\varphi_{n+1})\right) = b(t_n)\sin(\varphi_n)\left(1 + \delta\sin^2(\varphi_n)\right) + \sqrt{2\mathcal{E}_n}(t_{n+1} - t_n)\sin(\alpha_n + \theta_n),$$
$$\theta_n + \alpha_n + \theta^*_{n+1} - \alpha_{n+1} = 0,$$
$$\sqrt{2\mathcal{E}_{n+1}}\cos(\theta_{n+1}) = \sqrt{2\mathcal{E}_n}\cos(\theta^*_{n+1}),$$
$$\sqrt{2\mathcal{E}_{n+1}}\sin(\theta_{n+1}) = \sqrt{2\mathcal{E}_n}\sin(\theta^*_{n+1}) - 2u(\varphi_{n+1}, t_{n+1}). \tag{18}$$

The first pair of equations of (18) implicitly defines t_{n+1} and φ_{n+1}, while the last three give $\mathcal{E}_{n+1}$ and θ_{n+1} after expressing θ^*_{n+1} in terms of φ_{n+1}, θ_n and φ_n. The last pair of equations in (18) corresponds to (17) written in components normal and tangential to the boundary, and they give the expression for the change of energy

$$\mathcal{E}_{n+1} = \mathcal{E}_n - 2\sqrt{2E_n}u(\varphi_{n+1}, t_{n+1})\sin(\theta^*_{n+1}) + 2u^2(\varphi_{n+1}, t_{n+1}). \tag{19}$$

Let us denote by D the Euclidean distance between φ_n and φ_{n+1}, then we have

$$t_{n+1} = t_n + \frac{D}{\sqrt{2\mathcal{E}_n}}, \tag{20}$$

where the expression

$$D = \sqrt{\left[a(t_{n+1})\cos(\varphi_{n+1}) - a(t_n)\cos(\varphi_n)\right]^2 + \left[b(t_{n+1})\sin(\varphi_{n+1})\left(1 + \delta\sin^2(\varphi_{n+1})\right)\right) - b(t_n)\sin(\varphi_n)\left(1 + \delta\sin^2(\varphi_n)\right)\right]^2} \tag{21}$$

is obtained from the first two equations of (18).

We assume that the initial speed of the particle is much larger than the speed of the boundary, so that the shape of the billiard table does not change significantly from one impact to the next. This implies that the time interval between two consecutive collisions is small and the change in the speed of the particle due to a single collision is small compared to the initial speed of the particle. Motivated by this, let us write the billiard map in a 'slow-fast' form. Let us take an initial large value of speed w^* and introduce a small parameter $\varepsilon = \frac{1}{w^*}$ such that $0 < \varepsilon \ll 1$. Let us introduce the scaled speed v that is related to the original physical variable w through $v = \frac{w}{w*}$. In terms of ε this gives

$$w = \frac{v}{\varepsilon}. \tag{22}$$

This transformation is equivalent to $\mathcal{E} = \frac{E}{\varepsilon^2}$, where E is the *rescaled* energy. The billiard map in the rescaled energy and time variables becomes close to identity, since (19) transforms to

$$E_{n+1} = E_n - 2\varepsilon\sqrt{2E_n}u(\varphi_{n+1}, t_{n+1})\sin(\theta_{n+1}^*) + 2\varepsilon^2 u^2(\varphi_{n+1}, t_{n+1}), \tag{23}$$

or, in terms of v,

$$v_{n+1} = v_n - 2\varepsilon u(\varphi_{n+1}, t_{n+1})\sin(\theta_{n+1}^*) + O(\varepsilon^2) \tag{24}$$

Using (22) transforms the equation (20) to

$$t_{n+1} = t_n + \frac{\varepsilon D}{\sqrt{2E_n}}. \tag{25}$$

Note that in the limit $\varepsilon \to 0$ the variables (t, E) become constants, i.e. the billiard map coincides with the frozen billiard map in the domain bounded by (15).

Now, by virtue of C^r-smoothness of the boundary and smallness of ε and δ we write the time-dependent billiard map $B = B_{\varepsilon,\delta}(\varphi_n, \theta_n, E_n, t_n) \mapsto (\varphi_{n+1,\varepsilon,\delta}, \theta_{n+1,\varepsilon,\delta}, E_{n+1,\varepsilon,\delta}, t_{n+1,\varepsilon,\delta})$ in the form $B_{\varepsilon,\delta} = B_0 + \varepsilon B_1 + \delta B_2 + O(\varepsilon^2 + \delta^2)$. We define

$$\begin{aligned}
\varphi_{n+1,\varepsilon,\delta} &= \varphi_{n+1} + \varepsilon f_1(\varphi_n, \theta_n, E_n, t_n) + \delta g_1(\varphi_n, \theta_n, t_n) + O(\varepsilon^2 + \delta^2), \\
\theta_{n+1,\varepsilon,\delta} &= \theta_{n+1} + \varepsilon f_2(\varphi_n, \theta_n, E_n, t_n) + \delta g_2(\varphi_n, \theta_n, t_n) + O(\varepsilon^2 + \delta^2), \\
E_{n+1,\varepsilon} &= E_n + \varepsilon f_3(\varphi_n, \theta_n, E_n, t_n) + \varepsilon O(\varepsilon + |\delta|), \\
t_{n+1,\varepsilon} &= t_n + \varepsilon f_4(\varphi_n, \theta_n, E_n, t_n) + \varepsilon O(\varepsilon + |\delta|).
\end{aligned} \tag{26}$$

Here $B_1 = (f_1, f_2, f_3, f_4)^\top$ and $B_2 = (g_1, g_2, 0, 0)^\top$ (with $\top$ denoting the transpose). We use the notation $(\varphi_{n+1}, \theta_{n+1}) = B_s(\varphi_n, \theta_n)$, i.e. B_0 is the same as the static two-dimensional billiard map B_s in the ellipse (4), with the increase of the phase space dimension to account for the two 'frozen' variables E and t; so, $B_0(\varphi, \theta, E, t) = (B_s(\varphi, \theta), E, t)$.

Let us call B_0 the *unperturbed* time-dependent elliptic billiard map. The map B_0 is integrable with the first integral (9) where $b = b(t)$ and $c = c(t)$ are fixed, and two more trivial first integrals $I_2 = E$, $I_3 = t$. The phase space of B is $[0, \pi) \times (0, \pi) \times \mathbb{R}^+ \times \mathbb{S}^1$, with $\varphi \in [0, \pi)$, $\theta \in (0, \pi)$, $E \in \mathbb{R}^+$, and $t \in \mathbb{S}^1$.

Remark 3.2. Since we consider high billiard energies, we assume that the billiard reflection angle $\theta \in (0, \pi)$, i.e. the situations described in [3] where the billiard trajectory continues in a tangential or 'outward' direction to the boundary at the moment of collision do not occur.

By substituting the expression for $t_{n+1,\varepsilon,\delta}$ and $\varphi_{n+1,\varepsilon,\delta}$ from (26) into (21) and expanding in Taylor series, we find the zero-th order in ε and δ free-flight distance

$$D_0 = \sqrt{a^2[\cos(\varphi_n) - \cos(\varphi_{n+1})]^2 + b^2[\sin(\varphi_n) - \sin(\varphi_{n+1})]^2}. \tag{27}$$

Upon substituting expansion (26) into (18), and examining the coefficients of the order ε terms, we find that $B_1 = (f_1, f_2, f_3, f_4)^\top$ is given by the following expressions:

$$f_1 = \frac{D_0}{\sqrt{2E_n}}\left(\frac{\dot{a}\cos(\varphi_{n+1})\tan(\theta_n + \alpha_n) - \dot{b}\sin(\varphi_{n+1})}{a\sin(\varphi_{n+1})\tan(\theta_n + \alpha_n) + b\cos(\varphi_{n+1})}\right), \tag{28}$$

$$f_2 = \frac{-2u\cos(\theta_{n+1})}{\sqrt{2E_n}} + \frac{a^2\sin^2(\varphi_{n+1})}{a^2\sin^2(\varphi_{n+1}) + b^2\cos^2(\varphi_{n+1})}\left(\frac{D_0(\dot{a}a^{-1}b - \dot{b})}{a\sqrt{2E_n}\tan(\varphi_{n+1})} + \frac{bf_1}{a\sin^2(\varphi_{n+1})}\right),$$

(29)

$$f_3 = -2\sqrt{2E_n}u\sin(\theta_{n+1}),$$

(30)

$$f_4 = \frac{D_0}{\sqrt{2E_n}}.$$

(31)

The dot above a and b denotes the derivative with respect to time evaluated at time t_n. We also denote $a = a(t_n), b = b(t_n)$, and $u = \frac{\dot{a}b\cos^2(\varphi_{n+1}) + a\dot{b}\sin^2(\varphi_{n+1})}{\sqrt{a^2\sin^2(\varphi_{n+1}) + b^2\cos^2(\varphi_{n+1})}}$ is the normal speed of the boundary. Similarly, comparing the coefficients of the first order in δ, we find that the expressions for g_i for $i = 1,..,4$ are independent of E and $\dot{a}, \dot{b}$.

Remark 3.3. Observe that f_1, f_2, f_3, f_4 are written in a certain 'cross-form' as functions of the image of (φ, θ, E, t) under B_0 as well as the initial point (φ, θ, E, t) itself; however since B is a C^r diffeomorphism, one may express the functions f_i in form (26).

Remark 3.4. Observe that the Taylor series expansion of the map $B_{\varepsilon,\delta}$ consists of two perturbations that may be considered independently in the first order of the perturbation parameters: the $O(\varepsilon)$ perturbation terms that arise due to rescaling of energy (this is B_1), and the $O(\delta)$ perturbation terms that correspond to the polynomial perturbation of the boundary (this is B_2, which is independent of t).

Remark 3.5. For $\varepsilon = 0$, the variables E and t are constant and thus $B_{0,\delta}$ becomes a billiard map corresponding to an ellipse with a quartic polynomial perturbation, with the semi-axes lengths fixed at $a(t_n), b(t_n)$. This is a twist map and thus has a certain generating function $L(\varphi_n, \varphi_{n+1})$ [35] . The nonintegrability of such convex billiard curves and the relation between generating function, Melnikov function and Melnikov potential was investigated in detail in [35]. Thus, for the computation of the Melnikov function we do not require the knowledge of the explicit form of B_2, as we will be using the generating function formulation and Melnikov potential [35] for the polynomial part of the perturbation.

3.2. Phase space geometry of the time-dependent billiard

If a point in our billiard moves along the major semi-axis, it will continue to move along this semi-axis forever. In other words, this motion is confined to an invariant manifold Λ in the phase space of the billiard map B. It is given by $(\varphi \pmod{\pi}, \theta) = (0, \pi/2)$ and is parameterized by the energy $\mathcal{E} \in \mathbb{R}^+$ and the time $t \in \mathbb{S}^1$, so Λ is a cylinder. If we rescale the energy and take the limit $\varepsilon = 0$, the cylinder Λ corresponds to the hyperbolic saddle fixed point $z = (0, \pi/2)$ of the static billiard map B_s, so Λ is a normally-hyperbolic invariant manifold of $B_{0,\delta}$.

It has two branches of stable and unstable three-dimensional invariant manifolds $W_i^{s,u}(\Lambda)$ with $i = 1,2$. At $\delta = 0$, they coincide and form two symmetric three-dimensional homoclinic manifolds W_i that are inherited from one-dimensional separatrices of z (see (10)). Denoting $W = \cup_{i=1}^2 W_i$, we have

$$W = \{(\varphi, \theta, E, t) : \varphi \in [0, \pi), \theta \in (0, \pi), E \in \mathbb{R}^+, t \in \mathbb{S}^1; \quad \sin^2(\varphi) = \frac{b^2(t)}{c^2(t)\tan^2(\theta)}\}.$$

(32)

By the theory of Fenichel [40], the normal hyperbolicity of Λ persists as the map $B_{0,\delta}$ is perturbed. In particular, the stable and unstable manifolds of any subset of Λ that corresponds to a bounded set of values of the rescaled energy E persist at all small ε, and depend continuously on ε (and δ). If we return to the non-rescaled energy variable $\mathcal{E}$, this gives us, for all small δ, the normal hyperbolicity, for the time-dependent billiard, of the piece of Λ that corresponds to sufficiently large values of $\mathcal{E}$; with the stable and unstable manifolds $W^{s,u}(\Lambda)$ close to those for the frozen billiard map and depending continuously on δ.

At $\varepsilon = 0$, the stable and unstable manifolds $W_i^{s,u}$ are foliated by strong stable and strong unstable one-dimensional fibers $W_{(E,t);i}^{ss,uu} : (E,t) = const$. They form smooth invariant foliations transverse to Λ; such foliations are unique and persist at small smooth perturbations of the system [40]. Thus, these invariant foliations persist for the time-dependent billiard as well, and depend continuously on δ and, when the rescaling of the energy variable is introduced, on ε.

The closeness of B to identity (in the rescaled energy E and time t) implies a large spectral gap [31] for the normally-hyperbolic cylinder Λ. Thus, the stable and unstable manifolds $W^{s,u}(\Lambda)$ are C^r and their leaves $W^{ss,uu}$ are C^{r-1} in (φ, θ, E, t) and also in the parameters (ε, δ), for $r \geqslant 4$ [40].

Note also, that for the unperturbed system (i.e. at $\varepsilon = 0$ and $\delta = 0$) the fibers $W_{(E,t);i}^{ss}$ and $W_{(E,t);i}^{uu}$ coincide for each (E,t).

4. Inner and outer dynamics

In this section we will define and study the inner and scattering (outer) maps associated to Λ. Iterations of these maps will be the main tool for constructing a billiard orbit with growing energy in section 5.

4.1. Inner map

The inner map is the restriction of B to Λ. Physically, the inner map describes the billiard motion along the major axis of the billiard domain. Let us denote the inner map by Φ and hence $\Phi = B|_\Lambda$ writes as $B(0, \frac{\pi}{2}, E_n, t_n) = (0, \frac{\pi}{2}, E_{n+1}, t_{n+1})$ (recall that E is the rescaled energy). Therefore, using (23) and (25), the inner map can be given in an implicit form

$$E_{n+1} = E_n - 2\varepsilon\sqrt{2E_n}\dot{a}(t_{n+1}) + 2\varepsilon^2\dot{a}^2(t_{n+1}),$$

$$t_{n+1} = t_n + \frac{\varepsilon(a(t_n) + a(t_{n+1}))}{\sqrt{2E_n}}, \tag{33}$$

The map Φ which defines a C^r diffeomorphism $(E_n, t_n) \mapsto (E_{n+1}, t_{n+1})$ for small ε, preserves the symplectic form $\left(1 + \varepsilon\frac{\dot{a}(t)}{\sqrt{2E}}\right)dE \wedge dt$ that becomes standard form $dF \wedge dt$ upon defining $F = \sqrt{E}\left(\sqrt{E} + \varepsilon\frac{2\dot{a}(t)}{\sqrt{2}}\right)$. The symplecticity of the inner map and its closeness to identity imply that it may be approximated by the time-ε shift along a level curve of an autonomous Hamiltonian $H_{in}(t, E; \varepsilon) = H_{in}(t, E) + O(\varepsilon)$ defined on Λ. Let us find H_{in}. A series expansion in ε yields the following approximation of (33):

$$E_{n+1} = E_n - 2\varepsilon\dot{a}(t_n)\sqrt{2E_n} + O(\varepsilon^2), \qquad t_{n+1} = t_n + \frac{\sqrt{2}\varepsilon a(t_n)}{\sqrt{E_n}} + O(\varepsilon^2). \tag{34}$$

Since (34) gives

$$\frac{E_{n+1} - E_n}{t_{n+1} - t_n} = -\frac{2\dot{a}(t_n)E_n}{a(t_n)} + O(\varepsilon),$$

we see that Φ is approximated to $O(\varepsilon^2)$ by a time-ϵ shift along a trajectory of the solution of the differential equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2\dot{a}E}{a}.$$

Its integral $\sqrt{E}a(t) = \mathrm{const}$ gives the zero-th order approximation of H_{in}. Thus, map (33) up to $O(\varepsilon^2)$ is given by the time-ϵ shift along a level curve of the Hamiltonian

$$H_{\mathrm{in}}(t, E) = 2\sqrt{2E}a(t). \tag{35}$$

The corresponding Hamiltonian vector field is:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\partial H_{\mathrm{in}}}{\partial E} = \frac{\sqrt{2}a(t)}{\sqrt{E}}, \qquad \frac{\mathrm{d}E}{\mathrm{d}s} = -\frac{\partial H_{\mathrm{in}}}{\partial t} = -2\sqrt{2E}\dot{a}(t), \tag{36}$$

where s is an auxiliary time variable.

Let $(\bar{E}, \bar{t})$ be the image of (E, t) under Φ^p where $p = [\frac{1}{\varepsilon}]$. Let us show the following

Proposition 4.1. *The inner map* $\Phi^p : (E, t) \mapsto (\bar{E}, \bar{t})$ *satisfies the twist condition, i.e.* $\frac{\partial \bar{t}}{\partial E} \neq 0.$

Proof. Let us denote by ϕ the time-1 map of $H_{\mathrm{in}}(t, E)$. Observe that ϕ is $O(\varepsilon)$ close to Φ^p in (E, t) coordinates. Let us verify that ϕ has the twist property. Since H_{in} is integrable, ϕ is also integrable, i.e. it preserves H_{in}. From the first equation of (36), we have $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{a(t)}\sqrt{\frac{E}{2}}$. Expressing E in terms of H_{in} from (35) on this curve yields $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{H_{\mathrm{in}}}{4a^2(t)}$. Then we have

$$1 = H_{\mathrm{in}} \int_t^{\bar{t}} \frac{\mathrm{d}t}{4a^2(t)} = \sqrt{\frac{E}{2}}a(t) \int_t^{\bar{t}} \frac{\mathrm{d}t}{a^2(t)} \tag{37}$$

for the map ϕ. Differentiating both sides of (37) with respect to E yields $\frac{\partial \bar{t}}{\partial E} < 0$. Thus, by definition ϕ is a twist map. Since Φ^p is an $O(\varepsilon)$ perturbation of ϕ, it also has the twist property. $\square$

Since all orbits of H_{in} are invariant closed curves forming a continuous foliation of Λ, and ϕ is integrable twist map, it follows from the KAM theory that $\Phi^p : (E, t) \mapsto (\bar{E}, \bar{t})$ has closely spaced invariant curves on Λ. Hence, the energy of the billiard motion with the initial conditions on the major axis is always bounded.

Remark 4.2. It follows from standard results [33] that the inner map (33) coincides up to $O(\varepsilon^{r+1})$ with ε-time shift along a level curve of some Hamiltonian $H_{\mathrm{in}}(t, E; \varepsilon)$, which is given by (35) at $\varepsilon = 0$, a fact that we will make use of in section 5.

4.2. Scattering map: theory

The so-called outer dynamics on Λ is defined by the scattering map (also called the outer map), studied in detail in [22]. It is obtained by an asymptotic process: to construct it, one starts infinitesimally close to the normally hyperbolic invariant manifold, moves along its unstable manifold up to a homoclinic intersection and, then, back to Λ along its stable manifold.

Let us define the scattering map in a general setup. Let $T : M \mapsto M$ be a C^r (here we take $r \geqslant 1$) diffeomorphism on a compact manifold M, and let $\Lambda \subset M$ be a compact normally-hyperbolic invariant manifold of T. By the normal hyperbolicity, there exits stable $W^s(\Lambda)$ and unstable $W^u(\Lambda)$ manifolds of Λ with strong stable and strong unstable foliations $W^{ss,uu}(x)$ for each $x \in \Lambda$. Let us assume that stable and unstable manifolds of Λ intersect transversally along a homoclinic manifold $\Gamma \subset W^s(\Lambda) \cap W^u(\Lambda)$: for all $z \in \Gamma$ we have

$$T_z W^s_\Lambda + T_z W^u_\Lambda = T_z M, \qquad T_z W^s_\Lambda \cap T_z W^u_\Lambda = T_z \Gamma. \tag{38}$$

For a given $z \in \Gamma$, there are unique points $x_\pm \in \Lambda$ satisfying $z \in W^{ss}_{x_+}$ and $z \in W^{uu}_{x_-}$. Following [?] we call the homoclinic intersection at the point z *strongly transverse* if the leaf $W^{ss}_{x_+}$ is transverse to $W^s(\Lambda)$ and the leaf $W^{uu}_{x_-}$ is transverse to $W^u(\Lambda)$ at z:

$$T_z W^{ss}_{x_+} \bigoplus T_z \Gamma = T_z W^s_\Lambda, \qquad T_z W^{uu}_{x_-} \bigoplus T_z \Gamma = T_z W^u_\Lambda. \tag{39}$$

Conditions (39) are used to locally define the *scattering map* S_Γ. We say that Γ is a homoclinic channel if it satisfies (38) and (39). Let $\pi^s : \Gamma \mapsto \Lambda$ and $\pi^u : \Gamma \mapsto \Lambda$ be the projections by the strong stable leaves of stable manifolds, and strong unstable leaves of unstable manifolds, respectively. If we have a sufficiently large spectral gap (the expansion in the strong unstable directions is much larger than the possible expansion we have in the directions tangent to Λ), then the strong-stable and strong-unstable foliations are C^{r-1}-smooth [40]. This condition is obviously satisfied in our case, since the restriction of our billiard map to Λ is close to identity, i.e. the expansion in the directions tangent to Λ can be made as weak as we want. Thus, when conditions (39) are fulfilled, the projections $\pi^{s,u}$ are local C^{r-1} diffeomorphisms. Then, the scattering map $S_\Gamma : \Lambda \mapsto \Lambda$ defined as

$$S_\Gamma = \pi^s \circ (\pi^u)^{-1}$$

is a C^{r-1} diffeomorphism (which is symplectic if T is symplectic): [22].

The invariance property of the strong stable and strong unstable foliations (i.e. the property $T\left(W^{ss,uu}_x\right) = W^{ss,uu}_{T(x)}$ for all $x \in \Lambda$) implies that

$$\pi^{s,u} = T^{-1} \circ \pi^{s,u} \circ T, \qquad \pi^{s,u} = T^{-n} \circ \pi^{s,u} \circ T^n, \quad n \geqslant 1.$$

Hence, the dependence of the scattering map on the choice of a particular homoclinic channel obeys the rule

$$S_\Gamma = T^{-1} \circ S_{T(\Gamma)} \circ T.$$

4.3. Scattering map for B and splitting of invariant manifolds

Let us take a sufficiently large compact subset $\tilde{\Lambda} \subset \Lambda$ of the normally hyperbolic invariant cylinder Λ:

$$\tilde{\Lambda} = \{(\varphi, \theta, E, t) : (\varphi, \theta) = (0, \pi/2), E \in [E_1, E_2], t \in \mathbb{S}^1\}. \tag{40}$$

In this section we will find a subset $\bar{\Lambda} \subset \tilde{\Lambda}$ where the scattering map S_Γ is well-defined, i.e. we will find a set Γ consisting of homoclinic points for which the transversality conditions (38) and (39) are satisfied—then the projection $\pi^u(\Gamma)$ is the set $\bar{\Lambda}$. We will derive perturbatively an explicit formula for S_Γ up to $\varepsilon O(\varepsilon + |\delta|)$ and then provide an approximation to it by a time-ε shift along a level curve of a certain Hamiltonian H_{out}. In the process, we will also prove theorems 1.2 and 1.3.

To study the properties of the scattering map, we need to determine whether the perturbed stable and unstable manifolds $W_i^{s,u}(\tilde{\Lambda})$ of $\tilde{\Lambda}$ intersect transversally, and furthermore, if these intersections are strongly transverse. We will use Melnikov-type method to determine the existence of transverse intersections of $W_i^{s,u}(\tilde{\Lambda})$. Since unperturbed $W_1^{s,u}(\tilde{\Lambda})$ and $W_2^{s,u}(\tilde{\Lambda})$ are symmetric to each other, we will study $W_2^{s,u}(\tilde{\Lambda})$ only. From (32), the unperturbed invariant manifolds $W_2^{s,u}(\tilde{\Lambda})$ coincide and form a three-dimensional (unperturbed) homoclinic manifold W_2 given by

$$W_2(\tilde{\Lambda}) = \{(\varphi, \theta, E, t) : \varphi \in [0, \pi)\, \frac{\pi}{2}, \theta \in (0, \pi), E \in [E_1, E_2], t \in \mathbb{S}^1; \quad \sin(\varphi) = \frac{b(t)}{c(t)\tan(\theta)}\,\}. \tag{41}$$

We note that $W_2^{s,u}(\tilde{\Lambda})$ are co-dimension 1 manifolds. Therefore, in order to determine whether they split or not for nonzero $\varepsilon, \delta \neq 0$. we only need one measurement in the normal direction to the coincident tangent spaces of unperturbed $W_2^{s,u}(\tilde{\Lambda})$. Later in this section we will introduce the distance function $\bar{d}(x_0, \varepsilon, \delta)$ that measures the splitting of $W_2^{s,u}(\tilde{\Lambda})$ for $\varepsilon, \delta \neq 0$ in the normal direction to $x_0 = (\varphi_0, \theta_0, E_0, t_0) \in W_2(\tilde{\Lambda})$ and will show that if $\frac{\partial \bar{d}}{\partial \varphi_0} \neq 0$, then the strong transversality condition (39) is satisfied.

Let us consider strong stable and unstable foliations $W_{(\bar{E}, \bar{t}) \in \tilde{\Lambda}; 2}^{ss, uu}$ of the stable/unstable manifolds $W_2^{s,u}(\tilde{\Lambda})$ (with notation as in section 3.2). Since we study $W_2^{s,u}(\tilde{\Lambda})$, we will drop the subscript 2 and write $W_{(\bar{E}, \bar{t}) \in \tilde{\Lambda}}^{ss, uu}$ from now on.

Since B_0 is integrable and is identity in (E, t) variables, the unperturbed one-dimensional strong stable and unstable fibers $W_{(\bar{E}, \bar{t}) \in \tilde{\Lambda}}^{ss, uu}$ with the same basepoint $(\bar{E}, \bar{t})$ coincide, and may be trivially expressed as graphs over φ variable with $E = \bar{E} = $ const and $t = \bar{t} = $ const, with θ as a function of φ determined from (41). Because the spectral gap is large in our situation, the strong stable and unstable fibers are C^{r-1}-smooth functions of parameters [40]. Therefore, for small ε and δ, the fibers $W_{(\bar{E}, \bar{t}) \in \tilde{\Lambda}}^{ss, uu}$ and perturbed stable and unstable manifolds $W^{s,u}(\tilde{\Lambda})$ may be written as

$$E = E^{ss, uu}(\varphi; \bar{E}, \bar{t}, \varepsilon, \delta) = \bar{E} + \varepsilon \phi_1^{ss, uu}(\varphi; \bar{E}, \bar{t}, 0, 0) + \varepsilon O(\varepsilon + |\delta|), \tag{42}$$

$$t = t^{ss, uu}(\varphi; \bar{E}, \bar{t}, \varepsilon, \delta) = \bar{t} + \varepsilon \phi_2^{ss, uu}(\varphi; \bar{E}, \bar{t}, 0, 0) + \varepsilon O(\varepsilon + |\delta|), \tag{43}$$

$$\theta = \theta^{s,u}(\varphi, E, t, \varepsilon, \delta), \tag{44}$$

where $\phi_{1,2}^{ss, uu}$ are certain C^{r-1} functions, and the graphs of the C^r-functions $\theta^{s,u}$ define sufficiently large pieces of the stable and unstable manifolds $W_2^{s,u}(\tilde{\Lambda})$ at small ε and δ.

Observe that in the formulas (42) and (43) the $O(\delta)$ terms do not appear, since the billiard map is identity in (E, t) at $\varepsilon = 0$, so the strong-stable and strong-unstable fibers are given by $E = $ const and $t = $ const at $\varepsilon = 0$ even if $\delta \neq 0$. We note that if we know φ (and thus θ) as a function of (E, t), then the formulas (42) and (43) will provide us with a formula for the scattering map S_Γ.

In what follows, we start with the analysis of the splitting of $W_2^s(\tilde{\Lambda})$ and $W_2^u(\tilde{\Lambda})$ for the case $\delta = 0, \varepsilon \neq 0$.

Proof of theorem 1.2.   Let us set $\delta = 0$ in (15). It can be shown [41] that the distance $d(x_0, \varepsilon, 0)$ (measured in the normal direction at the point $x_0 = (\varphi_0, \theta_0, E_0, t_0) \in W_2(\tilde{\Lambda})$) between the perturbed invariant manifolds $W_2^{s,u}(\tilde{\Lambda})$ is given by

$$d(x_0, \varepsilon, 0) = \varepsilon M_1(x_0) + O\left(\varepsilon^2\right),$$

where M_1 is the Melnikov function corresponding to the time-dependent ellipse only (without the quartic polynomial part). The following formula for M_1 is derived in appendix A:

$$M_1(x_0) := \sum_{n=-\infty}^{\infty} \langle \nabla I(B_0(x_n), B_1(x_n)) \rangle, \tag{45}$$

where $x_n = (\varphi_n, \theta_n, E_n, t_n) \in W_2(\tilde{\Lambda})$ is the orbit of the point $x_0 \in W_2(\tilde{\Lambda})$ under the map B_0 (i.e. $x_n = B_0^n(x_0)$), and I is the first integral given by (9). Since the separatrices of the static billiard are one-dimensional, we have $\theta_n = \theta^0(\varphi_n)$, a smooth function of φ given by (41). Also observe that $E_n = E_0, t_n = t_0$. Hence we evaluate the series (45) with slow variables held constant and equal to their initial value.

Let us set $\bar{d}(x_0, \varepsilon, 0) = \frac{d(x_0, \varepsilon, 0)}{\varepsilon}$ for $\varepsilon \neq 0$, and $\bar{d}(x_0, 0, 0) = M_1(x_0)$. Then $\bar{d}(x_0, \varepsilon, 0) = M_1(x_0) + O(\varepsilon)$ and the zeroes of $\bar{d}(x, \varepsilon, 0)$ correspond to intersection of $W_2^s(\tilde{\Lambda})$ and $W_2^u(\tilde{\Lambda})$. The implicit function theorem implies that if $M_1(x_0) = 0$ and $DM_1(x_0) \neq 0$, then $W_2^{s,u}(\tilde{\Lambda})$ intersect transversally at x_0 along a 2-dimensional homoclinic manifold.

We are able to compute M_1 analytically (see appendix B). Note that since we express φ and θ through ξ on homoclinic manifolds using parametrisation (14), effectively we have $M_1(x) = M_1(\varphi, \theta(\varphi), E, t) = M_1(\xi, E, t)$.

We define $h = \log \lambda$ where λ from (8) is the largest eigenvalue of the linearization of the static map B_s at the saddle point z. For the map B, the value of λ depends on t and is given by $\lambda = \lambda(t) = \frac{a+c}{a-c}$ with $a = a(t)$, $b = b(t)$ and $c = c(t)$. We also define the variable τ in terms of ξ from (14): $\exp(\tau) := \xi := \tan(\varphi/2)$. The method developed in [35] enables us to compute the sum of the series (45) in terms of elliptic functions—the computations are in appendix B. Using the definitions of the elliptic functions dn, cn, sn and complete elliptic integrals E, E', K, K' depending on parameter $m \in [0, 1]$ whose dependence on h is $\frac{K'}{K} = \frac{\pi}{h}$ as in appendix B, we find:

$$M_1 = \frac{4b}{v}\left(-\dot{a}b + \dot{b}a\right)\left(\frac{2K}{h}\right)^2\left(\frac{E'}{K'} - 1 + \mathrm{dn}^2\left(\frac{2K\tau}{h}\right)\right), \tag{46}$$

where $v = \sqrt{2E}$ is the (rescaled) speed of the particle. Since $\left(\frac{2K}{h}\right)^2\left(\frac{E'}{K'} - 1 + \mathrm{dn}^2\left(\frac{2K\tau}{h}\right)\right) = \sum_{n=-\infty}^{n=\infty} \mathrm{sech}^2(\tau + nh) > 0$, as known from [42], the zeroes of M_1 only exist for values of time t^* satisfying $(-\dot{a}b + \dot{b}a) = 0$, i.e. when $\frac{d}{dt}\left(\frac{a}{b}\right) = 0$. For such t^*, the values of v and $\varphi = \varphi(\tau)$ for which the Melnikov function vanishes can be arbitrary. Hence the zeroes are of the form $(\varphi, \theta(\varphi), t^*, v)$. By implicit function theorem, if $DM_1 \neq 0$, then the zeros of the Melnikov function correspond to zeros of the splitting function d for all small ε, hence to the homoclinic intersections, and these intersections are transverse. The condition $DM_1 \neq 0$ implies $\frac{d^2}{dt^2}\left(\frac{a(t^*)}{b(t^*)}\right) \neq 0$ which is satisfied if (and only if) t^* is a nondegenerate critical point of $\frac{a(t^*)}{b(t^*)}$. $\qquad\square$

This gives us the existence, for all small ε, of a transverse intersection $\tilde{\Gamma}$ of $W^u(\tilde{\Lambda})$ and $W^s(\tilde{\Lambda})$ along a smooth two-dimensional surface close to the surface $\{t = t^*, \theta = \theta_{0,0}^{s,u}(\varphi, E, t^*)\}$.

Since ε is just an energy scaling parameter, we obtain the transverse homoclinic intersection Γ of $W^u(\Lambda)$ and $W^s(\Lambda)$ for all sufficiently large values of the non-rescaled energy $\mathcal{E}$.

In the limit $\varepsilon = 0$ (i.e. in the limit $\mathcal{E} \to +\infty$) the projection π^u of this homoclinic surface to the cylinder Λ by the strong-unstable fibers, i.e. the domain of definition of the scattering map $S_{\tilde{\Gamma}}$ shrinks to just the vertical line $t = t^*$ (recall that the strong-unstable fibers are close to the lines $(E, t) = const$ at small ϵ). In other words, we cannot define the scattering map at $\delta = 0$ by first order expansion in ε. We, therefore, proceed to the case $\delta \neq 0$.

Let us introduce the Melnikov function M_2 corresponding to the quartic polynomial perturbation $O(\delta)$ *only*, with the time and speed variables frozen. The following formula for M_2 is again derived in appendix A:

$$M_2(x_0) := \sum_{n=-\infty}^{\infty} \langle \nabla I(B_0(x_n)), B_2(x_n) \rangle, \tag{47}$$

where the series is evaluated as before over the orbit of $x_0 \in W_2(\tilde{\Lambda})$ under the map B_0 and I is the first integral given by (9). Using the same notation as for M_1 above, we find from appendix B that (see also [35]):

$$M_2 = -4m\frac{ab^2}{c^2}\left(\frac{2K}{h}\right)^3 \mathrm{dn}\left(\frac{2K\tau}{h}\right) \mathrm{sn}\left(\frac{2K\tau}{h}\right) \mathrm{cn}\left(\frac{2K\tau}{h}\right). \tag{48}$$

It is known [35] that for each fixed t, the function M_2 is h-periodic in τ and has two simple, in terms of τ, zeroes in the period $[0, h]$.

Before we proceed to prove theorem 1.3, let us introduce some notation. Rewrite (46) as $M_1 = \frac{f(t)g(\tau,t)}{v}$, and (48) as $M_2 = j(\tau, t)$, where

$$f(t) = -\dot{a}b + \dot{b}a,$$

$$g(\tau,t) = 4b\left(\frac{2K}{h}\right)^2 \left(\frac{E'}{K'} - 1 + \mathrm{dn}^2\left(\frac{2K\tau}{h}\right)\right) > 0,$$

$$j(\tau,t) = -4m\frac{ab^2}{c^2}\left(\frac{2K}{h}\right)^3 \mathrm{dn}\left(\frac{2K\tau}{h}\right) \mathrm{sn}\left(\frac{2K\tau}{h}\right) \mathrm{cn}\left(\frac{2K\tau}{h}\right). \tag{49}$$

Let us also define the function $\phi(t)$ by:

$$\phi(t) = \min_\tau \left|\frac{g(\tau,t)}{j(\tau,t)}\right|/\sqrt{2}. \tag{50}$$

Note that it is seen from the properties of elliptic functions dn, cn and sn that for each t, the function $\frac{g(\tau,t)}{j(\tau,t)}$ is non-zero, odd and periodic in τ, with one maximum and one minimum point in the period $\tau \in [0, h(t))$.

Proposition 4.3. *Let (v, t) be such that $\frac{\varepsilon|f(t)|}{|\delta|v} < \max_\tau |\frac{j(\tau,t)}{g(\tau,t)}| - \tilde{k}$ where $\tilde{k} > 0$ is constant and $v = \sqrt{2E}$ is the rescaled particle speed. Let $\bar{\Lambda}$ be the subset of $\tilde{\Lambda}$ corresponding to these values of (E, t). Then the stable and unstable manifolds $W_2^{s,u}(\bar{\Lambda})$ have a strong-transverse intersection along a two-dimensional homoclinic manifold Γ for all small $\varepsilon, \delta \neq 0$ such that $|\delta| \gg \varepsilon^2$.*

Proof. As shown in appendix A, the distance $d(x_0, \varepsilon, \delta)$ measured in the normal direction at the point $x_0 = (\varphi_0, \theta_0, E_0, t_0) \in W_2(\tilde{\Lambda})$ between the perturbed manifolds $W_2^{s,u}(\tilde{\Lambda})$ is given by

$$d(x_0, \varepsilon, \delta) = \varepsilon M_1(x_0) + \delta M_2(x_0) + O\left(\varepsilon^2 + \delta^2\right). \tag{51}$$

To deal with two small parameters ε and δ, let us consider

$$\frac{d(x_0, \varepsilon, \delta)}{\delta} = \frac{\varepsilon}{\delta} M_1(x_0) + M_2(x_0) + O(\frac{\varepsilon^2}{\delta} + \delta) = \frac{\varepsilon f(t) g(\tau, t)}{\delta v} + j(\tau, t) + O\left(\frac{\varepsilon^2}{\delta} + \delta\right). \tag{52}$$

We have from the assumption of the proposition that $\frac{\varepsilon M_1(x_0)}{\delta}$ is bounded for all small nonzero δ and that terms of order $O\left(\frac{\varepsilon^2}{\delta} + \delta\right)$ are uniformly small. Then, if we drop the $O\left(\frac{\varepsilon^2}{\delta} + \delta\right)$ terms in (52) and define

$$\bar{d} = \frac{\varepsilon f(t) g(\tau, t)}{\delta v} + j(\tau, t), \tag{53}$$

it follows from the implicit function theorem that if $\frac{\partial \bar{d}}{\partial \tau}|_{\bar{d}=0} \neq 0$, then the zeroes of $\bar{d}$ correspond to a transverse intersection of $W_2^{s,u}(\tilde{\Lambda})$ along a 2-dimensional homoclinic manifold Γ for all small $\frac{\varepsilon^2}{\delta}$ and δ. This manifold is a graph of a smooth function τ of (v, t), i.e. it is a graph of a smooth function φ of (E, t). Since the strong-stable and strong-unstable fibers are ε-close to $(E, t) = const$, we immediately have that the conditions (38) and (39) of the strong transverse intersection are fulfilled at the points of Γ.

Thus, to prove the proposition, we need to investigate for which values of v and t the function $\bar{d}(\tau)$ has simple zeroes. Rearranging (53) yields

$$\frac{\varepsilon f(t)}{\delta v} = -\frac{j(\tau, t)}{g(\tau, t)}. \tag{54}$$

Let us fix any t and v. Since $g(\tau, t) > 0$ and $v > 0$, solutions of this equation belong to the interval of values of τ for which the sign of $j(\tau, t)$ is the same as the sign of $f(t)$. Degenerate zeros correspond to critical points of the function fix$_t \frac{j(\tau,t)}{g(\tau,t)}$. It can be checked that it has exactly one (positive) maximum and one (negative) minimum in the period $\tau \in [0, h(t))$; since this function is odd, the maximum and minimum are the same in the absolute value. It follows that for any constant $\tilde{k} > 0$, given t and v such that

$$\frac{\epsilon |f(t)|}{|\delta| v} < \max_\tau |\frac{j(\tau, t)}{g(\tau, t)}| - \tilde{k}, \tag{55}$$

equation (54) has exactly two solutions $\tau(t, v)$, both of them non-degenerate.

As we explained above, these solutions correspond to strong-transverse homoclinic intersections, provided δ, ε and ε^2/δ are small enough. $\qquad \square$

Now we can prove theorem 1.3.

Proof. We have shown in proposition 4.3 that the values of (v, t) satisfying (55) correspond to a strong-transverse homoclinic intersection, i.e. they lie in the domain of definition of the scattering map S_Γ. It is also seen from (54) that if, for some $\tilde{k} > 0$,

$$\frac{\varepsilon |f(t)|}{|\delta| v} > \max_\tau |\frac{j(\tau, t)}{g(\tau, t)}| + \tilde{k}, \tag{56}$$

then the function d does not have zeros for small $\delta, \frac{\varepsilon^2}{\delta}$ and ε, so the region (56) is not included in the domain of the scattering map.

Now, by noting that $\varepsilon/v = 1/\sqrt{2\mathcal{E}}$ where $\mathcal{E}$ is the non-rescaled energy, we immediately obtain the statement of the theorem from estimates (55) and (56). $\qquad\qquad\square$

Now we proceed to obtain a first order expression in perturbation parameters ε, δ for S_Γ in terms of E and t. Let $(\bar{E}_0, \bar{t}_0) \in \bar{\Lambda}$ and $(\tilde{E}_0, \tilde{t}_0) \in \bar{\Lambda}$ be two points in the domain of S_Γ given by (2). We will obtain a perturbative expression up to $\varepsilon O(\varepsilon + |\delta|)$ for the scattering map $S_\Gamma : \bar{\Lambda} \mapsto \bar{\Lambda}$:

$$S_\Gamma : (\bar{E}_0, \bar{t}_0; \varepsilon, \delta) \mapsto (\tilde{E}_0, \tilde{t}_0; \varepsilon, \delta). \tag{57}$$

We will call the first order approximation of S_Γ the truncated scattering map.

Let us derive a formula for $\phi_{1,2}^{ss,uu}$ given by (42) and (43). Let us take a point $(\varphi_0, \theta_0, E_0, t_0) \in \Gamma \pitchfork W_{(\bar{E}_0, \bar{t}_0)}^{ss}$. Its energy component E_0 is given by (42):

$$E_0 = E^{ss}(\varphi_0; \bar{E}_0, \bar{t}_0, \varepsilon, \delta) = \bar{E}_0 + \varepsilon\phi_1^{ss}(\varphi_0; \bar{E}_0, \bar{t}_0, 0, 0) + \varepsilon O(\varepsilon + |\delta|).$$

Let $B\left(W_{(\bar{E}_0, \bar{t}_0)}^{ss}\right)$ denote the action of map B on the point $(\varphi_0, \theta_0, E_0, t_0)$ on the leaf $W_{(\bar{E}_0, \bar{t}_0)}^{ss}$. Let $(\varphi_{n,\varepsilon,\delta}, \theta_{n,\varepsilon,\delta}, E_{n,\varepsilon,\delta}, t_{n,\varepsilon,\delta})$ be the orbit of the point $(\varphi_0, \theta_0, E_0, t_0)$ under B, and let $(\bar{E}_{n,\varepsilon}, \bar{t}_{n,\varepsilon})$ be the orbit of $(\bar{E}_0, \bar{t}_0) \in \bar{\Lambda}$ under B (the notation is the same as in (26)). Note that on $\bar{\Lambda}$ we have $(\varphi_{n,\varepsilon}, \theta_{n,\varepsilon}) = (0, \frac{\pi}{2})$ for all n, and there is no dependence on δ of B restricted to Λ.

Consider the energy E component of $B\left(W_{(\bar{E}_0, \bar{t}_0)}^{ss}\right)$. Using the notation (26) gives

$$\begin{aligned}
E_{1,\varepsilon} &= E_{1,0} + \varepsilon f_3(\varphi_0, \theta_0(\varphi_0), E_0, t_0) + \varepsilon O(\varepsilon + |\delta|) \\
&= E_0 + \varepsilon f_3(\varphi_0, \theta_0(\varphi_0), E_0, t_0) + \varepsilon O(\varepsilon + |\delta|) \\
&= \bar{E}_0 + \varepsilon\phi_1^{ss}(\varphi_0(\varphi_0), \bar{E}_0, \bar{t}_0, 0) + \varepsilon f_3(\varphi_0, \theta_0(\varphi_0), E_0, t_0) + \varepsilon O(\varepsilon + |\delta|),
\end{aligned} \tag{58}$$

where $f_3(\varphi_0, \theta_0(\varphi_0), E_0, t_0)$ is evaluated on the unperturbed homoclinic trajectory on the unperturbed homoclinic manifold $W_2(\bar{\Lambda})$ given by (41). The invariance property of the stable and unstable foliations gives $B\left(W_{(\bar{E}_0, \bar{t}_0)}^{ss}\right) = W_{B(\bar{E}_0, \bar{t}_0)}^{ss}$ and, by definition of the map B, we have $W_{B(\bar{E}_0, \bar{t}_0))}^{ss} = W_{(\bar{E}_{1,\varepsilon}, \bar{t}_{1,\varepsilon})}^{ss}$. Therefore, we have

$$\begin{aligned}
E_{1,\varepsilon} &= E(\varphi_{1,\varepsilon}; \bar{E}_{1,\varepsilon}, \bar{t}_{1,\varepsilon}, \varepsilon, \delta) \\
&= \bar{E}_{1,\varepsilon} + \varepsilon\phi_1^{ss}(\varphi_1, \bar{E}_1, \bar{t}_1, 0, 0) + \varepsilon O(\varepsilon + |\delta|) \\
&= \bar{E}_0 + \varepsilon f_3(\cdot, \bar{E}_0, \bar{t}_0)|_{\bar{\Lambda}} + \varepsilon\phi_1^{ss}(\varphi_1, \bar{E}_0, \bar{t}_0, 0, 0) + \varepsilon O(\varepsilon + |\delta|), \tag{59}
\end{aligned}$$

where we used $\bar{E}_1 = \bar{E}_0$, $\bar{t}_1 = \bar{t}_0$ and $\bar{E}_{1,\varepsilon} = \bar{E}_0 + \varepsilon f_3(\cdot, \bar{E}_0, \bar{t}_0)|_{\bar{\Lambda}}$ in the last line above, with $f_3(\cdot, \bar{E}_0, \bar{t}_0)|_\Lambda$ denoting the restriction of f_3 to $\bar{\Lambda}$. The notation $f_3(\cdot, \bar{E}_0, \bar{t}_0)|_{\bar{\Lambda}}$ signifies that the variables (φ, θ) on $\bar{\Lambda}$ are fixed at $(0, \pi/2)$.

Examining the coefficients of $O(\varepsilon)$ terms in (58) and (59) gives

$$\phi_1^{ss}(\varphi_0; \bar{E}_0, \bar{t}_0, 0, 0) = \phi_1^{ss}(\varphi_1; \bar{E}_0, \bar{t}_0, 0, 0) + f_3(\cdot, \bar{E}_0, \bar{t}_0)|_\Lambda - f_3(\varphi_0, \theta_0(\varphi_0), E_0, t_0). \tag{60}$$

Upon iterating (60), we obtain an expression for $\phi_1^{ss}(\varphi_0; \bar{E}_0, \bar{t}_0, 0, 0)$:

$$\phi_1^{ss}(\varphi_0; \bar{E}_0, \bar{t}_0, 0, 0) = \phi_1^{ss}(\varphi_\infty; \bar{E}_\infty, \bar{t}_\infty, 0, 0) + \sum_{i=0}^{\infty} \{f_3(\cdot, \bar{E}_i, \bar{t}_i)|_\Lambda - f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\}, \tag{61}$$

where $\varphi_\infty = 0$, $\bar{E}_\infty = \bar{E}_0$ and $\bar{t}_\infty = \bar{t}_0$, and $f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)$ is evaluated over the unperturbed homoclinic trajectory on $W_2(\bar{\Lambda})$ given by (41). Therefore the slow variables (E, t) in the summation are held constant with $\bar{E} = E_i = E_0, \bar{t} = t_i = t_0$. Hence, the equation for the energy E component of the strong stable leaf $W^{ss}_{(\bar{E}_0, \bar{t}_0)}$ through (E_0, t_0) is

$$E_0 = \bar{E}_0 + \varepsilon \phi_1^{ss}(0; \bar{E}_0, \bar{t}_0, 0, 0) + \varepsilon \sum_{i=0}^{\infty} \{f_3(\cdot, \bar{E}_i, \bar{t}_i)|_\Lambda - f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\} + \varepsilon O(\varepsilon + |\delta|).$$
(62)

Similarly, we express the E component of the strong unstable leaf $W^{uu}_{(\tilde{E}_0, \tilde{t}_0)}$ through $(\varphi_0, \theta_0, E_0, t_0) \in \Gamma \pitchfork W^{uu}_{(\tilde{E}_0, \tilde{t}_0)}$ with the base point $(\tilde{E}_0, \tilde{t}_0)$ as

$$E_0 = \tilde{E}_0 + \varepsilon \phi_1^{uu}(0; \tilde{E}_0, \tilde{t}_0, 0, 0) - \varepsilon \sum_{i=-\infty}^{-1} \{f_3(\cdot, \tilde{E}_i, \tilde{t}_i)|_\Lambda - f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\} + \varepsilon O(\varepsilon + |\delta|).$$
(63)

Using that $\phi_1^{ss}(0; \bar{E}_0, \bar{t}_0, 0, 0) - \phi_1^{uu}(0; \tilde{E}_0, \tilde{t}_0, 0, 0) = 0$ since the unperturbed stable and unstable foliations coincide, and subtracting (63) from (62) gives an expression for the E component of the truncated scattering map up to $\varepsilon O(\varepsilon + |\delta|)$:

$$\tilde{E}_0 = \bar{E}_0 + \varepsilon \sum_{i=-\infty}^{\infty} \{f_3(\cdot, E_i, t_i)|_\Lambda - f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\}.$$
(64)

Analogously, for the t components of the truncated map S_Γ we obtain:

$$\tilde{t}_0 = \bar{t}_0 + \varepsilon \sum_{i=-\infty}^{\infty} \{f_4(\cdot, E_i, t_i)|_\Lambda - f_4(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\}.$$
(65)

We see that S_Γ is close to identity and may be approximated by a time-ε shift of some Hamiltonian flow $H_{out}(t, E)$ with the accuracy $\varepsilon O(\varepsilon + |\delta|)$. From (64) and (65), we have Hamilton's equations for $H_{out}(t, E)$ with $' = \frac{d}{ds}$ denoting differentiation with respect to auxiliary time s:

$$E' = \sum_{i=-\infty}^{\infty} \{f_3(\cdot, E_i, t_i)|_\Lambda - f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\}, \qquad t' = \sum_{i=-\infty}^{\infty} \{f_4(\cdot, E_i, t_i)|_\Lambda - f_4(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\}.$$

Let us compute the above infinite sums. Recalling the definitions of f_3 and f_4 given by (30) and (31) and that the summation is performed over the fast variables (φ, θ) while the slow variables (E, t) are held fixed at their initial value $(E_i = E_0, t_i = t_0)$, it turns out (see appendix C) that the sum for E' may be computed to give

$$\sum_{i=-\infty}^{\infty} \{f_3(\cdot, E_i, t_i)|_\Lambda - f_3(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\} = 2\sqrt{2E_0} \sum_{i=-\infty}^{\infty} \{-\dot{a}(t_{i+1}) + u_{i+1}\sin(\theta_{i+1})\} = 2\sqrt{2E_0} \left(\frac{b\dot{b} - a\dot{a}}{c}\right),$$
(66)

where u_{i+1} is the normal speed of the boundary at $(i+1)$th impact. The expression for t' may be computed using the geometric property of the ellipse [43] that the sum of homoclinic lengths converges to $-2c(t_0)$:

$$\sum_{i=-\infty}^{\infty} \{f_4(\cdot, E_i, t_i)|_{\bar{\Lambda}} - f_4(\varphi_i, \theta_i(\varphi_i), E_i, t_i)|_{W_2(\bar{\Lambda})}\} = \frac{1}{\sqrt{2E_0}} \sum_{i=-\infty}^{\infty} \{2a(t_i) - D_0|_{W_2(\bar{\Lambda})}\} = \frac{2c(t)}{\sqrt{2E}},$$
(67)

where $D_0|_{W_2(\tilde{\Lambda})}$ is flight distance D_0 (see (27)) evaluated on $W_2(\tilde{\Lambda})$. Therefore the scattering map S_Γ is approximated up to $O(\varepsilon(\varepsilon + |\delta|))$ by a time-ϵ shift along a trajectory of the solution of the differential equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2E\frac{\dot{c}(t)}{c(t)}.$$

This corresponds to the Hamiltonian vector field

$$t' = \frac{\partial H_{\text{out}}}{\partial E} = \frac{\sqrt{2}c(t)}{\sqrt{E}}, \qquad E' = -\frac{\partial H_{\text{out}}}{\partial E} = -2\sqrt{2E}\dot{c}(t). \tag{68}$$

Therefore we obtain the following Hamiltonian $H_{\text{out}}(t, E)$ defined on Λ that approximates S_Γ:

$$H_{\text{out}}(t, E) = 2\sqrt{2E}c(t). \tag{69}$$

5. Energy growth

In [31] it is proved (lemma 4.4) that if two points on a normally-hyperbolic invariant manifold of a symplectic diffeomorphism are connected by an orbit of the iterated function system (IFS) formed by the inner and scattering maps, then there exists a trajectory of the original diffeomorphism that connects arbitrarily small neighbourhoods of those two points. lemmas 3.11 and 3.12 of [32] show that the same is true when orbits of an IFS are infinite (in one direction).

In this section we prove theorem 1.1 using these facts. Namely, we consider an iterated function system $\{\Phi, S_\Gamma\}$ comprised of the inner map Φ defined by (33) and the scattering map S_Γ defined by (57), and show that it has an orbit with the energy $\mathcal{E}$ tending to infinity. By the above quoted results, the existence of the orbit of the map B for which the energy grows to infinity follows too.

We start with the analysis of the behaviour of the IFS in the rescaled coordinates.

Lemma 5.1. *For any initial condition* $(E, t) \in \tilde{\Lambda}$ *with* $|\delta| \gg \varepsilon^2$, *there exist positive integers* n_1, n_2, *where* $n_1 + n_2 = O\left(\frac{1}{\varepsilon}\right)$, *such that the gain* ΔH_{in} *of the Hamiltonian* $H_{\text{in}}(t, E; \varepsilon)$ *(given in remark 4.2) along the orbit* $S_\Gamma^{n_2} \circ \Phi^{n_1}$ *of the IFS* $\{\Phi\}, S_\Gamma\}$ *is* $\Delta H_{\text{in}} \geqslant K_1 \min\left(\frac{\delta^2}{\varepsilon^2}, 1\right)$ *where* K_1 *is a strictly positive constant.*

Proof. Suppose that t^* is a nondegenerate critical point of $\frac{a}{b}$. Then $f(t^*) = 0$, which corresponds to $\frac{\dot{c}}{c} = \frac{\dot{a}}{a}$, where $f(t)$ is defined in (49). Therefore $(E, t^*) \in \bar{\Lambda}$ where $\bar{\Lambda}$ is the domain of S_Γ as in (2) for all E. There exists an interval in t near t^*, denote it $[t_1, t_2]$, such that $\frac{\dot{c}}{c} < \frac{\dot{a}}{a}$ and S_Γ is defined at $t \in [t_1, t_2]$. Denote $\Delta t = t_2 - t_1$. Let us obtain the lower bound on Δt. Taking (2) and rewriting it in scaled variable v using the definition (22), then Taylor expanding about t^*, we obtain that small Δt can be chosen such that

$$v < \frac{c_2 \varepsilon \Delta t}{\delta} \tag{70}$$

for a constant $c_2 > 0$. In other words, we can always choose the interval $[t_1, t_2]$ such that

$$\Delta t = c_1 \min\left(\frac{\delta}{\varepsilon}, 1\right)$$

where $c_1 > 0$ is constant.

Let us take a point $(E, t) \in \tilde{\Lambda}$. Iterate (E, t) under the inner map Φ until the image $\Phi^{n_1}(E, t) = (E_{n_1}, t_{n_1})$ enters the domain $[t_1, t_2]$, i.e. $t_{n_1} \in [t_1, t_2]$, and thus the point $(E_{n_1}, t_{n_1}) \in \bar{\Lambda}$ (if t is originally in the interval $[t_1, t_2]$, then iterate until it gets out of this interval and, then, returns to it again). Note that the change in t during one iteration of Φ is of order ε which is much smaller than Δt (because $\delta/\varepsilon \gg \varepsilon$ by assumption). Therefore, the iterates of Φ cannot 'miss' $[t_1, t_2]$ and the number of iterations n_1 is bounded from above as $O(\varepsilon^{-1})$.

There exists a level curve h_0 of $H_{\mathrm{in}}(t, E; \varepsilon)$ passing through (E, t). The orbit of (E, t) under Φ will follow the level curve h_0. Indeed, the number of iterations n_1 of Φ is bounded by a number of order $O\left(\frac{1}{\varepsilon}\right)$. Denote by $\phi_{h_0}^{n_1\varepsilon}$ the time$-n_1\varepsilon$ shift along h_0 with the initial condition (E, t). It follows from standard mean value theorem estimates and the remark 4.2 that Φ^{n_1} coincides with $\phi_{h_0}^{n_1\varepsilon}$ up to $O(\varepsilon^r)$. Hence we have the following bound for the difference between the level of H_{in} at (E_{n_1}, t_{n_1}) and $h_0 = H_{\mathrm{in}}(\phi_{h_0}^{n_1\varepsilon}(E, t); \varepsilon)$:

$$\|H_{\mathrm{in}}(E_{n_1}, t_{n_1}; \varepsilon) - H_{\mathrm{in}}(\phi_{h_0}^{n_1\varepsilon}(E, t); \varepsilon)\| \leqslant \max_{(E,t)\in\tilde{\Lambda}} \|DH_{\mathrm{in}}(E, t; \varepsilon)\| \|(E_{n_1}, t_{n_1}) - \phi_{h_0}^{n_1\varepsilon}(E, t)\|$$
$$\leqslant \max_{(E,t)\in\tilde{\Lambda}} \|DH_{\mathrm{in}}(E, t; \varepsilon)\| \tilde{C}_1 \varepsilon^r$$
$$< C_2 \varepsilon^r, \tag{71}$$

where $\max_{(E,t)\in\tilde{\Lambda}} \|DH_{\mathrm{in}}(E, t; \varepsilon)\|$ is bounded as $\tilde{\Lambda}$ is compact; $\tilde{C}_1, C_2 > 0$ are constants. Since $r \geqslant 4$, we have that the error in the difference of $H_{\mathrm{in}}(t, E; \varepsilon)$ following Φ is maximum $O(\varepsilon^4)$.

When $t \in [t_1, t_2]$, following the level curve of $H_{\mathrm{out}}(t, E; \varepsilon, \delta)$ will give a greater gain of energy than following $H_{\mathrm{in}}(t, E; \varepsilon)$. Therefore we iterate S_Γ while $t \in [t_1, t_2]$ and its orbit will follow the level curve of $H_{\mathrm{out}}(t, E; \varepsilon, \delta)$. Following $H_{\mathrm{in}}(t, E; \varepsilon)$ will switch to following $H_{\mathrm{out}}(t, E; \varepsilon, \delta)$ when $t = t_1 + O(\varepsilon)$ and then switch back to following another level curve of $H_{\mathrm{in}}(t, E; \varepsilon)$ when $t = t_2 + O(\varepsilon)$. As $\Delta t \gg \varepsilon$, it follows that the number n_2 of iterates of S_Γ is $n_2 \sim O(\frac{\Delta t}{\varepsilon})$.

Let us consider the Hamiltonian flow given by $H_{\mathrm{out}}(t, E) = 2\sqrt{2E}c(t)$ as in (69). Since the scattering map S_Γ and the time-ε flow map $\phi_{H_{\mathrm{out}}}$ of $H_{\mathrm{out}}(t, E)$ coincide up to $O(\varepsilon(\varepsilon + |\delta|))$, we have an upper bound for the difference between the values of H_{in} evaluated at (E_{n_2}, t_{n_2}) and at the time-$n_2\epsilon$ shift by the flow of H_{out} with the initial condition (E_{n_1}, t_{n_1}) (below $C_3 > 0$ is an irrelevant constant):

$$\|H_{\mathrm{out}}(E_{n_2}, t_{n_2}) - H_{\mathrm{out}}(\phi_{H_{\mathrm{out}}}^{n_2\varepsilon}(E_{n_1}, t_{n_1}))\| < C_3 \Delta t(\varepsilon + |\delta|) = O(|\delta|(\varepsilon + |\delta|)). \tag{72}$$

Let us compute the change ΔH_{in} following the level curve of $H_{\mathrm{out}}(t, E)$. Using (69), we have

$$\Delta H_{\mathrm{in}} = \int_{s_1}^{s_2} \frac{\mathrm{d}H_{\mathrm{in}}(t, E; \varepsilon)}{\mathrm{d}s}\Big|_{H_{\mathrm{out}}=\mathrm{const}} \mathrm{d}s = \int_{t_1}^{t_2} \frac{\mathrm{d}H_{\mathrm{in}}(t, E; \varepsilon)}{\mathrm{d}s}\Big|_{H_{\mathrm{out}}=\mathrm{const}} \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t.$$

where $t_{1,2} = t_{1,2}(s_{1,2})$ and s is the auxiliary time variable. Since $\frac{\mathrm{d}H_{\mathrm{in}}(t,E;\varepsilon)}{\mathrm{d}s} = \frac{\partial H_{\mathrm{in}}(t,E;\varepsilon)}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}s} + \frac{\partial H_{\mathrm{in}}(t,E;\varepsilon)}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s}$ where we use (68) for $\frac{\mathrm{d}E}{\mathrm{d}s}$ and $\frac{\mathrm{d}t}{\mathrm{d}s}$, yielding $\frac{\mathrm{d}H_{\mathrm{in}}(t,E;\varepsilon)}{\mathrm{d}s} = 4(\dot{a}c - \dot{c}a) + O(\varepsilon)$. Thus

$$\Delta H_{\mathrm{in}} = \int_{t_1}^{t_2} (4(\dot{a}c - \dot{c}a) + O(\varepsilon)) \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2} \left[H_{\mathrm{in}}(t, E; 0)\left(\frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right) + O(\varepsilon)\right]\mathrm{d}t.$$

Note that $\frac{\dot{a}}{a} - \frac{\dot{c}}{c} > 0$ for all $t \in [t_1, t_2]$. We expand the above integral in Taylor series to obtain the estimate from below

$$\Delta H_{\text{in}} \sim H_{\text{in}}(t, E; 0)(\Delta t)^2 + O(\varepsilon)\Delta t > K_1 \min\left(\frac{\delta^2}{\varepsilon^2}, 1\right), \tag{73}$$

where $K_1 > 0$ is constant.

The error estimates (71) and (72) imply that for n_1 iterations of Φ followed by n_2 iterations of S_Γ, the increase of H_{in} is given by $\Delta H_{\text{in}} > K_1 \min\left(\frac{\delta^2}{\varepsilon^2}, 1\right) + O\left(\varepsilon^4, |\delta|(\varepsilon + |\delta|)\right)$. Since $|\delta| \gg \varepsilon^2$, the net gain of H_{in} is strictly positive and is given by (73) indeed. $\qquad\square$

Proof of theorem 1.1. Take any initial condition $(\mathcal{E}, t)$ on the cylinder Λ, where $\mathcal{E}$ is the non-rescaled kinetic energy of the billiard particle. If $\mathcal{E} \geqslant \frac{C}{|\delta|}$ for a sufficiently large constant $C > 0$, then one can find the scaling parameter ε such that the scaled energy $E = \mathcal{E}\varepsilon^2$ lies in the middle of the interval $[E_1, E_2]$ corresponding to the compact piece $\tilde{\Lambda}$ considered in the lemma above, and $|\delta| \gg \varepsilon^2$. Suppose that $\tilde{\Lambda}$ is sufficiently large, $E_2 - E_1$ is sufficiently large. Since the function $a(t)$ is bounded, the ratio of H_{in}/E is bounded away from zero and infinity. Therefore, by repeated application of lemma 5.1 we find that the iterated function system $\{\Phi, S_\Gamma\}$ has an orbit, starting with our initial conditions $(E = (E_1 + E_2)/2, t)$ for which the value of $H_{\text{in}}(t, E, \varepsilon)$ increases without bound until the orbit stays in $\tilde{\Lambda}$. In other words, this orbit will eventually get to the values of E larger than E_2.

For the non-rescaled energy $\mathcal{E}$ this means the multiplication at least to $2E_2/(E_1 + E_2) > 1$. Thus, we have shown that for every initial condition $(\mathcal{E}, t)$ with $\mathcal{E} \geqslant \frac{C}{|\delta|}$ there exists an orbit of the IFS with the end point $(\bar{\mathcal{E}}, \bar{t})$ such that $\bar{\mathcal{E}} > q\mathcal{E}$ where the factor $q > 1$ is independent of the initial point. By taking the end point of the orbit we just constructed as the new initial point, and so on, we continue the process up to infinity and obtain the orbit of the IFS for which the energy $\mathcal{E}$ tends to infinity.

The shadowing lemmas 3.11 and 3.12 of [32] imply the existence of a true orbit of the map B that shadows the orbit of the IFS $\{\Phi, S_\Gamma\}$, so the energy $\mathcal{E}$ tends to infinity along this true orbit too (the shadowing lemmas of [32] require compactness of Λ, but it is easy to see that the result remains valid also in the situation where every orbit of the inner map is bounded—so the Poincare recurrence theorem can be used, and this property holds true in our case, as the KAM curves bound every orbit of Φ). $\qquad\square$

Acknowledgments

Appendix A. Melnikov function derivation

In this section we provide a derivation of the Melnikov function M_1 given by (45); the Melnikov function M_2 given by (47) is derived in the same manner. Melnikov theory for n-dimensional diffeomorphisms with hetero-homoclinic connections to normally hyperbolic

invariant manifolds has been developed in [41]. Let us briefly review this construction and adapt it for our slow-fast setup. Since for the map B the invariant manifolds $W^{s,u}(\Lambda)$ are three-dimensional while the phase space is four-dimensional, one only needs a scalar Melnikov function to measure their splitting for small nonzero ε, δ. Let us consider the case $\varepsilon > 0$ and $\delta = 0$. Take $\tilde{\Lambda}$ as in (40). By symmetry we only need to consider the splitting of $W_2^{s,u}(\tilde{\Lambda})$. Theorem 3.4 in [41] gives the following expression for the Melnikov function M_1:

$$\sum_{n=-\infty}^{n=\infty} \langle DB_0^n(x_{-n})B_1(x_{-n-1}), \nu(x_0) \rangle, \tag{A.1}$$

where $x_0 = (\varphi_0, \theta_0, E_0, t_0) \in W_2(\tilde{\Lambda})$ and $\nu(x_0)$ is the vector forming a basis of an orthogonal space to the tangent space of the unperturbed three dimensional homoclinic manifold $W_2(\tilde{\Lambda})$. Since B_0 has a first integral I, we take $\nabla I(x_0) = \nu(x_0)$. By the property of first integrals, observe that $\nabla I(x_0) = (DB_0(x_0))^T \nabla I(x_1)$ and by induction $\nabla I(x_0) = (DB_0^n(x_0))^T \nabla I(x_n)$. Then rewriting (A.1) yields

$$M_1 = \sum_{n=-\infty}^{n=\infty} \langle DB_0^n(x_{-n})B_1(x_{-n-1}), \nabla I(x_0) \rangle \tag{A.2}$$

$$= \sum_{n=-\infty}^{n=\infty} \langle B_1(x_{-n-1}), (DB_0^n(x_{-n}))^T \nabla I(x_0) \rangle \tag{A.3}$$

$$= \sum_{n=-\infty}^{n=\infty} \langle B_1(x_{n-1}), (DB_0^{-n}(x_n))^T \nabla I(x_0) \rangle \tag{A.4}$$

$$= \sum_{n=-\infty}^{n=\infty} \langle B_1(x_{n-1}), (DB_0^{-n}(x_n))^T (DB_0^n(x))^T \nabla I(x_n) \rangle \tag{A.5}$$

$$= \sum_{n=-\infty}^{n=\infty} \langle B_1(x_{n-1}), (DB_0^n(x)DB_0^{-n}(x_n))^T \nabla I(x_n) \rangle \tag{A.6}$$

$$= \sum_{n=-\infty}^{n=\infty} \langle B_1(x_{n-1}), \nabla I(x_n) \rangle, \tag{A.7}$$

which gives (45). Note that since (E, t) are close to identity, effectively the summation above is performed only over the fast variables (φ, θ), while (E, t) are held at an initial value, hence they enter the sum as 'fixed coefficients'. For (45) to converge, we require the restriction of the perturbed components of fast variables (φ, θ) (i.e. f_1, f_2, g_1, g_2) to Λ to vanish, and the form of I given by (9) was chosen to ensure that $\nabla_{E,t}I(x) = 0$ on Λ (i.e. the gradient of I with respect to slow variables (E, t)).

The Melnikov function (47) is obtained by repeating the same steps above, by setting $\varepsilon = 0$, and $\delta \neq 0$. Note that the B_2 terms are independent of E and are evaluated at a given fixed moment of time t, hence effectively (47) corresponds to the Melnikov function for the δ-polynomial perturbation of the elliptic billiard that has been studied by Delshams and Ramirez Ros in [35]. Since at $O(\varepsilon + |\delta|)$ the components of B_1 and B_2 given by formula (26) simply add, the distance between perturbed invariant manifolds $W^{s,u}(\tilde{\Lambda})$ for $\varepsilon, \delta \neq 0$ is given by (51).

Appendix B. Computation of the Melnikov function M_1 for time-dependent ellipse

The Melnikov functions (45) and (47) can be computed analytically in terms of elliptic functions using the theory developed in [35] to give the formulas (46) and (48) respectively. In this appendix we will derive the formula (46). First, let us introduce some notation following [35] and quote the proposition 3.1 we use from [35].

Given a parameter $m \in [0, 1]$, we have the following complete elliptic integrals of the first and second kind respectively:

$$K = K(m) = \int_0^{\pi/2} (1 - m \sin(\theta))^{-1/2} d\theta,$$

$$E = E(m) = \int_0^{\pi/2} (1 - m \sin(\theta))^{1/2} d\theta.$$

The incomplete elliptic integral of the second kind is

$$E(u) = E(u \mid m) := \int_0^u dn^2(v \mid m) dv,$$

where the function dn is one of the Jacobian elliptic functions.

Further, $K' = K'(m) := K(1 - m)$, $E' = E'(m) := E(1 - m)$ and if one of $m, K, K', E, E', \frac{K'}{K}$ is given, all the rest are determined. We determine the parameter m for a given $T, h > 0$ by relation

$$\frac{K'}{K} = \frac{T}{h}.$$

From now on, we do not explicitly write the dependence of K, K', E, E', m on T and h. We introduce a function $\chi_T(z)$,

$$\chi_T(z) = \left(\frac{2K}{h}\right)^2 \left(\frac{E'}{K'} - 1\right) z + \left(\frac{2K}{h}\right) E \left(\frac{2Kz}{h} + K'i\right),$$

with the properties: (1): χ is meromorphic on $\mathbb{C}$, (2): χ is Ti-periodic with h-periodic derivative, (3): the poles of χ are in the set $h\mathbb{Z} + Ti\mathbb{Z}$, all simple with residue 1. The following formula is easily derived using the properties of elliptic functions:

$$\chi(i\pi/2 - \tau) - \chi(h + i\pi/2 - \tau) = -2. \tag{B.1}$$

It is also easily shown [35] that the following relation holds:

$$\chi(z + h) - \chi(z) = \frac{2\pi}{T}. \tag{B.2}$$

For an isolated singularity $z_0 \in \mathbb{C}$ of a function q, we denote by $a_j(q, z_0)$ the coefficient of $(z - z_0)^{-j}$ in the Laurent series of q around z_0.

Then the following result holds.

Proposition B.1. [35, 42] *Let q be a function satisfying:*

- *q is analytic in $\mathbb{R}$, with only isolated singularities in $\mathbb{C}$*
- *q is Ti-periodic for some $T > 0$,*
- *$|q(\tau)| \leqslant Ae^{-c|\Re\tau|}$ when $|\Re\tau| \to \infty$, for some constants $A, c \geqslant 0$.*

Then, $Q(\tau) = \sum_{n=-\infty}^{\infty} q(\tau + hn)$ is analytic in $\mathbb{R}$, has only isolated singularities in $\mathbb{C}$, and is doubly periodic with periods $h \neq 0$, where $h \in \mathbb{R}$ and Ti. Furthermore, $Q(\tau)$ may be expressed as

$$Q(\tau) = - \sum_{z \in \mathrm{Sing}_T(q)} \sum_{j \geqslant 0} \frac{a_{j+1}(q,z)}{j!} \chi_T^j(z - \tau), \tag{B.3}$$

where $\mathrm{Sing}_T(q)$ is the set of singularities of q in $I_T = \{z \in \mathbb{C} : 0 < \Im z < T\}$, and χ^j denotes the jth derivative of χ.

We note that if q is meromorphic, then $Q(\tau)$ is elliptic and can be computed analytically. Now we proceed to give the derivation of (46). We will show that the sum (45) is an elliptic function with two periods $\log \lambda$ and πi where λ given by (8), and then apply the above proposition to compute (45).

Let us denote by $I_{\varphi,\theta,E,t}$ the partial derivatives of integral I given by (9) with respect to φ, θ, E and t respectively. We will evaluate (45) over the unperturbed homoclinic $W_2(\tilde{\Lambda})$ for fixed (E,t), hence we use the parameter $\xi = \tan \frac{\varphi_n}{2}$ and the equation of $W_2(\tilde{\Lambda})$ given by (41). Then the formula (14) expresses θ as a function of φ to give $W_2(\Lambda)$ in terms of ξ, holding (E,t) fixed at some initial value $(E_0,t_0) = (E_n,t_n)$. Using $B_1 = (f_1,f_2,f_3,f_4)^T$ from (26) and expressing $f_{1,2,3,4}$ in terms of ξ we write (45) as

$$M_1 := \sum_{n=-\infty}^{\infty} \langle \nabla I(B_0(x_n), B_1(x_n)) \rangle = \sum_{n=-\infty}^{\infty} f_1(\xi_n)I_\varphi(\xi_{n+1}) + f_2(\xi_n)I_\theta(\xi_n) + f_4(\xi_n)I_t(\xi_{n+1}).$$

Note we have written φ and θ in terms of ξ by virtue of (14), and $x_n = (\varphi_n, \theta_n, E_n, t_n)$ as before. We have suppressed the dependence of functions $f_{1,2,3,4}$ and integral I on (E,t). Note that the term $f_3 I_E$ is identically zero since I is independent of energy E and thus we omit it from the above sum. Since parametrisation (14) yields $\xi_{n+1} = \lambda^{-1}\xi_n$, we may express the sum above purely in terms of ξ_n, i.e.

$$M_1 = \sum_{n=-\infty}^{\infty} f_1(\xi_n)I_\varphi(\lambda^{-1}\xi_n) + f_2(\xi_n)I_\theta(\lambda^{-1}\xi_n) + f_4(\xi_n)I_t(\lambda^{-1}\xi_n) = \sum_{n=-\infty}^{\infty} F(\xi_n), \tag{B.4}$$

for certain function F. Introduce a change of variables τ defined by $e^\tau = \xi$. Since $\xi \in (0,\infty)$, then $\tau \in (-\infty,\infty)$, and for brevity put $h = \log \lambda$, as in [35]. Then M_1 becomes $M_1 = \sum F(e^{\tau+nh}) = \sum \tilde{F}(\tau + nh)$, after swapping $n \mapsto -n$. Here $\tilde{F}(\tau) = f_1(\tau)I_\varphi(\tau) + f_2(\tau)I_\theta(\tau) + f_4(\tau)I_t(\tau)$. Using the formulae (27)–(30) and (9) together with (14), we obtain the following expressions for $f_1 I_\varphi|_{W_2}$, $f_2 I_\theta|_{W_2}$ and $f_4 I_t|_{W_2}$ in terms of τ:

$$f_1(\tau)I_\varphi(\tau)|_{W_2} = \frac{-8\dot{a}c^2\lambda(\lambda+1)e^{2\tau}(\lambda^2 - e^{2\tau})^2}{v_0(1 + e^{2\tau})(e^{2\tau} + \lambda^3)(e^{2\tau} + \lambda^2)^2} + \frac{16\dot{b}ac^2\lambda^2 e^{2\tau}(\lambda - e^{2\tau})(\lambda^2 - e^{2\tau})}{bv_0(1 + e^{2\tau})(e^{2\tau} + \lambda^3)(e^{2\tau} + \lambda^2)^2}, \tag{B.5}$$

$$f_2(\tau)I_\theta(\tau)|_{W_2} = -\underbrace{\frac{16a^2c\lambda^2(\dot{a}ba^{-1} - \dot{b})e^{2\tau}(\lambda^2 - e^{2\tau})(\lambda + e^{2\tau})}{v_0 b(\lambda^3 + e^{2\tau})(\lambda^2 + e^{2\tau})^2(e^{2\tau} + 1)}}_{1} + \underbrace{\frac{8ac\lambda e^{2\tau}(-\dot{a}(\lambda+1)(\lambda^2 - e^{2\tau}) + 2\lambda\dot{b}ab^{-1}(\lambda - e^{2\tau}))}{v_0(e^{2\tau} + \lambda^3)(e^{2\tau} + \lambda^2)(e^{2\tau} + 1)}}_{2}$$
$$+ \underbrace{\frac{16c^2\lambda^2 e^{2\tau}(\dot{a}b(\lambda^2 - e^{2\tau})^2 + 4\dot{b}a\lambda^2 e^{2\tau})}{v_0 b(e^{2\tau} + \lambda^3)(e^{2\tau} + \lambda^2)^2(e^{2\tau} + \lambda)}}_{3}, \tag{B.6}$$

$$f_4(\tau)I_t(\tau)|_{W_2} = \frac{16a^2\lambda^2 e^{2\tau}(a\dot{b}b^{-1} - \dot{a})(e^{2\tau} + \lambda)}{(e^{2\tau} + \lambda^3)(e^{2\tau} + \lambda^2)(e^{2\tau} + 1)}. \tag{B.7}$$

Since in this appendix we are only considering the orbit of map B restricted to W_2, we drop the subscript W_2 for brevity. It is clear that f_1I_φ, f_2I_θ, f_4I_t are analytic on $\mathbb{R}$, exponentially decay at infinity and are πi-periodic on $\mathbb{C}$ with isolated singularities that are poles, hence they are meromorphic, satisfying proposition 1. Therefore we may apply proposition 1 to compute $M_1 = \sum F(\tau)$. We therefore take $T = \pi$, $\frac{K'}{K} = \frac{\pi}{h}$ and $\chi_\pi(z) = \chi(z)$.

In the following computations, we will be using the following formula that can be shown using properties of elliptic functions:

$$\chi(z + lh + h/2) - \chi(z + lh) = 1 + \frac{2K}{h}m\operatorname{sn}\left(\frac{2K\tau}{h}\right)\operatorname{cd}\left(\frac{2K\tau}{h}\right) = 1 + Y(\tau), \tag{B.8}$$

where $l \in \mathbb{Z}$, and we defined $Y(\tau) = \frac{2K}{h}m\operatorname{sn}\left(\frac{2K\tau}{h}\right)\operatorname{cd}\left(\frac{2K\tau}{h}\right)$ for brevity. Also, we have

$$\chi'\left(\frac{\pi i}{2} - \tau\right) = \left(\frac{2K}{h}\right)^2\left(\frac{E'}{K'} - 1 + \operatorname{dn}^2\left(\frac{2K\tau}{h}\right)\right) = X(\tau), \tag{B.9}$$

where for brevity we put $X(\tau) = \left(\frac{2K}{h}\right)^2\left(\frac{E'}{K'} - 1 + \operatorname{dn}^2\left(\frac{2K\tau}{h}\right)\right)$ and ' denotes differentiation w.r.t. τ. In (B.8) and (B.9) we have used the identities $\operatorname{dn}(u) = dn(-u)$, $\operatorname{dn}(u) = -\operatorname{dn}(u + 2K + 2K'i)$, $E(-u) = -E(u)$, $E(u + 2K + 2K'i) = E(u) + 2E + 2i(K' - E')$, $E(K - u) = E - E(u) + m\operatorname{sn}(u)\operatorname{cd}(u)$ and the Legendre equality $EK' + E'K - K'K = \frac{\pi}{2}$ together with the formulas (B.1) and (B.2).

We will compute separately the three components $\sum f_1I_\varphi$, $\sum f_1I_\theta$ and $\sum f_4I_t$ of the expression for M_1 given by (B.4). Further, wherever it facilitates the computations, will consider individually the $\dot{a}$ and $\dot{b}$ components of the sums $\sum f_1I_\varphi$, $\sum f_1I_\theta$ and f_4I_t.

Consider $\sum f_1I_\varphi$. Take the $\dot{a}$ component of $\sum f_1I_\varphi$ with the coefficient $\frac{-8c^2\lambda(\lambda+1)}{v_0}$ factored out. Denote by $a_j(f_1I_\varphi, z_0; \dot{a})$ the corresponding Laurent series coefficient of the $\dot{a}$ component of f_1I_φ at z_0 (here $\frac{-8c^2\lambda(\lambda+1)}{v_0}$ is factored out). We have:

- Simple pole at $z = i\pi/2$ with $a_1(f_1I_\varphi, i\pi/2; \dot{a}) = \frac{1+2\lambda+\lambda^4}{2(-1+\lambda)^3(1+\lambda)^2(1+\lambda+\lambda^2)}$,

- simple pole at $z = 3h/2 + i\pi/2$ with $a_1(f_1I_\varphi, 3h/2 + i\pi/2; \dot{a}) = \frac{-1-2\lambda-\lambda^2}{2(-1+\lambda)^3(1+\lambda+\lambda^2)}$,

- double pole at $z = h + i\pi/2$ with $a_1(f_1I_\varphi, h + i\pi/2; \dot{a}) = \frac{2\lambda}{(-1+\lambda)^3(1+\lambda)^2}$ and $a_2(f_1I_\varphi, h + i\pi/2; \dot{a}) = \frac{1}{(\lambda-1)^2(1+\lambda)}$.

Observe that $a_1(f_1I_\varphi, i\pi/2; \dot{a}) + a_1(f_1I_\varphi, 3h/2 + i\pi/2; \dot{a}) = -a_1(f_1I_\varphi, h + i\pi/2; \dot{a})$. Thus we obtain the following formula for $\dot{a}$ component of $\sum f_1I_\varphi$:

$$\sum f_1I_\varphi|_{\dot{a}} = \frac{8c^2\lambda(\lambda+1)}{v_0}\left(-2a_1(i\pi/2) + a_1(3h/2 + i\pi/2)\left(1 + Y(\tau)\right) + a_2(h + i\pi/2)X(\tau)\right).$$

Similarly, considering the $\dot{b}$ component of $\sum f_1I_\varphi$ (see (B.5)), with factored out $\frac{16\dot{b}ac^2\lambda^2}{bv_0}$, we have:

- Simple pole at $z = \frac{i\pi}{2}$ with $a_1(f_1I_\varphi, i\pi/2; \dot{b}) = \frac{\lambda^2+1}{2(\lambda+1)(\lambda-1)^3(\lambda^2+\lambda+1)}$,
- simple pole at $z = 3h/2 + i\pi/2$ with $a_1(f_1I_\varphi, 3h/2 + i\pi/2; \dot{b}) = \frac{-(\lambda+1)(\lambda^2+1)}{2\lambda(\lambda-1)^3(\lambda^2+\lambda+1)}$,

- double pole at $h + i\pi/2$ with $a_1(f_1 I_\varphi, h + i\pi/2; \dot{b}) = \frac{\lambda^2+1}{2\lambda(\lambda+1)(\lambda-1)^3}$ and $a_2(f_1 I_\varphi, h + i\pi/2; \dot{b}) = \frac{1}{2\lambda(\lambda-1)^2}$.

Here $a_1(f_1 I_\varphi, i\pi/2; \dot{b}) + a_1(f_1 I_\varphi, 3h/2 + i\pi/2; \dot{b}) = -a_1(f_1 I_\varphi, h + i\pi/2; \dot{b})$. Thus we have the following formula for $\dot{b}$ component of $\sum f_1 I_\varphi$:

$$\sum f_1 I_\varphi|_{\dot{b}} = -\frac{16 \dot{b} a c^2 \lambda^2}{b v_0} \left(-2a_1(i\pi/2) + a_1(3h/2 + i\pi/2)(1 + Y(\tau)) + a_2(h + i\pi/2)X(\tau) \right).$$

Adding the $\dot{a}$ and $\dot{b}$ contributions and simplifying gives the formula

$$\sum f_1 I_\varphi = \frac{\dot{a}}{v} \left(\frac{-2b^2(3a^4 + 2a^2c^2 + c^4)}{ac(3a^2 + c^2)} + 2b^2 X(\tau) - \frac{4a^3 b^2}{c(3a^2 + c^2)} Y(\tau) \right)$$
$$+ \frac{\dot{b}}{bv} \left(\frac{2b^2(3a^2 - c^2)(a^2 + c^2)}{c(3a^2 + c^2)} - 2ab^2 X(\tau) + \frac{4a^2 b^2(a^2 + c^2)}{c(3a^2 + c^2)} Y(\tau) \right). \quad \text{(B.10)}$$

Next we will compute $\sum f_2 I_\theta$; the corresponding expression (B.6) consist of the sum of three parts. Take the part (1) defined by braces, and factor out $\frac{-16a^2 c \lambda^2(\dot{a} b a^{-1} - \dot{b})}{v_0 b}$. Denote by $a_j(f_1 I_\theta, z_0; (1))$ the corresponding Laurent series coefficient of the (1) component of $f_1 I_\varphi$ at z_0, (without coefficient $\frac{-16a^2 c \lambda^2(\dot{a} b a^{-1} - \dot{b})}{v_0 b}$). We have

- Simple pole at $z = i\pi/2$ with $a_1(f_2 I_\theta, i\pi/2; (1)) = \frac{\lambda^2+1}{2(\lambda^2-1)^2(\lambda^2+\lambda+1)}$,
- simple pole at $z = 3h/2 + i\pi/2$ with $a_1(f_2 I_\theta, 3h/2 + i\pi/2; (1)) = \frac{(\lambda+1)^2}{2\lambda(\lambda-1)^2(\lambda^2+\lambda+1)}$,
- double pole at $z = h + i\pi/2$ with $a_1(f_2 I_\theta, h + i\pi/2; (1)) = \frac{-1-4\lambda-\lambda^2}{2\lambda(\lambda^2-1)^2}$ and $a_2(f_2 I_\theta, h + i\pi/2; (1) = \frac{1}{2\lambda(\lambda^2-1)}$,

where $a_1(f_2 I_\theta, i\pi/2; (1)) + a_1(f_2 I_\theta, 3h/2 + i\pi/2; (1)) = -a_1(f_2 I_\theta, h + i\pi/2; (1))$. Hence the sum for the part (1) is:

$$\sum f_2 I_\theta|_{(1)} = \frac{16a^2 c \lambda^2(\dot{a} b a^{-1} - \dot{b})}{v_0 b} \left(-2a_1(f_2 I_\theta, i\pi/2; (1)) + a_1(f_2 I_\theta, 3h/2 + i\pi/2; (1)) (1 + Y(\tau)) \right)$$
$$+ \frac{16a^2 c \lambda^2(\dot{a} b a^{-1} - \dot{b})}{v_0 b} (a_2(f_2 I_\theta, h + i\pi/2; (1))X(\tau)). \quad \text{(B.11)}$$

Let us consider the $\dot{a}$ and $\dot{b}$ components of the second component (2) of (B.6) separately. Take contribution in $\dot{a}$. Factor out $\frac{-8a\dot{a}c\lambda(\lambda+1)}{v_0}$. Denote by $a_j(f_1 I_\theta, z_0; \dot{a}, (2))$ the corresponding Laurent series coefficient of the $\dot{a}$ coefficient of the (2) component of $f_1 I_\varphi$ at z_0. We have

- Simple pole at $z = \frac{i\pi}{2}$ with $a_1(f_2 I_\theta, i\pi/2; \dot{a}, (2)) = \frac{\lambda^2+1}{2(\lambda^2-1)(\lambda-1)(\lambda^2+\lambda+1)}$,
- simple pole at $z = 3h/2 + i\pi/2$ with $a_1(f_2 I_\theta, 3h/2 + i\pi/2; \dot{a}, (2)) = \frac{\lambda+1}{2(\lambda-1)^2(\lambda^2+\lambda+1)}$,
- Simple pole at $z = h + i\pi/2$ with $a_1(f_2 I_\theta, h + i\pi/2; \dot{a}, (2)) = \frac{-1}{(\lambda-1)^2(\lambda+1)}$,

where $a_1(f_2 I_\theta, i\pi/2; \dot{a}, (2)) + a_1(f_2 I_\theta, 3h + i\pi/2; \dot{a}, (2)) = -a_1(f_2 I_\theta, h + i\pi/2; \dot{a}, (2))$. Thus the corresponding sum for part (2) for $\dot{a}$ contribution is

$$\sum f_2 I_\theta|_{\dot{a},(2)} = \frac{8a\dot{a}c\lambda(\lambda + 1)}{v_0} \left(-2a_1(f_2 I_\theta, i\pi/2; \dot{a}) + a_1(f_2 I_\theta, 3h/2 + i\pi/2; \dot{a})(1 + Y(\tau)) \right).$$
$$\text{(B.12)}$$

For $\dot{b}$ component of (2) of (B.6), first factor out $\frac{16a^2 c\lambda^2 \dot{b}}{bv}$. Using the notation as above, we find:

- Simple pole at $z = \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, \mathrm{i}\pi/2; \dot{b}, (2)) = \frac{1}{2(\lambda-1)^2(\lambda^2+\lambda+1)}$,
- simple pole at $z = 3h/2 + \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{b}, (2)) = \frac{\lambda^2+1}{2\lambda(\lambda-1)^2(\lambda^2+\lambda+1)}$,
- simple pole at $z = h + \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{b}, (2)) = \frac{-1}{2\lambda(\lambda-1)^2}$,

where $\quad a_1(f_2 I_\theta, \mathrm{i}\pi/2; \dot{b}, (2)) + a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{b}, (2)) = -a_1(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{b}, (2))$. Thus the second component (2) sum for $\dot{b}$ is

$$\sum f_2 I_\theta|_{\dot{b},(2)} = -\frac{16a^2 c\lambda^2 \dot{b}}{bv}\left(-2a_1(f_2 I_\theta, \mathrm{i}\pi/2; \dot{b}) + a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{b})(1 + Y(\tau))\right).$$
(B.13)

Finally, let us consider the $\dot{a}$ and $\dot{b}$ coefficients of part (3) of (B.6) separately. For $\dot{a}$ component of (3), we factor out $\frac{16\dot{a}c^2\lambda^2}{v_0}$ and obtain

- Simple pole at $z = 3h/2 + \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{a}, (3)) = -a_1(h/2 + \mathrm{i}\pi/2) = \frac{-1-\lambda}{2\lambda(\lambda-1)^3}$,
- Simple pole at $z = h + \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{a}, (3)) = 0, a_2(h + \mathrm{i}\pi/2) = \frac{1}{\lambda(\lambda-1)^2}$.

Hence we get the corresponding sum

$$\sum f_2 I_\theta|_{\dot{a},(3)} = -\frac{16\dot{a}c^2\lambda^2}{v_0}\left(2a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{a}, (3)) + a_2(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{a}, (3))X(\tau)\right).$$
(B.14)

For third component (3) coefficient in b, we factor out $\frac{64a\dot{b}c^2\lambda^4}{bv_0}$ to yield

- Simple pole at $z = 3h/2 + \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{b}, (3)) = -a_1(h/2 + \mathrm{i}\pi/2) = \frac{1}{2\lambda^2(\lambda+1)(\lambda-1)^3}$,
- simple pole at $z = h + \mathrm{i}\pi/2$ with $a_1(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{b}, (3)) = 0, \; a_2(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{b}, (3)) = \frac{-1}{4\lambda^3(\lambda-1)^2}$,

which gives the sum

$$\sum f_2 I_\theta|_{\dot{b},(3)} = -\frac{64a\dot{b}c^2\lambda^4}{bv_0}\left(2a_1(f_2 I_\theta, 3h/2 + \mathrm{i}\pi/2; \dot{b}, (3)) + a_2(f_2 I_\theta, h + \mathrm{i}\pi/2; \dot{b}, (3))X(\tau)\right).$$
(B.15)

Adding the expressions (B.11)–(B.15) and simplifying gives the formula

$$\sum f_2 I_\theta = \frac{\dot{a}}{v}\left(\frac{2b^2(9a^4 + c^4)}{ac(3a^2 + c^2)} - 6b^2 X(\tau) + \frac{12a^3 b^2}{c(3a^2 + c^2)}Y(\tau)\right)$$
$$+ \frac{\dot{b}}{bv}\left(6ab^2 X(\tau) - \frac{2b^2(3a^2 - c^2)}{c} - \frac{4a^2 b^2}{c}Y(\tau)\right).$$
(B.16)

Finally we compute $\sum_{n=-\infty}^{\infty} f_4 I_t$. Let us take out the factor $16a^2\lambda^2(a\dot{b}b^{-1} - \dot{a})$. Then we obtain

- Simple pole at $z = \mathrm{i}\pi/2$ with $a_1(f_4 I_t, \mathrm{i}\pi/2) = \frac{1}{2(-1-\lambda+\lambda^3+\lambda^4)}$,
- simple pole at $z = 3h/2 + \mathrm{i}\pi/2$ with $a_1(f_4 I_t, 3\log\lambda/2 + \mathrm{i}\pi/2) = \frac{-1-\lambda}{2\lambda(-1+\lambda^3)}$,
- simple pole at $z = h + \mathrm{i}\pi/2$ with $a_1(f_4 I_t, \log\lambda + \mathrm{i}\pi/2) = \frac{1}{2\lambda(-1+\lambda^2)}$.

Observe   that $a_1(f_4I_t, \log\lambda + i\pi/2) + a_1(f_4I_t, 3\log\lambda/2 + i\pi/2) = -a_1(f_4I_t, i\pi/2)$.
Therefore the sum is

$$\sum f_4I_t = -\left(\frac{a\dot{b}}{bv} - \frac{\dot{a}}{v}\right)(16a^2\lambda^2)\left(a_1(3h/2 + i\pi/2)(3 + Y(\tau)) + 2a_1(h + i\pi/2)\right).$$

Upon simplifying, this yields

$$\sum f_4I_t = \left(\frac{a\dot{b}}{bv} - \frac{\dot{a}}{v}\right)\left(\frac{4ab^2(3a^2 - c^2)}{c(3a^2 + c^2)} + \frac{8a^3b^2}{c(3a^2 + c^2)}Y(\tau)\right). \tag{B.17}$$

Finally adding (B.10), (B.16) and (B.17) yields (46).

Appendix C. Computation of zero order distance D_0

In this appendix we show that the free flight distance (27) that is incorporated in the formulas
for f_1, f_2, f_4 turns out to be an elliptic function when evaluated on the homoclinic manifold
$W_2(\tilde{\Lambda})$ (see formula (41)).

Proposition C.2.   *The zero order in ε, δ free flight distance D_0 given by equation (27)
evaluated on $W_2(\tilde{\Lambda})$ expressed in terms of ξ is*

$$D_0|_{W_2(\tilde{\Lambda})} = \frac{2a(\lambda + \xi_n^2)^2}{(\lambda^2 + \xi_n^2)(1 + \xi_n^2)} = \frac{2a(1 + \lambda\xi_{n+1}^2)^2}{(1 + \xi_{n+1}^2)(1 + \lambda^2\xi_{n+1}^2)}. \tag{C.1}$$

Proof.   Recall that $W_2(\tilde{\Lambda})$ corresponds to the billiard orbit passing the focus at $(-c, 0)$, and
therefore each consecutive collision point on the boundary (16) is in alternate halves of the
ellipse. In terms of variable $\varphi \pmod{\pi}$ the negative signs in the square brackets in (27) change
to positive and gives

$$D_0|_{W_2(\tilde{\Lambda})} = \sqrt{a^2[\cos(\varphi_n) + \cos(\varphi_{n+1})]^2 + b^2[\sin(\varphi_n) + \sin(\varphi_{n+1})]^2}.$$

Taking the parametrisation (14) to express trigonometric functions of φ_{n+1} and φ_n in terms
of ξ_n and using the identity $\frac{b^2}{a^2}(1 + \lambda)^2 \equiv 4\lambda$, with λ given by (8), we obtain:

$$D_0|_{W_2(\tilde{\Lambda})} = \sqrt{a^2\left(\frac{4\lambda}{(1+\lambda)^2}\left(\frac{2\xi_n}{1 + \xi_n^2} + \frac{2\lambda\xi_n}{\lambda^2 + \xi_n^2}\right)^2 + \left(\frac{1 - \xi_n^2}{1 + \xi_n^2} + \frac{\lambda^2 - \xi_n^2}{\lambda^2 + \xi_n^2}\right)^2\right)}.$$

Taking a positive root of this expression since it is Euclidean distance and simplifying
yields:

$$D_0|_{W_2(\tilde{\Lambda})} = \sqrt{a^2\left(\frac{4(\lambda + \xi_n^2)^4}{(1 + \xi_n^2)^2(\lambda^2 + \xi_n^2)^2}\right)}.$$

$\square$

Remark C.1.   Note that in the limit of $n \to \infty$, (C.1) yields $D_0|_{W_2(\tilde{\Lambda})} = 2a$, i.e. the length
of the major axis of the ellipse, which corresponds the hyperbolic fixed point z.

Remark C.2. Analogously we may calculate $D_0|_{W_1(\tilde{\Lambda})}$, which is:

$$D_0|_{W_1(\tilde{\Lambda})} = \frac{2a(1 + \lambda\xi_n^2)^2}{(1 + \xi_n^2)(1 + \lambda^2\xi_n^2)} = \frac{2a(\lambda + \xi_{n+1}^2)^2}{(\lambda^2 + \xi_{n+1}^2)(1 + \xi_{n+1}^2)}. \tag{C.2}$$

Appendix D. Scattering map computations

We provide a derivation of truncated scattering map formula (66). From the E component of S_Γ given by (64), we need to compute the sum

$$\sum_{n=-\infty}^{\infty} f_3(\cdot, E_n, t_n)|_\Lambda - f_3(\varphi_n, \theta_n(\varphi_n), E_n, t_n)|_{W_2(\tilde{\Lambda})} = 2\sqrt{2E_0} \sum_{n=-\infty}^{\infty} -\dot{a}(t_{n+1}) + u_{n+1}\sin(\theta_{n+1}),$$

where $u_{n+1} = \frac{\dot{a}b\cos^2(\varphi_{n+1}) + a\dot{b}\sin^2(\varphi_{n+1})}{\sqrt{a^2\sin^2(\varphi_{n+1}) + b^2\cos^2(\varphi_{n+1})}}$ is the normal boundary speed at boundary point φ_{n+1} evaluated on the homoclinic manifold $W_2(\tilde{\Lambda})$. Rewriting φ in terms of ξ using (14) gives

$$u_{n+1} = \frac{\dot{a}b(1 - \xi_{n+1}^2)^2 + 4a\dot{b}\xi_{n+1}^2}{(1 + \xi_{n+1}^2)\sqrt{4c^2\xi_{n+1}^2 + b^2(1 + \xi_{n+1}^2)^2}}.$$

Now

$$4c^2\xi_{n+1}^2 + b^2(1 + \xi_{n+1}^2)^2 \equiv \frac{b^2(\xi_{n+1}^2 + \lambda)(\lambda\xi_{n+1}^2 + 1)}{\lambda},$$

and expressing $\sin(\theta)$ in terms of ξ using (14), we obtain

$$\begin{aligned} u_{n+1}\sin(\theta_{n+1}) - \dot{a}(t_{n+1}) &= \frac{\lambda(\dot{a}b(1 - \xi_{n+1}^2)^2 + 4a\dot{b}\xi_{n+1}^2)}{b(\xi_{n+1}^2 + \lambda)(\lambda\xi_{n+1}^2 + 1)} - \dot{a}(t_{n+1}) \\ &= \frac{\dot{a}\xi_{n+1}^2(\lambda + 1)^2}{(\xi_{n+1}^2 + \lambda)(\lambda\xi_{n+1}^2 + 1)} + \frac{4a\dot{b}\lambda\xi_{n+1}^2}{b(\xi_{n+1}^2 + \lambda)(\lambda\xi_{n+1}^2 + 1)}. \end{aligned}$$

It is clear that we need to calculate $\sum_{n=-\infty}^{\infty} \frac{\xi_n^2}{(\xi_n^2 + \lambda)(\lambda\xi_n^2 + 1)}$. After a change of variables $\xi = e^\tau, h = \log\lambda$ as before, the function $\frac{e^{2\tau}}{(e^{2\tau} + \lambda)(\lambda e^{2\tau} + 1)}$ is bounded for $\Re\tau \to \infty$, is periodic in the complex plane with period $i\pi$, and it has simple poles at $\tau = \pm\log\lambda/2 + i\pi/2$ with residues $\frac{\mp 1}{2(\lambda^2 - 1)}$, and by using the result $\chi(\frac{\log\lambda}{2} + i\pi/2 - \tau) - \chi(\frac{\log\lambda}{2} + i\pi/2 - \tau) = 2$, we have that $\sum_{n=-\infty}^{\infty} \frac{\xi_n^2}{(\xi_n^2 + \lambda)(\lambda\xi_n^2 + 1)} = \frac{1}{\lambda^2 - 1}$. Therefore after some manipulations we get

$$u_{n+1}\sin(\theta_{n+1}) - \dot{a}(t_{n+1}) = \frac{b\dot{b} - a\dot{a}}{c},$$

which gives (66).

The t component of S_Γ is computed using the geometric property of the ellipse, as given in section 4.3.

Appendix E. Computation of Melnikov function M_2

As remarked previously, M_2 corresponds to the Melnikov function for the $O(\delta)$ quartic polynomial perturbation of the ellipse, evaluated for the 'fixed' boundary at time $t = t_n$. Therefore

M_2 may be evaluated by standard methods developed for static perturbations of elliptic billiards using the generating function approach [35]. The parametrisation (16) of the boundary at fixed time t_n for first order in δ coincides with the parametrisation in [35] given by equation (4.6) up to the adjustment of a constant factor. Thus, it is easy to essentially re-write the derivation of the Melnikov function in [35] to obtain

$$M_2 = -4m\frac{ab^2}{c^2}\left(\frac{2K}{h}\right)^3 \operatorname{dn}\left(\frac{2K\tau}{h}\right)\operatorname{sn}\left(\frac{2K\tau}{h}\right)\operatorname{cn}\left(\frac{2K\tau}{h}\right). \tag{E.1}$$

Notice that (E.1) coincides with the Melnikov function for the perturbed ellipse in [35], (see section 4.4.2 of [35]), upon setting $c^2 = 1$, $c_1 = b$.

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