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Persistent heterodimensional cycles in periodic perturbations of Lorenz-like attractors

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Abstract

We prove that heterodimensional cycles can be created by unfolding a pair of homoclinic tangencies in a certain class of C^r-diffeomorphisms ($r = 3, \ldots, \infty, \omega$). This implies the existence of a C^2-open domain in the space of dynamical systems with a certain type of symmetry where systems with heterodimensional cycles are dense in C^r. In particular, we describe a class of three-dimensional flows with a Lorenz-like attractor such that an arbitrarily small time-periodic perturbation of any such flow can belong to this domain—in this case the corresponding heterodimensional cycles belong to a chain-transitive attractor of the perturbed flow.

Keywords: heterodimensional cycle, homoclinic bifurcation, homoclinic tangency, chaotic dynamics, Lorenz attractor

Mathematics Subject Classification numbers: 37G20, 37G25, 37G35

1. Introduction

1.1. Main results

A heterodimensional cycle is formed by intersections between invariant manifolds of hyperbolic periodic orbits of different indices (dimensions of unstable manifolds). By this definition, they only appear in dimension three or more for diffeomorphisms, or dimension four or higher if we consider systems of autonomous differential equations. Heterodimensional

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cycles in such dynamical systems create a basic mechanism that causes non-hyperbolicity and breaks structural stability. Early examples involving heterodimensional cycles were introduced by Abraham and Smale [1] and Shub [46]. Later on, a systematic study was carried out by Diaz and his collaborators in [6, 10–12]. In [7], Bonatti and Diaz built a comprehensive theory of C^1 diffeomorphisms having heterodimensional cycles of co-index one (i.e. when the difference between the indices is one). They also showed the C^1-robustness of heterodimensional cycles—a C^1-small perturbation of a system with a heterodimensional cycle can always be constructed such that the perturbed system gets into a C^1-open domain in the space of dynamical systems where systems with heterodimensional cycles are dense (in C^∞ or C^ω sense). A general higher smoothness version of this result is missing and a C^r theory (with $r > 1$) of perturbations of heterodimensional cycles is much less developed (see, however, [4, 5, 10–12, 26]).

The aim of this work is to provide more examples where heterodimensional cycles appear naturally in multidimensional systems. In particular, we show that heterodimensional cycles can be born out of a certain type of homoclinic tangencies (after a C^r-small perturbation, for an arbitrarily large r, including the case of perturbations small in the real-analytic sense). Since homoclinic tangencies persist in the so-called Newhouse domains (C^2-open regions in the space of dynamical systems where systems with homoclinic tangencies are C^r-dense for every $r \geqslant 2$ [20, 36]), this gives us the persistence of heterodimensional cycles in the corresponding type of the Newhouse domain.

We apply the result to periodically perturbed Lorenz-like systems, which is the main motivation of this work. We give a detailed discussion on this matter in section 1.2.

Denote by $\mathrm{Diff}^r(\mathcal{M})$ the space of C^r-diffeomorphisms on a D-dimensional manifold $\mathcal{M}$, where $r = 3, \ldots, \infty, \omega$ and $D \geqslant 3$. Let $F \in \mathrm{Diff}^r(\mathcal{M})$ satisfy the following conditions.

(C1) F has a saddle periodic point O with multipliers $\gamma, \lambda, \lambda_1, \ldots, \lambda_{D-2}$ such that λ and γ are real,

$$|\lambda_{D-2}| < \cdots < |\lambda_1| < |\lambda| < 1 < |\gamma| \tag{1}$$

and

$$|\lambda\gamma| > 1. \tag{2}$$

(C2) There exist two orbits Γ and $\tilde{\Gamma}$ of quadratic homoclinic tangency between the unstable and stable manifolds of O.

In order to formulate the next condition, recall some definitions. Denote by $W^{uE}(O)$ a two-dimensional invariant manifold tangent to the eigenspace corresponding to λ and γ—the unstable and weak stable multipliers of O, and call it the extended unstable manifold of O. This manifold is not unique, but it contains $W^u(O)$ and any two of these manifolds are tangent to each other at every point of $W^u(O)$. Recall also that for any diffeomorphism satisfying (C1) there is a unique strong-stable C^r-foliation $\mathcal{F}_0$ in the stable manifold $W^s(O)$ which includes, as a leaf, the strong-stable manifold $W^{ss}(O)$ (tangent at O to the eigenspace corresponding to the multipliers smaller than λ in the absolute value). Detailed discussion can be found in Chapter 13 of [45] or in [49].

Assume the diffeomorphism F satisfies the following non-degeneracy assumption.

(C3) The homoclinic orbits Γ and $\tilde{\Gamma}$ do not lie in $W^{ss}(O)$, and the manifold $W^{uE}(O)$ is transverse to the strong-stable foliation $\mathcal{F}_0$ at the points of Γ and $\tilde{\Gamma}$ (in particular, $W^{uE}(O)$ is transverse to the stable manifold $W^s(O)$ at the points of Γ and $\tilde{\Gamma}$).

Observe that if we add any C^2-small perturbation to F without destroying the homoclinic tangencies, the tangencies will remain quadratic and also condition (C3) will remain fulfilled.

Note that conditions (C1) and (C3) imply that the set consisting of the saddle O, and the two homoclinic orbits Γ and $\tilde{\Gamma}$ is partially hyperbolic. Therefore, the foliation $\mathcal{F}_0$ can be smoothly extended to a neighbourhood of $O \cup \Gamma \cup \tilde{\Gamma}$, see [49].

It should be noticed that a single homoclinic tangency is not enough for creating heterodimensional cycles in diffeomorphisms of the type considered in this paper, i.e. those having a saddle with real multipliers being closest to the imaginary axis. It is shown in [22] that periodic orbits of different indices can be obtained by unfolding a single orbit of homoclinic tangency. However, these points and O cannot form heterodimensional cycles since they all lie in a certain two-dimensional invariant manifold (see [49]) while heterodimensional cycles require at least 3-dimensional ambient space. Therefore, we must consider an interplay between two orbits of homoclinic tangency. This is similar to the results of [27, 28] where we obtained heterodimensional cycles by perturbations of a pair of homoclinic loops to a saddle-focus equilibrium state.

A way to make homoclinic tangencies come in pairs is to assume a symmetry in the system. Note that Lorenz-like systems that motivate this work do possess symmetry, so when such system has a homoclinic loop it also has a second one. When we add a periodic perturbation that keeps the symmetry, the pair of homoclinic loops can transform to a symmetric pair of homoclinic tangencies of the type we consider here.

The diffeomorphism F is $\mathbb{Z}_2$-symmetric if there exists a C^r-diffeomorphism $\mathcal{R}$ such that $\mathcal{R}^2 = id$ and $\mathcal{R} \circ F = F \circ \mathcal{R}$. In order to describe our assumptions on the involution $\mathcal{R}$, consider a small neighbourhood V of the point O. We assume that the point O is symmetric with respect to $\mathcal{R}$, so $\mathcal{R}O = O$. It is well-known that one can choose coordinates in V, with O at the origin, such that $\mathcal{R}$ will be linear in these coordinates (a nonlinear involution $v \mapsto \mathcal{R}(v)$ becomes linear: $v^{\text{new}} \mapsto \mathcal{R}_0 v^{\text{new}}$, after the coordinate transformation $v^{\text{new}} = (v + \mathcal{R}_0\mathcal{R}(v))/2$, where $\mathcal{R}_0$ is the derivative of $\mathcal{R}$ at zero). Choose such coordinates v. Let τ be the period of the point O. As the linear map $\mathcal{R}$ commutes with the derivative $\mathrm{D}F^\tau$ at O, the invariant subspaces of $\mathrm{D}F^\tau|_O$ are invariant with respect to $\mathcal{R}$ too. Denote $v = (x, y, z)$ where the x-, y-, and z- spaces are the eigenspaces of $\mathrm{D}F^\tau|_O$ corresponding to λ, γ, and the rest of the multipliers λ_i, respectively. As we mentioned, the x-, y- and z-spaces are invariant under $\mathcal{R}$. We assume that $\mathcal{R} : (x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z})$ in V acts in the following way:

$$\bar{x} = x, \quad \bar{y} = -y, \quad \bar{z} = \mathcal{S}z, \tag{3}$$

where $\mathcal{S}$ is a linear involution that changes the signs of some of z-coordinates.

Denote by $\mathrm{Diff}^r_s(\mathcal{M})$ the subspace of $\mathrm{Diff}^r(\mathcal{M})$ consisting of $\mathcal{R}$-symmetric diffeomorphisms. Maps that are close to F in $\mathrm{Diff}^r(\mathcal{M})$ (in particular, the maps that are close to F in $\mathrm{Diff}^r_s(\mathcal{M})$) have a saddle periodic point, a hyperbolic continuation of O, that continuously depends on the map; its stable and unstable manifolds also depend on the map continuously. Those of these maps that have orbits of homoclinic tangency close to Γ form a codimension-1 surface $\mathcal{H}$ in $\mathrm{Diff}^r(\mathcal{M})$. For the maps that belong to the surface $\mathcal{H} \cap \mathrm{Diff}^r_s(\mathcal{M})$ we also have a symmetric to Γ orbit $\tilde{\Gamma}$ of homoclinic tangency to O; conditions (C1)–(C3) are fulfilled for every map in this surface. One can define a functional μ in a neighbourhood of F in $\mathrm{Diff}^r(\mathcal{M})$ such that $d\mu(F_\varepsilon)/d\varepsilon \neq 0$ for any one-parameter family F_ε of maps in $\mathrm{Diff}^r(\mathcal{M})$, which is transverse to the surface $\mathcal{H}$, and $|\mu(F_\varepsilon)|$ measures the distance between the unstable and stable manifolds of O near a certain point of Γ. Thus, the surface $\mathcal{H}$ is given by the equation $\mu = 0$. Another functional we need is $\theta = -\ln|\lambda|/\ln|\gamma|$ (it is a modulus of topological conjugacy [33, 35] and is known to play an important role in bifurcations of homoclinic tangencies [13, 16–19, 21, 22]). We consider any two-parameter family $F_{\varepsilon_1,\varepsilon_2}$ of diffeomorphisms from

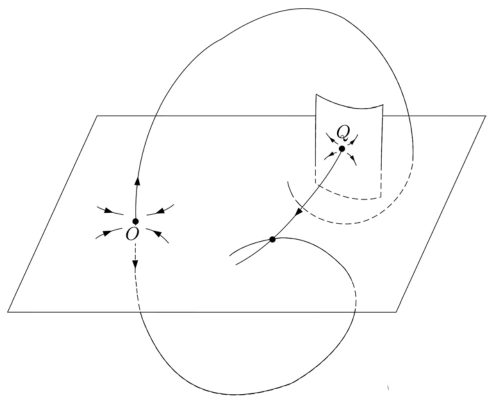

Figure 1. A heterodimensional cycle can be obtained by splitting the homoclinic tangencies while changing θ.

$\mathrm{Diff}^r_s(\mathcal{M})$ (so all diffeomorphisms in the family are symmetric) such that $F_{\varepsilon_1^*,\varepsilon_2^*}$ equals to the map F, and assume that

$$\det \frac{\partial(\mu(F_{\varepsilon_1,\varepsilon_2}),\theta(F_{\varepsilon_1,\varepsilon_2}))}{\partial(\varepsilon_1,\varepsilon_2)} \neq 0.$$

This condition means that we can consider $\mu(\varepsilon_1,\varepsilon_2)$ and $\theta(\varepsilon_1,\varepsilon_2)$ as new parameters, so we further use the notation $F_{\mu,\theta}$ for the chosen family. Let θ^* be the value of θ for the original diffeomorphism F, so $F = F_{0,\theta^*}$.

We also need one more (C^1-open) condition on the multipliers of O:

(C4) $|\lambda_1| < \lambda^2$ and $|\lambda||\gamma|^{\frac{1}{2}} < 1$.

We do not know if theorem 1 below holds without this condition, but our proof uses it in an essential way.

We can now state the main result of the paper.

Theorem 1. *Let $\{F_{\mu,\theta}\}$ be the two-parameter family of diffeomorphisms in $\mathrm{Diff}^r_s(\mathcal{M})$ such that F_{0,θ^*} satisfies conditions (C1)–(C4). Then, there exists a sequence $\{(\mu_j,\theta_j)\}$ accumulating on $(0,\theta^*)$ such that for any sufficiently large j the diffeomorphism F_{μ_j,θ_j} has a symmetric pair of heterodimensional cycles, each of which includes the index-1 saddle periodic point O and some index-2 saddle periodic point.*

Let us sketch the proof of this theorem. First, by changing μ, we destroy the original homoclinic tangency and obtain a new one, $\hat{\Gamma}$, such that transverse homoclinics to O will exist near $\hat{\Gamma}$ and also some additional properties are satisfied by $\hat{\Gamma}$ (see lemma 3). It is known (see [16]) that by changing θ one can create a saddle orbit Q of index 2 near $\hat{\Gamma}$ (condition $|\lambda\gamma| > 1$ is crucial here, as it implies expansion of areas transverse to the strongly contracting directions). By using the existence of transverse homoclinics to O, we prove that for any index-2 saddle periodic point near $\hat{\Gamma}$, its unstable manifold will intersect $W^s(O)$ (see lemma 11). Finally, we show that, by changing μ and θ together, the index-2 saddle periodic point Q can be found such that that $W^s(Q)$ intersects the piece of the unstable manifold of O near the orbit of homoclinic tangency which is symmetric to $\hat{\Gamma}$ (see lemma 12). In order to be able to do this, we need to have $W^s(Q)$ sufficiently 'straight', which we achieve using condition (C4). The obtained existence of both intersections of $W^s(Q)$ with $W^u(O)$ and $W^u(Q)$ with $W^s(O)$ means the existence of the heterodimensional cycle involving O and Q (see figure 1).

Recall that the Newhouse region in $\mathrm{Diff}^r(\mathcal{M})$ is an open set comprised by diffeomorphisms having the so-called wild-hyperbolic set [32]. Systems with homoclinic tangencies are dense in the Newhouse region. Moreover, any family of diffeomorphisms which is transverse to a codimension-1 surface filled by diffeomorphisms which have a saddle periodic point O with a qudratic homoclinic tangency which satisfies the non-degeneracy conditions described in (C3) intersects the Newhouse region over an open set of parameter values, so parameter values corresponding to the existence of quadratic homoclinic tangencies to the hyperbolic continuation of O are dense in these regions and the non-degeneracy conditions (C3) are fulfilled for these tangencies [20]. Since our family $F_{\mu,\theta}$ is transverse to the codimension-1 surface $\mathcal{H} \cap \mathrm{Diff}^r(\mathcal{M})$, it follows that we have open regions in the (μ, θ) plane where the parameter values are dense for which the map $F_{\mu,\theta}$ has a symmetric pair of homoclinic tangencies satisfying conditions (C1)–(C4). Thus, theorem 1 implies the following result on the Newhouse region in $\mathrm{Diff}^r_s(\mathcal{M})$:

Corollary 1. *There exist open sets in the plane of parameters (μ, θ) where parameter values corresponding to the existence of a pair of symmetric homoclinic tangencies to O are dense, and parameter values corresponding to the existence of heterodimensional cycles involving O and an index-2 saddle periodic point are dense in these sets.*

Let us now consider the case without symmetry. Then, the simultaneous existence of two homoclinic tangencies given by condition (C2) is a codimension-2 phenomenon. Each of these homoclinic tangencies can be split independently, so we can introduce two splitting parameters, μ_1 and μ_2, which measure the distance between the stable and unstable manifolds near a point of Γ and, respectively, a point of $\tilde{\Gamma}$. As we have more parameters which we can perturb independently, the result analogous to theorem 1 becomes easier to obtain. In particular, we do not make assumption (C4) in the non-symmetric case. However, we need one more condition, without which the birth of heterodimensional cycle from the pair of homoclinic tangencies satisfying (C1)–(C3) will be impossible.

Recall that a uniquely defined smooth strong-stable foliation $\mathcal{F}_0$ exists in the stable manifold of O. The homoclinic orbits Γ and $\tilde{\Gamma}$ lie in $W^s(O)$, so for each point of these orbits there is a uniquely defined leaf of $\mathcal{F}_0$ which passes through this point. Assume that the following 'coincidence condition' holds:

(C5) There is a leaf of $\mathcal{F}_0$ which contains, simultaneously, a point of Γ and a point of $\tilde{\Gamma}$.

Note that if condition (C5) is not satisfied, then both orbits of homoclinic tangency will be contained in the same three-dimensional invariant manifold [49] and, therefore, no heterodimensional cycles can be born near them. So, condition (C5) is necessary for the creation of heterodimensional cycles. This condition is automatically fulfilled in the symmetric case (when the involution $\mathcal{R}$ near O preserves the orientation in the weak stable direction x, as given by (3)). However, in the general case this is an additional equality-type condition, which makes the bifurcation under consideration a bifurcation of codimension 3. In principle, when we consider perturbations of systems satisfying conditions (C1)–(C3) and (C5), we may consider the distance between the nearest leaves of the foliation $\mathcal{F}_0$ passing through the points of Γ and $\tilde{\Gamma}$ as an independent bifurcation parameter. We, however, do not need this and consider an arbitrary 2-parameter unfolding F_ε, with $\varepsilon = (\varepsilon_1, \varepsilon_2)$, of the map F satisfying (C1)–(C3) and (C5), for which we require only that

$$\det \frac{\partial(\mu_1(F_\varepsilon), \mu_2(F_\varepsilon))}{\partial(\varepsilon_1, \varepsilon_2)} \neq 0.$$

Thus, we can choose (μ_1, μ_2) as new parameters.

The same strategy we use for the proof of theorem 1 gives us the following

Theorem 2. *Let $\{F_{\mu_1,\mu_2}\}$ be a two-parameter family of diffeomorphisms in* $\mathrm{Diff}^r(\mathcal{M})$ *such that $F_{0,0}$ satisfies conditions (C1)–(C3) and (C5). Then, there exists a sequence $(\mu_j^1, \mu_j^2) \to 0$ such that for every sufficiently large j the diffeomorphism $F_{\mu_j^1,\mu_j^2}$ has a heterodimensional cycle including a hyperbolic continuation of the index-1 saddle periodic point O and an index-2 saddle periodic point.*

1.2. Periodically perturbed Lorenz-like attractors

Our main application is the problem of a periodic perturbation of Lorenz-like attractors. There are several approaches to Lorenz attractors, the classical 'geometric models' by Guckenheimer–Williams [24] and Afraimovich–Bykov–Shilnikov [2, 3], and their modern generalisations (see e.g. [30]). The differences between the Guckenheimer–Williams (GW) and Afraimovich–Bykov–Shilnikov (ABS) models are not large: it is easy to check that the open set in the space of dynamical systems which is described by the GW model is a subset of the open set described by the ABS model. For the purposes of this paper, we understand the Lorenz attractor (or Lorenz-like attractor) as an object described by the ABS (we provide more details later). Importantly, the corresponding conditions for a system to have a Lorenz-like attractor are formulated in [2, 3] in an explicit form (they are a sort of cone conditions for the Poincare map on a certain cross-section), which make it possible to verify them numerically or analytically. Indeed, in [47, 48], it was checked with the use of rigorous numerics that the classical Lorenz system [29] does have a Lorenz attractor in this sense. The same is true for an open set of parameter values in the Morioka–Shimizu model [39] (as was checked by rigorous numerics in [8]) and for the so-called extended Lorenz model [34] (the latter result was obtained analytically, based on a Shilnikov criterion for the existence of a Lorenz-like attractor, which was proposed in [41] and later proven for several cases in [31, 34, 38]).

The Morioka–Shimizu model and the extended Lorenz model are, probably, even more important then the classical Lorenz model because they serve as normal forms for several codimension-3 bifurcations of equilibrium states which have three Lyapunov exponents simultaneously equal to zero, in systems with certain types of Z_2-symmetry [37, 43]. Therefore, the existence of a Lorenz-like attractor in these normal forms also implies that the Lorenz-like attractor is born at the unfolding of such 'triple instability' bifurcations in an arbitrary system of differential equations.

More importantly (see [43]), the same systems serve as normal forms for some codimension-3 bifurcations of periodic orbits (with 4 zero Lyapunov exponents—one Lyapunov exponent is always zero for a periodic orbit, so having 3 more zero Lyapunov exponents is a codimension-3 bifurcation). This means that some iteration of the Poincaré map near any periodic orbit undergoing such triple instability bifurcation is close (in appropriately chosen coordinates) to the time-1 map of the flow of the corresponding normal form. It is the same as to say that some iteration of the Poincare map is the period map of some time-periodic perturbation of this normal form. Since these particular normal forms, as we mentioned, have a Lorenz-like attractor for a certian region of parameter values, these bifurcations give rise to attractors obtained by applying a small time-periodic perturbation to a Lorenz-like attractor. Multidimensional systems of differential equations can have an unbounded number of periodic orbits, any of which can undergo the 'triple instability' bifurcations which we discuss here, provided there are at least three bifurcation parameters and the flow does not contract

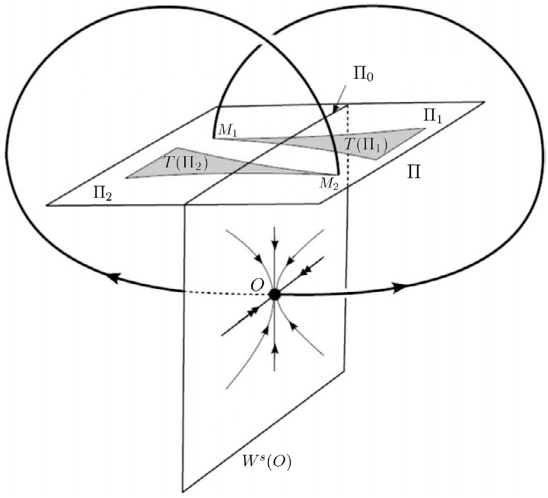

Figure 2. The Afraimovich–Bykov–Shilnikov model.

three-dimensional volumes (so there is no effective reduction to a low-dimensional case). Different scenarios where these bifurcations happen and the system acquires one or several periodically perturbed Lorenz-like attractors are presented in [14, 15, 21–23].

The question of a time-periodic perturbation of the Lorenz-like attractors is also interesting in its own right. To be precise, we define the term 'time-periodic perturbation' as follows. **Definition.** Let $\dot{x} = f(x)$ be an autonomous ODE and $g(x,t)$ be a function satisfying $g(x,t) = g(x, t+\tau)$ for some $\tau > 0$. Then, we call

$$\dot{x} = f(x) + \delta g(x,t)$$

a time-periodic perturbation of the original system, if δ is sufficiently small.

A general theory proposed in [50] asserts that after any sufficiently small time-periodic perturbation is applied to a system with a Lorenz-like attractor the period map will have a unique chain-transitive attractor $\mathcal{A}$. The equilibrium state of the non-perturbed system becomes the saddle fixed point of the period map, and this fixed point, along with its unstable manifold, belongs to $\mathcal{A}$. The unstable manifold may have homoclinic tangencies to the stable manifold. In this paper, we give conditions, under which an arbitrarily small perturbation of such tangencies can create a heterodimensional cycle that involves the fixed point (with the one-dimensional unstable manifold) and another saddle periodic orbit with a two-dimensional unstable manifold. It follows from the results of [50], that when the heterodimensional cycle containing the fixed point exists, it lies in $\mathcal{A}$, and the entire unstable manifolds of both its periodic points also lie in $\mathcal{A}$. This underscores very non-trivial dynamics in the attractor. In particular, since the attractor $\mathcal{A}$ contains saddles with different numbers of positive Lyapunov exponents (1 and 2), the relevance of Lyapunov exponents computations for the understanding of chaos represented by such attractors is questionable (e.g. the shadowing property could be violated [9]).

Let us now describe the ABS model in more detail. Consider a smooth system of differential equations having a saddle equilibrium state O with a one-dimensional unstable manifold $W^u(O)$. Assume also that the nearest to the imaginary axis characteristic exponent (an eigenvalue of the linearisation matrix) at O is real and negative. Take a compact cross-section Π (of codimension 1) transverse to a piece of the stable manifold $W^s(O)$, and let the two unstable separatrices Γ_1 and Γ_2 of $W^u(O)$ intersect Π at some points M_1 and M_2, respectively.

Denote by Π_0 the intersection of Π with $W^s_{\rm loc}(O)$, and by Π_1 and Π_2 the two parts separated by Π_0 so that we have $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$. Then, consider the Poincaré map T on Π induced by the orbits of the system—we assume that every orbit starting from $\Pi\backslash\Pi_0$ returns to Π, so the Poincare map is defined everywhere on $\Pi\backslash\Pi_0$ (the orbits that start on Π_0 tend to O as $t \to +\infty$ and do not return to Π). Let (u, v) be the coordinates on Π such that $\{u = 0\}$, $\{u > 0\}$ and $\{u < 0\}$ correspond to Π_0, Π_1 and Π_2, respectively (see figure 2). The map T is smooth outside Π_0, and for a point $M = (u, v)$ we have

$$\lim_{u\to 0^+} T(M) = M_1 \quad \text{and} \quad \lim_{u\to 0^-} T(M) = M_2.$$

We assume that the image $T(\Pi)$ lies strictly in the inner part of Π, so a small neighbourhood $\mathcal{D}$ of the set formed by forward orbits starting from Π is strictly forward-invariant, hence there is an attractor inside $\mathcal{D}$ (the Lorenz-like attractor). By the assumption on the characteristic exponents at O, the map T near Π_0 is expanding in the u-direction and contracting in the v-direction. In [2, 3], explicit conditions for extending the hyperbolicity property to the whole of Π are given. Under these conditions, there exists a smooth stable invariant foliation $\mathcal{F}$ on Π, which includes Π_0 as one of its leaves [42]. Furthermore, the quotient map of T obtained by taking quotient along the leaves of $\mathcal{F}$ is expansive. This allows for a detailed study of the structure of the attractor in $\mathcal{D}$.

We call the system Lorenz-like if it satisfies the above described properties of the ABS model. It is symmetric if the Poincare map is symmetric with respect to an involution that changes the sign of the expanding variable u. As mentioned before, examples of such systems are the classical Lorenz model [29]

$$\begin{cases} \dot{x} &=& \sigma(y - x), \\ \dot{y} &=& x(\rho - z) - y, \\ \dot{z} &=& xy - \beta z, \end{cases} \tag{4}$$

and the Morioka–Shimizu model [39]

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& x(1 - z) - \lambda y, \\ \dot{z} &=& -\alpha z + x^2. \end{cases} \tag{5}$$

A computer-assisted proof for the existence of the Lorenz attractor (in the sense of the ABS model) in system (4) for the values of parameters (σ, ρ, β) close to $\sigma = 10, \rho = 28, \beta = 8/3$ was given in [47, 48] and, in [8], for system (5) for an open set of (α, λ) near $\alpha = 0.606, \lambda = 1.045$.

Note that the equilibrium state O is a saddle fixed point for the time-t map of the system for any t. If we add a small τ-periodic perturbation to a Lorenz-like system, then O would continue as a saddle fixed point of the time-τ map. Theorem 7 in [50] states that for all small time-periodic perturbations of a Lorenz-like system the period map has a unique chain-transitive attractor $\mathcal{A} \subset \mathcal{D}$ which coincides with the set of all points attainable from O by ε-orbits for all $\varepsilon > 0$. In particular, the attractor $\mathcal{A}$ contains O and its unstable manifold. Therefore, when O is a part of the heterodimensional cycle, this heterodimensional cycle is in $\mathcal{A}$.

Recall that systems with homoclinic loops to O are C^∞-dense among Lorenz-like systems [2, 50]; systems with a symmetric pair of homoclinic loops to O are C^∞-dense among symmetric Lorenz-like systems. For the time-τ map of the system (without a periodic perturbation), the homoclinic loop corresponds to a continuous family of orbits homoclinic to the fixed point O, i.e. to a non-transverse intersection of its stable and unstable manifolds. Thus, given any symmetric Lorenz-like system, we can add an arbitrarily small time-independent perturbation (without destroying the symmetry) such that conditions (C1) and (C2) will be satisfied. The strong-stable invariant foliation in the Lorenz-like systems [2, 3] also persists at small time-periodic perturbations [50], which implies that the non-degeneracy condition (C3) will hold automatically.

Thus, in order to apply theorem 1, it remains to check condition (C4). The multipliers of O for the time-1 map of an autonomous flow are the exponents of the eigenvalues of the linearisation matrix of the system at O. Therefore, condition (C4) will be fulfilled by the time-τ map of a Lorenz-like flow (and, hence, by any sufficiently small perturbation of it) if

(C4′) $\mathrm{Re}\,\nu_1 < 2\nu_0$ and $\nu_0 + \frac{1}{2}\nu < 0$,

where ν_j and ν are the characteristic exponents of O such that

$$\cdots \leqslant \mathrm{Re}\,\nu_2 \leqslant \mathrm{Re}\,\nu_1 < \nu_0 < 0 < \nu.$$

Now, by theorem 1, we have the following

Theorem 3. *Let the equilibrium state of a symmetric Lorenz-like system satisfy condition (C4′). Then, there exists an arbitrarily small time-periodic perturbation (which keeps the symmetry of the system) such that the attractor $\mathcal{A}$ of the period map of the perturbed system contains a symmetric pair of heterodimensional cycles, each of which involves O and an index-2 saddle periodic point. Moreover, in an open neighbourhood of this map in* $\mathrm{Diff}^r_s(\mathcal{D})$, *these heterodimensional cycles are a part of the attractor $\mathcal{A}$ for a C^r-dense subset of this neighbourhood (for any $r \leqslant \infty$).*

Note that the C^ω case is not included here because we do not know whether the perturbation for a Lorenz-like system to have a pair of homoclinic loops can be made analytic (it should be possible, but we are not aware of a proof of such result). If condition (C4′) is not fulfilled, then a weaker statement follows from theorem 2.

Theorem 4. *For any symmetric Lorenz-like system, there exists an arbitrarily small (in C^r, for any $r \leqslant \infty$) time-periodic perturbation such that the attractor $\mathcal{A}$ of the period map of the perturbed system contains a heterodimensional cycle involving O and an index-2 saddle periodic point.*

Note that the Lorenz system (4) does not satisfy condition (C4′) at classical parameter values, while the Morioka–Shimizu system (5) fulfils this condition for the set of parameter values for which a proof of the existence of Lorenz attractor is obtained in [8]. Therefore, theorem 4 is applicable to time-periodic perturbations of the Lorenz attractor in the Lorenz system, and the stronger theorem 3 is applicable to the periodic perturbation of the Lorenz attractor in the Morioka–Shimizu system.

The rest of this paper is organised as follows. In section 2 we describe the dynamics near O and define the first return map. In section 3 we make perturbations which give us a homoclinic tangency with some special properties required to create heterodimensional cycles. Next, we give in section 5 the condition for having a periodic point of index 2. A formula for leaves of the strong-stable foliation $\mathcal{F}^s$ is derived in section 4. Finally, with all the preparation, we prove theorems 1 and 2 in section 6.

2. The first return map

Let a C^r-diffeomorphism F fulfil conditions (C1)–(C3). We embed it into a parametric family F_ε such that $F = F_{\varepsilon^*}$, where ε is the set of parameters defined in the previous section. Observe that this family is transverse to the surface of diffeomorphisms satisfying (C1)–(C3).

Let V be a small neighbourhood of O, and take two points $M^+, M^- \in \Gamma \cap V$ such that $M^+ \in W^s_{\rm loc}(O)$, $M^- \in W^u_{\rm loc}(O)$, $F^{-\tau}(M^+) \notin V$ and $F^\tau(M^-) \notin V$, where τ is the period of the point O. Let $\Pi_0, \Pi_1 \subset V$ be two small open sets containing M^+ and M^-, respectively. In what follows we consider the local map $T_0 \equiv F^\tau_\varepsilon|_V : V \to \mathcal{M}$ and the global map $T_1 \equiv F^l_\varepsilon|_{\Pi_1} : \Pi_1 \to \mathcal{M}$ where l is the positive integer such that $F^l(M^-) = M^+$ (it exists, because M^+ and M^- belong to the same orbit Γ).

Let C^r-coordinates $(x, y, z) \in \mathbb{R}^D$ be introduced in V such that the map T_0 takes the form

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)x + f_1(x, y, z, \varepsilon), \\ \bar{y} &= \gamma(\varepsilon)y + f_2(x, y, z, \varepsilon), \\ \bar{z} &= A(\varepsilon)z + f_3(x, y, z, \varepsilon), \end{aligned} \tag{6}$$

where the eigenvalues of the $(D-2) \times (D-2)$ matrix A are the multipliers $\lambda_1 \ldots \lambda_{D-2}$; the functions f_i $(i = 1, 2, 3)$ and their first derivatives vanish at the origin, and, furthermore,

$$\begin{aligned} &f_{1,3}(0, y, 0, \varepsilon) = 0, \qquad f_2(x, 0, z, \varepsilon) = 0, \qquad f_1(x, 0, z, \varepsilon) = 0, \qquad f_2(0, y, 0, \varepsilon) = 0, \\ &\frac{\partial f_{1,3}}{\partial(x,y)}(0, y, 0, \varepsilon) = 0, \qquad \frac{\partial f_2}{\partial y}(x, 0, z, \varepsilon) = 0 \end{aligned} \tag{7}$$

for all sufficiently small x, y and z. The existence of such coordinate transformation is shown in [22]. In the appendix we show that in the symmetric case (i.e. when $F \in \mathrm{Diff}^r_s$) this transformation can be done in such a way that the involution $\mathcal{R}$ is still locally linear and satisfies (3) in the new coordinates. Note that this coordinate transformation, and its first and second derivatives with respect to (x, y, z), are C^{r-2}-smooth functions of both the parameters ε and (x, y, z) [22]. Therefore, λ, γ, and A in (6) are C^{r-2}-smooth functions of ε, and the functions $f_{1,2,3}$, as well as the derivatives of $f_{1,2,3}$ with respect to (x, y, z) up to order 2, are C^{r-2}-smooth functions of (x, y, z, ε).

The first two identities in (7) mean that the local manifolds $W^s_{\rm loc}(O)$ and $W^u_{\rm loc}(O)$ are straightened, i.e. we have $W^s_{\rm loc}(O) = \{y = 0\}$ and $W^u_{\rm loc}(O) = \{x = 0, z = 0\}$. The third identity implies that the leaves of the strong-stable foliation $\mathcal{F}_0$ in $W^s_{\rm loc}(O)$ have the form $\{x = c, y = 0\}$ and the quotient map on $W^s_{\rm loc}(O)$ obtained by factorising over the leaves of $\mathcal{F}_0$ is linear. The forth identity corresponds to the linearisation of the map restricted to $W^u_{\rm loc}(O) : \{x = 0, z = 0\}$.

In order to obtain necessary formulas for the first return map to Π_0, we need, first, to consider iterates of T_0. Take any point $(x_0, y_0, z_0) \in V$, and let $(x_k, y_k, z_k) = T^k_0(x_0, y_0, z_0)$. The triple (x_k, y_0, z_k) is a uniquely defined function of x_0, y_k and z_0 on a small neighbourhood of (x^+, y^-, z^+) for any $k \geqslant 0$ (see e.g. [18, 40]). It follows from lemma 7 of [22] that if the map T_0 satisfies conditions (7), then the following relations hold for all sufficiently large k:

$$\begin{aligned} x_k &= \lambda(\varepsilon)^k x_0 + \phi_k(x_0, y_k, z_0, \varepsilon), \\ y_0 &= \gamma(\varepsilon)^{-k} y_k + \psi_k(x_0, y_k, z_0, \varepsilon), \\ z_k &= \hat{\phi}_k(x_0, y_k, z_0, \varepsilon), \end{aligned} \tag{8}$$

where $\phi_k, \psi_k, \hat{\psi}_k$ are smooth functions such that

$$\|\phi_k, \hat{\phi}_k\|_2 = o(|\lambda(\varepsilon)|^k), \qquad \|\psi_k\|_2 = o(|\gamma(\varepsilon)|^{-k}), \tag{9}$$

and also

$$\|\hat{\phi}_k\|_1 = o(\hat{\lambda}^k) \tag{10}$$

where $\hat{\lambda}$ is any number such that $\max\{\lambda^2, |\lambda_1|\} < \hat{\lambda} < |\lambda|$. We use the following notation in formulas (9) and (10): $\|\cdot\|_1$ stands for the maximum of the C^0-norms of the function and its first derivative with respect to (x_0, y_k, z_0), while $\|\cdot\|_2$ denotes the maximum of the C^0-norms of the function, its first derivative with respect to $(x_0, y_k, z_k, \varepsilon)$, and all its second derivatives except for the second derivative with respect to ε alone.

In the case where condition (C4) is fulfilled, we obtain stronger estimates. In appendix A.3 we show that when $|\lambda_1| < \lambda^2$ and $|\lambda\gamma| > 1$ there exists a C^2-smooth extended unstable invariant manifold $W^{uE}_{\text{loc}}(O)$ which contains the local unstable manifold $W^u_{\text{loc}}(O)$ and is tangent to $z = 0$ at the points of $W^u_{\text{loc}}(O)$, i.e. $W^{uE}_{\text{loc}}(O)$ is given by the equation $z = \eta(x, y, \varepsilon)$ where $\eta(0, y, \varepsilon) \equiv 0$, $\frac{\partial}{\partial x}\eta(0, y, \varepsilon) \equiv 0$. Furthermore, in $W^{uE}_{\text{loc}}(O)$ there is an invariant foliation $\mathcal{F}^{uE}$ with the leaves of the form $h(x, y, \varepsilon) = const$ where $h(x, 0, \varepsilon) \equiv x$ and $h(0, y, \varepsilon) \equiv 0$. The functions η and h are C^2, but if the coordinates are introduced where the map T_0 gets into the form (6) and (7), the second derivative with respect to ε alone may not exist. It is also shown in the Appendix that in the symmetric case the manifold $W^{uE}_{\text{loc}}(O)$ and the invariant foliation $\mathcal{F}^{uE}$ on it are invariant with respect to the involution $\mathcal{R}$, i.e. $\eta(x, -y, \varepsilon) \equiv \mathcal{S}\eta(x, y, \varepsilon)$ and $h(x, -y, \varepsilon) \equiv h(x, y, \varepsilon)$. From now on, we will omit ε in all expressions for simplicity.

We can now choose new coordinates $z^{\text{new}} = z - \eta(x, y)$ and $x^{\text{new}} = h(x, y)$. It is easy to see that the map keeps its form (6) and (7) in the new coordinates, and estimates (8)–(10) hold. In the symmetric case, we also have that formula (3) for the involution $\mathcal{R}$ remains unchanged.

In the new coordinates the invariant manifold $W^{uE}_{\text{loc}}(O)$ and foliation $\mathcal{F}^{uE}$ get straightened: $W^{uE}_{\text{loc}}(O)$ is given by $\{z = 0\}$ and the leaves of $\mathcal{F}^{uE}$ are $\{x = const, z = 0\}$. This implies that in the new coordinates

$$f_3(x, y, 0) = 0, \qquad f_1(x, y, 0) = 0 \tag{11}$$

(the first equation follows from the invariance of $W^{uE}_{\text{loc}}(O)$; the invariance of $\mathcal{F}^{uE}$ implies that $f_1(x, y, 0) = f_1(x, 0, 0)$, which gives the second equation of (11) by virtue of the third equation of (7)).

Lemma 1. *Once identities (7) and (11) are fulfilled, one can find positive constant $\lambda_0 < \lambda^2$ such that, for all $k \geqslant 0$,*

$$\left\|\frac{\partial x_k}{\partial z_0}\right\| \leqslant \lambda_0^k, \qquad \left\|\frac{\partial z_k}{\partial z_0}\right\| \leqslant \lambda_0^k. \tag{12}$$

Proof. We can rewrite formula (6) for T_0 as

$$\begin{aligned}
\bar{x} &= \lambda x + f_1(x, y, z),\\
y &= \gamma^{-1}\bar{y} - \gamma^{-1} f_2(x, y, z),\\
\bar{z} &= Az + f_3(x, y, z),
\end{aligned}$$

from which one deduces the following relation between (x_0, y_k, z_0) and its jth iterate (x_j, y_j, z_j) $(1 \leqslant j \leqslant k)$:

$$\begin{aligned}
x_j &= \lambda^j x_0 + \sum_{s=1}^{j} \lambda^{s-1} f_1(x_{j-s}, y_{j-s}, z_{j-s}),\\
y_j &= \gamma^{j-k} y_k - \sum_{s=j+1}^{k} \gamma^{-s+j} f_2(x_{k-s+j}, y_{k-s+j}, z_{k-s+j}),\\
z_j &= A^j z_0 + \sum_{s=1}^{j} A^{s-1} f_3(x_{j-s}, y_{j-s}, z_{j-s}).
\end{aligned} \tag{13}$$

By formulas A.18, A.20 and A.34 in [22], we have

$$\|y_j\| \leqslant C|y_k| \cdot |\gamma|^{j-k}, \qquad \left\|\frac{\partial y_j}{\partial z_0}\right\| \leqslant C|\gamma|^{j-k} \tag{14}$$

for some constant C. Since f_3 vanishes at $z = 0$ (see (11)), and its derivative vanishes at the origin, it follows that

$$\|f_3\| \leqslant \delta\|z\|$$

where δ can be made as small as we need by taking the neighbourhood V of the otigin sufficiently small. Therefore,

$$\|\bar{z}\| \leqslant (\|A\| + \delta)\|z\| \leqslant \lambda_0\|z\|$$

(we can always choose such λ_0 satisfying $\lambda_0 < \lambda^2$ because $|\lambda_1| < \lambda^2$ by the assumption of this lemma). This gives

$$\|z_j\| \leqslant \|z_0\|\lambda_0^j. \tag{15}$$

Now assume that the inequalities

$$\left\|\frac{\partial(x_s, z_s)}{\partial z_0}\right\| \leqslant \lambda_0^s \tag{16}$$

hold for all $s = 0, \ldots, j-1$ (they are, obviously true for $s = 0$) and prove that they remain true for $s = j$. By induction, this will prove the lemma.

By differentiating equation (13), we find

$$\begin{aligned} \frac{\partial x_j}{\partial z_0} &= \sum_{s=1}^{j} \lambda^{s-1}\left(\frac{\partial f_1}{\partial x}\frac{\partial x_{j-s}}{\partial z_0} + \frac{\partial f_1}{\partial y}\frac{\partial y_{j-s}}{\partial z_0} + \frac{\partial f_1}{\partial z}\frac{\partial z_{j-s}}{\partial z_0}\right), \\ \frac{\partial z_j}{\partial z_0} &= A^j + \sum_{s=1}^{j} A^{s-1}\left(\frac{\partial f_3}{\partial x}\frac{\partial x_{j-s}}{\partial z_0} + \frac{\partial f_3}{\partial y}\frac{\partial y_{j-s}}{\partial z_0} + \frac{\partial f_3}{\partial z}\frac{\partial z_{j-s}}{\partial z_0}\right). \end{aligned} \tag{17}$$

Recall that the C^2 function f_1 vanishes both at $z = 0$ and $y = 0$ while the C^2 function f_3 vanishes at $z = 0$ (see (7),(11)) and its derivative is zero at the origin. Therefore,

$$\left\|\frac{\partial f_1}{\partial(x,z)}\right\| \leqslant K\|y\|, \qquad \left\|\frac{\partial f_1}{\partial y}\right\| \leqslant K\|z\|,$$

$$\left\|\frac{\partial f_3}{\partial(x,y)}\right\| \leqslant K\|z\|, \qquad \left\|\frac{\partial f_3}{\partial z}\right\| \leqslant \delta,$$

where K and δ are some constants and δ can be chosen as small as we want (by choosing the neighbourhood V small enough). By plugging these inequalities into (17), we obtain

$$\begin{aligned} \left\|\frac{\partial x_j}{\partial z_0}\right\| &\leqslant K\sum_{s=1}^{j} |\lambda|^{s-1}\left(\|y_{j-s}\| \cdot \left\|\frac{\partial(x_{j-s}, z_{j-s})}{\partial z_0}\right\| + \|z_{j-s}\| \cdot \left\|\frac{\partial y_{j-s}}{\partial z_0}\right\|\right), \\ \left\|\frac{\partial z_j}{\partial z_0}\right\| &\leqslant \|A\|^j + \sum_{s=1}^{j} \|A\|^{s-1}\left(K\|z_{j-s}\| \cdot \left\|\frac{\partial x_{j-s}}{\partial z_0}\right\| + K\|z_{j-s}\| \cdot \left\|\frac{\partial y_{j-s}}{\partial z_0}\right\| + \delta\left\|\frac{\partial z_{j-s}}{\partial z_0}\right\|\right). \end{aligned}$$

Now, using estimates (14) and (15) (where one should replace j by $(j-s)$) and (16) (where one should change s to $(j-s)$), we obtain

$$
\begin{aligned}
\left\|\frac{\partial x_j}{\partial z_0}\right\| &\leqslant K\sum_{s=1}^{j}|\lambda|^{s-1}\left(C|y_k|\cdot|\gamma|^{j-s-k}\cdot\lambda_0^{j-s}+\|z_0\|\lambda_0^{j-s}\cdot C|\gamma|^{j-s-k}\|\right)\leqslant\\
&\leqslant \frac{KC}{|\lambda|}(|y_k|+|z_0|)\,\lambda_0^j\sum_{s=1}^{j}\left(\frac{|\lambda|}{|\gamma|\lambda_0}\right)^s,\\
\left\|\frac{\partial z_j}{\partial z_0}\right\| &\leqslant \|A\|^j+\sum_{s=1}^{j}\|A\|^{s-1}\left(K\|z_0\|\lambda_0^{j-s}\cdot\lambda_0^{j-s}+K\|z_0\|\lambda_0^{j-s}\cdot C|\gamma|^{j-s-k}+\delta\lambda_0^{j-s}\right)\leqslant\\
&\leqslant \lambda_0^j+\frac{K\|z_0\|(C+1)+\delta}{\|A\|}\,\lambda_0^j\sum_{s=1}^{j}\left(\frac{\|A\|}{\lambda_0}\right)^s.
\end{aligned}
\tag{18}
$$

Recall that we assume $|\lambda\gamma|>1$. Hence, if $\lambda_0<\lambda^2$ is chosen close enough to λ^2, we have $\frac{|\lambda|}{|\gamma|\lambda_0}<1$. Also, since $|\lambda_1|<\lambda^2$, where λ_1 is the largest, in the absolute value, eigenvalue of A, we have that $\lambda_0<\lambda^2$ can be chosen such that $\frac{\|A\|}{\lambda_0}<1$. This means that the sums $\sum_{s=1}^{j}\left(\frac{|\lambda|}{|\gamma|\lambda_0}\right)^s$ and $\sum_{s=1}^{j}\left(\frac{\|A\|}{\lambda_0}\right)^s$ in (18) are uniformly bounded for all j. Therefore, since $|y_k|$, $\|z_0\|$ and δ can be taken as small as we need by choosing the neighbourhood V small enough, the estimates (18) imply that the inequalities (16) hold for $s=j$. Therefore, by induction, they hold for all s. At $s=k$ we obtain the lemma. □

We now proceed to obtain necessary formulas for the global map T_1. Let us write its Taylor expansion near the point M^-. At $\varepsilon=\varepsilon^*$, the point M^- is homoclinic, so its image $M^+=T_1M^-$ belongs to the local stable manifold, and the curve $T_1W^u_{\rm loc}$ has a quadratic tangency to $W^s_{\rm loc}$. In the coordinate system where the local stable and unstable manifolds are straightened, i.e. they are given by the equations $\{y=0\}$ and, respectively, $\{x=0,z=0\}$, we have $M^-=(0,y^-,0)$ and $M^+=(x^+,0,z^+)$ and the Taylor expansion for $T_1:(x,y,z)\mapsto(x',y',z')$ is given by

$$
\begin{aligned}
x'-x^+ &= ax+b(y-y^-)+a_{13}z+h_1(x,y-y^-,z),\\
y' &= y^+(\varepsilon)+cx+d(y-y^-)^2+a_{23}z+h_2(x,y-y^-,z),\\
z'-z^+ &= a_{31}x+a_{32}(y-y^-)+a_{33}z+h_3(x,y-y^-,z),
\end{aligned}
\tag{19}
$$

where $d\neq 0$ and the Taylor expansions for functions $h_{1,2,3}$ start with quadratic terms (the term $d(y-y^-)^2$ is taken out of h_2, so h_2 does not contain it). We will use the coordinate system where the map T_0 is in the form (6) and the identities (7) hold.

When we vary ε, the map T_1 can be kept in the form (19) where the coefficients and the functions $h_{1,2,3}$ now depend on ε (e.g. we choose $y^-(\varepsilon)$ in such a way that there is no linear term in $(y-y^-(\varepsilon))$ in the equation for y' in (19)). We however take d independent of ε, so h_2 is allowed to include the $(y-y^-(\varepsilon))^2$-term with the coefficient which vanishes at $\varepsilon=\varepsilon^*$. Recall that the coordinates we use are of class C^2, but the second derivative with respect to ε alone may not exist. Thus, we have that all the coefficients, as well as the functions $h_{1,2,3}$ and their first derivatives with respect to (x,y,z) are at least C^1 functions of ε. So, we can write

$$
h_{1,3}=O(x^2+(y-y^-)^2+z^2),\qquad h_2=O(x^2+z^2+|x|\cdot|y-y^-|+\|z\|\cdot|y-y^-|)+o((y-y^-)^2)_{\varepsilon\to\varepsilon^*},
\tag{20}
$$

and

$$
\frac{\partial h_{1,2,3}}{\partial\varepsilon}=o(|x|+\|z\|+|y-y^-|),\qquad \frac{\partial^2 h_{1,2,3}}{\partial\varepsilon\partial(x,y,z)}=o(1)_{(x,y-y^-(\varepsilon),z)\to 0}.
\tag{21}
$$

By construction, the value of $y^+(\varepsilon)$ measures the magnitude of splitting between the curve $T_1W^u_{\rm loc}$ and the local stable manifold. Thus, $\mu(F_\varepsilon)=y^+(\varepsilon)$ can be taken as the parameter governing the splitting of the homoclinic tangency at the point M^+. It is our standing condition

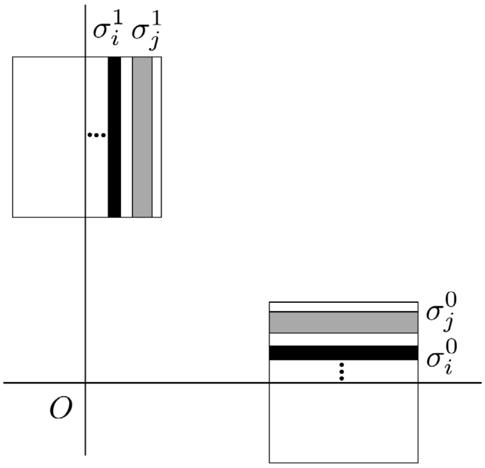

Figure 3. The projections of the countable sequences of disjoint sets σ_k^0 along the leaves of $\mathcal{F}^s$ onto $\{z = 0\}$.

that $\partial\mu/\partial\varepsilon \neq 0$, so we simply assume that μ is one of the parameters ε (see the explanation before theorem 1).

Note that our conditions in section 1 imply that

$$d \neq 0,\ x^+ \neq 0 \text{ and } bc \neq 0 \tag{22}$$

in formula (19). The first two inequalities come, respectively, from the facts that the tangency is quadratic and it is not in the strong-stable manifold of O. The third one follows from the transversality of the extended unstable manifold $W^{uE}(O)$ to the strong-stable foliation $\mathcal{F}_0$, see Condition (C3).

Indeed, the first identity in the second line of (7) implies that W^{uE}_{loc} is tangent to the plane $z = 0$ at the points of W^u_{loc} (see [22]); in particular, it is tangent to $z = 0$ at the homoclinic point M^-. So, the tangent plane to the image $T_1 W^{uE}_{\text{loc}}$ is given by

$$x' - x^+ = ax + b(y - y^-), \qquad y' = cx, \qquad z' = a_{31}x + a_{32}(y - y^-).$$

The transversality of $T_1 W^{uE}$ to $\mathcal{F}_0$ just means that this tangent plane intersects the strong-stable leaf $\{x' = x^+, y' = 0\}$ at a single point (the point M^+). This is equivalent to the requirement that the equation

$$0 = ax + b(y - y^-), \qquad 0 = cx$$

has only one solution ($x = 0, y = y^-$), which implies $bc \neq 0$.

We can now define the maps $T_1 T_0^k$ of the first return to Π_0. We fix the choice of the neighbourhoods Π_0 and Π_1 as follows: $\Pi_0 = \{(x, y, z) \mid |x - x^+| < \delta/2, |y| < \delta, \|z - z^+\| < \delta/2\}$ and $\Pi_1 = \{(x, y, z) \mid |x| < \delta, |y - y^-| < \delta/2, \|z\| < \delta\}$, where $\delta > 0$ is small such that $T_0(\Pi_0) \cap \Pi_0 = \emptyset$ and $T_0^{-1}(\Pi_1) \cap \Pi_1 = \emptyset$. Let k^* be the smallest number such that $T_0(\Pi_0) \cap \Pi_1 \neq \emptyset$. There are two countable sequences of disjoint subsets $\sigma_k^0 \subset \Pi_0$ and $\sigma_k^1 := T_0^k(\sigma_k^0) \subset \Pi_1$ such that $k \geqslant k^*$, and $\sigma_k^0 \to W^s_{\text{loc}}(O)$ and $\sigma_k^1 \to W^u_{\text{loc}}(O)$ as $k \to +\infty$ (see figure 3). Therefore, the first-return map $T : \Sigma^0 := \bigcup_{k_0}^{+\infty} \sigma_k^0 \to \Pi_0$ is defined as

$$T(M) = T_1 \circ T_0^k(M) \quad \text{if} \quad M \in \sigma_k^0. \tag{23}$$

For a point $M \in \Sigma^0$ we call the corresponding k in (23) *the stay number* of M. The image of Σ^0 under T may not be entirely contained in Π_0. However, throughout this paper, we only consider points sufficiently close to M^+ such that their images lie in Π_0.

In the same way, a global map $\tilde{T}_1$ and a first-return map $\tilde{T}$ are defined near the second orbit of homoclinic tangency, $\tilde{\Gamma}$. In the symmetric case, i.e. when $F_\varepsilon \in \mathrm{Diff}^r_s(\mathcal{M})$, the maps T_1 and $\tilde{T}_1$ are related by the symmetry $\mathcal{R}$. Namely, we denote by $\tilde{M}^+$ and $\tilde{M}^-$ the points that are $\mathcal{R}$-symmetric to M^+ and M^-. These two points satisfy $\tilde{M}^+ \in W^s_{\mathrm{loc}}(O) \cap \tilde{\Gamma}$, $\tilde{M}^- \in W^u_{\mathrm{loc}}(O) \cap \tilde{\Gamma}$, and have coordinates $(x^+, 0, \mathcal{S}z^+)$ and $(0, -y^-, 0)$. We can choose the neighbourhood Π_0 such that it will contain both points M^+ and $\tilde{M}^+$. In order to achieve this, note that the directions corresponding to coordinates z are strongly contracting, so we can just let Π_0 be the set $\{(x, y, z) \mid |x - x^+| < \delta/2, |y| < \delta, \|z\| < \delta\}$ and choose x^+ sufficiently small. When δ is small, the property $T_0(\Pi_0) \cap \Pi_0 = \emptyset$ and $T_0^{-1}(\Pi_1) \cap \Pi_1 = \emptyset$ holds. The neighbourhood $\tilde{\Pi}_1$ is defined as $\tilde{\Pi}_1 = \mathcal{R}\Pi_1 = \{(x, y, z) \mid |x| < \delta, |y + y^-| < \delta/2, \|\mathcal{S}z\| < \delta\}$, which implies $T_0^{-1}(\tilde{\Pi}_1) \cap \tilde{\Pi}_1 = \emptyset$.

The second global map $\tilde{T}_1 \equiv F^l|_{\tilde{\Pi}_1} : (x, y, z) \mapsto (x', y', z')$ takes the form

$$
\begin{aligned}
x' - x^+ &= ax - b(y + y^-) + a_{13}\mathcal{S}z + h_1(x, -y - y^-, \mathcal{S}z), \\
y' &= -\mu - cx - d(y + y^-)^2 - a_{23}\mathcal{S}z - h_2(x, -y - y^-, \mathcal{S}z), \\
z' - \mathcal{S}z^+ &= \mathcal{S}a_{31}x - \mathcal{S}a_{32}(y + y^-) + a_{33}z + \mathcal{S}h_3(x, -y - y^-, \mathcal{S}z),
\end{aligned} \tag{24}
$$

with the same coefficients and functions $h_{1,2,3}$ as in (19).

There is a countable sequence of disjoint subsets $\tilde{\sigma}^0_k \subset \Pi_0$ such that $\tilde{\sigma}^1_k = T_0^k(\tilde{\sigma}^0_k) \subset \tilde{\Pi}_1$, where $k \geqslant k^*$, and $\tilde{\sigma}^0_k \to W^s_{\mathrm{loc}}(O)$ and $\tilde{\sigma}^1_k \to W^u_{\mathrm{loc}}(O)$ as $k \to +\infty$. The first return map $\tilde{T} : \tilde{\Sigma}^0 = \bigcup_{k_0}^{+\infty} \tilde{\sigma}^0_k \to \tilde{\Pi}_0$ is defined as

$$
\tilde{T}(M) = \tilde{T}_1 \circ T_0^k(M) \quad \text{if} \quad M \in \tilde{\Sigma}^0_k. \tag{25}
$$

3. An adjustment to the homoclinic tangency

In order to create a heterodimensional cycle in the small neighbourhood U of $O \cup \Gamma \cup \tilde{\Gamma}$, we need the homoclinic tangency to satisfy the following conditions:

(a) the signs of cdx^+ and cx^+y^- are positive, where c and d are the coefficients in the global map (19); and
(b) there are two transverse homoclinic points in $W^u_{\mathrm{loc}}(O)$ close to M^- such that M^- lies between these two points.

In section 6.1, conditions (a) and (b) are used to show the existence of the non-transverse and, respectively, transverse intersections between the invariant manifolds of two periodic orbits of different indices. In this section we prove that unfolding the original homoclinic tangency produces new homoclinic tangencies satisfying the above conditions. Depending on the signs of c and d, the original homoclinic tangency falls into one of the four classes: (1) $cdx^+ < 0, dy^- < 0$, (2) $cdx^+ < 0, dy^- > 0$, (3) $cdx^+ > 0, dy^- < 0$, and (4) $cdx^+ > 0, dy^- > 0$. We start with showing that tangencies of classes (1), (3), and (4) can be replaced by tangencies of class (2).

Lemma 2. *Take any smooth one-parameter family F_μ of diffeomorphisms, where μ is the splitting parameter for the homoclinic tangency Γ, and F_0 fulfils conditions (C1)–(C3). Then, there exists a sequence $\{\mu_k\}$ accumulating on $\mu = 0$ such that the saddle O of F_{μ_k} has a class (2) homoclinic tangency and a tangency point $M^-_k \in W^u_{\mathrm{loc}}(O)$ satisfying $M^-_k \to M^-$ as $k \to +\infty$.*

Proof. We will assume $x^+ > 0$ and $y^- > 0$ throughout this section since this can be always achieved by changing signs of x and/or y at the very beginning. There is nothing to prove if the original tangency already belongs to class (2). For the remaining three cases, we first construct new tangencies, and then show that some of those tangencies belong to class (2).

Let us create a secondary homoclinic tangency by making the curve $T_1 \circ T_0^k \circ T_1(W^u_{\text{loc}}(O))$ intersect $W^s_{\text{loc}}(O)$ non-transversely. By formula (19) for T_1 (where one should take $y^+(\varepsilon) = \mu$), the image $(x_0, y_0, z_0) = T_1(x, y, z)$ of a point $(x, y, z) \in \Pi_1$ is given by

$$\begin{aligned} x_0 - x^+ &= ax + b(y - y^-) + a_{13}z + h_1(x, y - y^-, z), \\ y_0 &= \mu + cx + d(y - y^-)^2 + a_{23}z + h_2(x, y - y^-, z), \\ z_0 - z^+ &= a_{31}x + a_{32}(y - y^-) + a_{33}z + h_3(x, y - y^-, z). \end{aligned} \tag{26}$$

Consequently, the image $T_1(W^u_{\text{loc}}(O))$ has the form

$$y_0 = \mu + \frac{d}{b^2}(x_0 - x^+)^2 + h_2(0, \frac{x_0 - x^+}{b}, 0), \tag{27}$$

$$z_0 - z^+ = \frac{a_{32}}{b}(x_0 - x^+) + h_3(0, \frac{x_0 - x^+}{b}, 0), \tag{28}$$

where $h_2(0, (x_0 - x^+)/b, 0) = o((x_0 - x^+)^2)$ and $h_3(0, (x_0 - x^+)/b, 0) = o(|x_0 - x^+|)$. For any point $(x_0, y_0, z_0) \in T_1(W^u_{\text{loc}}(O)) \cap \sigma^0_k$, we can find its kth iterate $(x_k, y_k, z_k) = T_0^k(x_0, y_0, z_0)$ by formula (8):

$$x_k = \lambda^k x_0 + o(\lambda^k), \tag{29}$$

$$y_0 = \gamma^{-k} y_k + o(\gamma^{-k}), \tag{30}$$

$$z_k = O(\hat{\lambda}^k). \tag{31}$$

The point (x_0, y_0, z_0) is a homoclinic point if $T_1(x_k, y_k, z_k) = (\bar{x}, \bar{y}, \bar{z}) \in W^s(O)$, namely, the coordinate $\bar{y}$ equals zero. From the second equation in (19), we have

$$\bar{y} = \mu + cx_k + d(y_k - y^-)^2 + a_{23}z_k + h_2(x_k, y_k - y^-, z_k) = 0. \tag{32}$$

By plugging (27) and (31) into (30), and plugging (29) and (31) into (32), we obtain the following system whose solutions correspond to homoclinic points $(x_0, y_0, z_0) \in T_1(W^u_{\text{loc}}(O))$:

$$\begin{aligned} 0 &= \mu - \gamma^{-k}y^- - \gamma^{-k}(y_k - y^-) + \tfrac{d}{b^2}(x_0 - x^+)^2 + u_1(x_0, y_k, \mu) + u_2(x_0, \mu), \\ 0 &= \mu + c\lambda^k x^+ + c\lambda^k(x_0 - x^+) + d(y_k - y^-)^2 + u_3(x_0, y_k, \mu) + u_4(x_0, y_k, \mu), \end{aligned} \tag{33}$$

where $u_1 = o(\gamma^{-k}), u_2 = o(x_0^2), u_3 = o(\lambda^k)$, and $u_4 = o(|\lambda|^k + y_k^2)$. After letting $X = x_0 - x^+$ and $Y = y_k - y^-$, system (33) recasts as

$$\begin{aligned} 0 &= \mu - \gamma^{-k}y^- - \gamma^{-k}Y + \tfrac{d}{b^2}X^2 + \hat{u}_1(X, Y, \mu) + \hat{u}_2(X, \mu), \\ 0 &= \mu + c\lambda^k x^+ + c\lambda^k X + dY^2 + \hat{u}_3(X, Y, \mu) + \hat{u}_4(Y, \mu), \end{aligned} \tag{34}$$

where $\hat{u}_1 = o(\gamma^{-k}), \hat{u}_2 = o(X^2), \hat{u}_3 = o(\lambda^k)$ and $\hat{u}_4 = o(|\lambda|^k + Y^2)$.

A non-degenerate homoclinic tangency corresponds to a solution to system (34) with multiplicity two. This corresponds to the vanishing determinant of the Jacobian matrix. Now, by

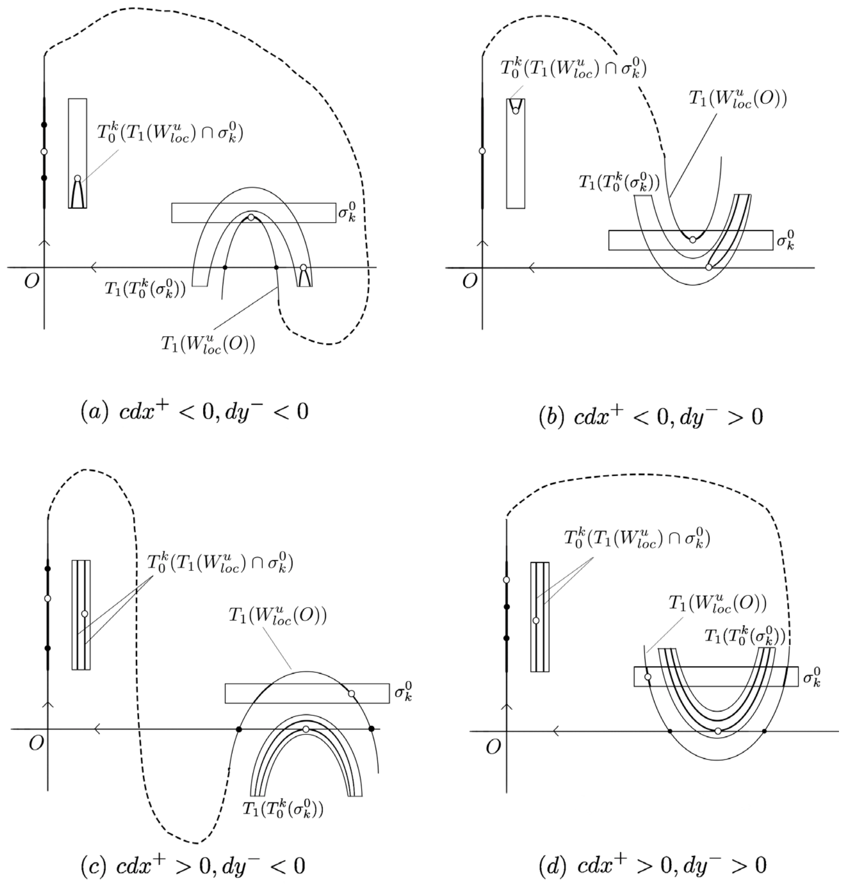

Figure 4. Creation of secondary homoclinic tangencies for $x^+, y^- > 0$. Here we project the iterates of $W^u_{\text{loc}}(O)$ and σ_k^0 onto the two-dimensional plane $\{z = 0\}$ along the leaves of $\mathcal{F}$ (note that such projection is well-defined by the non-degeneracy condition (C2)), and take $\mu = \mu_k^i$ for some $i \in \{1, 2\}$. The horizontal and the vertical strips are the projections of σ_k^0 and $T_0^k(\sigma_k^0)$, and the hollowed dots denote the points in the orbit of the homoclinic tangency while the solid dots denote those in the transverse homoclinic orbits.

letting the Jacoby matrix of system (34) have determinant zero, expressing μ as a function of X and Y from the first equation of (34), and plugging this expression for μ into the second one, we arrive at the following system:

$$\begin{aligned} 0 &= c\lambda^k\gamma^{-k} + 4\tfrac{d^2}{b^2}(X + v_1(X,Y))(Y + v_2(X,Y)) + o(\lambda^k\gamma^{-k}), \\ 0 &= c\lambda^k x^+ + \gamma^{-k}y^- + c\lambda^k X + \gamma^{-k}Y + dY^2 - \tfrac{d}{b^2}X^2 + o(\lambda^k\gamma^{-k}), \end{aligned} \tag{35}$$

where $v_1 = o(|\gamma|^{-k} + |X|)$ and $v_2 = o(|\lambda|^k + |Y|)$. With the further coordinate transformation

$$(\hat{X}, \hat{Y}) = (X + v_1(X,Y), Y + v_2(X,Y)), \tag{36}$$

we obtain

$$
\begin{array}{rcl}
0 & = & c\lambda^k\gamma^{-k} + 4\frac{d^2}{b^2}\hat{X}\hat{Y} + o(\lambda^k\gamma^{-k}), \\
0 & = & c\lambda^k x^+ + \gamma^{-k}y^- + c\lambda^k\hat{X} + \gamma^{-k}\hat{Y} + d\hat{Y}^2 - \frac{d}{b^2}\hat{X}^2 + o(|\lambda|^k + |\gamma|^{-k}).
\end{array} \tag{37}
$$

Quadratic tangencies of the original system correspond to non-degenerate solutions to (37), and the value of $\mu = \mu_k$ corresponding to such tangency can be found from either of the equations in (34).

In what follows, we find solutions to (37). Let k be even so that λ^k and γ^{-k} are always positive. Consider first class (1), where $cdx^+ < 0$ and $dy^- < 0$. We do the following scaling:

$$
(\hat{X}, \hat{Y}) \mapsto |\lambda|^{\frac{k}{2}} \sqrt{\left|\frac{cx^+}{d}\right|} \left(-\frac{b^2\gamma^{-k}}{4dx^+}U, V \right).
$$

In the new variables system (37) takes the form

$$
\begin{array}{rcl}
1 & = & UV + o(1)_{k\to+\infty}, \\
1 & = & V^2 + o(1)_{k\to+\infty}.
\end{array} \tag{38}
$$

For any sufficiently large k the above system has two non-degenerate solutions $(1+o(1), 1+o(1))$ and $(-1+o(1), -1+o(1))$, corresponding to two solutions to system (37):

$$
\begin{array}{rcl}
(\hat{X}_k^1, \hat{Y}_k^1) & = & \left(-\frac{b^2|\lambda|^{\frac{k}{2}}\gamma^{-k}}{4dx^+} \sqrt{\left|\frac{cx^+}{d}\right|} + o(|\lambda|^{\frac{k}{2}}\gamma^{-k}), |\lambda|^{\frac{k}{2}} \sqrt{\left|\frac{cx^+}{d}\right|} + o(|\lambda|^{\frac{k}{2}}) \right), \\
(\hat{X}_k^2, \hat{Y}_k^2) & = & \left(\frac{b^2|\lambda|^{\frac{k}{2}}\gamma^{-k}}{4dx^+} \sqrt{\left|\frac{cx^+}{d}\right|} + o(|\lambda|^{\frac{k}{2}}\gamma^{-k}), -|\lambda|^{\frac{k}{2}} \sqrt{\left|\frac{cx^+}{d}\right|} + o(|\lambda|^{\frac{k}{2}}) \right).
\end{array} \tag{39}
$$

These two solutions give us two homoclinic tangency points $M_k^1, M_k^2 \in T_1(W^u_{\text{loc}}(O)$ for two different μ values μ_k^1 and μ_k^2 (see figure 4(a)). From equations (28), (30) and (36), we find the coordinates of these tangency points as

$$
M_k^1 = (\hat{X}^1 + x^+ + o(\gamma^{-k}), \gamma^{-k}(\hat{Y}^1 + y^- + o(1)), z_1) \quad \text{and} \quad M_k^2 = (\hat{X}^2 + x^+, \gamma^{-k}(\hat{Y}^2 + y^- + o(1)), z_2), \tag{40}
$$

where we do not write the z-coordinates explicitly. Let $M_k^- = (0, y_k^-, 0) \in W^u_{\text{loc}}(O)$ be the pre-image of any of the points M_k^1 and M_k^2. By (26) and (40), we have $y_k^- - y^- = (\hat{X}_k^i + o(\hat{X}_k^i) + o(\gamma^{-k}))/b$. This immediately shows that $M_k^- \to M^-$ as $k \to +\infty$. The first equation in (34) yields the corresponding μ values, which are $\mu_k^i = \gamma^{-k}y^-(1+o(1))$ $(i = 1, 2)$.

Remark 1. Note that the condition $dy^- < 0$ has not been used in the above computation. In fact, we can also create new tangencies for class (2) in the same way (see figure 4(b)).

Now consider classes (3) and (4), where we have $cdx^+ > 0$. By using the scaling

$$
(\hat{X}, \hat{Y}) \mapsto b|\lambda|^{\frac{k}{2}} \sqrt{\frac{cx^+}{d}} \left(U, -\frac{\gamma^{-k}}{4dx^+}V \right),
$$

and dividing the first and second equation of (37) to $c\lambda^k\gamma^{-k}$ and $c\lambda^k x^+$, respectively, we arrive at the following system

$$\begin{array}{rcl} 1 & = & UV + o(1)_{k\to+\infty}, \\ 1 & = & U^2 + o(1)_{k\to+\infty}. \end{array} \tag{41}$$

For any sufficiently large k, system (41) has non-degenerate solutions $(1+o(1), 1+o(1))$ and $(-1+o(1), -1+o(1))$, which lead to two solutions to system (37) as

$$\begin{array}{rcl} (\hat{X}_k^1, \hat{Y}_k^1) & = & \left(b|\lambda|^{\frac{k}{2}}\sqrt{\frac{cx^+}{d}} + o(|\lambda|^{\frac{k}{2}}), -\frac{b|\lambda|^{\frac{k}{2}}\gamma^{-k}}{4dx^+}\sqrt{\frac{cx^+}{d}} + o(|\lambda|^{\frac{k}{2}}\gamma^{-k})\right), \\ (\hat{X}_k^2, \hat{Y}_k^2) & = & \left(-b|\lambda|^{\frac{k}{2}}\sqrt{\frac{cx^+}{d}} + o(|\lambda|^{\frac{k}{2}}), \frac{b|\lambda|^{\frac{k}{2}}\gamma^{-k}}{4dx^+}\sqrt{\frac{cx^+}{d}} + o(|\lambda|^{\frac{k}{2}}\gamma^{-k})\right). \end{array} \tag{42}$$

For each sufficiently large k, these two solutions give us two points of homoclinic tangency $M_k^1, M_k^2 \in T_1(W^u_{\mathrm{loc}}(O))$ (see figures 4(c) and (d)). Similar to the discussion for class (1), for the pre-image M_k^- of any of the points M_k^1 and M_k^2, we have $M_k^- \to M^-$ as $k \to +\infty$. The corresponding μ values can be found from the second equation in (34), which gives $\mu_k^i = -cx^+\lambda^k(1+o(1))$ $(i = 1, 2)$.

We proceed to compute the signs of the coefficients c and d corresponding to the new homoclinic tangencies. We have shown that for each sufficiently large k there exist two values of $\mu = \mu_k^i (i = 1, 2)$ that correspond to a homoclinic tangency. The associated global map for this tangency is

$$\hat{T} := T_1 \circ T_0^k \circ T_1 : (x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z}).$$

By denoting $T_1^{-1}(M_k^i) = (0, y_k^i, 0)$, the coefficients c_k^i and d_k^i of $\hat{T}$ are given by

$$c_k^i = \frac{\partial \bar{y}(0, y_k^i, 0)}{\partial x} \quad \text{and} \quad d_k^i = \frac{1}{2}\frac{\partial^2 \bar{y}(0, y_k^i, 0)}{\partial y^2}, \tag{43}$$

where $\bar{y}$ is related to $(x_k, y_k, z_k) = T_0^k(x_0, y_0, z_0) = T_0^k \circ T_1(x, y, z)$ by (32). We note from (33)–(35) that

$$\hat{Y} = Y + v_2 = Y + \frac{1}{2d}\left(\frac{\partial \hat{u}_3}{\partial Y} + \frac{\partial \hat{u}_4}{\partial Y}\right) = \frac{1}{2d}\left(2d(y_k - y^-) + \frac{\partial(cx_k + a_{23}z_k)}{\partial y_k} + \frac{\partial h_2}{\partial y_k}\right) = \frac{1}{2d}\frac{\partial \bar{y}}{\partial y_k}. \tag{44}$$

This fact along with equations (26) and (29)–(31) yields

$$\begin{array}{rcl} c_k^i & = & \left(\frac{\partial \bar{y}}{\partial x_k}\frac{\partial x_k}{\partial x} + \frac{\partial \bar{y}}{\partial y_k}\frac{\partial y_k}{\partial x} + \frac{\partial \bar{y}}{\partial z_k}\frac{\partial z_k}{\partial x}\right)\Big|_{(x,y,z)=(0,y_k^i,0)} \\ & = & ac\lambda^k + 2cd\gamma^k\hat{Y}_k^i + o(\hat{Y}_k^i) + o(\lambda^k), \end{array} \tag{45}$$

where $\hat{Y}_k^i$ is given by (39) or (42).

Let us now compute d_k^i which is given by

$$d_k^i = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \bar{y}}{\partial x_k}\frac{\partial x_k}{\partial y} + \frac{\partial \bar{y}}{\partial y_k}\frac{\partial y_k}{\partial y} + \frac{\partial \bar{y}}{\partial z_k}\frac{\partial z_k}{\partial y}\right)\Big|_{(x,y,z)=(0,y_k^i,0)}. \tag{46}$$

It can be easily seen from (26) and (29)–(31) that

$$\frac{\partial}{\partial y}\left(\frac{\partial \bar{y}}{\partial x_k}\frac{\partial x_k}{\partial y} + \frac{\partial \bar{y}}{\partial z_k}\frac{\partial z_k}{\partial y}\right)\Big|_{(x,y,z)=(0,y_k^i,0)} = o(\lambda^k). \tag{47}$$

Regarding the rest of the derivatives in (46), we note from the first equation of (26) that

$$y - y^- = \frac{(x_0 - x^+)}{b} + o(x_0 - x^+) = \frac{X}{b} + o(X) = \frac{\hat{X} + o(\gamma^{-k})}{b}(1 + o(1)),$$

see (35). Together with equations (26) and (30), this leads to

$$\frac{\partial y_k}{\partial y}\Big|_{(x,y,z)=(0,y_k^i,0)} = 2d\gamma^k(y_k^i - y^-) + o(y_k^i - y^-) = \frac{2d\gamma^k}{b}(\hat{X}_k^i + o(\gamma^{-k}))(1 + o(1)), \tag{48}$$

where $\hat{X}_k^i$ is given by (39) or (42). Now, with the help of (44) and (48), we obtain

$$\begin{aligned} d_k^i &= \frac{1}{2}\left(\frac{\partial^2 \bar{y}}{\partial y_k^2}\left(\frac{\partial y_k}{\partial y}\right)^2 + \frac{\partial \bar{y}}{\partial y_k}\frac{\partial^2 y_k}{\partial y^2}\right)\Big|_{(x,y,z)=(0,y_k^i,0)} + o(\lambda^k) \\ &= \frac{4d^3\gamma^{2k}}{b^2}(\hat{X}_k^i + o(\gamma^{-k}))^2(1 + o(1)) + 2d^2\gamma^k\hat{Y}_k^i + o(\lambda^k). \end{aligned} \tag{49}$$

For class (1), where $cdx^+ < 0$ and $dy^- < 0$, we plug the solutions (39) into the above equations and get

$$\begin{aligned} c_k^i &= (-1)^{(i+1)}2cd|\lambda|^{\frac{k}{2}}\gamma^k\sqrt{\left|\frac{cx^+}{d}\right|} + o(|\lambda|^{\frac{k}{2}}\gamma^k), \\ d_k^i &= (-1)^{(i+1)}2d^2|\lambda|^{\frac{k}{2}}\gamma^k\sqrt{\left|\frac{cx^+}{d}\right|} + o(|\lambda|^{\frac{k}{2}}\gamma^k), \end{aligned} \tag{50}$$

which implies $c_k^1 d_k^1 x^+ < 0$ and $d_k^1 y^- > 0$. Therefore, by taking $\mu_k = \mu_k^1$ and $M_k^- = T_1^{-1}(M_k^1)$, we obtain a homoclinic tangency that belongs to class (2), as required.

Let now $cdx^+ > 0$. With the corresponding solutions (42), equations (45) and (49) yield

$$\begin{aligned} c_k^i &= (-1)^i\frac{bc|\lambda|^{\frac{k}{2}}}{2x^+}\sqrt{\frac{cx^+}{d}}\,(1 + o(1)), \\ d_k^i &= 4cd^2\lambda^k\gamma^{2k}x^+ + o(\lambda^k\gamma^{2k}). \end{aligned} \tag{51}$$

Observe that $c_k^i (i = 1, 2)$ have different signs and d_k^i always have the same sign as d. It follows that for class (4) where $cdx^+ > 0$ and $dy^- > 0$ one can obtain the desired class (2) homoclinic tangency by picking i such that $c_k^i < 0$. If the original tangency belongs to class (3) where $cdx^+ > 0$ and $dy^- < 0$, then we can first obtain a class (1) tangency by choosing i such that $c_k^i > 0$. After this, repeat what we did for class (1) tangency. □

We are now in the position to show that a homoclinic tangency satisfying conditions (a) and (b) can be recovered from any kind of the original tangency.

Lemma 3. *For any smooth one-parameter family F_μ of diffeomorphisms with the diffeomorphism F_0 satisfying conditions (C1)–(C3), there exists a sequence $\{\mu_k\}$ accumulating on $\mu = 0$ such that the saddle O of F_{μ_k} has a new homoclinic tangency point M_k^- for which $cdx^+ > 0$ and $cx^+y^- > 0$, and in $W^u_{\mathrm{loc}}(O) \cap \Pi_1$ there exist two transverse homoclinic points N_k^1 and N_k^2 such that the y-coordinate of M_k^- lies between those of N_k^1 and N_k^2. The distance between the points $N_k^{1,2}$ and M_k^- tends to zero as $k \to +\infty$.*

Proof. By lemma 2, it is sufficient to prove lemma 3 only for the case where the homoclinic tangency of F_0 belongs to class (2), namely, we may assume that $cdx^+ < 0$ and $dy^- > 0$. We

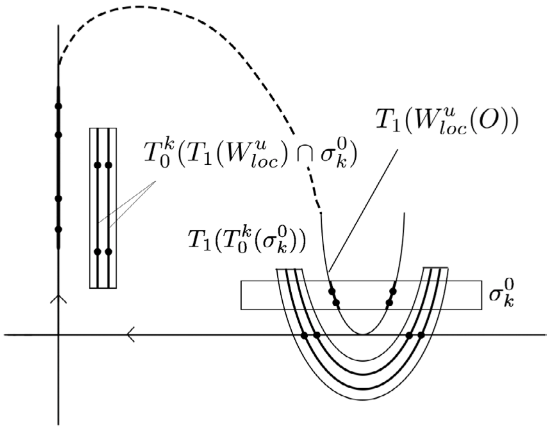

Figure 5. Transverse homoclinic points at $\mu = 0$.

start with showing that in this case there exist infinitely many transverse homoclinic points at $\mu = 0$. Indeed, non-degenerate solutions of system (34) correspond to transverse homoclinic points. By using the scaling

$$(X, Y) \mapsto \left(b|\gamma|^{-\frac{k}{2}} \sqrt{\frac{y^-}{d}} U, |\lambda|^{\frac{k}{2}} \sqrt{\frac{cx^+}{d}} V \right),$$

we rewrite system (34) at $\mu = 0$ as

$$\begin{aligned} 1 &= U^2 + o(1)_{k\to+\infty}, \\ 1 &= V^2 + o(1)_{k\to+\infty}. \end{aligned} \tag{52}$$

This gives four non-degenerate solutions to (34) at $\mu = 0$

$$(X, Y) = \left(\pm b|\gamma|^{-\frac{k}{2}} \sqrt{\frac{y^-}{d}} + o(|\gamma|^{-\frac{k}{2}}), \pm|\lambda|^{\frac{k}{2}} \sqrt{\frac{cx^+}{d}} + o(|\lambda|^{\frac{k}{2}}) \right) =: \left(\pm\tilde{X} + o(|\gamma|^{-\frac{k}{2}}), \pm\tilde{Y} + o(|\lambda|^{\frac{k}{2}}) \right) \tag{53}$$

for any sufficiently large k. These solutions correspond to four transverse homoclinic points in $T_1(W^u_{\text{loc}}(O))$:

$$\begin{aligned} N_k^1 &= \left(x^+ + \tilde{X} + o(|\gamma|^{-\frac{k}{2}}), \gamma^{-k}(y^- + \tilde{Y}) + o(\gamma^{-k}), z^1 \right), \\ N_k^2 &= \left(x^+ + \tilde{X} + o(|\gamma|^{-\frac{k}{2}}), \gamma^{-k}(y^- - \tilde{Y}) + o(\gamma^{-k}), z^2 \right), \\ N_k^3 &= \left(x^+ - \tilde{X} + o(|\gamma|^{-\frac{k}{2}}), \gamma^{-k}(y^- + \tilde{Y}) + o(\gamma^{-k}), z^3 \right), \\ N_k^4 &= \left(x^+ - \tilde{X} + o(|\gamma|^{-\frac{k}{2}}), \gamma^{-k}(y^- - \tilde{Y}) + o(\gamma^{-k}), z^4 \right). \end{aligned} \tag{54}$$

Denote $T_1^{-1}(N_k^i)$ by $\hat{N}_k^i = (0, \hat{y}_k^i, 0)$. It follows from the first equation of (19) that $\hat{y}_k^{1,2} > y^-$ and $\hat{y}_k^{3,4} < y^-$, which means that the tangency point M^- is bounded by the four transverse homoclinic points $\hat{N}_k^i$ (see figure 5). Moreover, we have from the second equation of (19) that $\hat{y}_k^1 > \hat{y}_k^2$ and $\hat{y}_k^3 > \hat{y}_k^4$. By transversality, for each fixed k, all four homoclinic intersections persist for all sufficiently small μ.

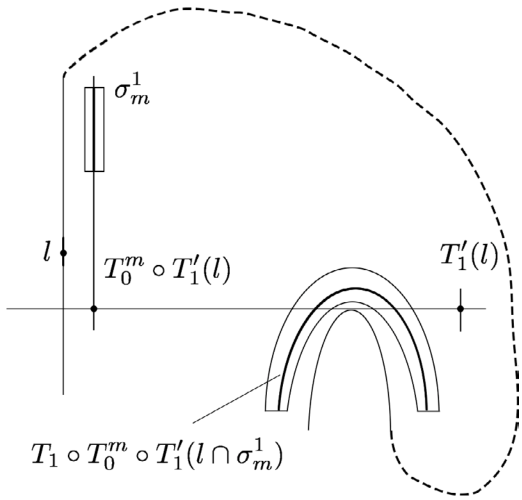

Figure 6. By changing μ, one can make $T_1 \circ T_0^m \circ T_1'(l)$ intersect $W^s_{\rm loc}(O)$ non-transversely. Here $l \in W^u_{\rm loc}(O)$ is a small piece containing the transverse homoclinic point.

In what follows we prove that there exists a sequence $\{\mu_m\}$ accumulating on $\mu = 0$ such that for each sufficiently large m the diffeomorphism F_{μ_m} has a non-transverse homoclinic point $M_m^- \in W^u_{\rm loc}(O)$ that belongs to class (4) and satisfies either $M_m^- \to \hat{N}_k^2$ or $M_m^- \to \hat{N}_k^3$ as $m \to +\infty$. This will complete the proof of the lemma after noting that class (4) tangencies satisfy condition (a), both $\hat{N}_k^2$ and $\hat{N}_k^3$ are bounded by the two transverse homoclinic points $\hat{N}_k^1$ and $\hat{N}_k^4$, and these points all tend to M^- as $k \to +\infty$.

We denote as T_1' the restriction of the global map T_1 to a small neighbourhood of the transverse homoclinic point $\hat{N}_k^2 = (0, \hat{y}_k, 0)$. We denote $T_1'(\hat{N}_k^2) = N_k^2 = (\hat{x}^+, 0, \hat{z}^+)$ and write the Taylor expansion of T_1' about the point $\hat{N}_k^2$ as

$$\begin{aligned} \bar{x} - \hat{x}^+ &= a'x + b'(y - \hat{y}_k) + a_{13}'z + h_1'(x,y,z), \\ \bar{y} &= c'x + d'(y - \hat{y}_k) + a_{23}'z + h_2'(x,y,z), \\ \bar{z} - \hat{z}^+ &= a_{31}'x + a_{32}'(y - \hat{y}_k) + a_{33}'z + h_3'(x,y,z), \end{aligned} \tag{55}$$

where $h_{1,2,3}' = O(x^2 + y^2 + z^2)$. The coefficients in these formula are obtained by evaluating, at $(0, \hat{y}_k, 0)$, the first derivatives of the map T_1 given by (19). Obviously,

$$a' = a + \dots, \quad b' = b + \dots, \quad c' = c + \dots, \quad d' = 2d(\hat{y}_k - y^-)(1 + \dots), \tag{56}$$

where the dots denote terms that tend to zero as $k \to +\infty$. We now create a homoclinic tangency by finding a point $M_m^- \in W^u_{\rm loc}(O)$ close to $\hat{N}_k^2$ such that $M_m^+ := T_1 \circ T_0^m \circ T_1'(M) \in W^s_{\rm loc}(O)$ for some m, and the curve $T_1 \circ T_0^m \circ T_1'(W^u_{\rm loc}(O))$ is tangent to $W^s_{\rm loc}(O)$ at the point M_m^+ as shown in figure 6.

Let m be even, so that λ^m and γ^{-m} are positive. The image $T_1'(W^u_{\rm loc}(O))$ is given by

$$\begin{aligned} y_0 &= \frac{d'}{b'}(x_0 - \hat{x}^+) + o(x_0 - \hat{x}^+), \\ z_0 - \hat{z}^+ &= \frac{a_{32}'}{b'}(x_0 - \hat{x}^+) + o(x_0 - \hat{x}^+). \end{aligned}$$

For any point $(x_0, y_0, z_0) \in T_1'(W^u_{\rm loc}(O)) \cap \sigma^0_m$, we can find its mth iterate $(x_m, y_m, z_m) = T_0^m(x_0, y_0, z_0)$ by using formula (8):

$$\begin{array}{rcl} x_m &=& \lambda^m x_0 + o(\lambda^m), \\ y_0 &=& \gamma^{-m} y_m + o(\gamma^{-m}), \\ z_m &=& O(\hat{\lambda}^m), \end{array} \tag{57}$$

The point (x, y, z) is a homoclinic point if and only if $T_1(x_m, y_m, z_m) \in W^s(O)$, namely,

$$0 = \mu + cx_m + d(y_m - y^-)^2 + a_{23} z_m + h_2(x_m, y_m, z_m).$$

Then, by repeating the same procedure as was used to find equation (34), we obtain

$$\begin{array}{rcl} 0 &=& -\gamma^{-m} y^- - \gamma^{-m} Y + \frac{d'}{b'} X + u_1(X, Y, \mu) + u_2(X, \mu), \\ 0 &=& \mu + c\lambda^m \hat{x}^+ + c\lambda^m X + dY^2 + u_3(X, Y, \mu) + u_4(Y, \mu), \end{array} \tag{58}$$

where $X = x - \hat{x}^+, Y = y_m - y^-, u_1 = o(\gamma^{-m}), u_2 = o(X), u_3 = o(\lambda^m)$, and $u_4 = o(\lambda^m + Y^2)$.

In order to have a homoclinic tangency, we need the Jacobian matrix of the right-hand side of (58) to have zero determinant, namely,

$$c\lambda^m \gamma^{-m} + \frac{2dd'}{b'}(Y + v(X, Y)) + o(Y) + o(\lambda^m \gamma^{-m}) = 0, \tag{59}$$

where $v = o(\lambda^m + |Y|)$. After the coordinate transformation

$$(\hat{X}, \hat{Y}) = (X, Y + v(X, Y)), \tag{60}$$

equation (58) keep their form, and equation (59) is recast as

$$c\lambda^m \gamma^{-m} + \frac{2dd'}{b'} \hat{Y} + o(\hat{Y}) + o(\lambda^m \gamma^{-m}) = 0. \tag{61}$$

The quadratic tangencies correspond to non-degenerate solutions to the system consisting of (58) and (61). With a straightforward computation one can find the solutions as

$$\begin{array}{rcl} \hat{X}_m &=& \frac{b' \gamma^{-m} y^-}{d'} + o(\gamma^{-m}), \\ \hat{Y}_m &=& -\frac{b' c \lambda^m \gamma^{-m}}{2dd'} + o(\lambda^m \gamma^{-m}), \\ \mu_m &=& -c\lambda^m \hat{x}^+ + o(\lambda^m), \end{array} \tag{62}$$

where m is sufficiently large, and each solution gives a non-transverse homoclinic point $M_m^- \in W^u_{\rm loc}(O)$ corresponding to a quadratic tangency at $\mu = \mu_m$.

The global map associated to M_m^- is $\hat{T} := T_1 \circ T_0^m \circ T_1' : (x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z})$, and the corresponding coefficients c_m and d_m are given by

$$\hat{c} = \frac{\partial \bar{y}}{\partial x}_{M_m^-} \quad \text{and} \quad \hat{d} = \frac{1}{2} \frac{\partial^2 \bar{y}}{\partial y^2}|_{M_m^-}. \tag{63}$$

Similar to the computation of such coefficients in the proof of lemma 2, by applying the chain rule to equations (19), (55) and (57), and using the formulas (59) and (62), we have

$$c_m = a' c \lambda^m + 2c' d \gamma^m \hat{Y}_m + o(\lambda^m) = c\lambda^m \left(\frac{a'd' - b'c'}{d'} \right) + o(\lambda^m), \tag{64}$$

$$d_m = dd'^2\gamma^{2m} + o(\gamma^m). \tag{65}$$

Equation (65) means that d_m has the same sign as d, which is positive. Equation (64) for c_m can be recast as

$$c_m = c\lambda^m \left(\frac{2ad(\hat{y}_k - y^-) - bc + \dots}{2d(\hat{y}_k - y^-)(1 + \dots)} \right) + o(\lambda^m). \tag{66}$$

Since $\hat{y}_k - y^-$ can be sufficiently small, the estimates in (56) imply that the sign of c_m is the same as $-b/(d(\hat{y}_k - y^-))$. It follows from $d > 0$ and $\hat{y}_k - y^- > 0$ that if $b < 0$, then we have $c_m > 0$, and this gives us the class (4) homoclinic tangency; if $b > 0$, then we just need to consider the point $\hat{N}_k^3$, for which $\hat{y}_k - y^- < 0$, instead of $\hat{N}_k^2$ in (55). $\square$

4. Invariant cone fields

In this section, we prove the existence of certain invariant cone fields in Π_0. These cone fields will help in two ways. First, estimates for the strong-stable leaves are obtained from stable invariant cones in lemmas 7 and 8. Second, we use the cones to obtain estimates for the multipliers of periodic orbits.

Recall that $\sigma_k^0 \subset \Pi_0$ $(k \geqslant k^*)$ are the sets of points whose images under T_0^k belong to Π_1, where k^* is the smallest integer such that $T_0^{k^*}(\Pi_0) \cap \Pi_1 \neq \emptyset$. Denote by Σ^0 the union of all σ_k^0 with $k \geqslant k^*$. For any $X \in \Sigma^0$, we have $T(X) = T_1 \circ T_0^k(X)$ where k is such that $X \in \sigma_k^0$.

Lemma 4. *If k^* is sufficiently large, then there exist constants $K > 0$ and $M > 0$ such that the cone field $\mathcal{C}^{cu}$ over Σ^0 (the center unstable cone filed) defined as*

$$\mathcal{C}^{cu}(X) = \{(\Delta x, \Delta y, \Delta z) \mid \|\Delta z\| \leqslant K(|\Delta x| + |\Delta y|)\} \tag{67}$$

is strictly forward-invariant under the derivative $\mathrm{D}T$ of the first-return map T (here, $(\Delta x, \Delta y, \Delta z)$ are coordinates in the tangent space to Σ^0). Moreover,

$$\|\mathrm{D}T(X)V\| \geqslant M|\lambda|^k\|V\| \tag{68}$$

for any $V \in \mathcal{C}^{cu}(X)$.

Proof. Take any $X \in \sigma_k^0$ and let $V_0 = (\Delta x_0, \Delta y_0, \Delta z_0)$ be a vector in the tangent space at the point X such that

$$\|\Delta z_0\| \leqslant K(|\Delta x_0| + |\Delta y_0|), \tag{69}$$

where $K > 0$ is some constant. Denote $\mathrm{D}T_0^k(X)V_0 = (\Delta x_1, \Delta y_1, \Delta z_1)$ and $\mathrm{D}T_1\mathrm{D}T_0^k(X)V_0 = (\Delta x_2, \Delta y_2, \Delta z_2)$. By formula (8) and noting that the first derivatives of the functions $\phi, \hat{\phi}$ and ψ in (8) are bounded, we have the following relations:

$$\Delta x_1 = \lambda^k \Delta x_0 + o(\lambda^k)(\Delta x_0 + \Delta y_1 + \Delta z_0), \tag{70}$$

$$\Delta y_0 = \gamma^{-k}\Delta y_1 + o(\gamma^{-k})(\Delta x_0 + \Delta y_1 + \Delta z_0), \tag{71}$$

$$\Delta z_1 = O(\hat{\lambda}^k)(\Delta x_0 + \Delta y_1 + \Delta z_0). \tag{72}$$

Equations (70) and (71) can be recast as

$$\Delta x_0 = \lambda^{-k}\Delta x_1(1+o(1)) + o(1)(\Delta y_1 + \Delta z_0), \tag{73}$$

$$\Delta y_0 = \gamma^{-k}\Delta y_1(1+o(1)) + o(\gamma^{-k})(\Delta x_0 + \Delta z_0). \tag{74}$$

By plugging these two equations into (69), we obtain

$$\|\Delta z_0\| \leqslant K|\lambda|^{-k}|\Delta x_1|(1+o(1)) + o(1)|\Delta y_1|,$$

where we denote by $o(1)$ the terms that go to zero as $k \to +\infty$. The above equation together with (72) and (73) implies

$$\|\Delta z_1\| \leqslant O(\hat{\lambda}^k\lambda^{-k})|\Delta x_1| + O(\hat{\lambda}^k)|\Delta y_1| \tag{75}$$

and

$$\|\Delta x_0\| + \|\Delta y_0\| = O(\lambda^{-k})(\|\Delta x_1\| + \|\Delta y_1\|). \tag{76}$$

The derivative $\mathrm{D}T_1$ is uniformly bounded in a small neighbourhood Π_1, so we have

$$\|\Delta z_2\| \leqslant \sup\|\mathrm{D}T_1\|(|\Delta x_1| + |\Delta y_1| + \|\Delta z_1\|).$$

Hence, when k^* is large enough, the above inequality together with (75) gives

$$\|\Delta z_2\| \leqslant (1+\sup\|\mathrm{D}T_1\|)(|\Delta x_1| + |\Delta y_1|). \tag{77}$$

Note that by (19) we have

$$\begin{pmatrix}\Delta x_2\\ \Delta y_2\end{pmatrix} = B_1\begin{pmatrix}\Delta x_1\\ \Delta y_1\end{pmatrix} + B_2\Delta z_1,$$

for some matrices B_1 and B_2, whose norm is uniformly bounded. In fact, B_1 is close to $\begin{pmatrix}a & b\\ c & 0\end{pmatrix}$, so, by (22), $\det(B_1) \neq 0$, i.e. B_1 is invertible. Thus, we have

$$\begin{pmatrix}\Delta x_1\\ \Delta y_1\end{pmatrix} = B_1^{-1}\begin{pmatrix}\Delta x_2\\ \Delta y_2\end{pmatrix} - B_1^{-1}B_2\Delta z_1. \tag{78}$$

By taking k^* sufficiently large, equations (75) and (78) imply

$$\left\|\begin{pmatrix}\Delta x_1\\ \Delta y_1\end{pmatrix}\right\| \leqslant 2\|B_1^{-1}\|\left\|\begin{pmatrix}\Delta x_2\\ \Delta y_2\end{pmatrix}\right\|. \tag{79}$$

We now combine the two inequalities (77) and (79). It follows that, by taking k^* sufficiently large, we have

$$\|\Delta z_2\| < 4\|B_1^{-1}\|(1+\sup\|\mathrm{D}T_1\|)(|\Delta x_2| + |\Delta y_2|), \tag{80}$$

which implies the lemma after letting $K = 4\|B_1^{-1}\|(1+\sup\|\mathrm{D}T_1\|)$; estimate (68) follows from (78) and (76). □

The existence of the center-unstable cone field $\mathcal{C}^{cu}$ implies that the areas of certain surfaces are expanded by the map T. We denote by $\mathcal{A}(S)$ the area of a surface S.

Lemma 5. *There exists $L > 0$ such that for any surface $S \subset \sigma_k^0$ such that its tangent space at every point lies in the cone field $\mathcal{C}^{cu}$, we have*

$$\mathcal{A}(T(S)) > L|\lambda\gamma|^k\mathcal{A}(S). \tag{81}$$

Proof. Since the cone field $\mathcal{C}^{cu}$ has the same form (67) at all points, we have that, for any surface S whose tangent space lies in $\mathcal{C}^{cu}$, its equation takes the form $z = S(x, y)$ and the derivatives $\partial S/\partial x$ and $\partial S/\partial x$ are uniformly bounded away from zero and infinity. Thus, there exist positive constants L_1 and L_2 such that

$$L_2\mathcal{A}(\pi_0(S)) < \mathcal{A}(S) < L_1\mathcal{A}(\pi_0(S)), \tag{82}$$

where π_0 is the projection onto the (x, y)-plane. Since C^{cu} is invariant under DT, the tangent space of $T(S)$ also lies in C^{cu}. Therefore,

$$\mathcal{A}(T(S)) > L_2\mathcal{A}(\pi_0(T(S))). \tag{83}$$

Let $G = \pi_0 \circ T|_{z=0} : (x, y) \mapsto (\bar{x}, \bar{y})$. We note that

$$\mathcal{A}(\pi_0(T(S))) = \int_{\pi_0(T(S))} \mathrm{d}x\mathrm{d}y = \int_{\pi_0(S)} |\det \mathrm{D}G|\mathrm{d}u\mathrm{d}v$$

and

$$\mathcal{A}(\pi_0(S)) = \int_{\pi_0(S)} \mathrm{d}u\mathrm{d}v.$$

Therefore, in order to prove the lemma, it is sufficient to show that there exists $L_3 > 0$ such that

$$|\det \mathrm{D}G| > L_3|\lambda\gamma|^k. \tag{84}$$

In what follows we prove inequality (84). By (8), the map $T_0^k|_{z=0}$ is given by

$$\begin{aligned} x_k &= \lambda^k x + \phi_k(x, y_k, 0), \\ y &= \gamma^{-k} y_k + \psi_k(x, y_k, 0), \\ z_k &= \hat{\phi}_k(x, y_k, 0). \end{aligned}$$

By (9) and (10), this map can be rewritten as

$$\begin{aligned} x_k &= \lambda^k x + \tilde{\phi}_k(x, y), \\ y_k &= \gamma^k y + \tilde{\psi}_k(x, y), \end{aligned} \tag{85}$$

where

$$\begin{aligned} &\tilde{\phi}_k = o(|\lambda|^k), \qquad \partial_x\tilde{\phi}_k = o(|\lambda|^k), \qquad \partial_y\tilde{\phi}_k = o(|\lambda\gamma|^k), \\ &\tilde{\psi}_k = o(1), \qquad \partial_x\tilde{\psi}_k = o(1), \qquad \partial_y\tilde{\psi}_k = o(|\gamma|^k). \end{aligned} \tag{86}$$

One can also express z_k as a function of x_k and y_k and see that this function satisfies

$$z_k = O(\hat{\lambda}^k), \qquad \partial_{x_k} z_k = O(\hat{\lambda}^k\lambda^{-k}), \qquad \partial_{y_k} z_k = O(\hat{\lambda}^k). \tag{87}$$

The map G can be written as the composition of the map (85) and the map $T_1|_{T_0^k(\{z=0\})}$ which, by (19), is given by

$$\begin{aligned} \bar{x} &= x^+ + ax_k + b(y_k - y^-) + a_{13}z_k + h_1(x_k, y_k, z_k), \\ \bar{y} &= \mu + cx_k + d(y_k - y^-)^2 + a_{23}z_k + h_2(x_k, y_k, z_k). \end{aligned}$$

The above formulas yield

$$\frac{\partial(x_k, y_k)}{(x, y)} = \begin{pmatrix} \lambda^k + o(\lambda^k) & o(\lambda^k\gamma^k) \\ o(1)_{k\to+\infty} & \gamma^k + o(\gamma^k) \end{pmatrix}, \tag{88}$$

and

$$\frac{\partial(\bar{x},\bar{y})}{(x_k,y_k)} = \begin{pmatrix} a+O(|\hat{\lambda}^k|\lambda|^{-k}|+|y_k-y^-|) & b+O(|\lambda|^k+|y_k-y^-|) \\ c+O(|\hat{\lambda}^k|\lambda|^{-k}|+|y_k-y^-|) & 2d(y_k-y^-)+O(|\lambda|^k+(y_k-y^-)^2) \end{pmatrix}. \tag{89}$$

A straightforward computation gives

$$|\det \mathrm{D}G| = \left|\det \frac{\partial(\bar{x},\bar{y})}{(x_k,y_k)} \det \frac{\partial(x_k,y_k)}{(x,y)}\right| = |bc+O(y_k-y^-)|\;|\lambda\gamma|^k + o(|\lambda\gamma|^k).$$

The term $y_k - y^-$ is bounded by the small number δ (the size of Π_0 and Π_1), and $bc \neq 0$ by (22). It follows that (84) holds indeed for some $L_3 > 0$. $\square$

We proceed to find a stable cone field.

Lemma 6. *There exists a stable cone field $\mathcal{C}^s$ over $\Sigma^0 \cap T(\Sigma^0)$ which is strictly backward-invariant under* $\mathrm{D}T$. *The cone at the point $X \in \sigma_k^0 \cap T(\Sigma^0)$ is given by*

$$\mathcal{C}^s(X) = \{(\Delta x,\Delta y,\Delta z) \mid |\Delta x| \leqslant K_1\hat{\lambda}^k|\lambda|^{-k}\|\Delta z\|, |\Delta y| \leqslant K_2\hat{\lambda}^k|\gamma|^{-k}\|\Delta z\|\}, \tag{90}$$

where K_1 and K_2 are some positive constants, independent of X. The restriction of $\mathrm{D}T$ *to $\mathcal{C}^s$ is contracting, i.e. there exists $M > 0$ such that*

$$\|\mathrm{D}T(X)V\| \leqslant M\hat{\lambda}^k\|V\| \tag{91}$$

for any $V \in \mathcal{C}^s(X)$.

Proof. Let $Y = T_1 \circ T_0^k(X)$ and let $V_2 = (\Delta x_2, \Delta y_2, \Delta z_2)$ be a vector in the tangent space at Y such that

$$|\Delta x_2| \leqslant S\|\Delta z_2\| \text{ and } |\Delta y_2| \leqslant S\|\Delta z_2\|, \tag{92}$$

where $S > 0$ is a constant. Denote $\mathrm{D}T_1^{-1}(Y)V_2 = (\Delta x_1, \Delta y_1, \Delta z_1)$. From (19), we have

$$\begin{aligned} (\Delta x_2, \Delta y_2) &= B_1(\Delta x_1, \Delta y_1) + B_2\Delta z_1, \\ \Delta z_2 &= B_3(\Delta x_1, \Delta y_1) + B_4\Delta z_1, \end{aligned} \tag{93}$$

where B_i $(i = 1\ldots 4)$ are some matrices whose norms are uniformly bounded. Note that B_1 is close to $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $bc \neq 0$ by (22), so the matrix B_1 is invertible.

Now, equation (93) can be rewritten as

$$\begin{aligned} (\Delta x_1, \Delta y_1) &= B_1^{-1}(\Delta x_2, \Delta y_2) - B_1^{-1}B_2\Delta z_1, \\ \Delta z_2 &= B_3B_1^{-1}(\Delta x_2, \Delta y_2) + (B_4 - B_1^{-1}B_2)\Delta z_1. \end{aligned} \tag{94}$$

By choosing S such that $S\|B_2B_1^{-1}\| < 1$, we obtain

$$\|B_3B_1^{-1}(\Delta x_2, \Delta y_2)\| \leqslant S\|B_3B_1^{-1}\|\;\|\Delta z_2\| < \hat{S}\|\Delta z_2\|,$$

where $\hat{S} < 1$ is a constant, independent of the choice of the point Y and the vector V_2.

Hence, the second equation in (94) implies $\|\Delta z_2\| = O(\|\Delta z_1\|)$, which by (92) further implies

$$|\Delta x_2| = O(\|\Delta z_1\|) \quad \text{and} \quad |\Delta y_2| = O(\|\Delta z_1\|). \tag{95}$$

Finally, from the first equation in (94) we find

$$|\Delta x_1| + |\Delta y_1| \leqslant l\|\Delta z_1\|, \tag{96}$$

where l is some positive constant, independent of the choice of Y and V_2.

Denote $\mathrm{D}T_0^{-k}\mathrm{D}T_1^{-1}(Y)V_2 = (\Delta x_0, \Delta y_0, \Delta z_0)$. By formula (8), noting that the first derivatives of $\phi, \hat{\phi}$ and ψ are bounded, we have the following relations:

$$\Delta x_1 = \lambda^k \Delta x_0 + o(\lambda^k)(\Delta x_0 + \Delta y_1 + \Delta z_0), \tag{97}$$

$$\Delta y_0 = \gamma^{-k}\Delta y_1 + o(\gamma^{-k})(\Delta x_0 + \Delta y_1 + \Delta z_0), \tag{98}$$

$$\Delta z_1 = O(\hat{\lambda}^k)(\Delta x_0 + \Delta y_1 + \Delta z_0). \tag{99}$$

Estimates (99) and (96) give

$$\|\Delta z_1\| = O(\hat{\lambda}^k)(|\Delta x_0| + \|\Delta z_0\|),$$
$$\|\Delta y_1\| = O(\hat{\lambda}^k)(|\Delta x_0| + \|\Delta z_0\|).$$

With these estimate and (96), equation (97) yields

$$|\Delta x_0| = O(\lambda^{-k})\|\Delta z_1\| + o(\|\Delta z_0\|).$$

By plugging the above equation into (99), we obtain

$$|\Delta z_1| = O(\hat{\lambda}^k)\|\Delta z_0\|, \tag{100}$$

which, along with (96), further implies

$$|\Delta x_1| + |\Delta y_1| = O(\hat{\lambda}^k)\|\Delta z_0\|. \tag{101}$$

Finally, the above equation together with (97) and (98) leads to

$$|\Delta x_0| = o(1)_{k\to+\infty}\|\Delta z_0\|,$$
$$|\Delta y_0| = o(\gamma^{-k})\|\Delta z_0\|.$$

This formula shows that the image by $\mathrm{D}T^{-1}$ of a vector satisfying (92) lies in the cone (90). If k^* was taken suffcently large, then every vector from the cone (90) satisfies (92), i.e. we have proven the required invariance of the cone field (90). Estimate (91) follows from (100) and (101) and the uniform boundedness of $\mathrm{D}T$. □

The strong-stable foliation $\mathcal{F}_0$ which exists in the stable manifold $W^s(O)$ extends to an invariant foliation $\mathcal{F}^s$ in a small neighborhood of the homoclinic cycle $O \cup \Gamma \cup \tilde{\Gamma}$ we consider here (see [49]). As the tangents to the leaves of the invariant foliation $\mathcal{F}^s$ must lie in the stable invariant cone $\mathcal{C}^s$, lemma 6 immediately implies the following formula for the leaves of $\mathcal{F}^s$.

Lemma 7. *The leaf of the strong-stable foliation $\mathcal{F}^s$ through a point $(x^*, y^*, z^*) \in \Sigma^0$ with a stay number k takes the form*

$$\begin{array}{rcl} x & = & x^* + \varphi_1(z; x^*, y^*, z^*), \\ y & = & y^* + \varphi_2(z; x^*, y^*, z^*), \end{array} \tag{102}$$

where

$$\begin{array}{ll} \varphi_1 = o(1)_{k\to+\infty}, & \frac{\partial\varphi_1}{\partial z} = o(1)_{k\to+\infty}, \\ \varphi_2 = o(\gamma^{-k}), & \frac{\partial\varphi_1}{\partial z} = o(\gamma^{-k}). \end{array}$$

Note that we do not estimate the derivatives of $\varphi_{1,2}$ with respect to (x^*, y^*, z^*) here.

In the proof of lemma 6, we have not used condition (C4) on the multipliers of O. Formula (102) will only be helpful in the non-symmetric case (theorem 2) where we have more parameters to do the bifurcation. When it comes to the symmetric case (theorem 1), we need a better estimate, which will be obtained by taking into account condition (C4).

Lemma 8. *If condition (C4) is satisfied, then the strong-stable leaf through a point $(x^*, y^*, z^*) \in \Sigma^0$ with a stay number k assumes the same form as in (102), but the function φ_1 now satisfies*

$$\varphi_1 = O(\lambda_0^k \lambda^{-k}), \qquad \frac{\partial\varphi_1}{\partial z} = O(\lambda_0^k \lambda^{-k}), \tag{103}$$

where λ_0 can be taken arbitrarily close to $|\lambda_1|$.

Proof. Take any point $X \in \sigma_k^0$ and consider a vector $(\Delta x_0, \Delta y_0, \Delta z_0)$ in the tangent space, at X, to the leaf of the invariant foliation $\mathcal{F}^s$ through X. We need to show that

$$|\Delta x| \leqslant K\lambda_0^k \lambda^{-k} \|\Delta z\| \tag{104}$$

for some constant K, independent of X.

Let $(x_k, y_k, z_k) = T_0^k X$ and $(\Delta x_1, \Delta y_1, \Delta z_1) = \mathrm{D}T_0^k(\Delta x_0, \Delta y_0, \Delta z_0)$. By formula (8), we have

$$\begin{array}{rcl} \Delta x_1 & = & \lambda^k(1 + \ldots)\Delta x_0 + o(\lambda^k)\Delta y_1 + \frac{\partial x_k}{\partial z_0}\Delta z_0, \\ \Delta z_1 & = & \frac{\partial z_k}{\partial z_0}\Delta z_0 + o(\lambda^k)\Delta x_0 + o(\lambda^k)\Delta y_1, \end{array} \tag{105}$$

where the dots denote terms that tend to zero as $k \to +\infty$. Since the vector $(\Delta x_0, \Delta y_0, \Delta z_0)$ is in the stable cone C^s, its image V by $\mathrm{D}(T_1 T_0^k)$ is also in C^s. So, as we have shown in the proof of lemma 6, the vector $(\Delta x_1, \Delta y_1, \Delta z_1) = \mathrm{D}T_1^{-1} V$ must satisfy

$$|\Delta x_1| + |\Delta y_1| = O(\|\Delta z_1\|),$$

see (96). Plugging this into (105) gives

$$\lambda^k(1 + \ldots)\Delta x_0 = O(\|\frac{\partial x_k}{\partial z_0}\| + \|\frac{\partial z_k}{\partial z_0}\|)\Delta z_0.$$

By lemma 1, this inequality implies (104). $\square$

5. The index-2 condition

In this section we find a condition which ensures that a period-2 point of T is a saddle of index 2. We will start with a result describing the multipliers of a periodic point (in a more general case where period-n orbits are considered).

Let $X \in \Sigma^0$ be a period-n point such that $X = T^n(X) = T_1 \circ T_0^{k_n} \circ T_1 \circ T_0^{k_{n-1}} \circ \cdots \circ T_1 \circ T^{k_1}(X)$, where $k_1, \ldots, k_n$ are the corresponding stay numbers. We sort the eigenvalues of $\mathrm{D}T^n$ the multipliers of X in decreasing order by their absolute values and denote them as $\nu_1, \ldots, \nu_D$. By lemmas (4) and (6), the derivative $\mathrm{D}T^n$ at X has a pair of invariant cones, which implies the existence of a two-dimensional invariant subspace E^{cu} (in the center-unstable cone) and a $(D-2)$-dimensional invariant subspace E^s in the stable cone. Estimates (68) and (91) for $\mathrm{D}T^n$ restricted to E^{cu} and, respectively, E^s immediately give the the following estimate on the multipliers of X.

Lemma 9. *The eigenvalues of $\mathrm{D}T^n|_{E^{cu}}$ are ν_1 and ν_2, and the eigenvalues of $\mathrm{D}T^n|_{E^s}$ are $\nu_3, \ldots, \nu_D$. Moreover, we have*

$$|\nu_i|^{-1} = O(|\lambda|^{k_1+\cdots+k_n}), \quad i = 1, 2, \tag{106}$$

and

$$|\nu_i| = O(\hat{\lambda}^{k_1+\cdots+k_n}), \quad i = 3, 4, \ldots, D. \tag{107}$$

We now consider orbits of period 2, and find the condition under which such point is an index-2 saddle, i.e. $|\nu_1| > 1$ and $|\nu_2| > 1$. Let $Q \in \Pi_0$ be a period-2 point of T with stay numbers k and m. Denote $Q_{01} = Q = (x_{01}, y_{01}, z_{01})$, $Q_{11} = T_0^k(Q) = (x_{11}, y_{11}, z_{11})$, $Q_{02} = T_1 \circ T_0^k(Q) = (x_{02}, y_{02}, z_{02})$ and $Q_{12} = T_0^m \circ T_1 \circ T_0^k(Q) = (x_{12}, y_{12}, z_{12})$.

Lemma 10. *There exist functions $r_{1,2,3,4}$, which depends on the integers m and k, parameters and the coordinates of the points Q_{ij}, such that the point Q is a saddle of index 2 if and only if there exists some number $s \in (-1, 1)$ such that*

$$(y_{11} - y^- + r_1)(y_{12} - y^- + r_2) = r_3 + r_4 s. \tag{108}$$

The functions $r_{1,2,3,4}$ satisfy

$$\begin{aligned} &r_1 = O((y_{11} - y^-)^2 + |\lambda|^k + |\gamma|^{-m}), \qquad r_2 = O((y_{12} - y^-)^2 + |\lambda|^m + |\gamma|^{-k}), \\ &r_3 = O(|\lambda|^k|\gamma|^{-k}| + |\lambda|^m|\gamma|^{-m} + |\lambda|^{(k+m)}), \qquad r_4 = O(\lambda^{(k+m)}). \end{aligned} \tag{109}$$

Proof. One can check that the condition $|\nu_1|, |\nu_2| > 1$ is equivalent to

$$|\nu_1\nu_2| > 1 \quad \text{and} \quad \frac{\nu_1 + \nu_2}{\nu_1\nu_2 + 1} = s, \qquad -1 < s < 1.$$

This can be written as

$$|\det \mathrm{D}T^2|_{E^{cu}}| > 1 \quad \text{and} \quad \frac{\operatorname{tr} \mathrm{D}T^2|_{E^{cu}}}{\det \mathrm{D}T^2|_{E^{cu}} + 1} = s, \qquad -1 < s < 1, \tag{110}$$

where E^{cu} is the two-dimensional invariant subspace introduced before lemma 9. In what follows, we use $(\Delta x, \Delta y, \Delta z)$ to denote a vector in E^{cu}. Note that Δz is a function of Δx and Δy. We, thus, need to compute the trace and the determinant of $\mathrm{D}T^2|_{E^{cu}} : (\Delta x, \Delta y) \mapsto (\Delta \bar{x}, \Delta \bar{y})$.

Denote

$$\eta_1 = y_{11} - y^- \quad \text{and} \quad \eta_2 = y_{12} - y^-. \tag{111}$$

Take a vector $V = (\Delta x_1, \Delta y_1, \Delta z_1) \in E^{cu}$. Formula (8) implies that

$$\mathrm{D}T_0^k|_{E^{cu}} V = A_1 \begin{pmatrix} \Delta x_1 \\ \Delta y_1 \end{pmatrix} = \begin{pmatrix} \lambda^k + o(\lambda^k) & o(\lambda^k\gamma^k) \\ o(1)_{k\to+\infty} & \gamma^k + o(\gamma^k) \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta y_1 \end{pmatrix} =: \begin{pmatrix} \Delta x_2 \\ \Delta y_2 \end{pmatrix}. \tag{112}$$

Note that the Δz_1 component is a bounded function of $(\Delta x_1, \Delta y_1)$ and its contribution to Δx_2 and Δy_2 goes into the small terms in A_1.

After noting $x_{11} = O(\lambda^k)$ and $z_{11} = O(\hat{\lambda}^k)$ from (8), we can write the matrix $\mathrm{D}T_1(Q_{11})$ as

$$\begin{pmatrix} a + O(|\lambda|^k + |\eta_1|) & b + O(|\lambda|^k + |\eta_1|) & a_{13} + O(|\lambda|^k + |\eta_1|) \\ c + O(|\lambda|^k + |\eta_1|) & 2d\eta_1 + O(|\lambda|^k + \eta_1^2) & a_{23} + O(|\lambda|^k + |\eta_1|) \\ a_{31} + O(|\lambda|^k + |\eta_1|) & a_{32} + O(|\lambda|^k + |\eta_1|) & a_{33} + O(|\lambda|^k + |\eta_1|) \end{pmatrix}. \tag{113}$$

Since the vector $\mathrm{D}T_0^k V_1 = (\Delta x_2, \Delta y_2, \Delta z_2)$ belongs to $\mathrm{D}T_0^k \mathcal{C}^{cu}$, we have from equation (75) that

$$\Delta z_1 = O(\hat{\lambda}^k \lambda^{-k})\Delta x_1 + O(\hat{\lambda}^k)\Delta y_1.$$

Along with (113), this leads to

$$\mathrm{D}(T_1 \circ T_0^k)|_{E^{cu}} V_1 = A_2 \begin{pmatrix} \Delta x_2 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} a + O(|\hat{\lambda}^k \lambda^{-k}| + |\eta_1|) & b + O(|\lambda|^k + |\eta_1|) \\ c + O(|\hat{\lambda}^k \lambda^{-k}| + |\eta_1|) & 2d\eta_1 + O(|\lambda|^k + \eta_1^2) \end{pmatrix} \begin{pmatrix} \Delta x_2 \\ \Delta y_2 \end{pmatrix} =: \begin{pmatrix} \Delta x_3 \\ \Delta y_3 \end{pmatrix}, \tag{114}$$

where the contribution of Δz_2 goes into the $O(\cdot)$ terms.

By repeating the same procedure, we also obtain the following formulas for $\mathrm{D}T_0^m|_{\mathrm{D}(T_1\circ T_0^k)E^{cu}}$ and $\mathrm{D}T_1|_{\mathrm{D}(T_0^m\circ T_1 \circ T_0^k)E^{cu}}$:

$$\mathrm{D}T_0^m|_{\mathrm{D}(T_1\circ T_0^k)E^{cu}} = A_3 = \begin{pmatrix} \lambda^m + o(\lambda^m) & o(\lambda^m\gamma^m) \\ o(1)_{m\to+\infty} & \gamma^m + o(\gamma^m) \end{pmatrix}, \tag{115}$$

and

$$\mathrm{D}T_1|_{\mathrm{D}(T_0^m\circ T_1 \circ T_0^k)E^{cu}} = A_4 = \begin{pmatrix} a + O(|\hat{\lambda}^m \lambda^{-m}| + |\eta_2|) & b + O(|\lambda|^m + |\eta_2|) \\ c + O(|\hat{\lambda}^m \lambda^{-m}| + |\eta_2|) & 2d\eta_2 + O(|\lambda|^m + \eta_2^2) \end{pmatrix}. \tag{116}$$

Now we can write the map $\mathrm{D}T^2_{E^{cu}}$ as the product $A_4A_3A_2A_1$. By equations (112) and (114)–(116), we have

$$A_2A_1 = \begin{pmatrix} o(1)_{k\to+\infty} & b\gamma^k + o(\gamma^k) \\ c\lambda^k + o(|\lambda|^k + |\eta_1|) & \gamma^k(2d\eta_1 + o(1)_{k\to+\infty}\eta_1 + O(|\lambda|^k + \eta_1^2)) \end{pmatrix}, \tag{117}$$

$$A_4A_3 = \begin{pmatrix} o(1)_{m\to+\infty} & b\gamma^m + o(\gamma^m) \\ c\lambda^m + o(|\lambda|^m + |\eta_2|) & \gamma^m(2d\eta_2 + o(1)_{m\to+\infty}\eta_2 + O(|\lambda|^m + \eta_2^2)) \end{pmatrix}, \tag{118}$$

which yields

$$\begin{aligned} \operatorname{tr} \mathrm{D}T^2_{E^{cu}} &= \operatorname{tr}(A_4A_3A_2A_1) = \\ &= \gamma^{k+m}(4d^2(\eta_1 + O(\eta_1^2))(\eta_2 + O(\eta_2^2))(1+\ldots) + \eta_1 O(|\lambda|^m + |\gamma|^{-k}) + \eta_2 O(|\lambda|^k + |\gamma|^{-m}) + \\ &\quad + bc\lambda^m\gamma^{-m}(1+\ldots) + bc\lambda^k\gamma^{-k}(1+\ldots)), \end{aligned}$$

where the dots stand for terms that tend to zero as $m, k \to +\infty$. This equation can be rewritten as

$$\begin{aligned} \operatorname{tr} \mathrm{D}T^2_{E^{cu}} &= \gamma^{k+m}(1+\dots)(4d^2(\eta_1 + O(\eta_1^2 + |\lambda|^k + |\gamma|^{-m})) \\ &\quad \cdot(\eta_2 + O(\eta_2^2 + |\lambda|^m + |\gamma|^{-k}) + O(|\lambda|^m|\gamma|^{-m} + |\lambda|^k|\gamma|^{-k} + |\lambda|^{k+m})). \end{aligned} \tag{119}$$

It follows immediately from (112) and (114)–(116) that

$$\begin{aligned} \det A_2A_1 &= -\lambda^k\gamma^k(bc + O(\eta_1) + o(1)_{k\to+\infty}), \\ \det A_4A_3 &= -\lambda^m\gamma^m(bc + O(\eta_2) + o(1)_{m\to+\infty}). \end{aligned} \tag{120}$$

Consequently, with the fact $bc \neq 0$ by (22), we obtain

$$\det \mathrm{D}T^2_{E^{cu}} = (\lambda\gamma)^{k+m}(bc)^2(1 + O(|\eta_1| + |\eta_2|) + o(1)_{k,m\to+\infty}), \tag{121}$$

and, since $|\lambda\gamma| > 1$,

$$|\det \mathrm{D}T^2_{E^{cu}}| > 1.$$

Therefore, by (119) and (121), condition (110) is indeed equivalent to (108) and (109). □

6. Proofs of theorems 1 and 2

We first prove theorem 1. It will be proved in two steps corresponding to finding the orbits of transverse and non-transverse heteroclinic intersections in a heterodimensional cycle. The proof of theorem 2 will be a modification of that of theorem 1.

6.1. Proof of theorem 1

Theorem 1 is a consequence of the following two lemmas. Recall that δ is the size of the neighbourhood Π_1 of M^-.

Lemma 11. *Let F satisfy conditions (C1)–(C3). If there exists two transverse homoclinic points $N_1, N_2 \in W^u_{\mathrm{loc}}(O)$ of O satisfying $0 < y^- - y_{N_1} < \delta/2$ and $0 < y_{N_2} - y^- < \delta/2$, then we can find an integer K such that, for any index-2 periodic point Q of F whose orbit lies in $\bigcup_K^{+\infty} \sigma_k^0$, the intersection $W^u(Q) \cap W^s(O)$ is non-empty. The result also holds for all diffeomorphisms sufficiently C^2-close to F.*

Lemma 12. *Consider a two-parameter family $\{F_{\mu,\theta}\}$ of diffeomorphisms in $\mathrm{Diff}^r_s(\mathcal{M}^{\mathcal{D}})$ where F_{0,θ^*} satisfies conditions (C1)–(C4). If $cx^+y^- > 0$ and $cdx^+ > 0$, then, for any sequence $\{(k_j, m_j)\}$ of pairs of even natural numbers satisfying $k_j, m_j \to +\infty$ and $m_j/k_j \to \theta^*$ as $j \to +\infty$, there exists a sequence $\{(\mu_j, \theta_j)\}$ accumulating on $(0, \theta^*)$ such that, for any sufficiently large j, the diffeomorphism F_{μ_j,θ_j} has an index-2 periodic orbit Q_j satisfying $T_1 \circ T_0^{m_j} \circ T_1 \circ T_0^{k_j}(Q_j) = Q_j$ and $W^s(Q_j) \cap W^u(O) \neq \emptyset$.*

Theorem 1 follows from these lemmas.

Proof of theorem 1. Lemma 3 gives us a sequence $\{\mu_i\}$ accumulating on $\mu = 0$ such that F_{μ_i,θ^*} has a new orbit Γ_i of homoclinic tangency to O. This orbit Γ_i has a point $M_i = (0, y_i, 0) \in W^u_{\mathrm{loc}}(O) \cap \Pi_1$ accompanied by two transverse homoclinic points

$N_i^1 = (0, y_i^1, 0)$ and $N_i^2 = (0, y_i^2, 0)$ such that $0 < y_i - y_i^1 < \delta/2$ and $0 < y_i^2 - y_i < \delta/2$. It follows that F_{μ_i,θ^*} has the property given by lemma 11.

Next, we fix a sufficiently large i. According to lemma 3, the global map associated to Γ_j has $cx^+y^- > 0$ and $cdx^+ > 0$. Obviously, F_{μ_i,θ^*} with a sufficiently large i fulfils conditions (C1)–(C4). Hence, lemma 12 gives a sequence $\{(\mu_i^n, \theta_i^n)\}_n$ accumulating on (μ_i, θ^*) such that the system $F_{\mu_i^n,\theta_i^n}$ has an index-2 periodic point Q_i^n satisfying $T_1 \circ T_0^{m(n,i)} \circ T_1 \circ T_0^{k(n,i)}(Q_i^n) = Q_i^n$ and $W^s(Q_i^n) \cap W^u(O) \neq \emptyset$, where T_0 and T_1 are the local and global maps of $F_{\mu_i^n,\theta_i^n}$. Since lemma 11 holds for F_{μ_i,θ^*} and all sufficiently C^2-close diffeomorphisms, the theorem follows by taking $(\mu_j, \theta_j) = (\mu_{i_j}^{n_j}, \theta_{i_j}^{n_j})$, where $\{n_j\}$ and $\{i_j\}$ are any sequences tending to positive infinity as $j \to +\infty$. □

We proceed to prove lemmas 11 and 12.

Proof of lemma 11. We will prove this lemma by using the fact that the map T expands two-dimensional areas, which follows from the assumption $|\lambda\gamma| > 1$.

Let us first define a quotient first-return map by the leaves of the invariant foliation $\mathcal{F}^s$. Recall that the first return map $T : \Sigma^0 \to \Pi_0$ (where $\Sigma^0 = \bigcup_{k^*}^{+\infty} \sigma_k^0$) takes the form $T(M) = T_1 \circ T_0^k(X)$ for any $M \in \sigma_k^0$ (see (23)). Let $\pi : U_0 \to \{z = 0\}$ be the projection map along the leaves of $\mathcal{F}^s$. Denote by $\hat{\Pi}_i, \hat{\sigma}_k^0$ and $\hat{\Sigma}^0$ the intersections of Π_i, σ_k^0 and Σ^0 with $\{z = 0\}$. The foliation $\mathcal{F}^s$ induces the quotient map from $\hat{\Sigma}^0$ to $\hat{\Pi}_0$:

$$\hat{T}(M) = \pi \circ T_1 \circ T_0^k(M),$$

for any $M \in \hat{\sigma}_k^0$.

Consider any surface $S_k \in \sigma_k^0$ whose tangents lie in the center-unstable cone field $\mathcal{C}^{cu}$. This surface is transverse to $\mathcal{F}^s$ and the angle between them are uniformly bounded. Therefore, by the absolute continuity of $\mathcal{F}^s$, there exist constants q_1 and q_2 which do not depend on the surface such that

$$q_1 \mathcal{A}(S_k) < \mathcal{A}(\pi(S_k)) < q_2 \mathcal{A}(S_k),$$

where we use $\mathcal{A}(\cdot)$ to denote the area. On the other hand, lemma 5 gives

$$\mathcal{A}(T(S_k)) > L|\lambda\gamma|^k \mathcal{A}(S_k), \tag{122}$$

where L is some positive constant. It follows that

$$\mathcal{A}(\pi \circ T(S_k)) > q_1 \mathcal{A}(T(S_k)) > q_1 L|\lambda\gamma|^k \mathcal{A}(S_k) > q_1 q_2^{-1} L|\lambda\gamma|^k \mathcal{A}(\pi(S_k)).$$

Thus, there exists k' such that for any $k > k'$ we have

$$\mathcal{A}(\hat{T}(S_k)) > q\mathcal{A}(\pi(S_k)), \tag{123}$$

for some $q > 1$.

Let $K = \max(k^*, k')$ and $Q \in \sigma_{k_0}^0$ $(k_0 > K)$ be any index-2 periodic point of T. Take any small piece W^u of the unstable manifold of Q. The tangent space of W^u lies in the cone field $\mathcal{C}^{cu}$. Inequality (123) implies that $\mathcal{A}(\pi(W^u))$ increases after every iteration by $\hat{T}$. This means that one can find n_0 such that $T^n(W^u) \in \sigma_{k_0}^0$ for all $n < n_0$ and $T^{n_0}(W^u)$ insects one of the boundaries $v_1 = \{x = x^+ - \delta/2\}$, $v_2 = \{x = x^+ + \delta/2\}$, $h_1 = \{y = \gamma^{-k_0}(y^- - \delta/2)\}$ and $h_2 = \{y = \gamma^{-k_0}(y^- + \delta/2)\}$ of $\sigma_{k_0}^0$. We claim that $T^{n_0}(W^u)$ intersects either h_1 or h_2. For that, we show that $T^{n_0}(W^u)$ cannot intersects v_1 and v_2. Indeed, formula (8) for the local map implies that x and z in (19) are of order of λ^{k_0}. Hence, the main contribution to the x-

coordinate in (19) is given by the term $b(y - y^+)$, which is of order δ_y (recall that we let $\Pi_1 = \{(x, y, z) \mid |x| < \delta, |y - y^-| < \delta_y, \|z\| < \delta\}$). It follows that, by taking K sufficiently large and δ_y sufficiently small, the image $T_1 \circ T_0^{k_0}(\sigma^0_{k_0})$ intersects neither v_1 nor v_2. The claim is proven.

We now take a special choice of the boundaries h_1 and h_2. Let $y = w_1(x,z)$ and $y = w_2(x,z)$ be the equations of the two pieces of $W^s(O)$ that go through the transverse homoclinic points N_1 and N_2, respectively. We replace Π_1 by its subset $\{(x, y, z) \in \Pi_1 \mid w_1(x, z) < y < w_2(x, z)\}$. Then, all the 'horizontal' boundaries of σ^0_k are pieces of $W^s(O)$. Lemma 11 follows by noticing that h_1 and h_2 are such boundaries.

The above computation goes through in the coordinate system where the local map T_0 assumes the form (6) and satisfies the identities in (7). This can be achieved when F has at least C^2-smoothness. Therefore, the above result holds for any diffeomorphism sufficiently C^2-close to F. $\square$

Proof of lemma 12. We start with finding a periodic point $Q \in \Pi_0$ of period 2 and index 2. We are searching for a point Q such that $T^2(Q) = T_1 \circ T_0^m \circ T_1 \circ T_0^k(Q) = Q$. Let $Q_{01} = Q = (x_{01}, y_{01}, z_{01}), Q_{11} = T_0^k(Q) = (x_{11}, y_{11}, z_{11}), Q_{02} = T_1 \circ T_0^k(Q) = (x_{02}, y_{02}, z_{02})$ and $Q_{12} = T_0^m \circ T_1 \circ T_0^k(Q) = (x_{12}, y_{12}, z_{12})$. Recall that $|\lambda\gamma| > 1$, hence $\theta = -\ln|\lambda|/\ln|\gamma| < 1$. Therefore, the condition $m/k \to \theta^*$ implies $k - m \gg 0$.

By formulas (8) and (19), the point Q is a period-2 point if

$$
\begin{array}{rclrcl}
x_{11} & = & \lambda^k x_{01} + \phi_k, & x_{02} - x^+ & = & a x_{11} + b(y_{11} - y^-) + a_{13} z_{11} + h_1, \\
y_{01} & = & \gamma^{-k} y_{11} + \psi_k, & y_{02} & = & \mu + c x_{11} + d(y_{11} - y^-)^2 + a_{23} z_{11} + h_2(, \\
z_{11} & = & \hat{\phi}_k, & z_{02} - z^+ & = & a_{31} x_{11} + a_{32}(y_{11} - y^-) + a_{33} z_{11} + h_3, \\
x_{12} & = & \lambda^m x_{02} + \phi_k, & x_{01} - x^+ & = & a x_{12} + b(y_{12} - y^-) + a_{13} z_{12} + h_1, \\
y_{02} & = & \gamma^{-m} y_{12} + \psi_k, & y_{01} & = & \mu + c x_{12} + d(y_{12} - y^-)^2 + a_{23} z_{12} + h_2, \\
z_{12} & = & \hat{\phi}_k, & z_{01} - z^+ & = & a_{31} x_{12} + a_{32}(y_{12} - y^-) + a_{33} z_{12} + h_3,
\end{array}
$$

which can be rewritten as

$$
\begin{aligned}
x_{01} - x^+ &= a\lambda^m x_{02} + b(y_{12} - y^-) + o(\lambda^m) + O((y_{12} - y^-)^2), \\
\gamma^{-k} y_{11} + o(\gamma^{-k}) &= \mu + c\lambda^m x_{02} + d(y_{12} - y^-)^2 + o(\lambda^m) + h_2(0, y_{12} - y^-, 0), \\
z_{01} - z^+ &= a_{31}\lambda^m x_{02} + a_{32}(y_{11} - y^-) + o(\lambda^m) + O((y_{12} - y^-)^2), \\
x_{02} - x^+ &= a\lambda^k x_{01} + b(y_{11} - y^-) + o(\lambda^k) + O((y_{11} - y^-)^2), \\
\gamma^{-m} y_{12} + o(\gamma^{-m}) &= \mu + c\lambda^k x_{01} + d(y_{11} - y^-)^2 + o(\lambda^k) + h_2(0, y_{11} - y^-, 0), \\
z_{02} - z^+ &= a_{31}\lambda^k x_{01} + a_{32}(y_{11} - y^-) + o(\lambda^k) + O((y_{11} - y^-)^2).
\end{aligned}
\tag{124}
$$

Note that it follows from the implicit function theorem that, at sufficiently large k, m, the variables x_{01}, x_{02}, z_{01} and z_{02} can be expressed as functions of y_{11} and y_{12}. Consequently, we need to consider only the equations for y_{11} and y_{12}. By introducing $\eta_1 = y_{11} - y^-$ and $\eta_2 = y_{12} - y^-$, finding a period-2 point becomes equivalent to solving the following system:

$$
\gamma^{-k}(\eta_1 + y^-) + o(\gamma^{-k}) = \mu + c\lambda^m x^+ + bc\lambda^m \eta_1 + d\eta_2^2 + o(\lambda^m) + h_2(0, \eta_2, 0), \tag{125}
$$

$$
\gamma^{-m}(\eta_2 + y^-) + o(\gamma^{-m}) = \mu + c\lambda^k x^+ + bc\lambda^k \eta_2 + d\eta_1^2 + o(\lambda^k) + h_2(0, \eta_1, 0). \tag{126}
$$

We will look for solutions $\eta_{1,2}$ which tend to zero as $k, m \to +\infty$. By lemma 10, the corresponding periodic point is of index-2 if, for some $s \in (-1, 1)$,

$$(\eta_1 + O(\eta_1^2 + \lambda^k + \gamma^{-m}))(\eta_2 + O(\eta_2^2 + \lambda^m)) = O(\lambda^m \gamma^{-m} + \lambda^{k+m}) \tag{127}$$

(recall that we assume $k - m \gg 0$, so $\lambda^k = o(\lambda^m)$ and $\gamma^{-k} = o(\gamma^{-m})$; condition $|\lambda\gamma| > 1$ also implies that $\gamma^{-m} = o(\lambda^{-m})$).

After expressing μ as a function of η_1 and η_2 from (125) and plugging the result into (126), we obtain

$$0 = cx^+ \lambda^m + d(\eta_1^2 - \eta_2^2) + o(\eta_1^2 + \eta_2^2) + o(\lambda^m). \tag{128}$$

Let

$$\hat{\eta}_1 = \eta_1 + O(\eta_1^2 + \lambda^k + \gamma^{-m}) \quad \text{and} \quad \hat{\eta}_2 = \eta_2 + O(\eta_2^2 + \lambda^m), \tag{129}$$

where the $O(\cdot)$ terms are exactly those in the left-hand side of (127). Consequently, equations (128) and (127) become

$$0 = cx^+ \lambda^m + d(\hat{\eta}_1^2 - \hat{\eta}_2^2) + o(\lambda^m + \hat{\eta}_1^2 + \hat{\eta}_2^2), \tag{130}$$

$$\hat{\eta}_1 \hat{\eta}_2 = (C_{k,m} + O(|\hat{\eta}_1| + |\hat{\eta}_2|))\lambda^{k+m}, \tag{131}$$

where $C_{k,m}$ is independent of $\hat{\eta}_{1,2}$ and uniformly bounded for all k and m.

After we rescale the variables as follows:

$$(\hat{\eta}_1, \hat{\eta}_2) = (\lambda^{k+\frac{m}{2}} \xi_1, \lambda^{\frac{m}{2}} \xi_2), \tag{132}$$

equations (130) and (131) transform to

$$\begin{aligned} 0 &= cx^+ - d\xi_2^2 + \dots, \\ \xi_1 \xi_2 &= C_{k,m} + \dots, \end{aligned} \tag{133}$$

where the dots denote terms that tend to zero as $k, m \to +\infty$. By noting $cdx^+ > 0$ from the assumption of the lemma, we find, for all sufficiently large k and m, two solutions

$$(\xi_1^*, \xi_2^*) = \pm \left(C_{k,m} \sqrt{\frac{d}{cx^+}} + o(1)_{k,m \to +\infty}, \sqrt{\frac{cx^+}{d}} + o(1)_{k,m \to +\infty} \right). \tag{134}$$

Then, the corresponding values of (η_1, η_2) can be found from (132), (129) and the corresponding values of μ can be found from either of the equations (125) and (126).

We proceed to seek for the intersection $W^s(Q) \cap W^u(O)$. For an index-2 point, its local stable manifold is a leaf of $\mathcal{F}^s$. In particular, a formula for the leaf through Q_{02} is given by lemma 8 as

$$\begin{aligned} x &= x_{02} + \varphi_1(z; x_{02}, y_{02}, z_{02}), \\ y &= y_{02} + \varphi_2(z; x_{02}, y_{02}, z_{02}), \end{aligned}$$

where $\varphi_1 = O(\lambda_0^m \lambda^{-m})$ and $\varphi_2 = o(\gamma^{-m})$. Here λ_0 is a value close to $|\lambda_1|$ such that $|\lambda_1| < \lambda_0$.

Now let $\mathcal{W} = \{(0, y, 0) \mid |y + y^-| < \varepsilon\}$ with $\varepsilon > 0$ be a small piece of $W^u_{\text{loc}}(O)$ containing the point $\tilde{M}^- = (0, -y^-, 0)$. By formula (24), the image $\tilde{T}_1(\mathcal{W})$ is given by

$$\begin{aligned} x - x^+ &= bt + h_1(0, t, 0), \\ -y &= \mu + dt^2 + h_2(0, t, 0), \\ \mathcal{S}^{-1} z - z^+ &= a_{32} t + h_3(0, t, 0), \end{aligned}$$

where $t \in (-\varepsilon, \varepsilon)$. Hence, we can write the condition for the intersection of $W^u(O) \cap W^s(Q)$ as

$$\begin{aligned} b\eta_1 + O(\lambda_0^m \lambda^{-m}) &= bt + h_1(0, t, 0), \\ -\gamma^{-m}(\eta_2 + y^-) + o(\gamma^{-m}) + o(\gamma^{-m}) &= \mu + dt^2 + h_2(0, t, 0), \\ \mathcal{S}^{-1}z - z^+ &= a_{32}t + h_3(0, t, 0), \end{aligned}$$

which can be rewritten as

$$-\gamma^{-m}(\eta_2 + y^-) + o(\gamma^{-m}) = \mu + d(\eta_1 + O(\lambda_0^m \lambda^{-m}))^2 + h_2(0, \eta_1 + O(\lambda_0^m \lambda^{-m}), 0). \tag{135}$$

Since the x- and z-coordinates of the points in the orbit of Q can be expressed as functions of η_1, η_2 and μ, the right-hand side of the above equation is just a function of η_1, η_2 and μ. We now express μ from (135) as a function of η_1 and η_2, and obtain

$$\mu + d\eta_1^2 + h_2(0, \eta_1, 0) = -\gamma^{-m}y^- + O(\eta_1 \lambda_0^m \lambda^{-m}) + O(\lambda_0^{2m} \lambda^{-2m}) + o(\gamma^{-m}), \tag{136}$$

which, along with (126), yields

$$\gamma^{-m}y^- = \frac{c}{2}\lambda^k x^+ + O(\eta_1 \lambda_0^m \lambda^{-m}) + O(\lambda_0^{2m} \lambda^{-2m}) + o(\gamma^{-m}) + o(\lambda^k). \tag{137}$$

Recall that condition (C4) gives $|\lambda|\, |\gamma|^{\frac{1}{2}} < 1$. This, together with the fact $\lambda_0 < \lambda^2$ given by lemma 1, implies $O(\lambda_0^{2m}\lambda^{-2m}) = O(\lambda^{2m}) = o(\gamma^{-m})$. By equations (129), (132) and (134), we have $\eta_1 = O(\lambda^k + \gamma^{-m})$, which implies $O(\eta_1 \lambda_0^m \lambda^{-m}) = O(\eta_1 \lambda^m) = o(\lambda^k) + o(\gamma^{-m})$. With these observations, equation (137) can be rewritten as

$$\gamma^{-m}y^- = \frac{c}{2}\lambda^k x^+ + o(\lambda^k) + o(\gamma^{-m}),$$

or

$$\lambda^k \gamma^m = \frac{2y^-}{cx^+} + o(\lambda^k \gamma^m) + o(1). \tag{138}$$

Recall the assumption $2y^-/cx^+ > 0$; we also have taken k and m even, so both sides of equation (138) are positive. We, therefore, may take logarithm on both sides, which gives

$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|} = \frac{m}{k} - \frac{C^*_{k,m}}{k}, \tag{139}$$

where $C^*_{k,m} = \ln(2y^-/cx^+ + o(\lambda^k\gamma^m))/\ln|\gamma|$ is uniformly bounded, for all sufficiently large k and m. Note that C^* is a function of θ—it depends, for example, on the coefficients of the global and local maps, which depend on θ as a parameter. It is important for us that C^* is continuous and bounded function of θ, so a value of θ that solves (139) can be found for each sufficiently large (k, m). It is also obvious, that if the sequence $\{(k_j, m_j)\}$ satisfies $k_j, m_j \to +\infty$ and $m_j/k_j \to \theta^*$ as $j \to +\infty$, then the values of θ_j we obtain from (139) accumulate on $\theta = \theta^*$. The corresponding μ values are obtained from (136) as $\mu_j = -\gamma^{m_j}y^- + o(\gamma^{m_j})$ and they tend to 0 as $j \to +\infty$. Therefore, for each sufficiently large j, the map F_{μ_j,θ_j} has an index-2 point of period 2 such that $W^s(Q_j) \cap W^u(O) \neq \emptyset$. □

6.2. Proof of theorem 2

In the non-symmetric case, we use a simpler construction than in theorem 1. In particular, we use a different version of lemma 12. Here we have two splitting parameters μ_1 and μ_2, which correspond to two different orbits Γ and $\tilde{\Gamma}$ of homoclinic tangency, respectively.

Lemma 13. *Consider a two-parameter family $\{F_{\mu_1,\mu_2}\}$ of diffeomorphisms in* $\mathrm{Diff}^r(\mathcal{M})$*, where $F_{0,0}$ satisfy conditions (C1)–(C3) and (C5). For every sufficiently large k there exist parameter values $\{(\mu_1,\mu_2)\}$, accumulating on 0 as $k\to+\infty$, such that the diffeomorphism F_{μ^1,μ^2} has an index-2 periodic point Q satisfying $T_1\circ T_0^k(Q)=Q$ and $W^s(Q)\cap W^u(O)\neq\emptyset$.*

Since lemma 11 remains true in the general case, theorem 2 follows immediately by replacing lemma 12 with lemma 13 in the proof of theorem 1.

In what follows we prove lemma 13. Here we consider the local map in the form (6) for which only identities in (7) satisfied (as we do not have condition (C4) here, we cannot assume identities (11)). Therefore, without the identities in (11), we do not have lemmas 1 and 8. We still can use lemma 7 which gives the equation for strong-stable leaves of a pointy (x^*,y^*,z^*) in the form

$$\begin{aligned} x &= x^*+\varphi_1(z;x^*,y^*,z^*),\\ y &= y^*+\varphi_2(z;x^*,y^*,z^*), \end{aligned} \tag{140}$$

where $\varphi_1=o(1)_{k\to+\infty}$ and $\varphi_2=o(\gamma^{-k})$.

Proof of lemma 13. The coincidence condition (C5) implies that the small neighbourhoods $\tilde{\Pi}_1$, Π_1 and Π_0, and the local and global maps associated to the two homoclinic tangency orbits Γ and $\tilde{\Gamma}$ can be defined in the same way as those in section 2. The local map for Γ and $\tilde{\Gamma}$ have the form of (8). The two global maps T_1 and $\tilde{T}_1$ are given by

$$\begin{aligned} x_0-x_i^+ &= a_ix_1+b_i(y_1-y_i^-)+a_{13}^iz_1+h_1^i,\\ y_0 &= \mu_i+c_ix_1+d_i(y_1-y_i^-)^2+a_{23}^iz_1+h_2^i,\\ z_0-z_i^+ &= a_{31}^ix_1+a_{32}^i(y_1-y_i^-)+a_{33}^iz_1+h_3^i, \end{aligned} \tag{141}$$

where T_1 corresponds to $i=1$ and $\tilde{T}_1$ corresponds to $i=2$.

Let Q be a periodic point such that $T_1\circ T_0^k(Q)=Q$. Denote $Q_0=Q=(x_0,y_0,z_0)$, $Q_1=T_0^k(Q)=(x_1,y_1,z_1)$. By formulas (8) and (141), the condition $Q=T_1\circ T_0^k(Q)$ is written as

$$\begin{aligned} x_1 &= \lambda^kx_0+\phi_k, & x_0-x^+ &= a_1x_1+b_1(y_1-y^-)+a_{13}^1z_1+h_1^1,\\ y_0 &= \gamma^{-k}y_1+\psi_k, & y_0 &= \mu+c_1x_1+d_1(y_1-y^-)^2+a_{23}^1z_1+h_2^1,\\ z_1 &= \hat{\phi}_k, & z_0-z^+ &= a_{31}^1x_1+a_{32}^1(y_1-y^-)+a_{33}^1z_1+h_3^1. \end{aligned}$$

We can express all variables here as functions of y_1, so the above equations reduce to

$$\gamma^{-k}(\eta+y^-)+o(\gamma^{-k})=\mu_1+c\lambda^kx^++bc\lambda^k\eta+d\eta_2^2+o(\lambda^k)+h_2(0,\eta_2,0), \tag{142}$$

where we denote $\eta=y_1-y^-$.

Like in the proof of lemma 10, this fixed point is a saddle of index-2 if, for some $s\in(-1,1)$,

$$\frac{\operatorname{tr}\mathrm{D}(T_1\circ T_0^k)|_{E^{cu}}}{\det\mathrm{D}(T_1\circ T_0^k)|_{E^{cu}}+1}=s, \tag{143}$$

where E^{cu} is the two-dimensional invariant subspace given in lemma 9. By formulas (112) and (114),

$$\mathrm{D}(T_1 \circ T_0^k)|_{E^{cu}} = \begin{pmatrix} o(1)_{k\to+\infty} & b\gamma^k + o(\gamma^k) \\ c\lambda^k + o(|\lambda|^k + |\eta|) & \gamma^k(2d\eta + o(1)_{k\to+\infty}\eta + O(|\lambda|^k + \eta^2)) \end{pmatrix}$$

(see (117)). Therefore, equation (143) gives us the value of

$$\eta = O(\lambda^k). \tag{144}$$

After that, we find μ from equation (142) as

$$\mu_1 = -c\lambda^k x^+ + o(\lambda^k). \tag{145}$$

Let us now construct the intersection $W^s(Q) \cap W^u(O)$. Like in the proof of theorem 1, this intersection is given by

$$\begin{aligned} x_0 - x^+ + o(1)_{k\to+\infty} &= b_2 t + h_1^2(0,t,0), \\ -y_0 + o(\gamma^{-k}) &= \mu_2 + d_2 t^2 + h_2^2(0,t,0), \\ z - z_0^+ &= a_{32}^2 t + h_3^2(0,t,0), \end{aligned} \tag{146}$$

which transforms into

$$-\gamma^{-k}(\eta + y_1^-) + o(\gamma^{-k}) = \mu_2 + d_2\eta^2 + O(\eta\hat{\lambda}^k\lambda^{-k}) + O(\hat{\lambda}^{2k}\lambda^{-2k}) + h_2^2(0,\eta,0). \tag{147}$$

This along with (144) gives us the corresponding value of

$$\mu_2 = o(1)_{k\to+\infty}. \tag{148}$$

The lemma follows immediately. □

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Appendix

Here we show that, for the map T_0 in the form (6), there exists a coordinate transformation $\mathcal{T}$ such that after this transformation the map T_0 will satisfy identities (7) and (11), and also keep the symmetry $\mathcal{R}$.

This transformation is constructed as a composition of

- $\mathcal{T}_1$ which straightens the local stable and unstable manifolds of O, thus giving the first two identities in (7);
- $\mathcal{T}_2$ which linearises both the restriction $T_0|_{W^u_{\mathrm{loc}}}$ and the quotient of $T_0|_{W^s_{\mathrm{loc}}}$ by the the strong-stable foliation—after that the third and forth identities in (7) become valid;
- $\mathcal{T}_3$ which gives the last two identities in (7); and
- $\mathcal{T}_4$ which straightens a certain local, $\mathcal{R}$-symmetric extended unstable manifold $W^{uE}_{\mathrm{loc}}(O)$ along with the foliation $\mathcal{F}^{uE}$ on it—this leads to identities (11).

In what follows we discuss the transformations $\mathcal{T}_i$, $(i = 1, 2, 3, 4)$ separately and show that they keep the system symmetric with respect to $\mathcal{R}$, i.e. they commute with $\mathcal{R}$.

A.1. Transformation $\mathcal{T}_1$

Let $(x = w_{ux}(y), z = w_{uz}(y))$ and $y = w_s(x,z)$ be the equations for the local unstable and stable invariant manifolds of O, respectively. The transformation $\mathcal{T}_1$ is defined as

$$(x^{\text{new}}, z^{\text{new}}) = (x - w_{ux}(y), z - w_{uz}(y)), \qquad y^{\text{new}} = y - w_s(x,z). \tag{A.1}$$

In the new coordinates, the manifolds W^u_{loc} and W^s_{loc} have equations $(x^{\text{new}}, z^{\text{new}}) = 0$ and, respectively $y^{\text{new}} = 0$. Thus, the first two identities in (7) follow immediately from the invariance of these manifolds with respect to T_0.

Let us show that the coordinate transformation given by (A.1) commutes with $\mathcal{R}$. Consider first the transformation $\psi : (x, y, z) \mapsto (x, y - w_s(x,z), z)$. By uniqueness of the stable manifold, $(R)W^s_{\text{loc}} = W^s_{\text{loc}}$. Therefore, for any x, z, the image by $\mathcal{R}$ of the point $(x, w_s(x,z), z) \in W^s_{\text{loc}}$ also lies in W^s_{loc}, i.e.

$$w_s(x, \mathcal{S}z) = -w_s(x,z). \tag{A.2}$$

(see formula (3) for $\mathcal{R}$). Similarly, by the uniqueness of the unstable manifold,

$$w_{ux}(-y) = w_{ux}(y), \qquad w_{uz}(-y) = \mathcal{S}w_{uz}(y). \tag{A.3}$$

Now let (x, y, z) be an arbitrary point in a neighbourhood of O. We have

$$\mathcal{R} \circ \mathcal{T}_1(x, y, z) = (x - w_{ux}(y), -y + w_s(x,z), \mathcal{S}z - \mathcal{S}w_{uz}(y)),$$

and

$$\mathcal{T}_1 \circ \mathcal{R}(x, y, z) = (x - w_{ux}(-y), -y - w_s(x, \mathcal{S}z), \mathcal{S}z - w_{uz}(-y)).$$

By (A.2),(A.2), this implies that $\mathcal{T}_1$ commutes with $\mathcal{R}$, as required.

A.2. Transformations $\mathcal{T}_2$ and $\mathcal{T}_3$

The construction of these two transformations is given in the proof of lemma 6 in [22]. Here we reconstruct them for our case and prove that they are $\mathcal{R}$-symmetric.

The transformation $\mathcal{T}_2$ in sought is in the form

$$x^{\text{new}} = x + h_1(x,z), \qquad y^{\text{new}} = y + h_2(y), \qquad z^{\text{new}} = z, \tag{A.4}$$

where $h_1(0,0) = 0, h_2(0) = 0, \partial h_1(0,0)/\partial(x,z) = 0$ and $\partial h_2(0)/\partial y = 0$ (hence the first two identities in (7) hold in the new coordinates). To obtain the identities

$$f_1(x,0,z) = 0 \quad \text{and} \quad f_2(0,y,0) = 0,$$

we must have $\bar{x}^{\text{new}} = \lambda x^{\text{new}}$ at $y = 0$, and $\bar{y}^{\text{new}} = \gamma y^{\text{new}}$ at $(x,z) = 0$, respectively. According to formula (6) for T_0, these conditions translate to

$$\begin{aligned} h_1(\bar{x}, \bar{z}) &= \lambda h_1(x,z) - f(x,0,z), \\ h_2(\bar{y}) &= \gamma h_2(y) - f(0,y,0), \end{aligned} \tag{A.5}$$

where we denote here $\bar{x} = \lambda x + f_1(x,0,z)$, $\bar{y} = \gamma y + f_2(0,y,0)$, and $\bar{z} = Az + f_3(x,0,z)$.

It has been shown in [22] that the above system has the following solution:

$$h_1(x,z) = \sum_{j=0}^{+\infty} \lambda^{-j-1} f_1(x_j, 0, z_j) \quad \text{and} \quad h_2(y) = -\sum_{j=1}^{+\infty} \gamma^{j-1} f_2(0, y_j, 0), \tag{A.6}$$

where $\{(x_j, z_j)\}$ is the forward orbit of $(x,z) =: (x_0, z_0)$ under the restriction of the local map (6) to to $W^s(O)$, and $\{y_j\}$ is the backward orbit of $y = :y_0$ under the restriction of the local map to $W^u(O)$. The functions h_1 and h_2 given by (A.6) are obviously $\mathcal{R}$-symmetric. We, therefore, proceed to the analysis of the transformation $\mathcal{T}_3$:

$$x^{\text{new}} = x + g_1(x, y), \qquad y^{\text{new}} = y + g_2(x, y, z), \qquad z^{\text{new}} = z + g_3(x, y), \tag{A.7}$$

where $g_{1,3}$ vanish at $x = 0$ and $y = 0$ while g_2 equals to zero at $(x, z) = 0$ and at $y = 0$. These conditions ensure that $\mathcal{T}_3$ keeps the identities obtained previously. We need to achieve that

$$\frac{\partial f_1}{\partial x}(0, y, 0) = 0,$$

in the new coordinates, which is equivalent to

$$\frac{\partial(\bar{x}^{\text{new}} - \lambda x^{\text{new}})}{\partial x^{\text{new}}}(0, y^{\text{new}}, 0) = 0. \tag{A.8}$$

Since the first identity in (7) ensures

$$\frac{\partial(\bar{x}^{\text{new}} - \lambda x^{\text{new}})}{\partial y^{\text{new}}}(0, y^{\text{new}}, 0) = 0,$$

equation (A.8) holds if and only if

$$\mathrm{d}(\bar{x}^{\text{new}} - \lambda x^{\text{new}}) = 0 \quad \text{when} \quad (x^{\text{new}}, z^{\text{new}}) = 0 \;\; \text{and} \;\; \mathrm{d}z^{\text{new}} = 0.$$

We have, from (A.7), that $(x, z) = 0$ at $(x^{\text{new}}, z^{\text{new}}) = 0$, and

$$\mathrm{d}z^{\text{new}} = \mathrm{d}z + \frac{\partial g_3}{\partial x}(0, y)\mathrm{d}x,$$

so $\mathrm{d}z^{\text{new}} = 0$ when

$$\mathrm{d}z = -\frac{\partial g_3}{\partial x}(0, y)\mathrm{d}x. \tag{A.9}$$

Equations (6), (A.7) and (A.9) imply that, when $(x^{\text{new}}, z^{\text{new}}) = 0$ and $\mathrm{d}z^{\text{new}} = 0$, we have

$$\begin{aligned}
& \mathrm{d}(\bar{x}^{\text{new}} - \lambda x^{\text{new}}) \\
= \; & \mathrm{d}(\bar{x} + g_1(0, \bar{y}) - \lambda x - \lambda g_1(0, y)) \\
= \; & \mathrm{d}(f_1(0, y, 0) + g_1(0, \bar{y}) - \lambda g_1(0, y)) \\
= \; & \tfrac{\partial f_1}{\partial x}(0, y, 0)\mathrm{d}x - \tfrac{\partial f_1}{\partial z}(0, y, 0)\tfrac{\partial g_3}{\partial x}(0, y)\mathrm{d}x \\
& + \tfrac{\partial g_1}{\partial x}(0, \bar{y})\left(\lambda \mathrm{d}x + \tfrac{\partial f_1}{\partial x}(0, y, 0)\mathrm{d}x - \tfrac{\partial f_1}{\partial z}(0, y, 0)\tfrac{\partial g_3}{\partial x}(0, y)\mathrm{d}x\right) - \lambda\tfrac{\partial g_1}{\partial x}(0, y)\mathrm{d}x.
\end{aligned} \tag{A.10}$$

We need to find functions g_1 and g_3 such that the right-hand side of (A.10) vanishes identically. Denote

$$\eta_1(y) = \frac{\partial g_1}{\partial x}(0, y) \quad \text{and} \quad \eta_3(y) = \frac{\partial g_3}{\partial x}(0, y), \tag{A.11}$$

and equate the right-hand side of (A.10) to zero. This gives the following condition:

$$\begin{aligned}
\eta_1(\bar{y}) \;\; = \;\; & \left(\lambda\eta_1(y) - \tfrac{\partial f_1}{\partial x}(0, y, 0) + \tfrac{\partial f_1}{\partial z}(0, y, 0)\eta_3(y)\right) \\
& \times \left(\lambda + \tfrac{\partial f_1}{\partial x}(0, y, 0) - \tfrac{\partial f_1}{\partial z}(0, y, 0)\eta_3(y)\right)^{-1},
\end{aligned} \tag{A.12}$$

where we have used the fact $\bar{x} = 0$ at $(x, z) = 0$, and $\bar{y} = \gamma y + f_2(0, y, 0)$.

Analogously, we will have identity

$$\frac{\partial f_3}{\partial z}(0,y,0)=0$$

satisfied in the new coordinates, if

$$\begin{aligned}\eta_3(\bar{y}) \;=\; & \left(A\eta_3(y)-\tfrac{\partial f_3}{\partial x}(0,y,0)+\tfrac{\partial f_3}{\partial z}(0,y,0)\eta_3(y)\right)\\ & \times\left(\lambda+\tfrac{\partial f_1}{\partial x}(0,y,0)-\tfrac{\partial f_1}{\partial z}(0,y,0)\eta_3(y)\right)^{-1},\end{aligned}\tag{A.13}$$

Equations (A.12) and (A.13) are solved by noticing that they can be viewed as the conditions for the manifold

$$w_1:\{u_1=\eta_1(y),u_3=\eta_3(y)\}\tag{A.14}$$

to be invariant under the map

$$\begin{aligned}\bar{y} &= \gamma y+f_2(0,y,0),\\ \bar{u}_1 &= \left(\lambda u_1-\tfrac{\partial f_1}{\partial x}(0,y,0)+\tfrac{\partial f_1}{\partial z}(0,y,0)u_3\right)\left(\lambda+\tfrac{\partial f_1}{\partial x}(0,y,0)-\tfrac{\partial f_1}{\partial z}(0,y,0)u_3\right)^{-1},\\ \bar{u}_3 &= \left(Au_3-\tfrac{\partial f_3}{\partial x}(0,y,0)+\tfrac{\partial f_3}{\partial z}(0,y,0)u_3\right)\left(\lambda+\tfrac{\partial f_1}{\partial x}(0,y,0)-\tfrac{\partial f_1}{\partial z}(0,y,0)u_3\right)^{-1}.\end{aligned}\tag{A.15}$$

Note that this map has a fixed point $(0,0,0)$. The multipliers of this point are the eigenvalues of the linearised map, which is given by

$$y\mapsto\gamma y,\qquad u_1\mapsto u_1-\frac{\partial^2 f_1}{\partial x\partial y}(0,0,0)\lambda^{-1}y,\quad u_3\mapsto\lambda^{-1}Au_3-\frac{\partial^2 f_3}{\partial x\partial y}(0,0,0)\lambda^{-1}y.$$

The spectrum of this map consists of the spectra of the following three operators: $y\mapsto\gamma y, u_1\mapsto u_1, u_3\mapsto\lambda^{-1}Au_3$. Therefore, the fixed point $(0,0,0)$ has one multiplier on the unit circle, one multiplier outside the unit circle and $(n-2)$ multipliers inside the unit circle. It has been known (see e.g. [25, 44]) that such fixed point lies in a unique one-dimensional unstable manifold that is tangent at zero to the eigenspace corresponding to the multiplier outside the unit circle. It follows that such unique manifold in our case is the sought manifold w_1.

The map (A.15) is symmetric with respect to $(y,u_1,u_3)\mapsto(-y,u_1,\mathcal{S}u_3)$. Indeed, this follows immediately from the relations

$$\begin{aligned}&\tfrac{\partial f_1}{\partial x}(0,-y,0)=\tfrac{\partial f_1}{\partial x}(0,y,0),\quad \tfrac{\partial f_1}{\partial z}(0,-y,0)\mathcal{S}=\tfrac{\partial f_1}{\partial z}(0,y,0),\\ &\tfrac{\partial f_3}{\partial x}(0,-y,0)=\mathcal{S}\tfrac{\partial f_3}{\partial x}(0,y,0),\quad \tfrac{\partial f_3}{\partial z}(0,-y,0)\mathcal{S}=\mathcal{S}\tfrac{\partial f_3}{\partial z}(0,y,0),\end{aligned}$$

which are, in turn, implied by the symmetry of T_0 with respect to $\mathcal{R}$. By uniqueness of w_1, it must be symmetric with respect to the transformation $(y,u_1,u_3)\mapsto(-y,u_1,\mathcal{S}u_3)$, which implies that $\eta_{1,3}$ are symmetric with respect to $y\mapsto -y$. Consequently, functions $g_{1,3}(x,z)$ can be any of those that vanish at $(x,z)=0$ and $y=0$ and satisfy (A.11). Due to the symmetry of $\eta_{1,2}$, it is easy to show that $g_{1,3}$ can be chosen symmetric with respect to $(x,y,z)\mapsto(x,-y,\mathcal{S}z)$, as required.

The next identity to be satisfied in the new coordinates is

$$\frac{\partial f_2}{\partial y}(x,0,z)=0.\tag{A.16}$$

Similarly to the above, by letting

$$\eta_2(x,z) = \frac{\partial g_2}{\partial y}(x,0,z), \tag{A.17}$$

the identity (A.16) is equivalent to

$$\eta_2(\bar{x},\bar{z}) = \left(\gamma\eta_2(x,z) - \frac{\partial f_2}{\partial y}(x,0,z)\mathrm{d}y\right)\left(\gamma + \frac{\partial f_2}{\partial y}(x,0,z)\mathrm{d}y\right)^{-1}. \tag{A.18}$$

This is the condition for the manifold $w_2 : v = \eta_2(x,z)$ to being invariant under the map

$$\bar{x} = \lambda x + f_1(x,0,z), \quad \bar{z} = Az + f_3(x,0,z) \quad \bar{v} = \left(\gamma v - \frac{\partial f_2}{\partial y}(x,0,z)\mathrm{d}y\right)\left(\gamma + \frac{\partial f_2}{\partial y}(x,0,z)\mathrm{d}y\right)^{-1}.$$

This map is symmetric with respect to $(x,z,v) \mapsto (x,\mathcal{S}z,v)$, and has a unique $(n-1)$-dimensional stable invariant manifold. It follows that η_2 exists and is symmetric with respect to $(x,z) \mapsto (x,\mathcal{S}z)$. The function $g_2(x,y,z)$ can be any of those that vanish at $(x,z)=0$ and $y=0$ and satisfy (A.17). The symmetry of η_2 implies that g_2 can be chosen symmetric with respect to $(x,y,z) \mapsto (x,-y,\mathcal{S}z)$, so we can now conclude that $\mathcal{T}_3$ is $\mathcal{R}$-symmetric.

A.3. Transformation $\mathcal{T}_4$

Recall that $\mathcal{T}_4$ is a transformation that straightens the extended-unstable invariant manifold $W^{uE}(O)$ of O and the foliation on it. This manifold is not unique and we choose a special one as follows.

We consider the following map G_0:

$$\begin{aligned}
\bar{x} &= \lambda x + f_1(x,y,z),\\
\bar{y} &= \gamma y + f_2(x,y,z),\\
\bar{z} &= Az + f_3(x,y,z),\\
\bar{u} &= \left(\left(\lambda + \tfrac{\partial f_1}{\partial x}\right)u + \tfrac{\partial f_1}{\partial y} + \tfrac{\partial f_1}{\partial z}v\right)\left(\gamma + \tfrac{\partial f_2}{\partial y} + \tfrac{\partial f_2}{\partial x}u + \tfrac{\partial f_2}{\partial z}v\right)^{-1},\\
\bar{v} &= \left(\left(A + \tfrac{\partial f_3}{\partial z}\right)v + \tfrac{\partial f_3}{\partial x}u + \tfrac{\partial f_3}{\partial y}\right)\left(\gamma + \tfrac{\partial f_2}{\partial y} + \tfrac{\partial f_2}{\partial x}u + \tfrac{\partial f_2}{\partial z}v\right)^{-1},
\end{aligned} \tag{A.19}$$

where the first three lines give the map T_0, e.g. the functions f_i satisfy all the identities in (7). Obviously, this map is C^{r-1} smooth and $\mathcal{R}$-symmetric. Note that G_0 is defined in $V_0 \times \mathbb{R}^{1+D}$, where V_0 is the domain of T_0. We now extend G_0 to the whole of $\mathbb{R}^{2D+3}$ by replacing the functions f_i $(i=1,2,3)$ in (A.19) with $f_i(\xi(x,y,z))$, were ξ is a C^r function such that, for two small numbers $\delta_1, \delta_2 > 0$ with $\delta_1 < \delta_2$, we have

$$\xi(x,y,z) = \begin{cases} (x,y,z) & \text{if } \|(x,y,z)\| < \delta_1 \\ 0 & \text{if } \|(x,y,z)\| > \delta_2 \end{cases}.$$

For simplicity we use the same notation for the new functions f_i so that the extension map, denoted by G, assumes the same form as (A.19). One can choose the function ξ such that the map G will be $\mathcal{R}$-symmetric.

It can be seen from (A.19) that G has a fixed point at zero, and the corresponding multipliers are λ, γ (these correspond to variables x and y), the eigenvalues of A (which corresponds to variables z) and also λ/γ (corresponding to the variable u) and the eigenvalues of A divided by γ (corresponding to the variables v). Since $|\gamma| > 1$, it follows that the eigenvalues corresponding to the variables z, u, v are smaller in absolute value that $\max\{\tilde{\lambda}, \lambda/\gamma\}$ where $\tilde{\lambda} > 0$ is a value close to $|\lambda_1|$, the largest absolute value of the eigenvalues of A. Since $|\gamma| > 1 > |\lambda|$ and $|\lambda| > \tilde{\lambda}$, we have that there is a spectrum dichotomy between x, y variables and z, u, v variables. The other assumption $|\lambda\gamma| > 1$ and $|\lambda_1| < \lambda^2$ (so, $\tilde{\lambda} < \lambda^2$) further implies

$$\left|\frac{\ln|\lambda\gamma^{-1}|}{\ln|\lambda|}\right| > \left|\frac{\ln\lambda^2}{\ln|\lambda|}\right| = 2 \quad \text{and} \quad \left|\frac{\ln\tilde{\lambda}}{\ln|\lambda|}\right| > \left|\frac{\ln\lambda^2}{\ln|\lambda|}\right| = 2. \tag{A.20}$$

Thus, the spectrum gap l between (x, y) and (z, u, v) is greater than 2. It follows that there exists a unique invariant C^2-manifold W_G for the map G, and it attracts all the orbits near it (see e.g. section 5 of [44] and [25]). This manifold has the form

$$\begin{aligned} z &= \eta_{uE}(x, y), \\ u &= \eta_1(x, y), \\ v &= \eta_2(x, y). \end{aligned} \tag{A.21}$$

We now take any surface w of the form (A.21) such that it is $\mathcal{R}$-symmetric and satisfies

$$\eta_2 = \frac{\partial\eta_{uE}}{\partial x}\eta_1 + \frac{\partial\eta_{uE}}{\partial y}. \tag{A.22}$$

This equation means that the line field given by $(\eta_1, 1, \eta_2)$ belongs to the tangent space of the surface $z = \eta_{uE}(x, y)$. Since W_G is attracting, the iterates $G^n(w)$ tend to W_G as $n \to +\infty$. It is easy to check that each iteration will be again $\mathcal{R}$-symmetric and will satisfy (A.22). Therefore, the limit W_G is $\mathcal{R}$-symmetric and satisfies (A.22). By our construction, the extended-unstable manifold $W^{uE}(O)$ is given by the C^2 function $z = \eta_{uE}(x, y)$. A C^2 foliation $\mathcal{F}^{uE}$ on $W^{uE}(O)$ can be found by integrating the line field given by $(\eta_1, 1, \eta_2)$. Namely, it consists of solutions to the system of differential equations $\dot{x} = \eta_1,\ \dot{y} = 1,\ \dot{z} = \eta_2$.

We can now define $\mathcal{T}_4$ as the composition of two transformations which straighten the manifold $W^{uE}(O)$ and the leaves of $\mathcal{F}^{uE}$, respectively. The former can be obtained by the same way as we did for $\mathcal{T}_1$, and it will be C^2 and $\mathcal{R}$-symmetric. Regarding the latter, we explain as follows.

Parametrize the leaves by its intersection with $\{y = 0\}$, which is denoted by c. Then, the leaf of $\mathcal{F}^{uE}$ that goes through the point $(c, 0, 0)$ is given by $(x, z) = h(y, c) =: (h_1(y, c), h_2(y, c))$, where h_i are C^2 functions. The foliation $\mathcal{F}^{uE}$ also induces a C^2 function

$$g : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto c,$$

where c satisfies $x = h_1(y,c)$. In order to linearise the quotient map along the leaves of this foliation (i.e. to straighten the leaves), we use the following C^2 transformation:

$$x^{\text{new}} = g(x, y), \quad y^{\text{new}} = y, \quad z^{\text{new}} = z. \tag{A.23}$$

Note that the foliation $\mathcal{F}^{uE}$ is $\mathcal{R}$-symmetric. This implies $h_1(y, c) = h_1(-y, c)$ and $g(x, y) = g(x, -y)$. Consequently, the above transformation is $\mathcal{R}$-symmetric. Therefore, the transformation $\mathcal{T}_4$ is C^2 and $\mathcal{R}$-symmetric.

Remark 2. The transformation $\mathcal{T}_4$ is C^1-smooth in parameters. This can be seen by letting ε be the vector of all parameters, and then adding $\bar{\varepsilon} = \varepsilon$ into system (A.19).

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