INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **16** (2003) 2163–2198 PII: S0951-7715(03)58683-2

Homoclinic bifurcations and dimension of attractors for damped nonlinear hyperbolic equations

Dmitry Turaev[1] **and Sergey Zelik**[2]

[1] WIAS, Mohrenstr. 39, D-10117 Berlin, Germany
[2] Université de Poitiers, Laboratoire d'Applications des Mathématiques—SP2MI, Boulevard Marie et Pierre Curie—Téléport 2, 86962 Chasseneuil Futuroscope Cedex, France

E-mail: turaev@wias-berlin.de and zelik@mathlabo.univ-poitiers.fr

Received 20 January 2003, in final form 4 July 2003
Published 15 September 2003
Online at stacks.iop.org/Non/16/2163

Recommended by P Constantin

Abstract
A new method of obtaining lower bounds for the attractor's dimension is suggested which involves analysis of homoclinic bifurcations. The method is applied for obtaining sharp estimates of the attractor's dimension for a class of abstract damped wave equations which are beyond the reach of the classical methods.

Mathematics Subject Classification: 35B41, 37G20, 35B45, 37D45

0. Introduction

It is well-known that the long-time behaviour of solutions of partial differential equations arising in mathematical physics can, in many cases, be described in terms of global attractors of the associated semigroups (see [1–4] and references therein). Moreover, it is also known that for a large class of equations of mathematical physics, including reaction–diffusion equations, Ginzburg–Landau equations, two-dimensional Navier–Stokes system, damped wave equations, etc, the corresponding attractor has finite Hausdorff and fractal dimensions. Thus, although the phase space for such problems is infinite-dimensional, the dynamics on the attractor happens to be finite-dimensional, hence it can possibly be understood by methods of the qualitative theory of ordinary differential equations. One of the crucial questions here is, of course, obtaining more or less realistic estimates for the dimension of the attractor.

The best known upper estimates here are usually obtained based on the Kaplan–Yorke concept of Lyapunov dimension $\dim_{\mathrm{L}}(\mathcal{A})$ of the attractor $\mathcal{A}$ (see [5]), and on the following estimate:

$$\dim_{\mathrm{H}}(\mathcal{A}) \leqslant \dim_{\mathrm{F}}(\mathcal{A}) \leqslant \dim_{\mathrm{L}}(\mathcal{A}), \tag{0.1}$$

where $\dim_H$ and $\dim_F$ denote the Hausdorff and the fractal dimension, respectively (see [7, 6, 3, 8–10]). The main point here is that the Lyapunov dimension, by its definition, can be explicitly estimated using sufficiently simple volume-contraction arguments (see [3], chapter 5, for details).

Lower bounds for the attractor's dimension are usually based on the observation that the unstable manifold of any equilibrium of the system is always contained in the global attractor $\mathcal{A}$. Consequently, the following estimate is valid:

$$\dim_F(\mathcal{A}) \geqslant \dim_H(\mathcal{A}) \geqslant \max_{u_0\in\mathcal{R}} N^+(u_0), \tag{0.2}$$

where $\mathcal{R}$ is the set of equilibria of the system, and $N^+(u_0)$ is the instability index of the equilibrium u_0 (see, e.g., [1, 2]).

We note that this method of obtaining the lower bounds for the attractor's dimension is perfect for the class of gradient systems (or, which is slightly more general, for systems possessing a global Lyapunov function). Indeed, the dynamics in the gradient case is, in a sense, trivial and the dimension of the attractor is determined by the instability indices of the equilibria only, no matter what the Lyapunov dimension of the attractor and what the volume-contraction properties. Namely, in this case we have the equality in the second part of (0.2):

$$\dim_H(\mathcal{A}) = \max_{u_0\in\mathcal{R}} N^+(u_0) \tag{0.3}$$

(see, e.g., [1, 11]).

There exists, however, a number of important equations of mathematical physics (such as two-dimensional Navier–Stokes system, Ginzburg–Landau equations, non-gradient systems of damped wave equations), for which the given methods of estimating the attractor's dimension from above and below yield *different* asymptotics for the dimension in terms of physical parameters of the system (see [3], chapter 6, [4] and references therein). Which asymptotics is then correct is a long-standing open problem in the theory of attractors. It is also worth noting that all systems mentioned above are far from being gradient and they usually demonstrate a very complicated (e.g. chaotic) dynamical behaviour.

In this paper, we present a new method of obtaining lower bounds for the attractor's dimension which exploits explicitly the recurrent (as opposed to gradient-like) nature of the system, and which is based on some general ideas from the theory of homoclinic bifurcations. Namely, we suggest estimating from below the attractor's dimension in terms of the maximum $M(\Gamma, u_0)$ of the dimension of the unstable manifold over the periodic orbits which can be born at a bifurcation of a homoclinic orbit Γ to an equilibrium u_0:

$$\dim_F(\mathcal{A}) \geqslant \dim_H(\mathcal{A}) \geqslant M(\Gamma, u_0). \tag{0.4}$$

To be more precise, one should consider a family of systems which depends on some set of parameters μ; then the global attractor is a function of μ as well, and (0.4) should be interpreted as

$$\limsup_{\mu\to\mu_0} \dim_F(\mathcal{A}_\mu) \geqslant \limsup_{\mu\to\mu_0} \dim_H(\mathcal{A}_\mu) \geqslant M(\Gamma, u_0),$$

where the bifurcational moment $\mu = \mu_0$ corresponds to the existence of the homoclinic loop Γ. Of course, one may use various homo/heteroclinic cycles with the same purposes—we take a homoclinic loop as the simplest possible construction.

The power of the homoclinic bifurcation theory is that it allows one to obtain a lot of knowledge about the behaviour of a dynamical system, based only on a very limited amount of information about it. The most prominent example is the Shilnikov homoclinic loop theorem: if a system has a homoclinic loop to an equilibrium whose nearest to the imaginary axis characteristic roots are complex, then the behaviour of orbits is chaotic, with a very detailed description of this behaviour available [12, 13]. The famous Lorenz attractor, as well as its higher-dimensional generalization—a wild spiral attractor—also appear as a result of a bifurcation of certain homoclinic loops (see [14–16]).

As found in [17], the common feature of various homoclinic bifurcations is that the behaviour near a homo/heteroclinic cycle is very much related to the Lyapunov dimension of equilibria and periodic orbits involved. In particular, for many classes of homoclinic loops the dimension $M(\Gamma, u_0)$ essentially coincides with the Lyapunov dimension of the corresponding equilibrium u_0:

$$M(\Gamma, u_0) \sim \dim_{\mathrm{L}}(u_0), \tag{0.5}$$

no matter how small the dimension $N^+(u_0)$ of the unstable manifold of u_0 is. Thus, under this approach, both upper and lower bounds for the attractor's dimension are given in terms of Lyapunov dimension. That is why we expect this method to be effective in order to obtain sharp bounds for the dimension. Of course, the existence of a homoclinic orbit and the possibility of perturbing it in the desired way within the class of systems under consideration is crucial for this method. However, the homoclinic phenomena are so typical for dynamical systems with non-trivial behaviour, that it would be natural to expect that in a wide class of equations of mathematical physics which demonstrate chaotic behaviour appropriate homoclinic bifurcations can indeed be detected (see, e.g., [18–21] where the existence of homoclinic loops was proven for the nonlinear Schrödinger and sine-Gordon equations).

We illustrate our method by a model example of an abstract damped wave equation

$$\partial_t^2 u + \gamma \partial_t u + \mathrm{A}u = F(u, \partial_t u) \tag{0.6}$$

in a Hilbert space H. We assume that $\mathrm{A} : D(\mathrm{A}) \to \mathrm{H}$ is a positive self-adjoint operator in H with compact inverse, whose eigenvalues satisfy the estimate

$$C_1 i^{2\kappa} \leqslant \lambda_i \leqslant C_2 i^{2\kappa}, \qquad i \in \mathbb{N} \tag{0.7}$$

for some positive C_1, C_2 and κ. Natural examples for A are provided by elliptic differential operators in a bounded domain, with $\mathrm{H} = L^2$. The quantity $\gamma > 0$ in (0.6) is a dissipation (or damping) parameter which is assumed to be small. To avoid technicalities, we have also assumed that the nonlinear operator $F = F(u, \partial_t u)$ belongs to some class $\mathbb{S}$ of very regular ('smoothing') operators which will be specified in section 2. Note that this class does not include superposition operators: we do not intend to apply our method to any specific equation in this paper, we only want to exemplify the basic ideas, so we restrict our consideration to this sufficiently simple setting.

We prove that under the above assumptions, equation (0.6) possesses a global attractor $\mathcal{A}$ in the corresponding energetic space E, and the Lyapunov dimension of the attractor satisfies

$$C_1' \gamma^{-1} \leqslant \dim_{\mathrm{L}}(\mathcal{A}) \leqslant C_2' \gamma^{-1} \tag{0.8}$$

for some positive constants $C_{1,2}'$ which are independent of γ. Consequently, due to (0.1), we have

$$\dim_{\mathrm{F}}(\mathcal{A}) \leqslant C_2' \gamma^{-1}. \tag{0.9}$$

On the other hand, when the nonlinearity belongs to our class $\mathbb{S}$ of very regular operators, we show that for every $\varepsilon > 0$ there exists a positive constant C_ε such that the instability index of any equilibrium state is less than $C_\varepsilon \gamma^{-\varepsilon}$. Thus, using the classical methods of estimating the dimension of the attractor $\mathcal{A}$ (which are based on (0.1) and (0.2)), we will necessarily have a huge gap between the asymptotics for upper and lower bounds of the attractor's dimension.

Nevertheless, using our 'homoclinic' method, we construct nonlinearities F belonging to the same class $\mathbb{S}$, for which we have

$$\dim_F(\mathcal{A}) \geqslant \dim_H(\mathcal{A}) \geqslant C_3\gamma^{-1} \tag{0.10}$$

for some positive constant C_3. Thus, at least in the case of damped hyperbolic equations with smoothing nonlinearities, the correct asymptotics for the dimension of the attractor is given by the corresponding asymptotics of the Lyapunov dimension, and the estimate (0.2) is not very relevant.

Note also that $\mathcal{A}$ is the so-called maximal attractor, so it could be possible, in principle, that the dimension of $\mathcal{A}$ can be decreased drastically by removing from $\mathcal{A}$ non-recurrent orbits (as in gradient-like systems where such an operation reduces $\mathcal{A}$ to a zero-dimensional set, typically). We show, however, that nonlinearities $F \in \mathbb{S}$ exist for which equation (0.6) has a minimal set whose dimension satisfies (0.10); therefore, the Lyapunov dimension (up to a constant factor) measures the complexity of the dynamics of damped hyperbolic equations correctly.

The examples which we are talking about are obtained as small perturbations of a decoupled system of second order ODEs (see (4.6)) which is an infinite collection of damped linear oscillators plus a single one degree of freedom Hamiltonian system describing a particle in a double-well potential on a straight line. Note that none of the modes here shows chaotic behaviour and, moreover, all of them but one are damped. We show, however, that for any fixed value of the damping parameter $\gamma > 0$, a (non-local) interaction of an arbitrarily small strength can be arranged between these modes such that extremely complicated behaviour is ignited, involving a huge ($\sim 1/\gamma$) number of modes (see remark 4.3). Note that we nowhere use the linear character of the oscillatory modes and our construction works for a chain of nonlinear damped oscillators as well. Therefore, our results should be applicable to perturbations of other integrable equations, such as the nonlinear Schrödinger equation, etc.

This paper is organized as follows. The existence of the global attractor for problem (0.6) is verified in section 1. The upper bounds for fractal and Lyapunov dimension of this attractor are obtained in section 2. The quantity $M(\Gamma, u_0)$ is computed for a special class of homoclinic loops in section 3. Finally, in section 4, we show that such homoclinic orbits really appear in equations (0.6) with the nonlinearities belonging to the class $\mathbb{S}$, then based on (0.4) we derive sharp lower bounds for the attractor's dimension.

1. Abstract nonlinear hyperbolic equation and its attractor

In this section, we study the following abstract nonlinear hyperbolic equation in a Hilbert space H:

$$\begin{aligned} &\partial_t^2 u + \gamma\partial_t u + \mathrm{A}u = F(u, \partial_t u), \\ &u|_{t=0} = u_0, \qquad \partial_t u|_{t=0} = u_0', \end{aligned} \tag{1.1}$$

where $u = u(t)$ is an unknown H-valued function, $\mathrm{A} : D(\mathrm{A}) \to \mathrm{H}$ is a given positive self-adjoint operator in H with compact inverse, $\gamma > 0$ is a given positive number which is assumed to be small and F is a given nonlinear operator.

As usual (see, e.g., [22] and [3], chapter 4), we define a scale H^s of Hilbert spaces associated with H via

$$\mathrm{H}^s := D((\mathrm{A})^{s/2}), \qquad \|u\|^2_{\mathrm{H}^s} := \|(\mathrm{A})^{s/2}u\|^2_{\mathrm{H}} = ((\mathrm{A})^s u, u) \tag{1.2}$$

(here and below $(\cdot, \cdot)$ denotes the inner product in H) and consider equation (1.1) as an evolution equation with respect to $\xi(t) = [u(t), \partial_t u(t)]$ in the corresponding energetic phase spaces

$$\mathrm{E}^s := \mathrm{H}^{s+1} \times \mathrm{H}^s, \qquad \xi(t) := [u(t), \partial_t u(t)] \in \mathrm{E}^s \tag{1.3}$$

(in fact, we will consider only the case where the initial data $[u_0, u_0']$ belong either to the space $\mathrm{E} := \mathrm{E}^0$, or to E^1).

We assume that the nonlinear term F belongs to the space

$$F \in C_b^1(\mathrm{E}, \mathrm{H}) \tag{1.4}$$

of smooth and globally bounded operators. It is also assumed that the partial derivatives F_u' and $F_{\partial_t u}'$ satisfy the following conditions:

$$\begin{aligned} &1.\ (F_v'(u, v)\theta, \theta) \leqslant \frac{\gamma}{8}\|\theta\|_{\mathrm{H}}^2 + C_\gamma\|\theta\|_{\mathrm{H}^{-1}}^2, \\ &2.\ |(F_u'(u, v)w, \theta)| \leqslant \frac{\gamma}{8}(\|w\|_{\mathrm{H}^1}^2 + \|\theta\|_{\mathrm{H}}^2) + C_\gamma(\|w\|_{\mathrm{H}}^2 + \|\theta\|_{\mathrm{H}^{-1}}^2), \end{aligned} \tag{1.5}$$

where $[u, v]$, $[w, \theta] \in \mathrm{E}$ are arbitrary, $\gamma > 0$ is the same as in equation (1.1), and the constant C_γ is independent of $[u, v]$ and $[w, \theta]$. These inequalities mean simply that the nonlinear dependence on $\partial_t u$ is subordinate to the linear term $\gamma\partial_t u$.

The following theorem shows that under the above assumptions equation (1.1) generates a dissipative semigroup in the energetic space E.

Theorem 1.1. *Let the assumptions (1.4) and (1.5) hold. Then, for every* $\xi(0) := [u(0), \partial_t u(0)] \in \mathrm{E}$, *equation (1.1) has a unique global solution* $\xi(t) \in C([0, \infty), \mathrm{E})$ *and the following estimate is valid:*

$$\|\xi(t)\|_{\mathrm{E}}^2 \leqslant C\|\xi(0)\|_{\mathrm{E}}^2 \mathrm{e}^{-\gamma t} + C_1, \tag{1.6}$$

where the constants C and C_1 depend only on F, A and γ. Consequently, equation (1.1) generates a semigroup

$$S_t : \mathrm{E} \to \mathrm{E}, \qquad by\ S_t\xi(0) := \xi(t). \tag{1.7}$$

Moreover, this semigroup is globally Lipschitz continuous with respect to the initial data $[u(0), \partial_t u(0)] \in \mathrm{E}$, *i.e.*

$$\|\xi_1(t) - \xi_2(t)\|_{\mathrm{E}}^2 \leqslant C\mathrm{e}^{Kt}\|\xi_1(0) - \xi_2(0)\|_{\mathrm{E}}^2, \tag{1.8}$$

where K and C depend only on A, *γ and F (and they are independent of the solutions $u_1(t)$ and $u_2(t)$ of problem (1.1)).*

If $\xi(0) \in \mathrm{E}^1$, then the corresponding solution $\xi(t)$ belongs to E^1 for every $t \geqslant 0$ and satisfies the following estimate:

$$\|\xi(t)\|_{\mathrm{E}^1}^2 \leqslant C_2\|\xi(0)\|_{\mathrm{E}^1}^2 \mathrm{e}^{-\gamma t/8} + C_3 \tag{1.9}$$

for some positive constants C_2 and C_3 which depend only on A, *F and γ.*

The proof of this theorem is quite standard, so we move it to appendix A.

Let us now verify that the semigroup $S_t : \mathrm{E} \to \mathrm{E}$ possesses a global compact attractor in the phase space E. Recall that the set $\mathcal{A} \subset \mathrm{E}$ is called a global attractor for the semigroup $S_t : \mathrm{E} \to \mathrm{E}$ if the following conditions are satisfied:

1. $\mathcal{A}$ is compact in E;
2. $\mathcal{A}$ is strictly invariant with respect to S_t, i.e. $S_t\mathrm{E} = \mathrm{E}$;
3. $\mathcal{A}$ attracts bounded subsets of E as $t \to \infty$, i.e. for every bounded $B \subset \mathrm{E}$ and every neighbourhood $\mathrm{O}(\mathcal{A})$ in E, there exists a number $T = T(\|B\|_{\mathrm{E}}, \mathrm{O})$ such that

$$S_t B \subset \mathcal{A}, \qquad \text{for } t \geqslant T \tag{1.10}$$

(see, e.g., [1,3], chapter 1, for details).

Theorem 1.2. *Let the assumptions of theorem 1.1 hold. Then semigroup (1.7) associated with the nonlinear hyperbolic problem (1.1) possesses a global attractor $\mathcal{A}$, which is bounded in* E^1. *This attractor is generated by all complete bounded solutions of (1.1):*

$$\mathcal{A} = \{\xi(0), \xi(t) := [u(t), \partial_t u(t)], t \in \mathbb{R} \text{ solves (1.1) and } \|\xi(t)\|_E \leqslant C_u, t \in \mathbb{R}\}.$$

Proof. According to the abstract attractor's existence theorem (see, e.g., [1]) the theorem will be proven if we verify the following conditions on the semigroup S_t:

1. $S_t : E \to E$ is continuous with respect to $\xi(0)$ for every fixed $t \geqslant 0$.
2. The semigroup S_t possesses a compact attracting (in the sense of (1.10)) set $K \subset\subset E$.

Let us verify these conditions. Indeed, the continuity of S_t is given by theorem 1.1. So, we are left to verify the second condition. To this end, we split the solution $u(t)$ of (1.1) as follows: $u(t) := v(t) + w(t)$, where $v(t)$ is a solution of the following problem:

$$\begin{aligned} &\partial_t^2 v + (\gamma + bA^{-1})\partial_t v + Av + Mv - F(v, \partial_t v) = Mu(t) + bA^{-1}\partial_t u(t), \\ &[v, \partial_t v]|_{t=0} = 0. \end{aligned} \tag{1.11}$$

Here $M \gg 1$ and $b \gg 1$ are sufficiently large positive constants which will be specified below.

Consequently, the remainder function $w(t)$ satisfies the equation

$$\begin{aligned} &\partial_t^2 w + (\gamma + bA^{-1})\partial_t w + Aw + Mw = l^1(t)w + l^2(t)\partial_t w, \\ &[w, \partial_t w]|_{t=0} = \xi(0), \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} l^1(t) &:= \int_0^1 F'_u(su(t) + (1-s)v(t), s\partial_t u(t) + (1-s)\partial_t v(t))\,ds, \\ l^2(t) &:= \int_0^1 F'_{\partial_t u}(su(t) + (1-s)v(t), s\partial_t u(t) + (1-s)\partial_t v(t))\,ds. \end{aligned} \tag{1.13}$$

We will prove that $\|w(t)\|_E$ tends uniformly (with respect to small variations in initial conditions) to zero, and $v(t)$ enters some fixed ball in E^1 as time grows. This ball is compact in E, so it can be taken as a desired attracting set K.

Lemma 1.1. *Let the assumptions of theorem 1.1 hold. Then, there exist large positive constants* $M = M(\gamma, F, A)$ *and* $b = b(\gamma, F, A)$ *such that the solution* $w(t)$ *of equation (1.12) satisfies the following estimate:*

$$\|[w(t), \partial_t w(t)]\|_E \leqslant C' e^{-\alpha t} \|\xi(0)\|_E \tag{1.14}$$

for appropriate positive constants C' *and* α *which are independent of* u.

Proof. Taking the inner product in H of equation (1.12) with $\partial_t w + [(\gamma + bA^{-1})/2]w(t)$, we derive the following relation:

$$\begin{aligned} &\partial_t\{\|\partial_t w\|_H^2 + \|w\|_{H^1}^2 + ((\gamma + bA^{-1})w, \partial_t w) + M\|w\|_H^2\} \\ &\quad + \frac{\gamma}{2}\{\|\partial_t w\|_H^2 + \|w\|_{H^1}^2 + M\|w\|_H^2 + ((\gamma + bA^{-1})w, \partial_t w)\} \\ &\qquad = -b\|\partial_t w\|_{H^{-1}}^2 - b\|w\|_H^2 - Mb\|w\|_{H^{-1}}^2 - \frac{\gamma}{2}(\|\partial_t w\|_H^2 + \|w\|_{H^1}^2 + M\|w\|_H^2) \\ &\qquad\quad + 2(l^1(t)w, \partial_t w) + 2(l^2(t)\partial_t w, \partial_t w) + (l^1(t)w, (\gamma + bA^{-1})w) \\ &\qquad\quad + (l^2(t)\partial_t w, (\gamma + bA^{-1})w) - \tfrac{1}{2}((\gamma + bA^{-1})\partial_t w, (\gamma + 2bA^{-1})w) \equiv h_w(t). \end{aligned} \tag{1.15}$$

We recall that, by conditions (1.4) and formulae (1.13),

$$\|l^2(t)\|_{\mathcal{L}(\mathrm{H},\mathrm{H})} + \|l^1(t)\|_{\mathcal{L}(\mathrm{H}^1,\mathrm{H})} \leqslant C, \tag{1.16}$$

where C is independent of u and t, and, by conditions (1.5),

$$(l^2(t)\partial_t w(t), \partial_t w(t)) \leqslant \frac{\gamma}{8}\|\partial_t w\|^2_{\mathrm{H}} + C_\gamma \|\partial_t w(t)\|^2_{\mathrm{H}^{-1}}. \tag{1.17}$$

Estimating the right-hand side $h_w(t)$ of (1.15) by Hölder inequality and taking into account estimates (1.16) and (1.17), we obtain

$$h_w(t) \leqslant \left(C_\gamma - \tfrac{1}{2}b\right)\|\partial_t w(t)\|^2_{\mathrm{H}^{-1}} + (C'_\gamma(1+b^3) - M)\|w\|^2_{\mathrm{H}}, \tag{1.18}$$

where C_γ and C'_γ are two positive constants which depend only on γ, F and A, but are independent of b, M and u. Fixing now the constants M and b in such a way that

$$b = 2C_\gamma, \qquad M \geqslant C'_\gamma(1+b^3),$$

we obtain the inequality $h_w(t) \leqslant 0$. Moreover, without loss of generality we assume that M is chosen in such a way that, in addition,

$$|((\gamma + b\mathrm{A}^{-1})w, \partial_t w)| \leqslant \tfrac{1}{2}(\|\partial_t w\|^2_{\mathrm{H}} + M\|w\|^2_{\mathrm{H}}).$$

Applying now Gronwall's inequality to (1.15), we obtain (1.14). Lemma 1.1 is proven.

Now we are ready to estimate the solution $v(t)$ of equation (1.11). We rewrite this equation in the following equivalent form:

$$\partial_t^2 v + \gamma\partial_t v + \mathrm{A}v - F(v, \partial_t v) = Mw(t) + b\mathrm{A}^{-1}\partial_t w(t) := h_{M,b}(t),$$

$$[v, \partial_t v]|_{t=0} := 0.$$

This equation is a non-autonomous analogue of equation (1.1). Moreover, due to theorem 1.1 and lemma 1.1, the function $h_{M,b}(t)$ can be estimated as follows:

$$\|h_{M,b}(t)\|^2_{\mathrm{H}^1} + \|\partial_t h_{M,b}(t)\|^2_{\mathrm{H}} \leqslant C_{M,b}(1 + \|\xi(0)\|^2_{\mathrm{E}}\mathrm{e}^{-\alpha t}) \tag{1.19}$$

for an appropriate constant $C_{M,b}$ which is independent of u. Consequently, using estimate (1.19) and the fact that $v(0) = 0$, $\partial_t v(0) = 0$, arguing exactly as in the proof of theorem 1.1 (see appendix A), we obtain that the solution $[v, \partial_t v]$ of (1.11) belongs to the space $C(\mathbb{R}_+, \mathrm{E}^1)$ and satisfies the estimate

$$\|[v(t), \partial_t v(t)]\|_{\mathrm{E}^1} \leqslant C_*(\|\xi(0)\|_{\mathrm{E}}\mathrm{e}^{-\alpha t} + 1) \tag{1.20}$$

for some positive constants α and C_* which depend on M and γ, but are independent of u.

Estimates (1.14) and (1.20) imply that the set

$$K := \{\xi \in \mathrm{E}^1, \|\xi\|_{\mathrm{E}^1} \leqslant 2C^*\}$$

is a compact (in E) attracting set for the semigroup S_t. Thus, all conditions of the abstract attractor's existence theorem are verified and theorem 1.2 is proven.

Remark 1.1. We recall that our conditions (1.4) and (1.5) imply that the operator F (along with its first derivatives F'_u and $F'_{\partial_t u}$) is globally bounded as $\|\xi\|_{\mathrm{E}} \to \infty$. This simplifying assumption is enough for our purposes because in our examples of sharp upper and lower bounds for the attractor's dimension (see section 4) the nonlinearity F has a bounded support. In fact, more general nonlinearities (see, e.g. [22, 23] and [3], chapter 4) can be treated in the same way.

Remark 1.2. We note that conditions (1.4) and (1.5) are, obviously, satisfied if

$$F \in C^1_b(\mathrm{E}^{-\delta}, \mathrm{H}) \tag{1.21}$$

for some positive exponent δ. In the following, we will often use this stronger condition (1.21) instead of conditions (1.4) and (1.5).

2. Upper bounds for the attractor's dimension

In this section we show, using the standard volume-contraction technique, that the attractor $\mathcal{A}$ of (1.1) constructed in the previous section has finite Hausdorff and fractal dimensions, and we obtain some estimates for this dimension in terms of the dissipation parameter γ. To this end, we need the following assumption: there is an exponent $\kappa > 0$ and two positive constants C_1 and C_2 such that

$$C_1 i^{2\kappa} \leqslant \lambda_i \leqslant C_2 i^{2\kappa}, \qquad i \in \mathbb{N}, \tag{2.1}$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the eigenvalues of the operator A.

Remark 2.1. We note that assumption (2.1) is always satisfied if A is an elliptic differential operator in a bounded domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary and $\mathrm{H} := L^2(\Omega)$ (see, e.g., [24]). Moreover, in this case $\kappa := k/2n$, where k is the order of A.

In order to formulate the abstract theorem for estimating the dimension of invariant sets, we need the following definition.

Definition 2.1. *A map $S : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is a subset of a certain Banach space* E *is called uniformly quasi-differentiable on $\mathcal{A}$ if for any $\xi \in \mathrm{E}$ there is a linear operator $S'(\xi) : \mathrm{E} \to \mathrm{E}$ (the quasi-differential) such that*

$$\|S(\xi_1) - S(\xi_2) - S'(\xi_1)(\xi_1 - \xi_2)\|_{\mathrm{E}} = \mathrm{o}(\|\xi_1 - \xi_2\|_{\mathrm{E}}) \tag{2.2}$$

holds uniformly with respect to $\xi_1, \xi_2 \in \mathcal{A}$. It is also assumed that

$$\sup_{\xi \in \mathcal{A}} \|S'(\xi)\|_{\mathcal{L}(\mathrm{E},\mathrm{E})} < \infty \qquad \text{and} \qquad S' \in C(\mathcal{A}, \mathcal{L}(\mathrm{E}, \mathrm{E})). \tag{2.3}$$

Theorem 2.1. *Let S_t be a semigroup in a certain Hilbert space* E *and let $\mathcal{A} \subset \mathrm{E}$ be a compact strictly invariant set of this semigroup ($S_t\mathcal{A} = \mathcal{A}$). Let us suppose also that the semigroup S_t is uniformly quasi-differentiable on $\mathcal{A}$ for some fixed $t = T$ and the following inequality holds for some positive integer d:*

$$\omega_d(\mathcal{A}) := \sup_{\xi \in \mathcal{A}} \omega_d(S'_T(\xi)) < 1, \tag{2.4}$$

where $\omega_d(L) := \|\Lambda^d L\|_{\Lambda^d \mathrm{E}}$ is the norm of the dth exterior power of the operator L in the Hilbert space $\Lambda^d\mathrm{E}$ (see, e.g., [3], chapter 5). Then the fractal dimension of $\mathcal{A}$ is finite in E. *Moreover,*

$$\dim_{\mathrm{H}}(\mathcal{A}, \mathrm{E}) \leqslant \dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \leqslant d. \tag{2.5}$$

For the proof of this theorem see [3], chapter 5, for the case of Hausdorff dimension and [8–10] for the fractal dimension.

Lemma 2.1. *Let the assumptions of theorem 1.1 hold. Then, semigroup (1.7), associated with hyperbolic equation (1.1), is uniformly quasi-differentiable on the attractor $\mathcal{A}$ and its quasi-differential $S'_t(\xi(0))$ at $\xi(0) \in \mathcal{A}$ is defined via the following standard expression:*

$$S'_t(\xi(0))\eta := [v(t), \partial_t v(t)], \tag{2.6}$$

where $\eta \in \mathrm{E}$ and $v(t)$ is the solution of the equation of variations

$$\begin{aligned} &\partial_t^2 v + \gamma\partial_t v + \mathrm{A}v = F'_u(u(t), \partial_t u(t))v(t) + F'_{\partial_t u}(u(t), \partial_t u(t))\partial_t v(t), \\ &[v, \partial_t v]|_{t=0} = \eta, \qquad [u(t), \partial_t u(t)] := S_t\xi(0). \end{aligned} \tag{2.7}$$

The assertion of the lemma is completely standard, so we move its proof into appendix A.

Thus, according to theorem 2.1, for estimating the dimension of $\mathcal{A}$ it is sufficient to estimate the norms of d-external powers for solving operator of equation of variations associated with the hyperbolic problem (1.1). To this end, following [22], we introduce a new variable $\theta(t) := \partial_t u + (\gamma/2)u(t)$ and rewrite system (1.1) in the equivalent form in variables $[u, \theta] \in \mathrm{E}$. We obtain

$$\partial_t \begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{2}u + \theta \\ F\left(u, \theta - \frac{\gamma}{2}u\right) + \frac{\gamma^2}{4}u - \mathrm{A}u - \frac{\gamma}{2}\theta \end{pmatrix}. \tag{2.8}$$

Now, instead of applying theorem 2.1 to the initial system in $[u, \partial_t u]$ variables we will use it for the transformed system (2.8) (since these systems are linearly equivalent, they have equivalent attractors whose dimensions coincide).

The equation of variations for the transformed system, obviously, has the form

$$\partial_t \eta(t) = \mathbb{L}(u(t), \partial_t u(t))\eta(t), \qquad [u(t), \partial_t u(t)] := S_t \xi(0), \tag{2.9}$$

where

$$\mathbb{L} := \begin{pmatrix} -\frac{\gamma}{2} & 1 \\ -\mathrm{A} + \frac{\gamma^2}{4} + F'_u(\xi) - \frac{\gamma}{2}F'_{\partial_t u}(\xi) & -\frac{\gamma}{2} + F'_{\partial_t u}(\xi) \end{pmatrix}. \tag{2.10}$$

In order to estimate the exterior powers of solving operator $S'_T(\xi(0)) : \eta \to \eta(T)$ of linear problem (2.9), we use the following standard lemma (see [22] or [3], chapter 5).

Lemma 2.2. *Let the assumptions of lemma 2.1 hold. Then*

$$\omega_d(S'_T(\xi(0))) \leqslant \mathrm{e}^{\int_0^T \mathrm{Tr}_d\{\mathbb{L}(\xi(t))\}\,\mathrm{d}t}, \qquad \xi(t) := S_t\xi(0), \tag{2.11}$$

where $\mathrm{Tr}_d(L)$ *means the d-dimensional trace of the operator* $L : \mathrm{E} \to \mathrm{E}$ *in* E, *i.e.*

$$\mathrm{Tr}_d(L) := \sup\left\{\sum_{i=1}^{d}(L\eta_i, \eta_i)_{\mathrm{E}} : \|\eta_i\|_{\mathrm{E}} = 1, (\eta_i, \eta_j)_{\mathrm{E}} = 0 \quad \text{for } i \neq j\right\}.$$

Now we are in a position to estimate the fractal dimension of the attractor $\mathcal{A}$ of hyperbolic equation (1.1). For simplicity, we assume that the nonlinearity F satisfies condition (1.21) and estimate the corresponding dimension in terms of parameters γ, κ (introduced in (2.1)) and δ (introduced in (1.21)). (In the general case of conditions (1.4) and (1.5), this dimension can be analogously estimated in terms of γ, κ and the constant C_γ defined in (1.5).)

Theorem 2.2. *Let the assumptions of theorem 1.1 hold and let, in addition, the nonlinearity F satisfy (1.21). Then, the fractal dimension of the attractor $\mathcal{A}$ is finite in* E *and can be estimated, as* $\gamma \to 0$, *as follows*

$$\dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \leqslant CN(\gamma) := C \begin{cases} \gamma^{-2/\kappa\delta}, & \kappa\delta < 1, \\ \gamma^{-2}\ln\frac{1}{\gamma}, & \kappa\delta = 1, \\ \gamma^{-2}, & \kappa\delta > 1, \end{cases} \tag{2.12}$$

where C is independent of $\gamma \to 0$, $\delta > 0$ *and* $\kappa > 0$.

Proof. According to theorem 2.1 and lemma 2.2, it is sufficient to prove that

$$\sup_{\xi\in\mathcal{A}} \mathrm{Tr}_d\{\mathbb{L}(\xi)\} < 0, \qquad \text{for some } d \leqslant CN(\gamma). \tag{2.13}$$

In order to show this, we first estimate the quadratic form associated with the operator $\mathbb{L}$, using the assumptions (1.21) and Schwartz inequality:

$$
\begin{aligned}
(\mathbb{L}\eta, \eta)_{\mathrm{E}} &= -\frac{\gamma}{2}\|\eta_u\|^2_{\mathrm{H}^1} + (\eta_\theta, \mathrm{A}\eta_u) - (\eta_\theta, \mathrm{A}\eta_u) + \frac{\gamma^2}{4}(\eta_u, \eta_\theta) \\
&\quad + \left(\left(F'_u - \frac{\gamma}{2}F'_{\partial_t u}\right)\eta_u, \eta_\theta\right) - \frac{\gamma}{2}\|\eta_\theta\|^2_{\mathrm{H}} + (F'_{\partial_t u}\eta_\theta, \eta_\theta) \\
&\leqslant -\frac{\gamma}{4}(\|\eta_u\|^2_{\mathrm{H}^1} + \|\eta_\theta\|^2_{\mathrm{H}}) + \tilde{C}\gamma^{-1}(\|\eta_u\|^2_{\mathrm{H}^{1-\delta}} + \|\eta_\theta\|^2_{\mathrm{H}^{-\delta}}) := (\mathcal{B}\eta, \eta),
\end{aligned} \tag{2.14}
$$

where $\eta := [\eta_u, \eta_\theta] \in \mathrm{E}$, the operator $\mathcal{B}$ is defined as

$$
\mathcal{B} := \frac{\gamma}{4}\begin{pmatrix} -\mathrm{Id} + 4\tilde{C}\gamma^{-2}\mathrm{A}^{-\delta/2} & 0 \\ 0 & -\mathrm{Id} + 4\tilde{C}\gamma^{-2}\mathrm{A}^{-\delta/2} \end{pmatrix}
$$

and the constant $\tilde{C}$ is independent of γ, δ and η.

It follows now from (2.14) that for any $d \in \mathbb{N}$,

$$
\mathrm{Tr}_d\{\mathbb{L}\} \leqslant \mathrm{Tr}_d\{\mathcal{B}\}. \tag{2.15}
$$

We now observe that the operator $\mathcal{B}$ is self-adjoint, hence, by the classical min–max principle (see, e.g., [3], chapter 5), its traces can be immediately expressed in terms of its eigenvalues, namely,

$$
\mathrm{Tr}_d\{\mathcal{B}\} = \frac{\gamma}{2}\left(-d + 4\tilde{C}\gamma^{-2}\sum_{i=1}^{d}\lambda_i^{-\delta/2}\right), \tag{2.16}
$$

where λ_i are the eigenvalues of A. The estimate (2.13) is an immediate corollary of (2.15), (2.16) and of the assumption (2.1) on the asymptotics of λ_i. Theorem 2.2 is proven.

The following theorem shows that the estimate (2.12) can be essentially improved if the additional regularity of the nonlinear term F is known.

Theorem 2.3. *Let the assumptions of theorem 1.1 hold and let, in addition,*

$$
F \in C^1_b(\mathrm{E}^{-s-1/2}, \mathrm{H}^{s+1/2}), \tag{2.17}
$$

where $s > -\frac{1}{2}$ is some regularity exponent. Then, the dimension of the corresponding attractor $\mathcal{A}$ can be estimated via

$$
\dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \leqslant C_1\begin{cases} \gamma^{-2/\kappa(2s+1)}, & \kappa(s+\frac{1}{2}) < 1, \\ \gamma^{-1}\ln\dfrac{1}{\gamma}, & \kappa(s+\frac{1}{2}) = 1, \\ \gamma^{-1}, & \kappa(s+\frac{1}{2}) > 1, \end{cases} \tag{2.18}
$$

where the constant C_1 depends on s and F, but is independent of γ.

Proof. Indeed, due to condition (2.17), we have the following estimates:

$$
\begin{aligned}
|(F'_u(u, \partial_t u)\eta_u, \eta_\theta)| &\leqslant \|F'_u(u, \partial_t u)\eta_u\|_{\mathrm{H}^{s+1/2}}\|\eta_\theta\|_{\mathrm{H}^{-s-1/2}} \\
&\leqslant C(\|\eta_u\|^2_{\mathrm{H}^{-s+1/2}} + \|\eta_\theta\|^2_{\mathrm{H}^{-s-1/2}})
\end{aligned}
$$

and, analogously,

$$
(F'_{\partial_t u}(u, \partial_t u)\eta_\theta, \eta_\theta) \leqslant \|F'_{\partial_t u}(u, \partial_t u)\eta_\theta\|_{\mathrm{H}^{s+1/2}}\|\eta_\theta\|_{\mathrm{H}^{-s-1/2}} \leqslant C\|\eta_\theta\|^2_{\mathrm{H}^{-s-1/2}},
$$

where the constant C depends only on F. These estimates allow (2.14) to be improved in the following way:

$$
(\mathbb{L}\eta, \eta)_{\mathrm{E}} \leqslant (\mathcal{B}_s\eta, \eta)_{\mathrm{E}},
$$

where

$$\mathcal{B}_s := \frac{\gamma}{4}\begin{pmatrix} -\mathrm{Id} + \tilde{C}\gamma^{-1}\mathrm{A}^{-(s+1/2)/2} & 0 \\ 0 & -\mathrm{Id} + \tilde{C}\gamma^{-1}\mathrm{A}^{-(s+1/2)/2} \end{pmatrix}$$

for some constant $\tilde{C} > 0$ which is independent of γ (the term $(\gamma/2)(F'_{\partial_t u}\eta_u, \eta_\theta)$ in (2.14) is of order γ and does not require additional estimates).

Computing now the d-dimensional trace of the operator $\mathcal{B}_s$ in terms of the eigenvalues λ_i, using asymptotics (2.1) for them and arguing as in the end of theorem 2.1 we derive the improved estimate (2.18).

Corollary 2.1. *Let the assumptions of theorem 1.1 hold and let, in addition, (2.17) be satisfied with the exponent $s > 1/\kappa - \frac{1}{2}$. Then, the dimension of the attractor $\mathcal{A}$ possesses the following upper bound:*

$$\dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \leqslant C\gamma^{-1} \qquad as\ \gamma \to 0, \tag{2.19}$$

where the constant C is independent of γ.

Let us now introduce the class $\mathbb{S}$ of smoothing nonlinearities F.

Definition 2.2. *A nonlinear operator $F : \mathrm{E} \to \mathrm{H}$ belongs to the class $\mathbb{S} = \mathbb{S}(C_{k,m})$ if, for every $m \in \mathbb{R}_+$, this operator belongs to $C^\infty(\mathrm{E}^{-m}, \mathrm{H}^m)$ and the following estimates valid, for every $k \in \mathbb{N} \cup \{0\}$:*

$$\|F\|_{C_b^k(\mathrm{E}^{-m},\mathrm{H}^m)} \leqslant C_{k,m} \tag{2.20}$$

for appropriate constants $C_{k,m}$.

Corollary 2.2. *Let the eigenvalues of the operator* A *satisfy condition (2.1) and let the nonlinearity F belong to the class $\mathcal{S}$. Then, the fractal dimension of the corresponding global attractor $\mathcal{A}$ associated with equation (1.1) possesses the following upper estimate:*

$$\dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \leqslant C\frac{1}{\gamma}, \tag{2.21}$$

where the constant C depends on constants $C_{1,m}$ (defined in (2.20)) and on constants C_1, C_2 and κ (defined in (2.1)), but are independent of γ.

Remark 2.2. In section 4, we will show that even in the case of extremely regular nonlinearities $F \in \mathbb{S}$, the dimension of the attractor $\mathcal{A}$ may indeed have the rate growth $\sim\gamma^{-1}$ as $\gamma \to 0$. So, estimate (2.21) is indeed sharp in the limit $\gamma \to 0$.

3. Bifurcations of a homoclinic loop and Lyapunov dimension

In this section, we consider bifurcations of a certain type of homoclinic loops; the results will be essentially used in the next section in order to obtain sharp lower bounds for the fractal dimension of the attractor $\mathcal{A}$ in the class $\mathcal{S}$.

In contrast to the previous sections, we consider here finite-dimensional systems of ODEs, namely, systems of the following form:

$$\dot{y} = \mathbb{A}y + \mathbb{F}(y), \qquad y \in \mathbb{R}^n, \tag{3.1}$$

where the nonlinearity $\mathbb{F}(y)$ belongs to $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies

$$\mathbb{F}(0) = \mathbb{F}'(0) = 0. \tag{3.2}$$

We assume that the matrix $\mathbb{A} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ has only one eigenvalue to the right of the imaginary axis. By scaling the time variable in (3.1) we can always make this eigenvalue equal to 1.

The rest of the spectrum consists of m pairs of complex eigenvalues $(-\lambda_1 \pm \omega_1, \ldots, -\lambda_m \pm \omega_m)$ and $(n-2m-1)$ eigenvalues whose real parts are less than some $-\lambda_{m+1} < -\lambda_m$. Here m is some positive integer such that $2m+1 \leqslant n$, and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m$ and $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m$ are given vectors satisfying the condition

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m < 1, \qquad \omega_k > 0, \quad k = 1, \ldots, m. \tag{3.3}$$

We assume that the matrix $\mathbb{A}$ can be brought to the following form by a linear transformation of coordinates:

$$\mathbb{A} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & R_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & R_m & 0 \\ 0 & \cdots & 0 & 0 & A \end{pmatrix}, \qquad R_k := \begin{pmatrix} -\lambda_k & \omega_k \\ -\omega_k & -\lambda_k \end{pmatrix}, \tag{3.4}$$

where the matrix $A \in \mathcal{L}(\mathbb{R}^{n-2m-1}, \mathbb{R}^{n-2m-1})$ satisfies the spectral assumption:

$$\operatorname{Re} \sigma(A) \leqslant -\lambda_{m+1} \qquad \text{with } \lambda_{m+1} > \lambda_m. \tag{3.5}$$

Such transformation can always be done when all ω are different. We, however, prefer not to make this assumption. Instead, we simply assume that the matrix $\mathbb{A}$ is in the form (3.4).

Accordingly, denoting $y = (z, (u_1, v_1), \ldots, (u_m, v_m), w)$, system (3.1) is written in the following form:

$$\begin{aligned} \dot{z} &= z + \cdots, \\ \dot{u}_k &= -\lambda_k u_k + \omega_k v_k + \cdots, \\ \dot{v}_k &= -\omega_k u_k - \lambda_k v_k + \cdots, \qquad k = 1, \ldots, m, \\ \dot{w} &= Aw + \cdots, \end{aligned} \tag{3.6}$$

where the dots in (3.6) stand for the nonlinearities, i.e. for the terms vanishing at the origin along with their first derivatives.

By construction, system (3.1) has a hyperbolic equilibrium at the origin $O : y = 0$. Moreover, the unstable manifold $W^{\mathrm{u}}(O)$ is one-dimensional here, and it is tangent to the z-axis at the origin. $W^{\mathrm{u}} \backslash O$ consists of two orbits (the separatrices) which leave the origin at $t = -\infty$ in opposite directions. We assume that one of the separatrices which leaves O towards positive z (we denote this separatrix as Γ) returns to the origin as $t \to +\infty$, thus it forms a homoclinic loop.

The system under consideration has an $(n-2m-1)$-dimensional smooth invariant strong-stable manifold W^{ss} which is tangent at O to the w-space and which consists of all orbits which tend to O faster than $\mathrm{e}^{-\lambda_m t}$ (see, e.g., [25], chapter 5). We assume that the homoclinic loop Γ belongs to W^{ss}, i.e. it enters O being tangent to the w-space. Note that this is a bifurcation of codimension $(2m+1)$, so we study here a problem which is, at large m, very degenerate from the point of view of the conventional bifurcation theory. However, the homoclinic loops of these type are quite typical for integrable equations with dissipation (see, e.g., system (4.6) in the next section).

Finally, we assume that

$$2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_m < 1, \qquad 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_m + \lambda_{m+1} > 1. \tag{3.7}$$

Remark 3.1. We note that inequalities (3.5) and (3.7) imply that the flow defined by (3.6) contracts $(2m+2)$-dimensional volumes near O while $(2m+1)$-dimensional volumes are not contracted. Thus, the Lyapunov dimension $\dim_L(\mathbb{A})$ of system (3.1) at the origin possesses the estimates (see, e.g., [3], chapter 5):

$$2m + 1 < \dim_L(\mathbb{A}_0) < 2m + 2.$$

The main result of this section is the following theorem.

Theorem 3.1. *Let the above assumptions hold. Let $\mathcal{R} := \{\mu_1, \dots, \mu_{2m}\}$ be an arbitrary set of $2m$ non-zero complex numbers such that if μ belongs to the set $\mathcal{R}$, then its complex-conjugate $\bar{\mu}$ belongs to $\mathcal{R}$ as well. Then, by an arbitrarily small C^∞-perturbation of system (3.1), a periodic orbit with $2m$ multipliers equal to $\mu_1, \dots, \mu_{2m}$ and with the rest of the multipliers inside the unit circle can be born from the homoclinic loop, i.e. for an arbitrarily small neighbourhood $\mathbb{V}$ of the homoclinic loop Γ, for every $\varepsilon > 0$ and every $r \in \mathbb{N}$, there exists a C^∞-function $\mathbb{G}_\varepsilon$ satisfying the inequality*

$$\|\mathbb{G}_\varepsilon - (\mathbb{A}_0 + \mathbb{F})\|_{C^k(\mathbb{R}^n, \mathbb{R}^n)} \leqslant \varepsilon,$$

such that the perturbed system $\dot{y} = \mathbb{G}_\varepsilon(y)$ possesses a periodic orbit of the type described above, which lies in $\mathbb{V}$.

Proof. Let us first locally straighten invariant manifolds W^{u}, W^{s} and $W^{\mathrm{ss}} \subset W^{\mathrm{s}}$, i.e. we make a coordinate transformation in a small neighbourhood of the origin such that the system takes, locally, the form

$$\begin{aligned} \dot{z} &= z(1 + p(y)), \\ \dot{u}_k &= -\lambda_k u_k + \omega_k v_k + f_k(y) \cdot (u, v, w), \\ \dot{v}_k &= -\omega_k u_k - \lambda_k v_k + g_k(y) \cdot (u, v, w), \qquad k = 1, \dots, m, \\ \dot{w} &= (A + q(y))w, \end{aligned} \tag{3.8}$$

where the functions $f_k(y), g_k(y), p(y), q(y)$ vanish at the origin. In these coordinates we have $W^{\mathrm{u}}_{\mathrm{loc}} = \{(u_k, v_k) = 0\ (k = 1, \dots, m),\ w = 0\}$, $W^{\mathrm{s}}_{\mathrm{loc}} = \{z = 0\}$, $W^{\mathrm{ss}}_{\mathrm{loc}} = \{z = 0, (u_k, v_k) = 0\ (k = 1, \dots, m)\}$, so the invariant manifolds are straightened indeed. When the system is brought to this form, we can freely change the characteristic exponents (i.e. $-\lambda_k \pm \mathrm{i}\omega_k$ and the eigenvalues of A) by localized small perturbations, without destroying the homoclinic loop. Indeed, we may arbitrarily add small localized perturbations to the coefficients λ_k, ω_k and A in (3.8), and this will not move the local invariant manifolds $W^{\mathrm{u}}_{\mathrm{loc}}$, $W^{\mathrm{s}}_{\mathrm{loc}}$, $W^{\mathrm{ss}}_{\mathrm{loc}}$. Thus, by applying perturbations of this kind, we will still have a homoclinic loop which enters O lying in $W^{\mathrm{ss}}_{\mathrm{loc}}$. So, we may always assume that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m < \lambda_{m+1}. \tag{3.9}$$

Moreover, we may always achieve by an arbitrarily small such perturbation that the set of characteristic exponents is non-resonant. After that is done, Sternberg's theorem [26] is applied which means that we can make a smooth coordinate transformation which makes the system linear in a small neighbourhood of O, i.e. the system takes, locally, the form

$$\begin{aligned} \dot{z} &= z, \\ \dot{u}_k &= -\lambda_k u_k + \omega_k v_k, \\ \dot{v}_k &= -\omega_k u_k - \lambda_k v_k, \qquad k = 1, \dots, m, \\ \dot{w} &= Aw. \end{aligned} \tag{3.10}$$

In other words, after the above transformations, our equation reads

$$\dot{y} = \mathbb{A}(\omega) y + \mathbb{F}(y), \qquad y := (z, u_1, v_1, \dots, u_n, v_n, w) \in \mathbb{R}^n \tag{3.11}$$

with the matrix $\mathbb{A}$ given by (3.4) (from now on, we fix λ satisfying (3.7) and (3.9), but we will vary the values of $\omega_1, \ldots, \omega_m$, therefore we indicate the dependence of $\mathbb{A}$ on ω explicitly). The smooth nonlinear function $\mathbb{F}$ vanishes in some neighbourhood O of the origin

$$\mathbb{F}(y) \equiv 0 \qquad \text{for } y \in \mathrm{O}. \tag{3.12}$$

Thus, by construction, the intersection of the homoclinic loop Γ with O consists of two pieces. The first piece, corresponding to large negative times, coincides with the positive local z-axis, and the second piece, corresponding to large positive t, lies in the w-space.

The solution of (3.10) which starts in O at a point $(z^0, u_1^0, v_1^0, \ldots, u_m^0, v_m^0, w^0)$ is written as

$$\begin{aligned} z(t) &= z^0 \mathrm{e}^t, \\ u_k(t) &= \mathrm{e}^{-\lambda_k t}(u_k^0 \cos\omega_k t + v_k^0 \sin\omega_k t), \\ v_k(t) &= \mathrm{e}^{-\lambda_k t}(-u_k^0 \sin\omega_k t + v_k^0 \cos\omega_k t), \\ w(t) &= \mathrm{e}^{At} w^0. \end{aligned} \tag{3.13}$$

We take some small $d > 0$ and consider two cross-sections to the homoclinic loop: $\Pi^{\mathrm{out}} = \{z = d\}$ and $\Pi^{\mathrm{in}} = \{\|w\|_A = d\}$ where the metric $\|w\|_A$ in the w-space $\mathbb{R}^{n-2m-1}$ is defined as follows:

$$\|w\|_A^2 := \int_0^\infty \|\mathrm{e}^{At} w\|^2 \, \mathrm{d}t$$

and $\|\cdot\|$ is a standard norm in $\mathbb{R}^{n-2m-1}$. Then, obviously,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|_A^2) = -2\|w\|^2 < 0$$

and, consequently, every non-zero solution $w = w(t)$ of equation $\dot{w} = Aw$ intersects transversely the ellipsoid $\|w\|_A = d$ at a unique point. Therefore, the Poincaré section Π^{in} is, indeed, well defined. Let also w_0 correspond to the intersection point of the homoclinic loop with Π^{in}, and let $\alpha \in \mathbb{R}^{n-2m-2}$ be local coordinates on the ellipsoid $\|w\|_A = d$ near w_0, i.e. there is a smooth function $\mathcal{W} : \mathbb{R}^{n-2m-2} \to \mathbb{R}^{n-2m-1}$ such that $\|\mathcal{W}(\alpha)\|_A \equiv d$, $\mathcal{W}(0) = w_0$ and $\mathcal{W}'(0) = \mathrm{Id}$. We introduce the local coordinates for $M \in \Pi^{\mathrm{in}}$ and $\bar{M} \in \Pi^{\mathrm{out}}$ as follows

$$M(Z, u_1, v_1, \ldots, u_m, v_m, \alpha) := (d \cdot Z, u_1, v_1, \ldots, u_m, v_m, \mathcal{W}(\alpha)),$$

$$\bar{M}(\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_m, \bar{v}_m, \bar{w}) := (d, \bar{u}_1, \bar{v}_1, \ldots, \bar{u}_m, \bar{v}_m, \bar{w}).$$

According to (3.13), the orbit of a point $M \in \Pi^{\mathrm{in}}$ with $Z > 0$ reaches Π^{out} at the moment of time $t = -\ln Z$, and the intersection of the orbit with Π^{out} is the point

$$\bar{M} := T_\omega^{\mathrm{loc}}(M) := \begin{pmatrix} Z^{\lambda_1}(u_1 \cos\omega_1 \ln Z - v_1 \sin\omega_1 \ln Z) \\ Z^{\lambda_1}(u_1 \sin\omega_1 \ln Z + v_1 \cos\omega_1 \ln Z) \\ \vdots \\ Z^{\lambda_k}(u_k \cos\omega_k \ln Z - v_k \sin\omega_k \ln Z) \\ Z^{\lambda_k}(u_k \sin\omega_k \ln Z + v_k \cos\omega_k \ln Z) \\ Z^{-A}\mathcal{W}(\alpha) \end{pmatrix}, \tag{3.14}$$

which defines the local Poincaré map $T_\omega^{\mathrm{loc}} : \Pi^{\mathrm{in}} \cap \{Z > 0\} \to \Pi^{\mathrm{out}}$.

Analogously, the orbits starting on Π^{out} close to the origin follow the homoclinic loop, so they come to the cross-section Π^{in} in finite time. These orbits define a global Poincaré map $T_0^{\mathrm{glo}} : \Pi^{\mathrm{out}} \to \Pi^{\mathrm{in}}$ which is a diffeomorphism (since it is defined by orbits of a smooth flow in a finite time and since the trajectories intersect Π^{in} transversely). Thus, the linear operator

$$\mathcal{T}_0 := \frac{\mathrm{d}}{\mathrm{d}\bar{M}} T_0^{\mathrm{glo}}(0) \tag{3.15}$$

is invertible and, due to our choice of coordinates in Π^{in} and Π^{out},

$$T_0^{\text{glo}}(0) = 0. \tag{3.16}$$

Moreover, without loss of generality we assume that $\mathcal{T}_0 \in \mathcal{L}(\mathbb{R}^{n-2m-2}, \mathbb{R}^{n-2m-2})$ can be represented as follows:

$$\mathcal{T}_0 = L_0 \cdot U_0, \tag{3.17}$$

where U_0 and L_0 are upper- and lower-triangular matrices, respectively:

$$L_0 = \begin{pmatrix} L_{11}^0 & 0 & \dots & & \\ L_{21}^0 & L_{22}^0 & 0 & \dots & \\ L_{31}^0 & L_{32}^0 & L_{33}^0 & 0 & \dots \\ & & & \ddots & \\ \dots & \dots & \dots & & \end{pmatrix}, \qquad U_0 = \begin{pmatrix} U_{11}^0 & U_{12}^0 & U_{13}^0 & \dots \\ 0 & U_{22}^0 & U_{23}^0 & \dots \\ \dots & 0 & U_{33}^0 & \dots \\ & & 0 & \ddots \\ \dots & \dots & \dots & \end{pmatrix}. \tag{3.18}$$

Indeed, decomposition (3.17) is well-known for generic invertible matrices $\mathcal{T}_0$ (and can be obtained, e.g., via a classical Gauss diagonalization procedure). If $\mathcal{T}_0$ is not generic, we can always put it in a general position by an arbitrarily small perturbation of the system, which is localized outside the d-neighbourhood of the origin and preserves the homoclinic loop (using the standard flow-box technique). Note that

$$U_{ii}^0 \neq 0, \qquad L_{ii}^0 \neq 0 \qquad \text{for } i = 1, \dots, n-2m-2, \tag{3.19}$$

since $\mathcal{T}_0$ is invertible.

We now consider an $(m+n-1)$-parameter family of small smooth perturbations of the system (3.11), namely for every $\omega \in \mathbb{R}^m$ which is sufficiently close to the original vector $\omega := \omega^0$ and for every sufficiently small $\theta \in \mathbb{R}^{n-1}$, we consider the following family of equations:

$$\dot{y} = \mathbb{A}(\omega)y + \mathbb{F}_{\theta,\omega}(y). \tag{3.20}$$

We assume that the function $\mathbb{F}_{\theta,\omega}$ is smooth with respect to all variables and satisfies the assumptions

$$\mathbb{F}_{\theta,\omega}(y) \equiv 0 \qquad \text{for } y \in \mathrm{O}, \quad \text{and} \quad \mathbb{F}_{0,0}(y) \equiv \mathbb{F}(y), \tag{3.21}$$

where $\mathbb{F}(y)$ is defined in (3.11). Then, for sufficiently small θ and $(\omega - \omega^0)$, the global Poincaré map $T_{\theta,\omega}^{\text{glo}} : \Pi^{\text{out}} \to \Pi^{\text{in}}$ is well defined and smooth. We assume that the perturbation (3.20) is such that this global Poincaré map is written, in a small neighbourhood of the origin in Π^{out}, as follows:

$$T_{\theta,\omega}^{\text{glo}}(\bar{M}) = \theta + T_0^{\text{glo}}(\bar{M}), \tag{3.22}$$

i.e. the only effect of the perturbation in the nonlinearity $\mathbb{F}$ is an additive term in the global map (see (3.16)). Obviously, such a family of perturbations exists (one can construct it by the flow-box technique). Since the global map is insensitive to changes in ω we will further use the notation T_θ^{glo}.

It is obvious that for every frequency vector ω which is sufficiently close to ω^0 and for every $M \in \Pi^{\text{in}} \cap \{Z > 0\}$ which is sufficiently close to 0 (in our local coordinates on Π^{in}) there exists a perturbation parameter θ such that system (3.20) possesses a periodic orbit which intersects with Π^{in} at the given point M. Indeed, note that, due to our construction, fixed points M with $Z > 0$ of the first-return map

$$T_{\theta,\omega}(M) := T_\theta^{\text{glo}}(T_\omega^{\text{loc}}(M)), \qquad T_{\theta,\omega} : \Pi^{\text{in}} \to \Pi^{\text{in}}, \tag{3.23}$$

correspond to periodic orbits of the system (3.20). Thus, we must find the value of θ for which the given point M is the fixed point of map (3.23). It remains to note that the fixed point equation $T_{\theta,\omega}M = M$ is recast, by virtue of (3.22) and (3.23), as the following relation

$$\theta = M - T_0^{\mathrm{glo}}(T_\omega^{\mathrm{loc}}(M)), \tag{3.24}$$

which indeed defines θ uniquely, given ω and M.

Our next step is to compute the multipliers of the periodic orbit in dependence on ω and M. By definition, these multipliers are the eigenvalues of the derivative with respect to M of the first-return map (3.23):

$$P(\omega, M) := \frac{\mathrm{d}}{\mathrm{d}M} T_{\theta,\omega}(M)|_{\theta=\theta(\omega,M)} = \mathcal{T}_{\omega,M} \circ \mathcal{Z}(\omega, M), \tag{3.25}$$

where

$$\mathcal{T}_{\omega,M} := \frac{\mathrm{d}}{\mathrm{d}\bar{M}} T_\theta^{\mathrm{glo}}(\bar{M})|_{\theta=\theta(\omega,M),\bar{M}=T_\omega^{\mathrm{loc}}(M)}, \qquad \mathcal{Z}(\omega, M) := \frac{\mathrm{d}}{\mathrm{d}M} T_\omega^{\mathrm{loc}}(M).$$

As M tends to the point $(\mathcal{Z}, u_1, v_1, \ldots, u_m, v_m, \alpha) = 0$ (this is the point of intersection of the homoclinic loop Γ with the cross-section Π^{in} at $\theta = 0$), we have $\theta \to 0$, by virtue of (3.24), (3.14) and (3.16). Thus, as $\omega \to \omega^0$, $M \to 0$, the matrix $\mathcal{T}_{\omega,M}$ tends to the matrix $\mathcal{T}_0$ defined by (3.15).

On the other hand, differentiating (3.14) with respect to M, we obtain

$$\mathcal{Z}(\omega, M) = \begin{pmatrix} Z^{\lambda_1-1}\cos(\omega_1 \ln Z + \varphi_1)\rho_1 & & & & \\ & \mathcal{R}_1(Z) & 0 & \ldots & 0 \\ Z^{\lambda_1-1}\sin(\omega_1 \ln Z + \varphi_1)\rho_1 & & & & \\ Z^{\lambda_2-1}\cos(\omega_2 \ln Z + \varphi_2)\rho_2 & & & & \\ & 0 & \mathcal{R}_2(Z) & \ldots & 0 \\ Z^{\lambda_2-1}\sin(\omega_2 \ln Z + \varphi_2)\rho_2 & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -AZ^{A-I} & 0 & 0 & \ldots & Z^{-A} \end{pmatrix}, \tag{3.26}$$

where we denote

$$\mathcal{R}_k(Z, \omega) = Z^{\lambda_k} \begin{pmatrix} \cos\omega_k \ln Z & -\sin\omega_k \ln Z \\ \sin\omega_k \ln Z & \cos\omega_k \ln Z \end{pmatrix} \tag{3.27}$$

and use, notationally, polar coordinates (ρ_k, φ_k) instead of (u_k, v_k) in the following way:

$$u_k = \frac{\rho_k}{r_k}\cos(\varphi_k - \psi_k), \qquad v_k = \frac{\rho_k}{r_k}\sin(\varphi_k - \psi_k) \tag{3.28}$$

with

$$r_k := (\lambda_k^2 + \omega_k^2)^{1/2}, \qquad \cos\psi_k = \frac{\lambda_k}{r_k} \qquad \text{and} \qquad \sin\psi_k = \frac{\omega_k}{r_k}. \tag{3.29}$$

In the following, we will study the eigenvalues of the matrix $P(\omega, M)$ defined by (3.25) for (ω, M) of some special form only. To be more precise, we fix some positive numbers β_k, $k = 1, \ldots, m$, such that

$$\begin{aligned} 1 - 2\lambda_1 - \cdots - 2\lambda_m &> \beta_1 > \beta_2 > \cdots > \beta_m \\ &> \beta_1 - \min\{\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \ldots, \lambda_m - \lambda_{m-1}, \lambda_{m+1} - \lambda_m\} \end{aligned} \tag{3.30}$$

(such numbers exist due to assumption (3.7)). Then, we fix

$$\rho_k = Z^{\beta_k}. \tag{3.31}$$

Moreover, we consider the perturbations of the frequency vector ω^0 in the form

$$\omega(\bar{\omega}) := \omega^0 + (\ln Z)^{-1}\bar{\omega}, \tag{3.32}$$

where $\bar{\omega} \in [-\pi, 3\pi]^m$. Then, for every small positive $Z \ll 1$, every $\bar{\omega} \in [-\pi, 3\pi]^m$ and every $\phi = (\phi_1, \ldots, \phi_m) \in [-\pi, 3\pi]^m$, the point $M = M(Z, \bar{\omega}, \phi)$ is defined as follows:

$$M(Z, \bar{\omega}, \phi) := y = \Bigg(d \cdot Z, \frac{\rho_1}{r_1}\cos(\phi_1 - \psi_1), \frac{\rho_1}{r_1}\sin(\phi_1 - \psi_1), \ldots, \frac{\rho_m}{r_m}\cos(\phi_m - \psi_m), \frac{\rho_m}{r_m}\sin(\phi_m - \psi_m), \mathcal{W}(0)\Bigg), \tag{3.33}$$

where the parameters $\rho_k = \rho_k(Z)$, $\omega = \omega(\bar{\omega})$, $r_k = r_k(\bar{\omega})$ and $\psi_k = \psi_k(\bar{\omega})$ are defined by (3.31), (3.32) and (3.29).

Thus, we consider finally the $(2m+1)$-parameter family of perturbations of equation (3.11) which corresponds to the choice of M of the form (3.33) and ω in the form (3.32) with small positive $Z \ll 1$ and arbitrary $\bar{\omega}, \phi \in [-\pi, 3\pi]^m$ and study the matrix (3.25) only for such (ω, M). In order to simplify the notation, we write in the following $P(Z, \bar{\omega}, \phi)$ instead of $P(\omega(\bar{\omega}), M(Z, \bar{\omega}, \phi))$, $\mathcal{Z}(Z, \bar{\omega}, \phi)$ instead of $\mathcal{Z}(\omega(\bar{\omega}), M(Z, \bar{\omega}, \phi))$ and so on. It is obvious that our family of perturbations is, indeed, arbitrarily small when $Z \to +0$.

The following lemma gives the principal part of the asymptotic expansions of the coefficients of the characteristic polynomial for the matrix $P(Z, \bar{\omega}, \phi)$ as $Z \to +0$.

Lemma 3.1. *Let*

$$\begin{aligned}\mathbb{P}_{Z,\bar{\omega},\phi}(\mu) &:= \det(\mu \mathrm{Id} - P(Z, \bar{\omega}, \phi)) \\ &:= \mu^n - \mathcal{M}_1\mu^{n-1} + \mathcal{M}_2\mu^{n-2} + \cdots + (-1)^n\mathcal{M}_n\end{aligned} \tag{3.34}$$

be the characteristic polynomial of the matrix $P(Z, \bar{\omega}, \phi)$ defined by (3.25). Then the following formulae are valid for the coefficients $\mathcal{M}_k$ of this polynomial:

$$\begin{aligned}\mathcal{M}_{2k-1}(Z, \bar{\omega}, \phi) &= \left(\prod_{j=1}^{2k-1} L^0_{jj}\right) \cdot \left(\prod_{j=1}^{2k-2} U^0_{jj}\right) Z^{-1+2\lambda_1+\cdots+2\lambda_{k-1}+\lambda_k+\beta_k} \\ &\quad\times\Big[U^0_{2k-1,2k-1}\cos(\omega^0_k \ln Z + \varphi_k + \bar{\omega}_k) \\ &\quad+U^0_{2k-1,2k}\sin(\omega^0_k \ln Z + \varphi_k + \bar{\omega}_k) + \mathbb{M}_{2k-1}(Z, \bar{\omega}, \phi)\Big],\end{aligned} \tag{3.35}$$

$$\begin{aligned}\mathcal{M}_{2k}(Z, \bar{\omega}, \phi) &= \left(\prod_{j=1}^{2k-1} L^0_{jj}\right) \cdot \left(\prod_{j=1}^{2k} U^0_{jj}\right) Z^{-1+2\lambda_1+\cdots+2\lambda_k+\beta_k} \\ &\quad\times\left[-L^0_{2k,2k}\sin\varphi_k + L^0_{2k+1,2k}\cos\varphi_k + \mathbb{M}_{2k}(Z, \bar{\omega}, \phi)\right]\end{aligned}$$

for $k = 1, \cdots, m$, and

$$\mathcal{M}_k = \mathbb{M}_k(Z, \bar{\omega}, \phi) \tag{3.36}$$

for $k > 2m$. Here L^0_{ij} and U^0_{ij} are the entries of the lower- and, respectively, upper-triangular matrices defined by (3.17). The functions $\mathbb{M}_k$ are smooth with respect to $(\bar{\omega}, \phi)$ and $Z > 0$, and they tend to zero, along with their derivatives with respect to $(\bar{\omega}, \phi)$, as $Z \to +0$.

Essentially, this lemma says that the first $2m$ coefficients of the characteristic equation can take arbitrary values for sufficiently small Z and carefully taken ϕ and $\bar{\omega}$, while the rest of the coefficients tend to zero as $Z \to +0$ (see (3.7) and (3.30)). Hence, $2m$ roots of the characteristic equation may indeed take arbitrary prescribed values (see the arguments after the proof of the lemma) and the rest of the roots tend to zero, which gives us the theorem.

The computations in the proof of the lemma are sufficiently lengthy, so we start with the example of the simplest possible case. Namely, let us assume $n = 4$, $m = 2$ and let $\mathcal{T}_0 = \mathrm{id}$ in (3.25). Then (see (3.26), (3.27) and (3.31)) the matrix P under consideration is written as

$$P = \begin{pmatrix} Z^{\lambda_1-1}\cos(\omega_1 \ln Z + \varphi_1)Z^{\beta_1} & Z^{\lambda_1}\cos\omega_1 \ln Z & -Z^{\lambda_1}\sin\omega_1 \ln Z \\ Z^{\lambda_1-1}\sin(\omega_1 \ln Z + \varphi_1)Z^{\beta_1} & Z^{\lambda_1}\sin\omega_1 \ln Z & Z^{\lambda_1}\cos\omega_1 \ln Z \\ -\lambda_2 Z^{\lambda_2-1} & 0 & 0 \end{pmatrix}.$$

It is immediately seen that the coefficients of the characteristic equation for this matrix are

$$\begin{aligned} \mathcal{M}_1 &= Z^{\lambda_1+\beta_1-1}\cos(\omega_1 \ln Z + \varphi_1) + Z^{\lambda_1}\sin\omega_1 \ln Z, \\ \mathcal{M}_2 &= -Z^{2\lambda_1+\beta_1-1}\sin\varphi_1 - \lambda_2 Z^{\lambda_2+\lambda_1-1}\sin\omega_1 \ln Z, \\ \mathcal{M}_3 &= -\lambda_2 Z^{2\lambda_1+\lambda_2-1}. \end{aligned}$$

By (3.7) and (3.30), we have full agreement with the statement of the lemma.

Proof of lemma 3.1. We recall that the matrix $\mathcal{T}_{Z,\bar\omega,\phi} := \mathcal{T}_{(\omega(\bar\omega),M(Z,\bar\omega,\phi))}$ in (3.25) is close to the matrix $\mathcal{T}_0$, hence it can be decomposed, analogously to (3.17):

$$\mathcal{T}_{Z,\bar\omega,\phi} = L_{Z,\bar\omega,\phi} \cdot U_{Z,\bar\omega,\phi}, \tag{3.37}$$

where U and L are upper- and lower-triangular matrices, respectively:

$$L = \begin{pmatrix} L_{11} & 0 & \dots & & \\ L_{21} & L_{22} & 0 & \dots & \\ L_{31} & L_{32} & L_{33} & 0 & \dots \\ & & & \ddots & \\ \dots & \dots & \dots & & \end{pmatrix}, \qquad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & \dots \\ 0 & U_{22} & U_{23} & \dots \\ \dots & 0 & U_{33} & \dots \\ & & 0 & \ddots \\ \dots & \dots & \dots & \end{pmatrix}. \tag{3.38}$$

Moreover, the entries $U_{ij} = U_{ij}(Z, \bar\omega, \phi)$ and $L_{ij} = L_{ij}(Z, \bar\omega, \phi)$ are smooth with respect to all arguments and are close to the corresponding entries U^0_{ij} and L^0_{ij} as $Z \ll 1$.

We now note that the matrix

$$C = C(Z, \bar\omega, \phi) := U \cdot \mathcal{Z}(Z, \bar\omega, \phi) \cdot L \tag{3.39}$$

is similar to $P(Z, \bar\omega, \phi) = L \cdot U \cdot Z(Z, \bar\omega, \phi)$ and, consequently, these two matrices have the same characteristic polynomials. So, we compute below the characteristic polynomial of the matrix C defined by (3.39). In order to do so, we recall that assumptions (3.9), (3.30) and (3.31) imply the following inequalities:

$$\begin{aligned} &Z \ll \rho_1 \ll \rho_2 \ll \dots \ll \rho_m \ll 1, \\ &Z^{\lambda_1}\rho_1 \gg Z^{\lambda_2}\rho_2 \gg \dots \gg Z^{\lambda_m}\rho_m \gg Z^{\lambda_{m+1}}, \\ &Z^{\lambda_1} \gg Z^{\lambda_2} \gg \dots \gg Z^{\lambda_m} \gg Z^{\lambda_{m+1}}, \end{aligned} \tag{3.40}$$

if $Z \ll 1$. Since the matrices U and L are upper- and lower-triangular, the entries of the matrix C are given by the following formula:

$$C_{ij} = \sum_{m=i}^{n} \sum_{k=j}^{n} \mathcal{Z}_{mk} U_{im} L_{kj}. \qquad (3.41)$$

Computing by (3.26) and (3.41), and using (3.40), it is easy to verify that the matrix C defined by (3.39) is estimated as follows:

$$\begin{pmatrix} O(Z^{\lambda_1-1}\rho_1) & O(Z^{\lambda_1}) & O(Z^{\lambda_1}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \dots \\ O(Z^{\lambda_1-1}\rho_1) & O(Z^{\lambda_1}) & O(Z^{\lambda_1}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \dots \\ O(Z^{\lambda_2-1}\rho_2) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \dots \\ O(Z^{\lambda_2-1}\rho_2) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_2}) & O(Z^{\lambda_3}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \qquad (3.42)$$

i.e. its entries are estimated as follows:

$$C_{i1} = O(Z^{\lambda_{k(i)}-1}\rho_{k(i)}), \qquad C_{ij} = O(Z^{\lambda_{s(i,j)}}) \quad (j \geqslant 2), \qquad (3.43)$$

where we denote

$$k(i) = \begin{cases} 1 & \text{at } i = 1, 2, \\ 2 & \text{at } i = 3, 4, \\ \vdots & \vdots \\ m & \text{at } i = 2m-1, 2m, \\ m+1 & \text{at } i > 2m \end{cases} \qquad (3.44)$$

and

$$s(i,j) = \begin{cases} m+1 & \text{at } i > 2m, \text{ or (at } i \leqslant 2m) \\ k(i) & \text{at } j = 2, \dots, 2k(i)+1, \\ k(i)+1 & \text{at } j = 2k(i)+2, 2k(i)+3, \\ \vdots & \vdots \\ m & \text{at } j = 2m, 2m+1, \\ m+1 & \text{at } j > 2m+1. \end{cases} \qquad (3.45)$$

We also denote here $\rho_{m+1} = 1$.

We recall now that the pth coefficient $\mathcal{M}_p = \mathcal{M}_p(Z, \bar{\omega}, \phi)$, $p = 1, \dots, n$ of the characteristic polynomial (3.34) can be represented as a sum of all main (i.e. diagonal) minors of order p of the matrix C defined by (3.39), i.e.

$$\mathcal{M}_p = \sum_{1 \leqslant i_1 < i_2 < \dots < \dots < i_p \leqslant n} M_{i_1,\dots,i_p}(C), \qquad (3.46)$$

where the minor $M_{i_1,\dots,i_p}(C)$ is the determinant of the matrix obtained as the intersection of the rows with the numbers $i_1, \dots, i_p$ and the columns with the same numbers.

Our task now is to show that the major contribution to $\mathcal{M}_p$ at $Z \ll 1$ is given by the minor $M_{1,2,\dots,p}$. Indeed, it follows from (3.43) that all the entries C_{ij} with $i > 1$ vanish at $Z = 0$ and, consequently, all the diagonal minors $M_{i_1,\dots,i_p}(C)$ with $i_1 > 1$ tend to zero as $Z \to 0$. For the minors $M_{i_1=1,i_2,\dots,i_p}(C)$ we use the following formula:

$$M_{i_1=1,i_2,\dots,i_p} = Z^{\lambda_{k_1}+\lambda_{k_2}+\dots+\lambda_{k_p}-1}\rho_{k_p} \cdot \det \mathcal{C}_{1,i_2,\dots,i_p}, \qquad (3.47)$$

where we denote $k_q \equiv k(i_q)$ (see (3.44)), and

$$\mathcal{C}_{1,i_2,\dots,i_p} = \begin{pmatrix} Z^{1-\lambda_1}\rho_{k_p}^{-1}C_{11} & Z^{-\lambda_1}C_{1i_2} & \dots & Z^{-\lambda_1}C_{1i_p} \\ Z^{1-\lambda_{k_2}}\rho_{k_p}^{-1}C_{i_21} & Z^{-\lambda_{k_2}}C_{i_2i_2} & \dots & Z^{-\lambda_{k_2}}C_{i_2i_p} \\ \vdots & \vdots & \ddots & \vdots \\ Z^{1-\lambda_{k_p}}\rho_{k_p}^{-1}C_{i_p1} & Z^{-\lambda_{k_p}}C_{i_pi_2} & \dots & Z^{-\lambda_{k_p}}C_{i_pi_p} \end{pmatrix}. \tag{3.48}$$

By (3.40) and (3.42), all the entries of the matrix $\mathcal{C}$ are bounded from above, so we have

$$M_{1,i_2,\dots,i_p}(C) = \mathrm{O}(Z^{\lambda_{k_1}+\lambda_{k_2}+\dots+\lambda_{k_p}-1}\rho_{k_p}). \tag{3.49}$$

If $p \geqslant 2m+1$, this estimate gives us

$$M_{1,i_2,\dots,i_p}(C) = \mathrm{O}(Z^{2\lambda_1+2\lambda_2+\dots+2\lambda_m+\lambda_{m+1}-1}), \tag{3.50}$$

so, by virtue of our assumption (3.7), all the diagonal minors of order $2m+1$ and larger tend to zero as $Z \to +0$, which proves (3.36).

Let us now consider the case $p \leqslant 2m$. We note that when i decreases at least on 2, the corresponding value of $\lambda_{k(i)}$ will also decrease. Thus, it follows from (3.40) and (3.49) that the main contribution to the coefficient $\mathcal{M}_p$ $(p \leqslant 2m)$ is given by the minor $M_{1,2,\dots,p}(C)$ in the case p is even, and by the two minors $M_{1,2,\dots,p-1,p}(C)$ and $M_{1,2,\dots,p-1,p+1}(C)$ in the case p is odd (and $p > 1$). Moreover, we claim that

$$\mathcal{M}_{1,2,\dots,2(l-1),2l} = Z^{-1+2\lambda_1+\dots+2\lambda_{l-2}+2\lambda_{l-1}+\lambda_l+\beta_l}\mathrm{O}(Z^{\varepsilon}) \tag{3.51}$$

for some $\varepsilon > 0$. Indeed, according to (3.30), (3.31) and (3.44),

$$\rho_{k(i)}\rho_l^{-1} = \mathrm{O}(Z^{\beta_{k(i)}-\beta_l}) \ll 1 \qquad \text{and} \qquad Z^{-\lambda_{k(i)}}C_{i,2l} = \mathrm{O}(Z^{\lambda_l-\lambda_{k(i)}}) \ll 1$$

for $i \leqslant 2(l-1)$. Consequently, the matrix $\mathcal{C}_{1,\dots,2(l-1),2l}(C)$ defined via (3.48) can be rewritten as follows:

$$\mathcal{C}_{1,\dots,2(l-1),2l} = \begin{pmatrix} 0 & \mathrm{O}(1) & \cdots & \mathrm{O}(1) & 0 \\ 0 & \mathrm{O}(1) & \cdots & \mathrm{O}(1) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \mathrm{O}(1) & \cdots & \mathrm{O}(1) & 0 \\ \mathrm{O}(1) & \mathrm{O}(1) & \cdots & \mathrm{O}(1) & \mathrm{O}(1) \end{pmatrix} + \mathrm{O}(Z^{\varepsilon}),$$

which implies (3.51), since the determinant of the matrix on the right-hand side of the last formula is, obviously, zero. Thus, we have proved that

$$\begin{aligned} \mathcal{M}_{2k-1} &= M_{1,\dots,2k-1}(C) + Z^{-1+2\lambda_1+\dots+2\lambda_{k-1}+\lambda_k+\beta_k}\mathrm{O}(Z^{\varepsilon}), \\ \mathcal{M}_{2k} &= M_{1,2,\dots,2k}(C) + Z^{-1+2\lambda_1+\dots+2\lambda_k+\beta_k}\mathrm{O}(Z^{\varepsilon}) \qquad (k = 1,\dots,m) \end{aligned} \tag{3.52}$$

for some small positive constant $\varepsilon > 0$. It remains to compute the determinants $M_{1,\dots,p}$ for $p = 1,\dots,2m$.

To this end, according to (3.40) and (3.43), we rewrite the formula (3.48) for $\mathcal{C}_{1,\dots,2l-1}$ and $\mathcal{C}_{1,\dots,2l}$ $(l = 1, \dots, m)$ as follows:

$$\mathcal{C}_{1,\dots,2l-1} = \begin{pmatrix} 0 & Z^{-\lambda_1}C_{12} & Z^{-\lambda_1}C_{13} & 0 & 0 & 0 & \cdots \\ 0 & Z^{-\lambda_1}C_{22} & Z^{-\lambda_1}C_{23} & 0 & 0 & 0 & \cdots \\ 0 & \mathrm{O}(1) & \mathrm{O}(1) & Z^{-\lambda_2}C_{34} & Z^{-\lambda_2}C_{35} & 0 & \cdots \\ 0 & \mathrm{O}(1) & \mathrm{O}(1) & Z^{-\lambda_2}C_{44} & Z^{-\lambda_2}C_{45} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Z^{1-\lambda_l}\rho_l^{-1}C_{2l-1,1} & \mathrm{O}(1) & \cdots & \cdots & \cdots & \cdots & \mathrm{O}(1) \end{pmatrix} + \mathrm{O}(Z^{\varepsilon}) \tag{3.53}$$

and

$$\mathcal{C}_{1,\dots,2l} = \begin{pmatrix} 0 & Z^{-\lambda_1}C_{12} & Z^{-\lambda_1}C_{13} & 0 & 0 & 0 & \cdots \\ 0 & Z^{-\lambda_1}C_{22} & Z^{-\lambda_1}C_{23} & 0 & 0 & 0 & \cdots \\ 0 & \mathrm{O}(1) & \mathrm{O}(1) & Z^{-\lambda_2}C_{34} & Z^{-\lambda_2}C_{35} & 0 & \cdots \\ 0 & \mathrm{O}(1) & \mathrm{O}(1) & Z^{-\lambda_2}C_{44} & Z^{-\lambda_2}C_{45} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Z^{1-\lambda_l}\rho_l^{-1}C_{2l-1,1} & \mathrm{O}(1) & \cdots & \cdots & \cdots & \mathrm{O}(1) & Z^{-\lambda_l}C_{2l-1,2l} \\ Z^{1-\lambda_l}\rho_l^{-1}C_{2l,1} & \mathrm{O}(1) & \cdots & \cdots & \cdots & \mathrm{O}(1) & Z^{-\lambda_l}C_{2l,2l} \end{pmatrix} + \mathrm{O}(Z^{\varepsilon}). \tag{3.54}$$

Since all the entries of $\mathcal{C}_{1,\dots,p}$ are bounded, we obtain from (3.47), (3.53) and (3.54)

$$M_{1,\dots,2k-1} = C_{2k-1,1} \times \prod_{l=1}^{k-1} \begin{vmatrix} C_{2l-1,2l} & C_{2l-1,2l+1} \\ C_{2l,2l} & C_{2l,2l+1} \end{vmatrix} + Z^{-1+2\lambda_1+\cdots+2\lambda_{k-1}+\lambda_k+\beta_k}\mathrm{O}(Z^{\varepsilon}) \tag{3.55}$$

and

$$\mathcal{M}_{1,\dots,2k} = \begin{vmatrix} C_{2k-1,1} & C_{2k-1,2k} \\ C_{2k,1} & C_{2k,2k} \end{vmatrix} \times \prod_{l=1}^{k-1} \begin{vmatrix} C_{2l-1,2l} & C_{2l-1,2l+1} \\ C_{2l,2l} & C_{2l,2l+1} \end{vmatrix} + Z^{-1+2\lambda_1+\cdots+2\lambda_k+\beta_k}\mathrm{O}(Z^{\varepsilon}) \tag{3.56}$$

for $k = 1, \dots, m$, where $\varepsilon > 0$ is a small positive number.

Now, it remains to express the right-hand sides of (3.55) and (3.56) in terms of the entries of the matrices $\mathcal{Z}(Z, \bar{\omega}, \phi)$, $U(Z, \bar{\omega}, \phi)$ and $L(Z, \bar{\omega}, \phi)$. One can easily see that, according to (3.26), (3.27), (3.40), (3.41) and (3.43),

$$\begin{aligned} C_{2k-1,1} &= L_{11}U_{2k-1,2k-1}\mathcal{Z}_{2k-1,1} + L_{11}U_{2k-1,2k}\mathcal{Z}_{2k,1} + Z^{-1+\lambda_k}\rho_k\mathrm{O}(Z^{\varepsilon}) \\ &= Z^{-1+\lambda_k+\beta_k}L_{11}^0\Big[U^0_{2k-1,2k-1}\cos(\omega_k^0 \ln Z + \phi_k + \bar{\omega}_k) \\ &\quad + U^0_{2k-1,2k}\sin(\omega_k^0 \ln Z + \phi_k + \bar{\omega}_k) + \mathrm{o}(1)\Big]. \end{aligned}$$

Analogously,

$$\begin{vmatrix} C_{2l-1,2l} & C_{2l-1,2l+1} \\ C_{2l,2l} & C_{2l,2l+1} \end{vmatrix} = Z^{2\lambda_l} \begin{vmatrix} U_{2l-1,2l-1} & U_{2l-1,2l} \\ 0 & U_{2l,2l} \end{vmatrix} \times \begin{vmatrix} \cos\omega_l \ln Z & -\sin\omega_l \ln Z \\ \sin\omega_l \ln Z & \cos\omega_l \ln Z \end{vmatrix}$$
$$\times \begin{vmatrix} L_{2l,2l} & 0 \\ L_{2l+1,2l} & L_{2l+1,2l+1} \end{vmatrix} + Z^{2\lambda_l} \mathrm{O}(Z^\varepsilon)$$
$$= Z^{2\lambda_l}[U^0_{2l-1,2l-1} U^0_{2l,2l} L^0_{2l,2l} L^0_{2l+1,2l+1} + \mathrm{o}(1)].$$

And, finally,

$$\begin{vmatrix} C_{2k-1,1} & C_{2k-1,2k} \\ C_{2k,1} & C_{2k,2k} \end{vmatrix} = L_{11} \begin{vmatrix} U_{2k-1,2k-1} & U_{2k-1,2k} \\ 0 & U_{2k,2k} \end{vmatrix}$$
$$\times \left(L_{2k,2k} \begin{vmatrix} \mathcal{Z}_{2k-1,1} & \mathcal{Z}_{2k-1,2k} \\ \mathcal{Z}_{2k,1} & \mathcal{Z}_{2k,2k} \end{vmatrix} + L_{2k+1,2k} \begin{vmatrix} \mathcal{Z}_{2k-1,1} & \mathcal{Z}_{2k-1,2k+1} \\ \mathcal{Z}_{2k,1} & \mathcal{Z}_{2k,2k+1} \end{vmatrix} \right)$$
$$+ Z^{-1+2\lambda_k+\beta_k} \mathrm{O}(Z^\varepsilon)$$
$$= L^0_{11} U^0_{2k-1,2k-1} U^0_{2k,2k} Z^{-1+2\lambda_k+\beta_k} [-L^0_{2k,2k} \sin\phi_k + L^0_{2k+1,2k} \cos\phi_k + \mathrm{o}(1)].$$

Inserting these formulae into (3.55), (3.56) and (3.52), we obtain expansions (3.35). Lemma 3.1 is proven.

We are now ready to finish the proof of theorem 3.1. Indeed, let us consider only such sequence of values of $Z \to +0$ for which

$$\left\{ \omega^0_k \frac{\ln Z}{2\pi} \right\} \to 0$$

for all $k = 1, \ldots, m$ (here $\{\cdot\}$ denotes the fractional part). It is easy to see then, that given any fixed values of the coefficients $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ of the characteristic polynomial of the derivative matrix $P(Z, \bar\omega, \phi)$ of the Poincaré map of the periodic orbit under consideration, the system of equations (3.35) for these coefficients can be resolved with respect to ϕ and $\bar\omega$. Moreover, ϕ and $\bar\omega$ depend on $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ smoothly and have finite limits as $Z \to +0$, along with the derivatives with respect to $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$.

Indeed, system (3.35), recast as

$$\begin{aligned} &U^0_{2k-1,2k-1} \cos\left(2\pi \left\{\omega^0_k \frac{\ln Z}{2\pi}\right\} + \phi_k + \bar\omega_k\right) + U^0_{2k-1,2k} \sin\left(2\pi \left\{\omega^0_k \frac{\ln Z}{2\pi}\right\} + \phi_k + \bar\omega_k\right) \\ &\quad = \mathcal{M}_{2k-1} \left(\prod_{j=1}^{2k-1} L^0_{jj}\right)^{-1} \cdot \left(\prod_{j=1}^{2k-2} U^0_{jj}\right)^{-1} Z^{1-2\lambda_1-\cdots-2\lambda_{k-1}-\lambda_k-\beta_k} \\ &\quad\quad -\mathbb{M}_{2k-1}(Z, \bar\omega, \phi), \qquad (k = 1, \ldots, m), \\ &-L^0_{2k,2k} \sin\phi_k + L^0_{2k+1,2k} \cos\phi_k = \mathcal{M}_{2k} \left(\prod_{j=1}^{2k-1} L^0_{jj}\right)^{-1} \cdot \left(\prod_{j=1}^{2k} U^0_{jj}\right)^{-1} \\ &\quad\quad \times Z^{1-2\lambda_1-\cdots-2\lambda_k-\beta_k} - \mathbb{M}_{2k}(Z, \bar\omega, \phi), \qquad (k = 1, \ldots, m), \end{aligned} \tag{3.57}$$

has a regular limit as $Z \to +0$:

$$\begin{aligned} &U^0_{2k-1,2k-1} \cos(\phi_k + \bar\omega_k) + U^0_{2k-1,2k} \sin(\phi_k + \bar\omega_k) = 0, \\ &-L^0_{2k,2k} \sin\phi_k + L^0_{2k+1,2k} \cos\phi_k = 0. \end{aligned} \tag{3.58}$$

Here we have used the fact that due to our assumptions (3.7) and (3.30),

$$1 - 2\lambda_1 - \cdots - 2\lambda_k - \beta_k > 0$$

for every $k = 1, \dots, m$. By (3.19), $U^0_{2k-1,2k-1} \neq 0$ and $L^0_{2k,2k} \neq 0$, so we may resolve the limit system (3.58) as follows:

$$\phi_k = \arctan \frac{L^0_{2k+1,2k}}{L^0_{2k,2k}}, \qquad \bar{\omega}_k = \frac{\pi}{2} - \arctan \frac{U^0_{2k-1,2k}}{U^0_{2k-1,2k-1}} - \phi_k. \tag{3.59}$$

Now, according to the implicit function theorem, we have indeed the functions $\phi(Z, \mathcal{M}_1, \dots, \mathcal{M}_{2m})$, $\bar{\omega}(Z, \mathcal{M}_1, \dots, \mathcal{M}_{2m})$, close to those given by (3.59), which satisfy (3.57) at small Z (hence, they satisfy (3.35)) and which depend smoothly on $\mathcal{M}_1, \dots, \mathcal{M}_{2m}$.

We now fix $\phi = \phi(Z, \mathcal{M}_1, \dots, \mathcal{M}_{2m})$, $\bar{\omega} = \bar{\omega}(Z, \mathcal{M}_1, \dots, \mathcal{M}_{2m})$, so we choose now $\mathcal{M}_1, \dots, \mathcal{M}_{2m}$ to parametrize our family of small perturbations. As we just have shown, $\mathcal{M}_1, \dots, \mathcal{M}_{2m}$ can be taken from an arbitrarily large domain in $\mathbb{R}^{2m}$. Let $\mathcal{M}_1, \dots, \mathcal{M}_{2m}$ be uniformly bounded and let $\mathcal{M}_{2m}$ stay bounded away from zero. As $Z \to +0$, the coefficients $\mathcal{M}_{2m+1}, \dots, \mathcal{M}_n$ of the characteristic polynomial tend uniformly to zero, according to (3.36). Thus, the characteristic equation

$$\mu^n - \mathcal{M}_1\mu^{n-1} + \dots + \mathcal{M}_{2m}\mu^{n-2m} - \mathcal{M}_{2m+1}\mu^{n-2m-1} + \dots + (-1)^n\mathcal{M}_n = 0 \tag{3.60}$$

has $(n - 2m)$ roots which tend to zero as $Z \to +0$, and $2m$ roots (we denote them as $\mu_1, \dots, \mu_{2m}$) which are bounded away from zero and tend to the roots of the polynomial

$$\mu^{2m} - \mathcal{M}_1\mu^{2m-1} + \dots + \mathcal{M}_{2m}. \tag{3.61}$$

Define the real numbers $\tilde{\mathcal{M}}_1, \dots, \tilde{\mathcal{M}}_{2m}$ such that $\mu_1, \dots, \mu_{2m}$ were the roots of the polynomial

$$\mu^{2m} - \tilde{\mathcal{M}}_1\mu^{2m-1} + \dots + \tilde{\mathcal{M}}_{2m}, \tag{3.62}$$

i.e.

$$\prod_{j=1}^{2m}(\mu - \mu_j) = \mu^{2m} - \tilde{\mathcal{M}}_1\mu^{2m-1} + \dots + \tilde{\mathcal{M}}_{2m}. \tag{3.63}$$

By construction, $(\tilde{\mathcal{M}}_1, \dots, \tilde{\mathcal{M}}_{2m})$ tend to $(\mathcal{M}_1, \dots, \mathcal{M}_{2m})$ as $Z \to +0$. Let us show that $\tilde{\mathcal{M}}_1, \dots, \tilde{\mathcal{M}}_{2m}$ depend on $\mathcal{M}_1, \dots, \mathcal{M}_{2m}$ smoothly, and that

$$\left.\frac{d(\tilde{\mathcal{M}}_1, \dots, \tilde{\mathcal{M}}_{2m})}{d(\mathcal{M}_1, \dots, \mathcal{M}_{2m})}\right|_{Z=0} = \mathbb{I}d. \tag{3.64}$$

Indeed, consider the linear operator $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & \cdots & \cdots & 0 & -1 \\ \mathcal{M}_n & \mathcal{M}_{n-1} & \cdots & \mathcal{M}_2 & \mathcal{M}_1 \end{pmatrix}. \tag{3.65}$$

Its characteristic equation is also given by (3.60), so it has, as well, $(n - 2m)$ eigenvalues close to zero and $2m$ eigenvalues which are bounded away from zero and are the roots of the polynomial (3.62). Hence, the operator $\mathcal{Q}$ has two invariant eigenspaces, one corresponds to the close to zero eigenvalues and the other corresponds to the eigenvalues which are bounded away from zero. The coefficients of the characteristic polynomial of $\mathcal{Q}$ restricted onto the second subspace are exactly the coefficients $\tilde{\mathcal{M}}_1, \dots, \tilde{\mathcal{M}}_{2m}$. Since all the entries of the matrix (3.65) are bounded and since it depends smoothly on $\mathcal{M}_1, \dots, \mathcal{M}_n$, the invariant subspaces depend on $\mathcal{M}_1, \dots, \mathcal{M}_n$ smoothly as well. This gives us the smooth dependence of $\tilde{\mathcal{M}}_1, \dots, \tilde{\mathcal{M}}_{2m}$ on $\mathcal{M}_1, \dots, \mathcal{M}_{2m}, \mathcal{M}_{2m+1}, \dots, \mathcal{M}_n$. Recall now that the coefficients $\mathcal{M}_{2m+1}, \dots, \mathcal{M}_n$ depend on $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ smoothly, and they are continuous in Z along with the derivatives with respect to $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$. Thus, the required smooth dependence of $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ on $(\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ at all small Z, including $Z = 0$, follows immediately. Identity (3.64) follows now from the fact that $(\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}) = (\mathcal{M}_1, \ldots, \mathcal{M}_{2m})$ at $Z = 0$.

Now, by implicit function theorem, we have that given any values $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ with $\tilde{\mathcal{M}}_{2m} \neq 0$, the corresponding values of $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$ are defined uniquely. In turn, the coefficients $\tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2m}$ are defined uniquely by (3.63), given any (symmetric with respect to the complex conjugation) set $\mathcal{R}$ of the non-zero roots $\mu_1, \ldots, \mu_{2m}$. Hence, given any such set $\mathcal{R}$, we find the corresponding values of $\mathcal{M}_1, \ldots, \mathcal{M}_{2m}$, and then the values of the perturbation parameters ϕ and $\bar{\omega}$, for arbitrarily small values of Z. Theorem 3.1 is proven.

By taking in theorem 3.1 the values of the multipliers $\mu_1, \ldots, \mu_{2m}$ outside the unit circle, we arrive at the following corollary.

Corollary 3.1. *Let the assumptions of theorem 3.1 hold. Then, by an arbitrarily small C^∞-perturbation of system (3.1), a periodic orbit P the instability index $N^+(P)$ of which satisfies*

$$N^+(P) = 2m \tag{3.66}$$

can be born in an arbitrarily small neighbourhood of the homoclinic loop under consideration.

Remark 3.2. We note that the unstable manifold $W^{\mathrm{u}}(P)$ of the periodic orbit P constructed in corollary 3.1 has dimension $2m + 1$. Thus, if every solution of the perturbed system (3.20) can be extended globally for positive $t \in \mathbb{R}_+$, then this unstable manifold is, obviously, a $(2m + 1)$-dimensional invariant submanifold for the system under consideration. Moreover, due to remark 3.1, we have

$$\dim W^{\mathrm{u}}(P) = [\dim_{\mathrm{L}}(\mathbb{A})], \tag{3.67}$$

where $[v]$ denotes the integral part of v. Since such invariant manifolds always belong to the attractor (if the system possesses a global attractor) then corollary 3.1 and formula (3.67) present a possibility of obtaining lower bounds for the attractor's dimension in terms of their Lyapunov dimension. This possibility will indeed be used in the next section in order to obtain sharp lower bounds for the attractor's dimension for the abstract hyperbolic equation (1.1).

It is also interesting to consider the case where the multipliers $\mu_1, \ldots, \mu_{2m}$ in theorem 3.1 are all equal to 1 in the absolute value. In this case, a small perturbation of the periodic orbit P with the multipliers $\mu_1, \ldots, \mu_{2m}$ can produce an $(m + 1)$-dimensional invariant torus (see a proof in appendix B). This gives us the following corollary.

Corollary 3.2. *Let the assumptions of theorem 3.1 hold. Then, by an arbitrarily small C^∞-perturbation of system (3.1), an $(m + 1)$-dimensional smooth invariant torus, densely filled by a quasi-periodic trajectory, can be born in an arbitrarily small neighbourhood of the homoclinic loop under consideration.*

4. Lower bounds for the dimension of the attractor

In this concluding section, we obtain sharp lower bounds for the attractor's dimension for the damped hyperbolic equation (1.1) with the nonlinearity in the class $\mathbb{S}$ (see definition 2.2). The main result here is the following theorem.

Theorem 4.1. *Let* $\mathrm{A} : D(\mathrm{A}) \to \mathrm{H}$ *be any linear self-adjoint operator whose eigenvalues satisfy (2.1), and let* $\{e_i\}_{i=1}^{\infty}$ *be the corresponding orthonormal system of eigenvectors. Then there exist two smooth nonlinear operators* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *in the form*

$$\mathbb{F}_i(u) := F_i^1((u, e_1), (u, e_2))e_1 + F_i^2((u, e_1), (u, e_2))e_2, \qquad u \in \mathrm{H}, \quad i = 1, 2 \tag{4.1}$$

(where $F_i^j \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$*,* $i, j = 1, 2$*) and a smoothing operator* $\Phi = \Phi_{\varepsilon,\gamma,k,m}$ *defined for every* $\varepsilon > 0$*, every small* $\gamma > 0$ *and every* $k, m \in \mathbb{N}$*, belonging to the class* $\mathbb{S}$ *and satisfying the estimate*

$$\|\Phi\|_{C_b^k(\mathrm{E}^{-m},\mathrm{H}^m)} \leqslant \varepsilon, \tag{4.2}$$

such that the fractal dimension of the attractor $\mathcal{A} = \mathcal{A}_{\gamma,\varepsilon,k,m}$ *of the equation*

$$\partial_t^2 u + \gamma\partial_t u + \mathrm{A}u = \mathbb{F}_1(u) + \gamma\mathbb{F}_2(u) + \Phi(u, \partial_t u) \tag{4.3}$$

possesses the following estimates:

$$C_1\frac{1}{\gamma} \leqslant \dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \leqslant C_2\frac{1}{\gamma}, \tag{4.4}$$

where the positive constants C_1 *and* C_2 *are independent of* γ*,* ε*,* k *and* m*.*

Proof. First, take a second order ODE in the form:

$$\partial_t^2 U = U - F_0(U), \qquad U \in \mathbb{R}, \tag{4.5}$$

where $F_0 \in C^\infty(\mathbb{R})$ vanishes at the origin together with its first derivative. We assume that equation (4.5) possesses a homoclinic orbit $U_0(t)$ to the equilibrium $U = 0$ (as an example, take $F_0(U) = U^3$). Let us fix a sufficiently small $\gamma > 0$, $n := [1/(2\gamma)] - 1$ and a frequency vector $\omega := (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ and consider the following decoupled system of second order ODEs:

$$\begin{aligned}
&\partial_t^2 U(t) = U(t) - F_0(U(t)),\\
&\partial_t^2 \bar{u}_1(t) + \gamma\partial_t\bar{u}_1(t) + \omega_1^2\bar{u}_1(t) = 0,\\
&\vdots\\
&\partial_t^2 \bar{u}_n(t) + \gamma\partial_t\bar{u}_n(t) + \omega_n^2\bar{u}_n(t) = 0.
\end{aligned} \tag{4.6}$$

By construction, this system has a homoclinic loop of the type studied in section 3, so one can expect an analogue of theorem 3.1 for the perturbations of system (4.6), as is indeed given by the following lemma.

Lemma 4.1. *Let the above assumptions hold and let, in addition,*

$$\omega_i > \frac{\gamma}{2}, \qquad \textit{for every } i = 1, \ldots, n. \tag{4.7}$$

Then, for every $\varepsilon > 0$ *and every* $k \in \mathbb{N}$*, there exist* C_0^∞*-functions* $\Phi_i : \mathbb{R}^{2n+2} \to \mathbb{R}$*,* $i = 0, 1, \ldots, n$*, satisfying*

$$\|\Phi_i\|_{C_b^k(\mathbb{R}^{2n+2},\mathbb{R})} \leqslant \varepsilon, \tag{4.8}$$

such that the system

$$\begin{aligned}
&\partial_t^2 U(t) = U(t) - F_0(U(t)) + \Phi_0(U(t), \partial_t U(t), \bar{u}(t), \partial_t\bar{u}(t)),\\
&\partial_t^2 \bar{u}_1(t) + \gamma\partial_t\bar{u}_1(t) + \omega_1^2\bar{u}_1(t) = \Phi_1(U(t), \partial_t U(t), \bar{u}(t), \partial_t\bar{u}(t)),\\
&\vdots\\
&\partial_t^2 \bar{u}_n(t) + \gamma\partial_t\bar{u}_n(t) + \omega_n^2\bar{u}_n(t) = \Phi_n(U(t), \partial_t U(t), \bar{u}(t), \partial_t\bar{u}(t)),
\end{aligned} \tag{4.9}$$

possesses a periodic orbit P *with the instability index* $N^+(p) = 2n$*.*

Proof. Let us introduce new variables

$$z(t) := \frac{U(t) + \partial_t U(t)}{2}, \qquad w(t) := \frac{U(t) - \partial_t U(t)}{2},$$
$$\bar{u}_i(t) := \bar{u}_i(t), \qquad \bar{v}_i(t) := (\omega_i^0)^{-1}\left(\partial_t \bar{u}_i(t) + \frac{\gamma}{2}\bar{u}_i(t)\right), \tag{4.10}$$

where $\omega_i^0 := (\omega_i^2 - \gamma^2/4)^{1/2} > 0$, $i = 1, \ldots, n$. In these variables, system (4.6) reads

$$\begin{aligned}
&\partial_t z = z - \tfrac{1}{2}F_0(z + w),\\
&\partial_t \bar{u}_1 = -\frac{\gamma}{2}\bar{u}_1 + \omega_1^0 \bar{v}_1,\\
&\partial_t \bar{v}_1 = -\frac{\gamma}{2}\bar{v}_1 - \omega_1^0 \bar{u}_1,\\
&\vdots\\
&\partial_t \bar{u}_n = -\frac{\gamma}{2}\bar{u}_n + \omega_n^0 \bar{v}_n,\\
&\partial_t \bar{v}_n = -\frac{\gamma}{2}\bar{v}_n - \omega_n^0 \bar{u}_n,\\
&\partial_t w = -w + \tfrac{1}{2}F_0(z + w).
\end{aligned} \tag{4.11}$$

Note that system (4.11) has the form of (3.1) and that all assumptions of theorem 3.1 are, obviously, satisfied. Consequently, according to this theorem, for every given $\varepsilon > 0$ and $k \in \mathbb{N}$, there are C_0^∞-functions $\tilde{\Phi}_i : \mathbb{R}^{2n+2} \to \mathbb{R}$, $i = 0, \ldots, 2n + 1$, satisfying

$$\|\tilde{\Phi}_i\|_{C_b^k(\mathbb{R}^{2n+2},\mathbb{R})} \leqslant \varepsilon, \tag{4.12}$$

such that the following perturbation of system (4.11)

$$\begin{aligned}
&\partial_t z = z - \tfrac{1}{2}F_0(z + w) + \tilde{\Phi}_0(z, w, \bar{u}, \bar{v}),\\
&\partial_t \bar{u}_i = -\frac{\gamma}{2}\bar{u}_i + \omega_i^0 \bar{v}_i + \tilde{\Phi}_{2i-1}(z, w, \bar{u}, \bar{v}),\\
&\partial_t \bar{v}_i = -\frac{\gamma}{2}\bar{v}_i - \omega_i^0 \bar{u}_i + \tilde{\Phi}_{2i}(z, w, \bar{u}, \bar{v}), \qquad i = 1, \ldots, n,\\
&\partial_t w = -w + \tfrac{1}{2}F_0(z + w) + \tilde{\Phi}_{2n+1}(z, w, \bar{u}, \bar{v}),
\end{aligned} \tag{4.13}$$

possesses a periodic orbit P with the instability index $N^+(P) = 2n$. This periodic orbit lies in a small neighbourhood of the homoclinic loop of the unperturbed system, so we may assume without loss of generality that all the functions $\tilde{\Phi}_i$ have finite supports.

It remains to rewrite system (4.13) as a system of second order ODEs for the variables $U(t) := z(t) + w(t)$ and $\bar{u}_i(t)$. To this end, we take the sum of the first and last equations of (4.13) and differentiate it with respect to t and, analogously, we differentiate the equations for $\bar{u}_i(t)$ in (4.13). This gives us

$$\begin{aligned}
&\partial_t^2 U(t) = U(t) - F_0(U(t)) + \bar{\Phi}_0(z(t), w(t), \bar{u}(t), \bar{v}(t)),\\
&\partial_t^2 \bar{u}_i(t) + \gamma \partial_t \bar{u}_i(t) + \omega_i^2 \bar{u}_i(t) = \bar{\Phi}_i(z(t), w(t), \bar{u}(t), \bar{v}(t)), \qquad i = 1, \ldots, n,
\end{aligned} \tag{4.14}$$

where the C_0^∞ functions $\bar{\Phi}_i : \mathbb{R}^{2n+2} \to \mathbb{R}$ satisfy

$$\|\bar{\Phi}_i\|_{C_b^{k-1}(\mathbb{R}^{2n-2},\mathbb{R})} \leqslant C_k \varepsilon \tag{4.15}$$

and the constant C_k is independent of ε. To finish the proof of the lemma, it remains to express the variables $z, w, \bar{v}_i$ in terms of U, $\partial_t U$, $\bar{u}_i$ and $\partial_t \bar{u}_i$ from the system

$$\begin{aligned}
&z + w = U,\\
&z - w = \partial_t U - \tilde{\Phi}_0(z, w, \bar{u}, \bar{v}) - \tilde{\Phi}_{2n+1}(z, w, \bar{u}, \bar{v}),\\
&\bar{v}_i = (\omega_i^0)^{-1}\left(\partial_t \bar{u}_i + \frac{\gamma}{2}\bar{u}_i\right) - (\omega_i^0)^{-1}\tilde{\Phi}_{2i}(z, w, \bar{u}, \bar{v}), \qquad i = 1, \ldots, n.
\end{aligned} \tag{4.16}$$

Note that in the case where all $\tilde{\Phi}_i$ are equal to zero identically, system (4.16) reduces to a non-degenerate linear system. Hence, due to (4.12), the system of equations (4.16) can indeed be solved in a unique way (by virtue of the implicit function theorem) if ε is small enough:

$$\begin{aligned} z &= \Theta_0(U, \partial_t U, \bar{u}, \partial_t \bar{u}), \\ \bar{v}_i &= \Theta_i(U, \partial_t U, \bar{u}, \partial_t \bar{u}), \qquad i = 1, \dots, n, \\ w &= \Theta_{n+1}(U, \partial_t U, \bar{u}, \partial_t \bar{u}) \end{aligned} \tag{4.17}$$

for some C^∞-functions Θ_i, $i = 0, \dots, n$. Inserting (4.17) into the right-hand side of (4.14) finishes the proof of lemma 4.1.

Let us now finish the proof of theorem 4.1. Indeed, let A be a self-adjoint positive operator in a Hilbert space H with a compact inverse, such that its eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ satisfy condition (2.1) for a certain positive constant κ. Let $\{e_i\}_{i=1}^\infty$ be the corresponding orthonormal system of eigenvectors. Then, every H-valued function $u(t)$, $t \in \mathbb{R}_+$, can be expanded as follows:

$$u(t) := \sum_{i=1}^{\infty} u_i(t) e_i, \qquad u_i(t) := (u(t), e_i). \tag{4.18}$$

Moreover, due to (1.2),

$$\|u(t)\|^2_{\mathrm{H}^s} := \sum_{i=1}^{\infty} \lambda_i^s |u_i(t)|^2, \qquad s \in \mathbb{R}. \tag{4.19}$$

We rewrite now equation (4.3) in the following equivalent form:

$$\partial_t^2 u_i(t) + \gamma \partial_t u_i(t) + \lambda_i u_i(t) = \bar{\Phi}_i(u(t), \partial_t u(t)), \qquad i = 1, 2, \dots, \tag{4.20}$$

where $u(t) = (u_1(t), u_2(t), \dots) \in \mathbb{R}^\infty$ (see (4.18)) and $\{\bar{\Phi}_i(u, \partial_t u)\}_{i=1}^\infty$ are defined as

$$\bar{\Phi}_i(u, \partial_t u) := (\mathbb{F}_1(u) + \gamma \mathbb{F}_2(u) + \Phi(u, \partial_t u), e_i). \tag{4.21}$$

We will construct the desired equation (4.3) in the form (4.20). The main idea is to construct the nonlinearities in such a way that the components $u_i(t)$ of the corresponding solution will satisfy system (4.9). Then, by lemma 4.1, this equation will possess a periodic orbit P such that $N^+(P) = 2n$, $n + 1 := [1/(2\gamma)]$ and, consequently, the fractal dimension of its attractor will be larger than $(2\gamma)^{-1}$. Indeed, let $\gamma > 0$, $\varepsilon > 0$ and let $k \in \mathbb{N}$ be arbitrary. Let us also fix $n := [1/(2\gamma)] - 1$ as in lemma 4.1. We need to rewrite system (4.9) constructed in lemma 4.1 in the form (4.20). To this end, we fix the frequencies $\omega_i^2 = \lambda_{i+2}$, $i = 1, \dots, n$, where λ_i are the eigenvalues of A, and introduce the variables

$$u_1(t) := U(t), \qquad u_2(t) := \partial_t U(t), \qquad u_3(t) := \bar{u}_1(t), \dots, \qquad u_{n+2}(t) := \bar{u}_n(t). \tag{4.22}$$

In these variables, system (4.9) is written as follows:

$$\begin{aligned} &\partial_t^2 u_1 + \gamma \partial_t u_1 + \lambda_1 u_1 = \{u_1 - F_0(u_1) + \lambda_1 u_1\} \\ &\qquad\qquad + \gamma\{u_2\} + \Phi_1(u_1, \dots, u_{n+2}, \partial_t u_1, \dots, \partial_t u_{n+2}), \\ &\partial_t^2 u_2 + \gamma \partial_t u_2 + \lambda_2 u_2 = \{u_2 - F_0'(u_1) u_2 + \lambda_2 u_2\} \\ &\qquad\qquad + \gamma\{u_1 - F_0(u_1)\} + \Phi_2(u_1, \dots, u_{n+2}, \partial_t u_1, \dots, \partial_t u_{n+2}), \\ &\partial_t^2 u_3 + \gamma \partial_t u_3 + \lambda_3 u_3 = \Phi_3(u_1, \dots, u_{n+2}, \partial_t u_1, \dots, \partial_t u_{n+2}), \\ &\vdots \\ &\partial_t^2 u_{n+2} + \gamma \partial_t u_{n+2} + \lambda_{n+2} u_{n+2} = \Phi_{n+2}(u_1, \dots, u_{n+2}, \partial_t u_1, \dots, \partial_t u_{n+2}), \end{aligned} \tag{4.23}$$

where the C_0^∞-functions Φ_i satisfy

$$\|\Phi_i\|_{C^{k-2}(\mathbb{R}^{n+2},\mathbb{R})} \leqslant C'_k\varepsilon. \tag{4.24}$$

Since every solution of (4.9) is, obviously, a solution of (4.23) as well, system (4.23) also has a periodic orbit P with $N^+(P) = 2n$.

We complete system (4.23) as follows:

$$\partial_t^2 u_i + \gamma\partial_t u_i + \lambda_i u_i = 0, \qquad i = n+3, n+4, \ldots. \tag{4.25}$$

Then, system (4.23) and (4.25) has the form (4.20) indeed. Moreover, since only the existence of a periodic orbit P with $N^+(P) = 2n$ is important for our purposes, we may cut off all the nonlinearities outside this orbit and define finally

$$\mathbb{F}_1(u) := \phi_0 \cdot \begin{pmatrix} u_1 - F_0(u_1) + \lambda_1 u_1 \\ u_2 - F'_0(u_1)u_2 + \lambda_2 u_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad \mathbb{F}_2(u) := \phi_0 \cdot \begin{pmatrix} u_2 \\ u_1 - F_0(u_1) \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{4.26}$$

and

$$\Phi(u, \partial_t u) := \begin{pmatrix} \Phi_1(u_1, \ldots, u_{n+2}, \partial_t u_1, \ldots, \partial_t u_{n+2}) \\ \vdots \\ \Phi_{n+2}(u_1, \ldots, u_{n+2}, \partial_t u_1, \ldots, \partial_t u_{n+2}) \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{4.27}$$

where $\phi_0 := \phi_0(|u_1|^2 + |u_2|^2)$ is an appropriate cut-off function.

Thus, the desired operators $\mathbb{F}_1$, $\mathbb{F}_2$ and Φ are defined. Let us now verify that they satisfy all conditions of theorem 4.1. Indeed, (4.1) is obvious. Since the operator Φ has a finite rank (see (4.27)), then, obviously, $\Phi \in \mathbb{S}$. Moreover, it follows from (2.1), (4.19) and (4.24) that, for every $m \in \mathbb{N}$

$$\|\Phi\|_{C^{k-2}(\mathrm{E}^{-m},\mathrm{H}^m)} \leqslant Cn^{4\kappa m}\varepsilon. \tag{4.28}$$

Hence, by rescaling, if necessary, ε, we may always satisfy (4.2) (for every fixed m). Furthermore, due to our construction, system (4.23) and (4.25) possesses a saddle periodic orbit P with $N^+(P) = 2n$. Therefore,

$$\dim_{\mathrm{F}}(\mathcal{A}, \mathrm{E}) \geqslant 2n \geqslant \frac{1}{2\gamma}. \tag{4.29}$$

The upper bounds for the fractal dimension in (4.4) is an immediate corollary of proposition 2.1 and corollary 2.2. Theorem 4.1 is proven.

Remark 4.1. We emphasize that the operators $\mathbb{F}_1$ and $\mathbb{F}_2$ from theorem 4.1 have a very simple structure (see (4.26)) and can be computed explicitly. It is also worth to emphasize that the unperturbed system (4.3)

$$\partial_t^2 u + \gamma\partial_t u + \mathrm{A}u = \mathbb{F}_1(u) + \gamma\mathbb{F}_2(u) \tag{4.30}$$

possesses a four-dimensional inertial manifold

$$M := \{(u_1, u_2, \partial_t u_1, \partial_t u_2) \in \mathbb{R}^4, (u_i, \partial_t u_i) = 0, i = 3, 4, \ldots\}$$

and, consequently, the fractal dimension of its attractor $\mathcal{A}_0$ satisfies

$$\dim_F(\mathcal{A}_0, E) \leqslant 4, \qquad \text{for every } \gamma > 0, \tag{4.31}$$

whereas its Lyapunov dimension, obviously, satisfies

$$\dim_L(\mathcal{A}_0, E) \sim \gamma^{-1}. \tag{4.32}$$

Theorem 4.1 shows, however, that we may drastically increase the fractal dimension of the attractor $\mathcal{A}$ by an arbitrarily small perturbation of equation (4.30), and achieve

$$\dim_F(\mathcal{A}, E) \sim \dim_L(\mathcal{A}, E) \sim \gamma^{-1}. \tag{4.33}$$

This example confirms that the Lyapunov dimension is a more robust qualitative characteristic of the global attractor than its fractal dimension.

Remark 4.2. We have constructed in theorem 4.1 the examples of attractors $\mathcal{A}$ of equations of the form (1.1) which depend explicitly on the first derivative $\partial_t u$ of the unknown function u. Differentiating, however, equation (4.3) by t and denoting $v = \partial_t u$ and $w(t) := (u(t), v(t)) \in \tilde{H} := H \times H$, we obtain the equation of the form

$$\partial_t^2 w + \gamma \partial_t w + \tilde{A} w = \tilde{\mathbb{F}}_1(w) + \gamma \tilde{\mathbb{F}}_2(w) + \tilde{\Phi}(w), \tag{4.34}$$

where the nonlinearities are already independent of $\partial_t w$. Therefore, the phenomena described in theorem 4.1 can appear in hyperbolic equations of the form (1.1) where the nonlinearity F is independent of $\partial_t u$ ($F(u, \partial_t u) \equiv F(u)$). Unfortunately, this reduction leads to linear operators $\tilde{A}$ in a special form

$$\tilde{A} := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}. \tag{4.35}$$

In order to avoid this restriction, we permit the explicit dependence of the nonlinearity F on $\partial_t u$ in our abstract model (1.1).

We recall that the usual way of obtaining lower bounds for an attractor's fractal dimension is to estimate the instability index for some equilibrium of the equation under consideration, see [1,2] and references therein (see also [27], where lower estimates for the instability index of a linear non-autonomous equation of type (1.1) with periodic coefficients were given based on the parametric resonance phenomena). In our next proposition, we show that it is, in principle, impossible, using this method, to obtain reasonable lower bounds for the attractor's dimension of equation (1.1) with nonlinearities belonging to $\mathbb{S}$.

Proposition 4.1. *Let* A *be a linear self-adjoint operator with a compact inverse in a Hilbert space* H *whose eigenvalues satisfy (2.1) and let the nonlinearity F in equation (1.1) belong to the class $\mathbb{S}$. Then, for every $\varepsilon > 0$, there exists a positive constant C_ε such that the instability index of any equilibrium u_0 of equation (1.1) is estimated as follows:*

$$N^+(u_0) \leqslant C_\varepsilon \gamma^{-\varepsilon}. \tag{4.36}$$

Proof. Indeed, due to the trick described in remark 4.2, it is sufficient to prove estimate (4.36) only for the case where

$$F(u, \partial_t u) \equiv F(u). \tag{4.37}$$

Let now u_0 be an arbitrary equilibrium of equation (1.1). Then, the corresponding equation of variations reads

$$\partial_t^2 w + \gamma \partial_t w + A_{u_0} w = 0, \qquad A_{u_0} := A - F'(u_0) \tag{4.38}$$

and, consequently, the spectrum of the linearization $D_{u_0}S_t$ of the semigroup (1.7) at the point $(u_0, 0)$ can be expressed as follows:

$$\sigma(D_{u_0}S_t) = \left\{ \mathrm{e}^{t\mu_\pm^k},\ \mu_\pm^k := -\frac{\gamma}{2} \pm \left(\frac{\gamma^2}{4} - \theta_k\right)^{1/2} \right\}, \tag{4.39}$$

where $\{\theta_k\}_{k=1}^\infty \in \mathbb{C}$ are the eigenvalues of the operator A_{u_0}.

We recall now that the operator F belongs to the class $\mathbb{S}$. Therefore,

$$\|F'(u_0)\|_{\mathcal{L}(\mathrm{H}^{-m},\mathrm{H}^m)} \leqslant C_m \tag{4.40}$$

for every $m \in \mathbb{N}$, where the constant C_m is independent of $u_0 \in \mathrm{H}$. Thus, due to the classical theory of compact perturbations of self-adjoint operators (see, e.g., [28]), we derive from (4.40) that for every $N \in \mathbb{N}$, there exists a constant C_N such that

$$|\theta_k - \lambda_k| \leqslant C_N k^{-N}, \qquad k \in \mathbb{N},$$

where λ_k are the corresponding eigenvalues of the unperturbed operator A. It follows that all but finitely many θ_k have positive real parts, and their imaginary parts tend to zero faster than any power of k as $k \to +\infty$. Hence, according to formula (4.39), the number of eigenvalues $\mu_\pm^k$ with non-negative real parts satisfies (4.36) indeed. Proposition 4.1 is proven.

Remark 4.3. We stress that our 'homoclinic' method of obtaining lower estimates for the attractor's dimension gives, in fact, more than an estimate from below of the maximal attractor. Indeed, in absolutely the same way as theorem 4.1 was deduced from corollary 3.1 to theorem 3.1, we obtain from corollary 3.2 that the nonlinearities $\mathbb{F}_{1,2}$ and Φ can be constructed in such a way that equation (4.3) would have an invariant torus of dimension $\sim C/\gamma$, where C is a certain constant, densely filled by quasi-periodic trajectories (at $\gamma = 0$ this system may be integrable, nevertheless all its invariant tori are killed at non-zero γ due to the way we introduce the dissipation in our examples; hence, the invariant torus which we obtain is of a different nature).

In other words, we show that equations under consideration may have minimal sets whose dimension is of the same order as the Lyapunov dimension of the maximal attractor. Recall also that, according to [29], any quasi-periodic flow on a smooth $(m+1)$-dimensional invariant torus can be perturbed in such a way that the torus would contain an invariant m-dimensional manifold, homeomorphic to $D^{m-1} \times S^1$, the flow on which is smoothly conjugate to a suspension over any aforehand given diffeomorphism of D^{m-1}. Hence, by corollary 3.2, every dynamics which is possible in a phase space of dimension $\sim C/\gamma$ can be encountered in equation (4.3), after an appropriate choice of the nonlinearities.

Acknowledgments

This research was partially supported by INTAS project no 00-899 and CRDF grant no 10545. We are also grateful to Messoud Effendiev who introduced us to each other.

Appendix A. Proof of theorem 1.1 and lemma 2.1

In this appendix, we prove the existence of a solution for problem (1.1) under the assumptions of theorem 1.1. We also prove the Lipschitz property of the corresponding semigroup S_t, as well as the quasi-differentiability of S_t on the attractor $\mathcal{A}$ (lemma 2.1). As usual (see [1], [3] chapter 4 and [4]), the proof is done via the Galerkin approximation method, based on the *a priori* estimates (1.6), (1.8) and (1.9).

We start with the proof of the *a priori* estimate (1.6). Let $\xi \in C(\mathbb{R}_+, \mathrm{E})$ be a solution of (1.1). Then, according to assumption (1.4), the nonlinear term $F(u, \partial_t u)$ belongs to the space $C(\mathbb{R}_+, \mathrm{H})$ and is globally bounded in it. Consequently, we may take a scalar product of equation (1.1) with $\partial_t u(t) + \alpha u(t)$, where $\alpha > 0$ is a sufficiently small number, and derive the following relation (see, e.g., [3], lemma II.4.1):

$$\tfrac{1}{2}\partial_t[\|\partial_t u(t)\|^2_{\mathrm{H}} + \|u(t)\|^2_{\mathrm{H}^1} + 2\alpha(u(t), \partial_t u(t))] + (\gamma - \alpha)\|\partial_t u(t)\|^2_{\mathrm{H}} + \alpha\|u(t)\|^2_{\mathrm{H}}$$
$$= (F, u + \alpha\partial_t u) \leqslant C_\varepsilon + \varepsilon(\|\partial_t u(t)\|^2_{\mathrm{H}} + \|u(t)\|^2_{\mathrm{H}^1}),$$

where $\varepsilon > 0$ can be arbitrarily small. Fixing now $\varepsilon \ll 1$ and $\alpha \ll 1$ in the last inequality and applying Gronwall's inequality, we obtain (1.6) indeed.

Let us now verify estimate (1.8). Let $\xi_{u_1}(t)$ and $\xi_{u_2}(t)$ be two solutions of (1.1) and let $\eta(t) := [v(t), \partial_t v(t)] := \xi_1(t) - \xi_2(t)$. Then $v(t)$ satisfies the following equation:

$$\partial_t^2 v + \gamma\partial_t v + \mathrm{A}v = L_1(t)v + L_2(t)\partial_t v, \qquad \eta|_{t=0} = \xi_1(0) - \xi_2(0),$$

where

$$L_1(t) := \int_0^1 F'_u(su_1 + (1-s)u_2, s\partial_t u_1)\,\mathrm{d}s$$

and

$$L_2(t) := \int_0^1 F'_{\partial_t u}(su_1 + (1-s)u_2, s\partial_t u_1)\,\mathrm{d}s.$$

We note that, according to (1.4), these operators are globally bounded as operators from $\mathrm{H}^1 \to \mathrm{H}$ and $\mathrm{H} \to \mathrm{H}$, respectively. Consequently, the right-hand side of the equation for v belongs to the space $C(\mathbb{R}_+, \mathrm{H})$ and we may take a scalar product of this equation with $\partial_t v$ and derive the following estimate:

$$\tfrac{1}{2}[\|\partial_t v(t)\|^2_{\mathrm{H}} + \|v(t)\|^2_{\mathrm{H}^1}] + \gamma\|\partial_t v(t)\|^2_{\mathrm{H}}$$
$$= (L_1(t)v(t), \partial_t v(t)) + (L_2(t)\partial_t v(t), \partial_t v(t)) \leqslant K(\|v(t)\|^2_{\mathrm{H}^1} + \|\partial_t v(t)\|^2_{\mathrm{H}}),$$

where K is independent of u_1 and u_2 (since the derivatives F_u and $F'_{\partial_t u}$ are assumed to be globally bounded). Applying Gronwall's inequality to this relation, we obtain (1.8).

In order to prove the *a priori* estimate (1.9), let us differentiate equation (1.1) with respect to t and introduce the function $z(t) := \partial_t u(t)$. Then, this function satisfies the following equation:

$$\begin{aligned}&\partial_t^2 z + \gamma\partial_t z + \mathrm{A}z = F'_u(u, \partial_t u)z + F'_{\partial_t u}(u, \partial_t u)\partial_t z,\\ &z|_{t=0} = u'_0, \qquad \partial_t z|_{t=0} = \partial_t^2 u(0) = -\mathrm{A}u_0 - \gamma u'_0 + F(u_0, u'_0).\end{aligned} \tag{A.1}$$

Denote $\eta(t) := [z(t), \partial_t z(t)]$. Since $\xi(0) := [u(0), \partial_t u(0)] \in \mathrm{E}^1$ and the nonlinearity F satisfies (1.4), it follows that $\eta(0) \in \mathrm{E}$ and

$$\|\eta(0)\|^2_{\mathrm{E}} \leqslant C(\|\xi(0)\|^2_{\mathrm{E}^1} + 1) \tag{A.2}$$

for an appropriate constant C depending on F and γ. Moreover, it follows from estimate (1.6) and from equation (1.1) that

$$\|\eta(t)\|^2_{\mathrm{E}^{-1}} \leqslant C'\|\xi(0)\|^2_{\mathrm{E}}\mathrm{e}^{-\gamma t} + C'_1, \tag{A.3}$$

where C' and C'_1 depend only on A, F and γ.

Taking the inner product in H of equation (A.1) with the function $\partial_t z(t) + (\gamma/2)z(t)$, we have

$$\begin{aligned}&\partial_t\{\|\partial_t z\|^2_{\mathrm{H}} + \|z\|^2_{\mathrm{H}^1} + \gamma\,(z(t), \partial_t z(t))\} + \gamma\|\partial_t z\|^2_{\mathrm{H}} + \gamma\|z\|^2_{\mathrm{H}^1}\\ &\quad = 2(F'_u(u, \partial_t u)z, \partial_t z) + 2(F'_{\partial_t u}(u, \partial_t u)\partial_t z, \partial_t z)\\ &\qquad + \gamma(F'_u(u, \partial_t u)z, z) + \gamma(F'_{\partial_t u}(u, \partial_t u)\partial_t z, z).\end{aligned} \tag{A.4}$$

Using conditions (1.5), we derive that

$$2(F'_u(u,\partial_t u)z,\partial_t z)+2(F'_{\partial_t u}(u,\partial_t u)\partial_t z,\partial_t z)\\ \leqslant \frac{\gamma}{2}(\|\partial_t z(t)\|^2_{\mathrm{H}}+\|z(t)\|^2_{\mathrm{H}^1})+4C_\gamma\|[z(t),\partial_t z(t)]\|^2_{\mathrm{E}^{-1}}. \tag{A.5}$$

Analogously, using assumption (1.4) on the derivatives of F, we estimate

$$(F'_u(u,\partial_t u)z,z)+(F'_{\partial_t u}(u,\partial_t u)\partial_t z,z)\leqslant \tfrac{1}{8}(\|\partial_t z(t)\|^2_{\mathrm{H}}+\|z(t)\|^2_{\mathrm{H}^1})+C'\|[z(t),\partial_t z(t)]\|^2_{\mathrm{E}^{-1}}. \tag{A.6}$$

Inserting estimates (A.5) and (A.6) into the right-hand side of (A.4), we obtain, for a sufficiently small $\gamma > 0$,

$$\partial_t\{\|\partial_t z\|^2_{\mathrm{H}}+\|z\|^2_{\mathrm{H}^1}+\gamma(z(t),\partial_t z(t))\}+\frac{\gamma}{8}\{\|\partial_t z\|^2_{\mathrm{H}}+\|z\|^2_{\mathrm{H}^1}+\gamma(z(t),\partial_t z(t))\}\leqslant C''_\gamma\|\eta(t)\|^2_{\mathrm{E}^{-1}}$$

for an appropriate constant C''_γ which depends only on γ, A and F. Applying here Gronwall's inequality, and using (A.2) and (A.3), we arrive at the estimate

$$\|\eta(t)\|^2_{\mathrm{E}}\leqslant C_4 e^{-\gamma t/8}\|\xi(0)\|^2_{\mathrm{E}^1}+C_4. \tag{A.7}$$

It remains to note that due to equation (1.1) and condition (1.4) we have

$$\|\xi(t)\|^2_{\mathrm{E}^1}\leqslant C(\|\eta(t)\|^2_{\mathrm{E}}+1),$$

so (1.9) follows from (A.7).

Now we can prove the existence of the solutions of (1.1). Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis in H generated by the eigenvectors of the self-adjoint operator A and let Π_N be an orthoprojector to the first N eigenvectors in H, $\mathrm{H}_N := \Pi_N\mathrm{H}$ and $\mathrm{E}_N := \Pi_N\mathrm{E}$. For every $N\in\mathbb{N}$, we consider the following problem in the finite-dimensional space E_N, which approximates the initial infinite-dimensional problem (1.1):

$$\partial_t^2 u_N+\gamma\partial_t u_N+\mathrm{A}u_N=\Pi_N F(u_N,\partial_t u_N),\\ \xi_N(t):=[u_N(t),\partial_t u_N(t)]\in\mathrm{E}_N,\qquad \xi_N(0)=\xi_N^0. \tag{A.8}$$

We recall that $(\Pi_N F, v_N)=(F,v_N)$ for every $v_N\in\mathrm{E}_N$ and, consequently, repeating word by word the derivation of estimates (1.6) and (1.9), we obtain the following uniform (with respect to N) *a priori* estimates for the solutions of (A.8):

$$1.\ \|\xi_N(t)\|^2_{\mathrm{E}}\leqslant C\|\xi_N(0)\|_{\mathrm{E}}e^{-\gamma t}+C_1,\\ 2.\ \|\xi_N(t)\|^2_{\mathrm{E}^1}\leqslant C_1\|\xi_N(0)\|_{\mathrm{E}^1}e^{-\gamma/8}t+C_3, \tag{A.9}$$

where the constants C, C_1, C_2 and C_3 are the same as in (1.6) and (1.9) (and, in particular, they are independent of N). On the other hand, equation (A.8) is a system of ODEs with smooth (C^1) nonlinearity and, consequently, estimates (A.9) give the global existence of a solution $\xi_N(t)\in\mathrm{E}_N$ of problem (A.8). Our task now is to pass to the limit $N\to\infty$ in (A.8) and construct the solution $u(t)$ of (1.1) as a limit of Galerkin solutions $u_N(t)$. To this end, we first assume that the initial conditions $\xi(0)$ for problem (1.1) belong to E^1:

$$\xi(0)\in\mathrm{E}^1\qquad\text{and set}\qquad \xi_N^0:=\Pi_N\xi(0). \tag{A.10}$$

Then, according to (A.9)(2) and equation (A.8), we have the uniform with respect to N estimate

$$\|\partial_t^2 u_N\|_{L^\infty([0,T],\mathrm{H})}+\|\xi_N\|_{L^\infty([0,T],\mathrm{E}^1)}\leqslant C(\|\xi(0)\|_{\mathrm{E}^1}), \tag{A.11}$$

which is valid for every $T > 0$. Consequently, without loss of generality, we may assume that, for every $T > 0$,

$$\xi_N\to\xi\qquad *\text{—weakly in } L^\infty([0,T],\mathrm{E}^1),\\ \partial_t^2 u_N\to\partial_t^2 u\qquad *\text{—weakly in } L^\infty([0,T],\mathrm{H}) \tag{A.12}$$

for some function $\xi := [u, \partial_t u] \in L^\infty(\mathbb{R}_+, \mathrm{E}^1)$. We claim that u solves the initial problem (1.1). To this end, we need to pass to the limit $N \to \infty$ in equations (A.8). Indeed, passing to the limit $N \to \infty$ in the linear terms of (A.8) is evident due to (A.12). In order to pass to the limit in the nonlinear term, we recall that the embedding

$$L^\infty([0, T], \mathrm{E}) \cap \{\partial_t^2 u \in L^\infty([0, T], \mathrm{H})\} \subset C([0, T], \mathrm{E})$$

is compact, for every $T > 0$ (see, e.g., [3], chapter 4) and, consequently, (A.12) implies the strong convergence $\xi_N \to \xi$ in $C([0, T], \mathrm{E})$. Since the operator F is continuous, it follows that $\Pi_N F(u_N, \partial_t U_N) \to F(u, \partial_t u)$, and u is indeed a solution of problem (1.1).

Thus, for every $\xi(0) \in \mathrm{E}^1$, we have constructed a solution $\xi \in L^\infty(\mathbb{R}_+, E^1) \cap C(\mathbb{R}_+, \mathrm{E})$ of problem (1.1) (moreover, arguing in a standard way as, e.g., in [3], chapter 4, one may verify that $\xi \in C(\mathbb{R}_+, \mathrm{E}^1)$, in fact). It is not difficult now to extend this result to the initial data from E. Indeed, let $\xi(0) \in \mathrm{E}$ be an arbitrary initial condition. Let us consider a sequence $\xi^n(0) \in \mathrm{E}^1$ such that

$$\xi^n(0) \to \xi(0) \qquad \text{as } n \to \infty. \tag{A.13}$$

Let also $\xi^n(t) \in \mathrm{E}^1$ be the corresponding solutions of problem (1.1), the existence of which has just been proven. Then, according to estimate (1.8), there exists some function $\xi \in C(\mathbb{R}_+, \mathrm{E})$ such that

$$\xi^n \to \xi \qquad \text{in } C([0, T], \mathrm{E}) \text{ strongly} \tag{A.14}$$

for every $T > 0$. As above, the *strong* convergence law (A.14) allows us to pass to the limit $n \to \infty$ in the equations for u^n and verify that the limit function $u(t)$ also satisfies equation (1.1). Thus, the existence of a solution of problem (1.1) is now proven under the assumptions of theorem 1.1.

It remains to prove the quasi-differentiability of the corresponding semigroup S_t on the attractor $\mathcal{A}$. Let $\xi_1(t)$ and $\xi_2(t)$ be two solutions of problem (1.1) belonging to $\mathcal{A}$ and let $v(t)$ be a solution of equation of variations (2.7) (computed along the trajectory $\xi_1(t)$) with $[v(0), \partial_t v(0)] = \xi_1(0) - \xi_2(0)$. Then, arguing as in the derivation of estimate (1.8), we obtain the following estimate:

$$\|[v(t), \partial_t v(t)]\|_{\mathrm{E}}^2 \leqslant C\|[v(0), \partial_t v(0)]\|_{\mathrm{E}}^2 \mathrm{e}^{Kt}, \tag{A.15}$$

where the constants C and K are independent of u_1 and u_2. Moreover, the function $w(t) := u_1(t) - u_2(t) - v(t)$ obviously satisfies the following linear non-homogeneous equation

$$\begin{aligned} &\partial_t^2 w + \gamma\partial_t w + \mathrm{A}w - F_u'(u_1, \partial_t u_1)w - F_{\partial_t u}'(u_1, \partial_t u_1)\partial_t w = h_{u_1,u_2}(t), \\ &[w, \partial_t w]|_{t=0} = 0, \end{aligned} \tag{A.16}$$

where

$$\begin{aligned} h_{u_1,u_2}(t) := &\left\{\int_0^1 (F_u'(su_1 + (1-s)u_2, s\partial_t u_1) - F_u'(u_1, \partial_t u_1)\,\mathrm{d}s\right\} v(t) \\ &+ \left\{\int_0^1 (F_{\partial_t u}'(su_1 + (1-s)u_2, s\partial_t u_1) - F_u'(u_1, \partial_t u_1)\,\mathrm{d}s\right\} \partial_t v(t). \end{aligned} \tag{A.17}$$

Consequently, analogously to (1.8) and (A.15), we have

$$\|[w(T), \partial_t w(T)]\|_{\mathrm{E}} \leqslant C\mathrm{e}^{KT} \int_0^T \|h_{u_1,u_2}(t)\|_{\mathrm{H}}^2\,\mathrm{d}t, \tag{A.18}$$

where the constants C and K are independent of u_1 and u_2. On the other hand, since the attractor $\mathcal{A}$ is compact in E, the set

$$\mathcal{A}_{1,2} := \{s\xi_1 + (1-s)\xi_2, \quad \xi_1, \xi_2 \in \mathcal{A}, \quad s \in [0, 1]\}$$

is also compact in E and, due to assumption (1.4), the derivatives F'_u and $F'_{\partial_t u}$ are uniformly continuous on this set. Consequently, we have an estimate

$$\|h_{u_1,u_2}(t)\|_{\mathrm{H}} \leqslant \alpha(\|\xi_1(t) - \xi_2(t)\|_{\mathrm{E}})\|[v(t), \partial_t v(t)]\|_{\mathrm{E}}, \tag{A.19}$$

where the function $\alpha(z)$ tends to zero as $z \to 0^+$ and is independent of t, u_1 and u_2. Estimates (1.8), (A.15), (A.18) and (A.19) imply that

$$\|[w(T), \partial_t w(T)]\|^2_{\mathrm{E}} \leqslant \alpha_T(\|\xi_1(0) - \xi_2(0)\|_{\mathrm{E}})\|\xi_1(0) - \xi_2(0)\|_{\mathrm{E}}, \tag{A.20}$$

where the function $\alpha_T(z)$ tends to zero as $z \to 0^+$, it depends on $T \geqslant 0$ but it is independent of $\xi_1, \xi_2 \in \mathcal{A}$. Thus, estimate (2.2) is verified. The continuity of $S'(\xi)$ on the attractor is verified analogously. End of the proof.

Appendix B. Proof of corollary 3.2

Let P be a periodic orbit, born by a small perturbation of a homoclinic loop of system (3.1), which has $2m$ multipliers equal to 1 in absolute value (such an orbit can indeed be born according to theorem 3.1). Let us prove that a smooth $(m + 1)$-dimensional invariant torus, filled by quasi-periodic trajectories each of which is dense in the torus, can be born at the bifurcations of P.

Consider a Poincaré map $(x \mapsto \bar{x})$ for the periodic orbit P:

$$\bar{x} = Bx + \mathrm{o}(x), \tag{B.1}$$

here $x = 0$ is the fixed point which corresponds to the orbit P. The eigenvalues of the matrix B are the multipliers of P. By assumption, $2m$ of these eigenvalues lie on the unit circle. Our goal is to prove that this map can be perturbed in such a way that an m-dimensional invariant torus would appear in a small neighbourhood of the origin.

Fix any integer $r \geqslant 1$. It is obvious that, by perturbations, small in C^r-topology, one can arrange arbitrary small changes in any entries of the matrix B in (B.1). Hence, we can always achieve that B would have exactly m pairs of complex-conjugate eigenvalues on the unit circle: $\mathrm{e}^{\pm\mathrm{i}\omega_1}, \ldots, \mathrm{e}^{\pm\mathrm{i}\omega_m}$, all ω are rationally independent and the factors ω_j/π are irrational; and the rest of the multipliers does not lie on the unit circle.

By the centre manifold theorem, our map has a $2m$-dimensional smooth invariant manifold which is tangent at $x = 0$ to the eigenspace of B corresponding to the multipliers on the unit circle. It is well-known that since the numbers $\{\pi, \omega_1, \ldots, \omega_m\}$ are rationally independent, there exist local coordinates $(z_1, \ldots, z_m) \in \mathbb{C}^m$ in which the map on the centre manifold takes the following (normal) form:

$$\bar{z}_j = Q_j(z_1 z_1^*, \ldots, z_m z_m^*)z + \mathrm{o}(z^r) \qquad (j = 1, \ldots, m), \tag{B.2}$$

where * means complex conjugation, and Q_j are complex polynomials of degree $\leqslant (r-1)/2$,

$$Q_j(0) = \mathrm{e}^{\mathrm{i}\omega_j}. \tag{B.3}$$

It is obvious that by an arbitrary small (in the C^r-topology) perturbation, one can make map (B.2) coincide with the polynomial map

$$\bar{z}_j = Q_j(z_1 z_1^*, \ldots, z_m z_m^*)z \qquad (j = 1, \ldots, m) \tag{B.4}$$

in a sufficiently small neighbourhood of zero. Thus, it is enough to prove that a small perturbation of map (B.4) can produce an m-dimensional invariant torus arbitrarily close to $z = 0$.

Let us, first, introduce polar coordinates $\rho_j \mathrm{e}^{\mathrm{i}\varphi_j} := z_j$, $j = 1, \ldots, m$. Map (B.4) can be recast as

$$\bar{\rho}_j = R_j(\rho_1^2, \ldots, \rho_m^2)\rho_j, \qquad \bar{\varphi}_j = \varphi_j + \Omega_j(\rho_1^2, \ldots, \rho_m^2) \qquad (j = 1, \ldots, m) \tag{B.5}$$

where $Q_j \equiv R_j e^{i\Omega_j}$ $(j = 1, \ldots, m)$, so R_j, Ω_j are real analytic functions, and

$$R_j(0) = 1, \qquad \Omega_j(0) = \omega_j \qquad (j = 1, \ldots, m). \tag{B.6}$$

Let $a_1, \ldots, a_m, \theta_1, \ldots, \theta_m$ be arbitrary small numbers. Then the map

$$\bar{\rho}_j = (a_j + R_j(\rho_1^2, \ldots, \rho_m^2))\rho_j, \qquad \bar{\varphi}_j = \varphi_j + \theta_j + \Omega_j(\rho_1^2, \ldots, \rho_m^2) \qquad (j = 1, \ldots, m) \tag{B.7}$$

is a small (real analytic) perturbation of (B.5). The amplitude map

$$\bar{\rho}_j = (a_j + R_j(\rho_1^2, \ldots, \rho_m^2))\rho_j \qquad (j = 1, \ldots, m) \tag{B.8}$$

is independent here of the phases $\varphi_1, \ldots, \varphi_m$. Therefore, a fixed point of (B.8) with all $\rho_1, \ldots, \rho_m$ non-zero corresponds to an m-dimensional invariant torus of (B.7).

Take now some sufficiently small strictly positive numbers $\rho_1^0, \ldots, \rho_m^0$. Put

$$a_j = 1 - R_j((\rho_1^0)^2, \ldots, (\rho_m^0)^2), \qquad \theta_j = \omega_j - \Omega_j((\rho_1^0)^2, \ldots, (\rho_m^0)^2) \qquad (j = 1, \ldots, m). \tag{B.9}$$

By (B.6), the numbers $a_1, \ldots, a_m, \theta_1, \ldots, \theta_m$ are indeed small when $\rho_1^0, \ldots, \rho_m^0$ are small. With $a_1, \ldots, a_m$ given by (B.9), $(\rho_1^0, \ldots, \rho_m^0)$ is a fixed point of (B.8), i.e. the torus $(\rho_1 = \rho_1^0, \ldots, \rho_m = \rho_m^0)$ is a smooth m-dimensional invariant torus of (B.7). By construction, map (B.7) is a C^r-small perturbation of the restriction of the original Poincaré map on the centre manifold, so we have indeed constructed the perturbation under which an invariant torus is born from our periodic orbit. By (B.6) and (B.7), the restriction of the Poincaré map on this torus is given by

$$\bar{\varphi}_j = \varphi_j + \omega_j \qquad (j = 1, \ldots, m).$$

Since the numbers $\pi, \omega_1, \ldots, \omega_m$ are rationally independent, it follows that every orbit of this map is indeed quasi-periodic and fills the torus densely. End of the proof.

References

[1] Babin A and Vishik M 1989 *Attractors of Evolutionary Equations* (Moscow: Nauka)

[2] Hale J 1987 *Asymptotic Behavior of Dissipative Systems* (*Math. Surveys and Monographs* vol 25) (Providence, RI: AMS)

[3] Temam R 1997 *Infinite Dimensional Dynamical Systems in Physics and Mechanics* 2nd edn (New York: Springer)

[4] Chepyzhov V and Vishik M 2002 *Attractors for Equations of Mathematical Physics* (Providence, RI: AMS)

[5] Kaplan L and Yorke J A 1979 *Functional Differential Equations and Approximation of Fixed Points* (*Lecture Notes in Mathematics* vol 730) (Berlin: Springer) pp 204–27

[6] Constantin P and Foias C 1985 Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractors for 2D Navier–Stokes equations *Commun. Pure Appl. Math.* **38** 1–27

[7] Douady A and Oesterlé J 1980 Dimension de Hausdorff des attracteurs *C. R. Acad. Sci. Paris* A **290** 1135–8

[8] Hunt B 1996 Maximum local Lyapunov dimension bounds the box dimension of chaotic attractors *Nonlinearity* **9** 845–52

[9] Blinchevskaya M and Ilyashenko Yu 1999 Estimate for the entropy dimension of the maximal attractor for k-contracting maps in an infinite dimensional space *Russ. J. Math. Phys.* **6** 20–6

[10] Chepyzhov V and Ilyin A 2001 A note on the fractal dimension of attractors of dissipative dynamical systems *Nonlin. Anal. Theory Methods Appl.* **44A** 811–9

[11] Sell G 1989 Hausdorff and Lyapunov dimension for gradient systems *Contemporary Math.* **99** 85–92

[12] Shilnikov L 1965 A case of the existence of a countable number of periodic motions *Sov. Math. Dokl.* **6** 163–6

[13] Shilnikov L 1970 A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type *Math. USSR, Sb.* **10** 91–102

[14] Afrajmovich V, Bykov V and Shilnikov L 1983 On structurally unstable attracting limit sets of Lorenz attractor type *Trans. Mosc. Math. Soc.* **1983** 153–216

[15] Shilnikov A, Shilnikov L and Turaev D 1993 Normal forms and Lorenz attractors *Bifurca. Chaos* **3** 1123–39

[16] Turaev D and Shilnikov L 1998 An example of a wild strange attractor *Sb. Math.* **189** 137–60
[17] Turaev D 1996 On dimension of non-local bifurcational problems *Bifurca. Chaos* **6** 123–56
[18] Li Y, McLaughlin D, Shatah J and Wiggins S 1996 Persistent homoclinic orbits for a perturbed nonlinear Schrödinger equation *Commun. Pure Appl. Math.* **49** 1175–255
[19] Li Y 1999 Smale horseshoes and symbolic dynamics in perturbed nonlinear Schrödinger equation *J. Nonlinear Sci.* **9** 363–415
[20] Shatah J and Zeng C 2000 Homoclinic orbits for the perturbed sine-Gordon equation *Commun. Pure Appl. Math.* **53** 283–99
[21] Shatah J and Zeng C 2000 Homoclinic orbits for a perturbed nonlinear Schrödinger equation *Commun. Pure Appl. Math.* **53** 1222–83
[22] Ghidaglia J and Temam R 1987 Attractors for damped nonlinear hyperbolic equations *J. Math. Pure Appl.* **66** 273–319
[23] Feireisl E 1995 Global attractors for semilinear damped wave equations with supercritical exponent *J. Diff. Eqns* **116** 431–47
[24] Triebel H 1978 *Interpolation Theory, Functional Spaces, Differential Operators* (Amsterdam: North-Holland)
[25] Shilnikov L, Shilnikov A, Turaev D and Chua L 1998 *Methods of Qualitative Theory in Nonlinear Dynamics* part I (Singapore: World Scientific)
[26] Sternberg S 1958 On the structure of local homeomorphisms of Euclidean n-space II *Am. J. Math.* **80** 623–31
[27] Zelik S 1997 The Mathieu–Hill operator equation with dissipation and estimates for its instability index *Math. Notes* **61** 451–64
[28] Ghohberg I and Krein M 1969 *Introduction to the Theory of Linear Non-selfadjoint Operators in Hilbert Space* (Providence, RI: AMS) (translated from Russian)
[29] Newhouse S, Ruelle D and Takens F 1978 Occurrence of strange Axiom A attractors near quasi-periodic flows on T^m, $m \geqslant 3$ *Commun. Math. Phys.* **64** 35–40