

## AN EXAMPLE OF A RESONANT HOMOCLINIC LOOP OF INFINITE CYCLICITY

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*Dedicated to Professor Yu. S. Ilyashenko on the occasion of his 60th birthday*

**ABSTRACT.** We describe a codimension-3 bifurcation surface in the space of  $C^r$ -smooth ( $r \geq 3$ ) dynamical systems (with phase space of dimension 4 or higher) having an attractive two-dimensional invariant manifold with an infinite sequence of periodic orbits of alternating stability which converge to a homoclinic loop.

2000 MATH. SUBJ. CLASS. 37G20, 34C07, 34C37.

KEY WORDS AND PHRASES. Codimension-3 homoclinic bifurcation, invariant manifold, swallow tail, limit cycle.

Consider an  $(n+1)$ -dimensional  $C^r$ -smooth ( $r \geq 3$ ) dynamical system with a saddle equilibrium state  $O$ . Suppose that the stable manifold  $W^s$  of  $O$  is  $n$ -dimensional and the unstable manifold  $W^u$  is one-dimensional. The unstable manifold consists of the point  $O$  and two orbits, called separatrices, leaving  $O$  at  $t = -\infty$  in opposite directions. We assume that one of the separatrices,  $\Gamma$ , tends to  $O$  as  $t \rightarrow +\infty$ , forming a homoclinic loop (this means that  $\Gamma$  is an orbit of intersection of  $W^u$  and  $W^s$ ).

Under certain assumptions, which we formulate below, such a system has a two-dimensional invariant manifold  $\mathcal{M}$  which contains the equilibrium state  $O$  and the homoclinic loop  $\Gamma$ . This manifold persists for every  $C^r$ -close system, even when the homoclinic loop splits. It is an attractive manifold: every forward orbit which stays in a small neighborhood  $U$  of  $\Gamma$  tends to  $\mathcal{M}$  as  $t \rightarrow +\infty$ , and every whole orbit which entirely lies in  $U$  must belong to  $\mathcal{M}$ .

The first results of this kind were obtained in [23], [2]; invariant manifold theorems for various classes of homoclinic loops can also be found in [8], [16], [24], [17], [22]. The significance of this result is obvious: it shows that the dynamics near the homoclinic loop  $\Gamma$  is essentially two-dimensional. We cannot expect chaotic dynamics, for example; and the only bifurcation we can expect is the birth of a certain number of limit cycles (in the case when  $\mathcal{M}$  is a Möbius band, one more bifurcation is possible — the formation of a double homoclinic loop). Therefore, the main questions are what is the number of limit cycles which can be born from  $\Gamma$ , can they coexist with  $\Gamma$ , etc.

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Received April 7, 2003.

For sufficiently smooth systems in the plane, the answers are known due to the works of Dulac and Leontovich. Thus, it was shown in [5] that, in the case of finite codimension (i. e., unless the system satisfies an infinite set of independent conditions of equality type), a homoclinic loop to a saddle in the plane is either stable (an  $\omega$ -limit set) or unstable (an  $\alpha$ -limit set). In [11], a sharp estimate on the number of limit cycles which can be born from the homoclinic loop in the plane was given (it was rediscovered in [14]).

For a large class of heteroclinic cycles of sufficiently smooth systems in the plane, the finiteness of the number of periodic orbits which can be born from such heteroclinic cycles in the case of finite codimension was proven by Ilyashenko and Yakovenko [10] (see [9] for an overview).

The aim of the present paper is to demonstrate that, in the case of planar systems obtained by reduction of a multidimensional system onto the two-dimensional invariant manifold, the situation is quite different. Namely, we give an example of a codimension-3 homoclinic loop  $\Gamma$  for which the attractive two-dimensional invariant manifold  $\mathcal{M}$  exists possessing an infinite sequence of periodic orbits of alternating stability.

As mentioned, such a situation is impossible in the case of sufficiently smooth planar systems. The main reason why this phenomenon happens in the case under consideration is that the smoothness of the non-local invariant manifold  $\mathcal{M}$  is always very limited. In general, this manifold is only  $C^{1+\varepsilon}$ , with  $\varepsilon < 1$ . Another important ingredient of our construction is the presence of complex characteristic exponents of the saddle  $O$ . The corresponding two-dimensional invariant subspace of the system linearized at  $O$  is transverse to  $\mathcal{M}$ , so our example is at least four-dimensional.

If  $O$  is at the origin of the coordinate frame, then the system near  $O$  is written as

$$\dot{z} = Bz + o(z),$$

where the matrix  $B$  has  $n$  eigenvalues to the left of the imaginary axis and one eigenvalue to the right. The eigenvalues of  $B$  are called characteristic exponents; we denote them as  $\lambda_1, \lambda_2, \dots, \lambda_n$ , and  $\gamma$ , assuming that

$$\gamma > 0 > \operatorname{Re} \lambda_1 \geq \dots \geq \operatorname{Re} \lambda_n.$$

We assume also that  $\lambda_1$  is real and simple, so that

$$\mathbf{(A)} \quad \gamma > 0 > \lambda_1 > \operatorname{Re} \lambda_2 \geq \dots \geq \operatorname{Re} \lambda_n.$$

We can introduce coordinates  $(x, y, w)$  (where  $x \in \mathbb{R}^1$ ,  $y \in \mathbb{R}^1$ , and  $w \in \mathbb{R}^{n-1}$ ) such that the system near  $O$  takes the form

$$\dot{y} = \gamma y + \dots, \quad \dot{x} = \lambda_1 x + \dots, \quad \dot{w} = Cw + \dots,$$

where the spectrum of the matrix  $C$  is  $\lambda_2, \dots, \lambda_n$  and the dots stand for nonlinearities. In this case, the unstable manifold  $W^u$  is tangent to the  $y$ -axis at  $O$ , and the  $(x, w)$ -space is tangent to the stable manifold  $W^s$  at  $O$ .

In  $W^s$ , there exists a unique  $(n-1)$ -dimensional smooth invariant manifold  $W^{ss}$  (the strong stable manifold) which is tangent at  $O$  to the  $w$ -space. The orbits which do not lie in  $W^{ss}$  tend to  $O$  along the leading direction (the  $x$ -axis) as  $t \rightarrow +\infty$  (see [22] for more details about the strong-stable manifold, as well as about the hierarchy

of various extended unstable manifolds mentioned below to the  $(x, y)$ -plane). We shall assume that the same holds true for the homoclinic orbit  $\Gamma$ , i. e.,

**(B)**  $\Gamma \not\subset W^{ss}$ .

The unstable manifold  $W^u$  lies within the so-called extended unstable manifold  $W^{ue}$ , which is a two-dimensional  $C^1$ -smooth invariant manifold tangent at  $O$  to the eigenspace corresponding to the characteristic exponents  $\lambda_1, \gamma$ , i. e., to the  $(x, y)$ -plane. Since the orbit  $\Gamma$  is the intersection of  $W^u$  and  $W^s$ , it also lies in the intersection of  $W^{ue}$  and  $W^s$ . We make the following assumption:

**(C)** the manifold  $W^{ue}$  is transverse to  $W^s$  at the points of the homoclinic orbit  $\Gamma$ .

Since  $W^{ue}$  is two-dimensional and  $W^s$  is a manifold of codimension 1, they can indeed intersect transversely along a one-dimensional trajectory. Although the manifold  $W^{ue}$  is not defined uniquely, any of  $W^{ue}$  contains  $W^u$  and all of them are tangent to each other at every point of  $W^u$ . In particular, all of them are tangent at every point of  $\Gamma$ , so the transversality condition above is well-posed.

Conditions (B) and (C) are necessary and sufficient (see [24], [22]) for the existence of a two-dimensional attracting invariant  $C^1$ -manifold  $\mathcal{M}$  which is transverse to  $W^{ss}$  at  $O$  and contains the homoclinic loop  $\Gamma$ . Moreover, it contains all orbits which always stay in a small neighborhood of  $\Gamma$  for all times.

Conditions (A), (B), and (C) are of inequality type, so the systems with homoclinic loops satisfying these conditions form bifurcational surfaces of codimension 1 in the space of  $C^r$ -smooth systems. Now, we impose two additional constraints on the system, which define a codimension-3 manifold in this surface. Namely, we assume that the saddle  $O$  is resonant, i. e.,

**(D)** the saddle value  $\sigma = \lambda_1 + \gamma$  equals zero.

We also assume that

**(E)** the separatrix value  $A$  introduced below (see (13)) equals 1.

As we shall show below, this condition is equivalent to the vanishing of the integral  $\int_{-\infty}^{+\infty} \operatorname{div} X(z(t))dt$ , where  $\{z(t)\}_{t \in (-\infty, +\infty)}$  denotes the homoclinic solution  $\Gamma$  and  $X$  denotes the vector field of the system on the two-dimensional invariant manifold  $\mathcal{M}$ .

In the case where the saddle value  $\sigma = \lambda_1 + \gamma$  is non-zero, the bifurcations of the homoclinic loop under consideration were studied in [3] for systems in the plane and in [18], [20] for multidimensional systems. In this case, only one periodic orbit can be born from the loop. The finiteness of the number of limit cycles which can be born from the homoclinic loop with  $\sigma = 0$  (in the case of finite codimension) was established in [11] for systems in the plane and in [15], [7] for three-dimensional systems. When  $|A| \neq 1$  (i. e., condition (E) is violated), the bifurcations of the resonant homoclinic loop on the plane were studied in [12]; for multidimensional systems, the corresponding bifurcation diagrams were constructed in [4] and the final proof was obtained in [21] (a three-dimensional case was also considered in [6]). It follows from these works that no more than two limit cycles can be born from the resonant homoclinic loop if  $|A| \neq 1$ .

The main result of this paper is the following theorem.

**Theorem.** *Suppose that a  $C^r$ -smooth ( $r \geq 3$ ) dynamical system on  $\mathbb{R}^{n+1}$  ( $n \geq 3$ ) has a homoclinic loop  $\Gamma$  and conditions (A)–(E) hold. Suppose also that the characteristic exponent next to  $\lambda_1$  is complex, i. e.,  $0 > \lambda_1 > \operatorname{Re} \lambda_2 = \operatorname{Re} \lambda_3 > \operatorname{Re} \lambda_k$  ( $3 < k \leq n$ ) and  $\operatorname{Im} \lambda_2 = -\operatorname{Im} \lambda_3 \neq 0$ ; moreover, we assume that*

$$\operatorname{Re} \lambda_2 > 2\lambda_1. \quad (1)$$

*Suppose that the general position conditions (F), (G) formulated below are satisfied. Then the homoclinic loop  $\Gamma$  is the limit of a sequence of isolated periodic orbits.*

Note that the fact that the presence of complex characteristic exponents can lead to an infinite number of single-round periodic orbits near a homoclinic loop has been known since [19], where it was shown that the dynamics near  $\Gamma$  is chaotic if  $\lambda_1$  is complex and  $\gamma + \operatorname{Re} \lambda_1 > 0$ . In our example, the dynamics is simple (it is restricted to the two-dimensional invariant manifold  $\mathcal{M}$ ), but still, we have infinitely many limit cycles. Note that condition (1) (along with condition(F)) prevents the manifold  $\mathcal{M}$  from being even  $C^2$ -smooth.

Now, let us formulate the remaining conditions (F) and (G) of the theorem. By assumption, the tangent space to  $W^s$  at  $O$  splits into two subspaces invariant with respect to the linearized system: one, the  $x$ -axis, corresponds to the characteristic exponent  $\lambda_1$ , and the second, the  $w$ -subspace, corresponds to the characteristic exponents  $\lambda_2, \dots, \lambda_n$ . We shall write  $w = (u, v)$ , where  $u \in \mathbb{R}^2$  is the projection onto the invariant subspace corresponding to the characteristic exponents  $\lambda_2$  and  $\lambda_3$  and  $v$  is the projection onto the invariant subspace corresponding to the rest of the characteristic exponents  $\lambda$ . We shall show below that condition (1) guarantees that the stable manifold  $W^s$  has a uniquely determined  $(n-3)$ -dimensional  $C^{r-1}$ -smooth invariant submanifold  $W^{s0}$  which is tangent at  $O$  to the  $(x, v)$ -subspace. Recall that we assume  $r \geq 3$ , so the manifold  $W^{s0}$  is at least  $C^2$ . It is this smoothness condition that ensures the uniqueness of  $W^{s0}$ : we shall see below that when (1) holds, every other manifold tangent to the  $(x, v)$ -space is only  $C^1$ . By condition B, the homoclinic loop  $\Gamma$  is tangent to the  $x$ -axis when it enters  $O$  at  $t = +\infty$ , so it is tangent to  $W^{s0}$  at  $O$ . We, however, assume that

$$(F) \quad \Gamma \not\subset W^{s0}.$$

Another invariant object we should mention is the invariant four-dimensional  $C^1$ -manifold  $W^{uee}$ , which is tangent at  $O$  to the  $(x, y, u)$ -space. This manifold includes the unstable manifold  $W^u$ ; the family of tangents  $N^{uee}$  to  $W^{uee}$  at the points of  $W^u$  is a uniquely defined continuous family of three-dimensional spaces, which is invariant with respect to the linearized flow and tends to the  $(x, y, u)$ -space when approaching the point  $O$ . The space  $N^{uee}$  contains a two-dimensional subspace  $N^{ue}$  which is tangent to the manifold  $W^{ue}$  (the invariant two-dimensional manifold tangent at  $O$  to the  $(x, y)$ -space; see condition (C)). The family of subspaces  $N^{ue}$  is also invariant with respect to the linearized flow, continuous, and it is defined uniquely. We shall show below that condition (1) guarantees the existence of another two-dimensional subspace  $N^{u0}$  of  $N^{uee}$ , which is transverse to  $N^{ue}$  at every point of the unstable manifold  $W^u$ ; the family of the subspaces  $N^{u0}$  is continuous, invariant with respect to the linearized flow, and it is uniquely defined by these conditions. Our last genericity assumption is that

(G) the subspace  $N^{u0}$  is transverse to  $W^s$  at every point of  $\Gamma$ .

We proceed to prove the theorem. Let us locally straighten the stable and unstable manifolds near the point  $O$ , i. e., make a  $C^r$ -transformation of the coordinates after which the equation of  $W_{loc}^s$  in a small neighborhood of  $O$  becomes  $y = 0$  and the equation of  $W_{loc}^u$  becomes  $(x, u, v) = 0$ . In these coordinates, the system near  $O$  is written as follows:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x \\ u \end{pmatrix} &= (D_1 + f_{11}(x, u, v, y)) \begin{pmatrix} x \\ u \end{pmatrix} + f_{12}(x, u, v, y)v, \\ \dot{v} &= D_2v + f_{21}(x, u, v, y) \begin{pmatrix} x \\ u \end{pmatrix} + f_{22}(x, u, v, y)v, \\ \dot{y} &= \gamma y(1 + g(x, u, v, y)). \end{aligned} \tag{2}$$

Here

$$D_1 = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\rho & -\omega \\ 0 & \omega & -\rho \end{pmatrix},$$

where  $\lambda_1 = -\lambda$  and  $\lambda_{2,3} = -\rho \pm i\omega$ , so the spectrum of  $D_1$  is  $\{\lambda_1, \lambda_2, \lambda_3\}$ ; the spectrum of the matrix  $D_2$  consists of the characteristic exponents  $\lambda_k$  with  $3 < k \leq n$ , so we may assume that

$$\|e^{D_2t}\| = o(e^{-\beta t}) \quad \text{as } t \rightarrow +\infty \tag{3}$$

for some  $\beta > \rho$ . Recall that, by assumption,

$$0 < \gamma = \lambda < \rho < 2\lambda \tag{4}$$

and  $\omega \neq 0$ . It will be also convenient for us to take  $\beta < 2\lambda$ .

The functions  $f_{ij}$  and  $g$  in (2) are  $C^{r-1}$ -functions vanishing at zero. By scaling time, we can always make

$$g \equiv 0,$$

which we assume hereafter. Importantly, the coordinates  $(x, u, v, y)$  can be chosen in such a way that the functions  $f_{ij}$  satisfy the identities

$$f_{i1}(x, u, v, 0) \equiv 0, \quad f_{1j}(0, 0, 0, y) \equiv 0. \tag{5}$$

The transformation which brings the system near  $O$  to the form (2), (5) is of class  $C^{r-1}$ . The existence and smoothness of this transformation is proven in [22] (following [1], [13]; note that the proof in [22] is carried out for the case where all the eigenvalues of the matrix  $D_1$  have the same real parts, but it remains valid without any modifications in the case under consideration, where the spectrum of  $D_1$  lies strictly in the strip  $-\beta < \text{Re}(\cdot) < -\beta'$  with  $\beta' > 0$  and  $\beta < \min(2\beta', \beta' + \gamma)$ ).

Note that the terms in the functions  $f_{ij}$  which do not satisfy identities (5) are always non-resonant. So the possibility to achieve these identities means that these particular non-resonant terms can be killed by a single  $C^{r-1}$ -transformation of coordinates. Identities (5) have also certain geometrical meaning.

Thus, it is easy to see that the first of identities (5) imply that the evolution of the  $(x, u)$ -variables on the stable manifold (i. e., in the system obtained by substituting  $y = 0$  into (2)) is independent of the  $v$  variables. Moreover, it is linear, i. e.,

$$\dot{x} = -\lambda x, \quad \dot{u}_1 = -\rho u_1 - \omega u_2, \quad \dot{u}_2 = -\rho u_2 + \omega u_1$$

at  $y = 0$ . It is obvious from this equation, that the manifold  $\{u = 0, y = 0\}$  is invariant; moreover, since  $\rho \in (\lambda, 2\lambda)$ , it is the only invariant manifold which is tangent to the  $\{u = 0, y = 0\}$ -space and is at least  $C^2$ -smooth. Thus, the invariant manifold  $W^{s0}$  from condition (F) is given by the equation  $\{u = 0, y = 0\}$  in our coordinates.

Similarly, the only invariant submanifold of  $W^s$  which is transverse to the  $x$ -axis is the manifold  $\{x = 0, y = 0\}$ ; i. e., it is the manifold  $W^{ss}$  mentioned in condition (B).

Let  $(x = 0, u = 0, v = 0, y = y^\circ(t))$  be a trajectory on the unstable manifold. Taking into account the second of identities (5), we see that the linearization of system (2) (with  $g \equiv 0$ ) is written for such a trajectory as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} X \\ U \end{pmatrix} &= D_1 \begin{pmatrix} X \\ U \end{pmatrix} + f_{12}(0, 0, 0, y^\circ(t))V, \\ \dot{V} &= (D_2 v + f_{22}(0, 0, 0, y^\circ(t)))V, \\ \dot{Y} &= \gamma Y, \end{aligned} \tag{6}$$

where  $(X, U, V, Y)$  are the coordinates on the tangent space. We see that the space  $V = 0$  is invariant with respect to the linearized system. By uniqueness,  $V = 0$  is the space  $N^{uee}$ , i. e., the tangent space to the invariant manifold  $W^{uee}$ . Within the space  $V = 0$  the system (6) reduces to

$$\frac{d}{dt} \begin{pmatrix} X \\ U \end{pmatrix} = D_1 \begin{pmatrix} X \\ U \end{pmatrix};$$

this system has exactly two invariant subspaces,  $X = 0$  and  $U = 0$ . The space  $(U = 0, V = 0)$  is the space  $N^{ue}$ , which is tangent to the invariant manifold  $W^{ue}$ ; hence the space  $(X = 0, V = 0)$  is the invariant space  $N^{u0}$  from condition (G).

We see that the invariant manifolds and subspaces mentioned in the genericity conditions on the homoclinic loop under consideration have especially simple equations when identities (5) hold. In particular, the manifold  $W_{loc}^{ue}$  is tangent to the space  $(u, v) = 0$  at every point of the local unstable manifold. Hence on this invariant manifold the system can be written as follows:

$$\dot{y} = \gamma y, \quad \dot{x} = -\lambda x + p(x, y),$$

where the  $C^1$ -function  $p$  vanishes identically, along with its first derivative with respect to  $x$ , both at  $y = 0$  and at  $x = 0$  (see (2), (5)). Thus, the divergence of the vector field on  $W_{loc}^{ue}$  vanishes (recall that  $\gamma = \lambda$  by assumption) at the points of  $W_{loc}^u$  and  $W_{loc}^{ue} \cap W_{loc}^s$ . This means that the flow on  $W_{loc}^{ue}$ , when linearized at the points of any orbit from  $W_{loc}^u$  or  $W_{loc}^{ue} \cap W_{loc}^s$ , preserves area (the manifold  $W_{loc}^{ue}$  is not uniquely defined, but this area-preservation property holds for any of these manifolds).

By assumption, the loop  $\Gamma$  coincides locally with one of the  $y$ -semiaxes when it leaves  $O$  at  $t = -\infty$ . Since  $\Gamma \not\subset W^{ss}$ , it enters  $O$  as  $t \rightarrow +\infty$  along the  $x$ -axis. We assume that  $\Gamma$  approaches  $O$  from the side of positive  $y$  and positive  $x$  as  $t \rightarrow -\infty$  and  $t \rightarrow +\infty$ , respectively.

Take two cross-sections  $S_0$  and  $S_1$  to the loop  $\Gamma$ . To be more precise,  $S_1$  is  $\{y = d\}$  and  $S_0$  is  $\{x = d\}$  for some small  $d > 0$ . We denote the coordinates on  $S_1$  as  $(x_1, u_{11}, u_{12}, v_1)$  and the coordinates on  $S_0$  as  $(y_0, u_{01}, u_{02}, v_0)$ .

Since  $g \equiv 0$  in (2), the last equation of (2) is easy to integrate:

$$y(t) = e^{\gamma t} y_0. \tag{7}$$

Thus, the orbit of a point on  $S_0$  intersects the cross-section  $S_1$  when leaving the  $d$ -neighborhood of  $O$  if and only if  $y_0 > 0$ ; the time of flight from  $S_0$  to  $S_1$  is equal to

$$\tau = -\frac{1}{\gamma} \ln \frac{y_0}{d}. \tag{8}$$

The time- $\tau(y_0)$  map of the upper part  $S_0^+$  (where  $y_0 > 0$ ) of  $S_0$  into  $S_1$  is called the local map  $T_0$ .

The flow outside a small neighborhood of  $O$  defines the global map  $T_1: S_1 \rightarrow S_0$ , which takes any point from a small neighborhood of zero in  $S_1$  into the first intersection point of the forward orbit of this point with  $S_0$ . The composition  $T = T_1 T_0$  is the Poincaré map near the homoclinic loop  $\Gamma$ ; its fixed points correspond to periodic orbits of the flow. Thus, to prove the theorem, we have to show that the map  $T: S_0^+ \rightarrow S_0$  has an infinite sequence of isolated fixed points converging to  $y_0 = 0$ .

In the same way as in [13], we can show that the fulfillment of identities (5) implies the following estimates for the solution of the system starting at the point with the coordinates  $(x_0, u_0, v_0)$  at  $t = 0$  and reaching  $\{y = d\}$  at some  $t = \tau$ :

$$\begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} = e^{D_1 \tau} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} + \xi_2(x_0, u_0, v_0, \tau), \quad v(\tau) = \xi_2(x_0, u_0, v_0, \tau), \tag{9}$$

where

$$\|\xi_{1,2}\|_{C^{r-2}} = o(e^{-\beta \tau}). \tag{10}$$

Here  $\beta > 0$  is the constant such that the spectrum of the matrix  $D_2$  lies strictly to the left of the line  $\text{Re}(\cdot) = -\beta$  (i. e., (3) holds) and the spectrum of the matrix  $D_1$  lies in the strip  $\mathcal{R}_{\beta, \beta'}: -\beta < \text{Re}(\cdot) < -\beta'$  with  $\beta' > 0$  and  $\beta < \min(2\beta', \beta' + \gamma)$  (i. e.,  $\beta \in (\rho, 2\lambda)$  in the case under consideration; see (4)). A detailed proof of estimates (9), (10) can be found in [21] (formally, only the case where all the eigenvalues of  $D_1$  have the same real parts is considered in [21]; however, the proof given there is transferred to the case under consideration, where the real parts of the eigenvalues of  $D_1$  are spread in the small strip  $\mathcal{R}_{\beta, \beta'}$ , without any modifications).

Substituting the expression (8) for the flight time into (9), (10), we obtain the following estimate for the local map  $T_0: S_0^+ \rightarrow S_1$  (recall that  $\gamma = \lambda$  by assumption

and that  $x_0 = d$  on  $S_0$ ):

$$\begin{aligned}
 x_1 &= y_0 + \varphi_1(y_0, u_0, v_0), \\
 u_{11} &= \left(\frac{y_0}{d}\right)^\nu \left(u_{01} \cos \Omega \ln \frac{y_0}{d} - u_{02} \sin \Omega \ln \frac{y_0}{d}\right) + \varphi_2(y_0, u_0, v_0), \\
 u_{12} &= \left(\frac{y_0}{d}\right)^\nu \left(u_{01} \sin \Omega \ln \frac{y_0}{d} + u_{02} \cos \Omega \ln \frac{y_0}{d}\right) + \varphi_3(y_0, u_0, v_0), \\
 v_1 &= \varphi_4(y_0, u_0, v_0),
 \end{aligned}
 \tag{11}$$

where  $\Omega = \omega/\gamma$  and  $\nu = \rho/\gamma$ , so  $\Omega \neq 0$  and  $1 < \nu < 2$ ; the functions  $\varphi_j$ ,  $j = 1, \dots, 4$ , satisfy the estimates

$$\begin{aligned}
 \varphi &= o(y_0^\nu), \\
 \frac{\partial^{p+q}\varphi}{\partial(u_0, v_0)^p \partial y_0^q} &= o(y_0^{\nu-q}) \quad (p+q = 1, \dots, r-2).
 \end{aligned}
 \tag{12}$$

The global map  $T_1: S_1 \rightarrow S_0$  is a diffeomorphism of a small neighborhood of the point  $M^-(0, 0, 0, 0) = \Gamma \cap S_1$  into a small neighborhood of the point  $M^+(0, u_1^+, u_2^+, v^+) = \Gamma \cap S_0$ . Hence it can be written as

$$\begin{aligned}
 y_0 &= a_{11}x_1 + a_{12}u_{01} + a_{13}u_{02} + a_{14}v_1 + \dots, \\
 u_{01} - u_1^+ &= a_{21}x_1 + a_{22}u_{01} + a_{23}u_{02} + a_{24}v_1 + \dots, \\
 u_{02} - u_2^+ &= a_{31}x_1 + a_{32}u_{01} + a_{33}u_{02} + a_{34}v_1 + \dots, \\
 v_0 - v^+ &= a_{41}x_1 + a_{42}u_{01} + a_{43}u_{02} + a_{44}v_1 + \dots,
 \end{aligned}
 \tag{13}$$

where the  $a_{ij}$  are certain coefficients, and the dots stand for the quadratic and higher-order terms.

The coefficient  $a_{11}$  is called the separatrix value (see [21]); it is exactly the value  $A$  from condition (E). It can be shown that, in the case under consideration, where  $\lambda = \gamma$ , the value of  $a_{11}$  is invariant with respect to the smooth coordinate transformations under which the system retains the form (2) with  $g \equiv 0$  and with identities (5) satisfied. This can be verified by a direct computation, but we give a different proof below.

Note that the two-dimensional invariant manifold  $\mathcal{M}$ , which contains the homoclinic loop  $\Gamma$ , is transverse at  $O$  to the strong stable manifold  $W^{ss}$ ; therefore, this manifold coincides with some local extended unstable manifold  $W^{ue}$  near  $O$  (see [22] for more details). We can choose  $(x, y)$  as the coordinates on  $\mathcal{M}$  near  $O$ . For the flow on  $\mathcal{M}$ , the global map  $T_1: S_1 \cap \mathcal{M} \rightarrow S_0 \cap \mathcal{M}$  is written as

$$y_0 = a_{11}x_1 + o(x_1).$$

Thus,  $a_{11}$  is the coefficient of expansion (or contraction) of distances at the point  $M^-$  by the global map restricted to  $\mathcal{M}$ . In our coordinates, the phase velocity vectors  $\dot{y} = \gamma y$  and  $\dot{x} = -\lambda x$  for the flow on  $\mathcal{M}$  taken at the points  $M^-(y = d)$  and  $M^+(x = d)$ , respectively, have the same length (recall that  $\gamma = \lambda$ ). Therefore,  $a_{11}$  is also the coefficient of expansion/contraction of areas by the flow on  $\mathcal{M}$  linearized at the points of the homoclinic orbit  $\Gamma$  on the segment from the point  $M^-$  to the point  $M^+$ . It follows that after a smooth coordinate transformation the coefficient  $a_{11}$  is multiplied by  $J(M^+)/J(M^-)$ , where  $J$  is the Jacobian of the coordinate

transformation on  $\mathcal{M}$ . Since  $\mathcal{M}$  coincides locally with some manifold  $W^{ue}$ , the flow on  $\mathcal{M}$  is divergence free at the points of  $W_{loc}^u$  and of  $W_{loc}^s \cap \mathcal{M}$ , provided that  $g = 0$  in (2) and identities (5) hold. Hence, when linearized at the points of  $\Gamma$ , the flow on  $\mathcal{M}$  become area-preserving near  $O$ . This implies that, in our coordinates, the coefficient  $a_{11}$  is independent of the choice of the points  $M^+$  and  $M^-$ , i. e., of the choice of the small constant  $d$ . Therefore, for the smooth coordinate transformations under which the system retains the form (2) with  $g \equiv 0$  and with identities (5) satisfied, the factor  $J(M^+)/J(M^-)$  must be independent of  $d$  as well. As  $d \rightarrow +0$ ,  $M^+$  and  $M^-$  converge to the same point  $O$ , which gives  $J(M^+)/J(M^-) \equiv 1$  for the coordinate transformations under consideration. Thus,  $a_{11}$  is indeed an invariant of such transformations.

By virtue of condition E, we have  $a_{11} = 1$ . By combining (11) and (13), we obtain the following equation on the fixed points  $(y, u, v)$  (we omit the subscript “0”) of the Poincaré map  $T = T_1 T_0$  on  $S_0^+$ :

$$\begin{aligned} y &= y + \left(\frac{y}{d}\right)^\nu |u| \left( a_{12} \cos \left( \Omega \ln \frac{y}{d} + \theta \right) + a_{13} \sin \left( \Omega \ln \frac{y}{d} + \theta \right) \right) + o(y^\nu), \\ u &= u^+ + O(y), \quad v = v^+ + O(y), \end{aligned} \tag{14}$$

where  $u = (|u| \cos \theta, |u| \sin \theta)$ . For all small  $y$ , the last equations of this system can be resolved with respect to  $u$  and  $v$ , so the system reduces to the following single equation in the  $y$ -variable:

$$0 = y^\nu |u^+| \left( a_{12} \cos \left( \Omega \ln \frac{y}{d} + \theta \right) + a_{13} \sin \left( \Omega \ln \frac{y}{d} + \theta \right) \right) + o(y^\nu). \tag{15}$$

This equation has an infinite sequence of isolated positive roots converging to zero, namely,

$$y_m = d e^{-\frac{\pi}{\Omega} m} e^{(-\theta - \arctan \frac{a_{12}}{a_{13}})} (1 + o(1)),$$

provided that  $u^+ \neq 0$  and  $a_{12}^2 + a_{13}^2 \neq 0$ . It remains to note that these two inequalities are, in fact, conditions (F) and (G), respectively. Indeed, by condition (F), the point  $M^+(0, u^+, v^+)$  does not belong to the manifold  $W^{s0}$ . The latter is given by the equation  $(y = 0, u = 0)$  in our coordinates, so condition (F) reads as  $u^+ \neq 0$  indeed. In turn, condition (G) reads in our coordinates as the transversality of the image of the plane  $(x_0 = 0, v_0 = 0)$  from  $S_0$  under the map  $T_1$  to the space  $y_1 = 0$  at the point  $M^+$  in  $S_1$ . It follows immediately from (13) that this transversality condition is equivalent to  $a_{12}^2 + a_{13}^2 \neq 0$ .

Thus, we have proved the existence of an infinite sequence of isolated fixed points of the Poincaré map, converging to  $W_{loc}^s \cap S_0$ . The fixed points of the Poincaré map correspond to periodic orbits of the flow. This completes the proof.

An interesting question is how the obtained family of limit cycles bifurcates. We deal with a codimension-3 bifurcation, so we need at least three governing parameters. We denote them as  $(\mu, \delta, \alpha)$ . The parameter  $\mu$  governs the splitting of the homoclinic loop  $\Gamma$ ; we take it equal to the  $y$ -coordinate of the point  $M^+ = T_1 M^-$  where  $M^- = W_{loc}^u \cap S_1$ . We also take  $\delta = \lambda/\gamma - 1$  and  $\alpha = a_{11} - 1$ . Then, following the same lines of reasoning as in the proof of the theorem, we can show that the fixed point of the Poincaré map satisfy the equation

$$y = \mu + (1 + \alpha)y^{1+\delta} + Ky^\nu \cos(\Omega \ln y + \theta) + o(y^\nu)$$

for some constants  $K \neq 0$  and  $\theta$ . This equation is bound to produce a rich bifurcation diagram. Thus, it can be shown that an infinite sequence of swallow tails exists in the parameter space.

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