Sbornik: Mathematics **189**:2 291–314

Matematicheskiĭ Sbornik **189**:2 137–160

UDC 517.95

An example of a wild strange attractor

D. V. Turaev and L. P. Shil'nikov

Abstract. It is proved that in the space of C^r-smooth ($r \geqslant 4$) flows in $\mathbb{R}^n$ ($n \geqslant 4$) there exist regions filled by systems that each have an attractor (here: a completely stable chain-transitive closed invariant set) containing a non-trivial basic hyperbolic set together with its unstable manifold, which has points of non-transversal intersection with the stable manifold. A construction is given for such a wild attractor containing an equilibrium state of saddle-focus type.

Bibliography: 22 titles.

§ 1. Statement of the problem and main results

In this paper we single out a class of dynamical systems with a strange attractor of a new type distinguished by the condition that it can contain a wild hyperbolic set (in the sense of [1]). Namely, we determine regions in the space of C^r-smooth ($r \geqslant 4$) flows in $\mathbb{R}^n$ ($n \geqslant 4$) that are filled by systems that each have an attractor (here this is a completely stable chain-transitive closed invariant set) containing a non-trivial basic hyperbolic set together with its unstable manifold, which has points of non-transversal intersection with the stable manifold.

As shown in [1]–[5], the presence of a wild set causes highly non-trivial behaviour of the orbits. We show, in particular, that the attractor we construct can contain periodic orbits with non-transversal homoclinic curves, and the order of tangency of the stable and unstable manifolds along a homoclinic curve can be arbitrarily high. Since a description of the bifurcations of an nth-order tangency requires n independent parameters, the result means, in particular, that a complete description of the dynamics and bifurcations of the attractor constructed is impossible in any finite-parameter family. Another reflection of this property is that the attractor can contain non-hyperbolic periodic orbits of arbitrarily high orders of degeneracy in the non-linear terms. Moreover, it can simultaneously contain periodic orbits with different dimensions of the unstable manifolds ($\dim W^u = 2$ and $\dim W^u = 3$).

Our construction is close to that of the geometric model of the Lorenz attractor constructed in [6] and [7], with the difference that it is essentially non-three-dimensional and that here the equilibrium state is not a saddle but a *saddle-focus*.

This work was carried out with the partial financial support of the Russian Foundation for Fundamental Research (grant no. 96-01-01135) and of INTAS (grant no. 930570).

AMS 1991 *Mathematics Subject Classification.* Primary 58F14; Secondary 58F13, 58F15, 58F25, 58F14, 58F12, 58F10, 34C37, 34C35.

Namely, let X be a C^r-smooth ($r \geqslant 4$) flow in $\mathbb{R}^n$ ($n \geqslant 4$) having an equilibrium state O of saddle-focus type whose characteristic exponents are γ, $-\lambda \pm i\omega, -\alpha_1, \dots, -\alpha_{n-3}$, where $\gamma > 0$, $0 < \lambda < \operatorname{Re}\alpha_j$, and $\omega \neq 0$. We assume that

$$\gamma > 2\lambda \tag{1}$$

when one of the separatrices returns. This condition was introduced in [8], where, in particular, it was shown to be necessary in order that, in the case when the separatrix returns to the saddle-focus as $t \to +\infty$ (forms a *homoclinic loop*), no stable periodic orbits can arise in a neighbourhood of the loop.

Let us introduce coordinates (x, y, z) ($x \in \mathbb{R}^1$, $y \in \mathbb{R}^2$, $z \in \mathbb{R}^{n-3}$) so that the equilibrium state is at the origin of coordinates, the one-dimensional unstable manifold of the point O is tangent to the x-axis, and the $(n-1)$-dimensional stable manifold is tangent to the plane $\{x = 0\}$; further, the coordinates y_1 and y_2 correspond to the leading exponents $\lambda \pm i\omega$, while the coordinates of z correspond to the non-leading exponents α.

We shall assume that the flow possesses a *cross-section*, say, the surface $\Pi = \{\|y\| = 1, \|z\| \leqslant 1, |x| \leqslant 1\}$. Since the stable manifold W^s is tangent at O to the surface $\{x = 0\}$, it is given locally by an equation of the form $x = h^s(y, z)$, where h^s is some smooth function and $h^s(0,0) = 0$. We suppose that such a representation is valid at least for ($\|y\| \leqslant 1$, $\|z\| \leqslant 1$) and that $|h^s| < 1$. Thus, the surface Π serves as a cross-section for W^s_{loc}, and the intersection of W^s_{loc} with Π has the form Π_0: $x = h_0(\varphi, z)$, where φ is an angular coordinate, $y_1 = \|y\| \cos\varphi$, $y_2 = \|y\| \sin\varphi$, and h_0 is a smooth function with $-1 < h_0 < 1$. By a change of coordinates we can make $h_0 \equiv 0$ and we shall assume that this is satisfied.

We assume that all orbits starting on $\Pi \setminus \Pi_0$ return to Π, thereby determining the Poincaré maps: $T_+ \colon \Pi_+ \to \Pi$ and $T_- \colon \Pi_- \to \Pi$, where $\Pi_+ = \Pi \cap \{x > 0\}$ and $\Pi_- = \Pi \cap \{x < 0\}$. It is clear that if P is a point on Π with coordinates (x, φ, z), then

$$\lim_{x \to -0} T_-(P) = P_-^1, \qquad \lim_{x \to +0} T_+(P) = P_+^1,$$

where P_-^1 and P_+^1 are the points where the one-dimensional unstable separatrices of the point O intersect Π. Correspondingly, we can extend the maps T_+ and T_- by continuity, so that

$$T_-(\Pi_0) = P_-^1, \qquad T_+(\Pi_0) = P_+^1. \tag{2}$$

Obviously, the region $\mathcal{D}$ filled by the orbits starting on Π (plus the point O and its separatrices) is an *absorbing region* for the system X in the sense that the orbits starting on $\partial\mathcal{D}$ enter $\mathcal{D}$ and remain there for all positive values of the time t. By construction, $\mathcal{D}$ is the cylinder $\{\|y\| \leqslant 1,\ \|z\| \leqslant 1,\ |x| \leqslant 1\}$ with two attached handles enclosing the separatrices (Fig. 1).

We assume that the (semi)flow in $\mathcal{D}$ is *pseudohyperbolic.* (Here it is more convenient for us to include in this concept a meaning stronger than the usual one [9].) Namely, we propose the following.

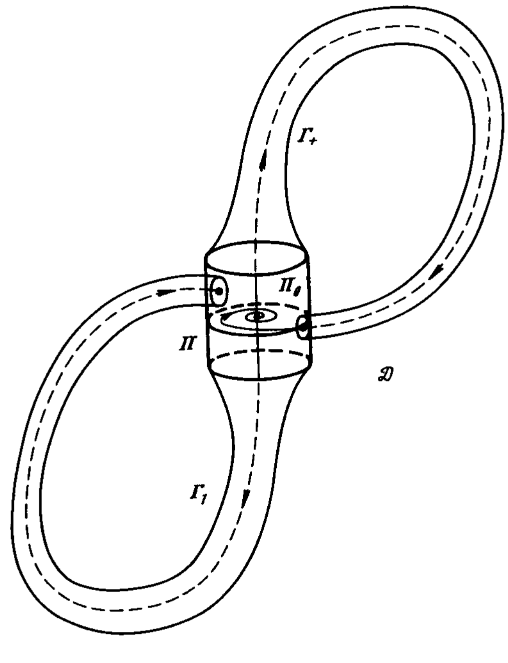

Figure 1

Definition. A semiflow is said to be *pseudohyperbolic* if the following two conditions hold:

(A) at each point of phase space the tangent space decomposes into a direct sum of subspaces N_1 and N_2 depending continuously on the point, in an invariant way with respect to the linearized semiflow and so that the maximal Lyapunov exponent corresponding to N_1 is strictly less than the minimal Lyapunov exponent corresponding to N_2, that is, for any point M and for any non-zero vectors $u \in N_1(M)$ and $v \in N_2(M)$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \frac{\|u_t\|}{\|u\|} < \liminf_{t\to+\infty} \frac{1}{t} \ln \frac{\|v_t\|}{\|v\|},$$

where u_t and v_t are the shifts of u and v by the semiflow, linearized along the orbit of M;

(B) the linearized semiflow *exponentially expands volume* in restriction to N_2.

The novelty here lies in the introduction of the condition (B), which guarantees the absence of stable periodic orbits. Generally speaking, the definition does not exclude the case when the maximal Lyapunov exponent corresponding to N_1 is everywhere non-negative: then the linearized semiflow is, a fortiori, expanding in restriction to N_2, and the condition (B) holds trivially. By contrast, this paper will take up the case when the linearized semiflow is *exponentially contracting* in restriction to N_1, and here the condition (B) is essential.

We remark that the property of being pseudohyperbolic is preserved under small smooth perturbations of the system: according to [9], the invariant decomposition of the tangent space is preserved under small perturbations and the subspaces N_1 and N_2 vary continuously. It is obvious that the property of exponential expansion of volumes in N_2 is also stable under small perturbations.

This definition is sufficiently broad; in particular, it includes hyperbolic flows, for which we can set $(N_1, N_2) = (N^s, N^u \oplus N_0)$ or $(N_1, N_2) = (N^s \oplus N_0, N^u)$, where N^s and N^u are the stable and unstable invariant subspaces, and N_0 is the one-dimensional invariant subspace spanned by the phase velocity vector. The geometric model of the Lorenz attractor in [7] or [10] also relates to this class: here N_1 serves as the tangent to a contracting invariant foliation of codimension two, and the expansion of areas in the two-dimensional subspace N_2 is ensured by the fact that the Poincaré map is expanding in a direction transversal to the contracting foliation.

In this article we assume that the subspace N_1 has codimension three, that is, $\dim N_1 = n - 3$ and $\dim N_2 = 3$, and that for $t \geqslant 0$ the linearized flow is exponentially contracting on N_1. Here the condition (A) means that even if contraction takes place for vectors in N_2, it is weaker than on N_1. To underscore the last circumstance, we say that N_1 is the *strongly contracting* subspace, while N_2 is the *centre* subspace, and we denote them by N^{ss} and N^c, respectively. In addition we assume that the coordinates (x, y, z) in $\mathbb{R}^n$ are introduced in such a way that at each point of the region $\mathcal{D}$ the space N^{ss} has non-zero projection on the z-coordinate space and N^c has non-zero projection on the (x, y)-coordinate space.

We note that the requirements of being pseudohyperbolic are initially satisfied at the point O: here N^{ss} coincides with the z-coordinate space and N^c with the (x, y)-coordinate space, and the condition (1) just ensures the expansion of volumes in the invariant (x, y)-subspace. These properties of the linearized flow are automatically inherited by the orbits in a small neighbourhood of O. In essence, we have required that the properties be inherited in the large neighbourhood $\mathcal{D}$ of the point O.

According to [9], exponential contraction in N^{ss} implies the existence of an invariant contracting foliation $\mathcal{N}^{ss}$ with C^r-smooth leaves tangent to N^{ss}. It can be shown similarly ([11], [12]) that this foliation is absolutely continuous. When the quotient by the leaves is taken, the region $\mathcal{D}$ becomes a branched three-dimensional manifold: since $\mathcal{D}$ is bounded, and since the quotient semiflow expands volume, the orbits of the quotient semiflow must obviously be pasted together on certain surfaces in order to stay bounded (cf. [10]).

The properties of pseudohyperbolicity and expansion of volumes are inherited in a natural way by the Poincaré map $T \equiv (T_+, T_-)$ on the cross-section Π. Here the following hold.

(A*) There is a foliation with smooth leaves of the form $(x, \varphi) = h(z)\big|_{-1 \leqslant z \leqslant 1}$, where the derivative $h'(z)$ is uniformly bounded, and it has the following properties: it is *invariant* in the sense that $T_+^{-1}(l \cap T_+(\Pi_+ \cup \Pi_0))$ and $T_-^{-1}(l \cap T_-(\Pi_- \cup \Pi_0))$ are leaves of the foliation if l is a leaf (provided that they are non-empty sets), it is *absolutely continuous* in the sense that the projection map along leaves from one transversal of the foliation onto another transversal changes areas by a finite multiple, with the corresponding ratios of areas bounded away from infinity and zero, and it is *contracting* in the sense that if two points belong to a single leaf, then the distance between their iterates under the action of the map T tends exponentially to zero.

(B*) The quotient maps $\widetilde{T}_+$ and $\widetilde{T}_-$ exponentially expand area.

Satisfaction of the properties (A*) and (B*) and the relation (2) for the Poincaré map T is sufficient for the validity of Theorems 1–3 below. As in [7], where an analogous result was proved for the Lorenz attractor, it is possible to establish conditions sufficient for (A*) and (B*) to hold.

Lemma 1. *Let the map T be written in the form*

$$(\overline{x},\overline{\varphi}) = g(x,\varphi,z), \qquad \overline{z} = f(x,\varphi,z),$$

where the functions f and g are smooth for $x \neq 0$ and have discontinuities at $x = 0$:

$$\lim_{x\to -0}(g,f) = (x_-,\varphi_-,z_-) \equiv P^1_-, \qquad \lim_{x\to +0}(g,f) = (x_+,\varphi_+,z_+) \equiv P^1_+.$$

Suppose that

$$\det \frac{\partial g}{\partial(x,\varphi)} \neq 0. \tag{3}$$

Let

$$A = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial(x,\varphi)}\left(\frac{\partial g}{\partial(x,\varphi)}\right)^{-1}\frac{\partial g}{\partial z}, \quad B = \frac{\partial f}{\partial(x,\varphi)}\left(\frac{\partial g}{\partial(x,\varphi)}\right)^{-1},$$
$$C = \left(\frac{\partial g}{\partial(x,\varphi)}\right)^{-1}\frac{\partial g}{\partial z}, \qquad D = \left(\frac{\partial g}{\partial(x,\varphi)}\right)^{-1}.$$

If

$$\lim_{x\to 0} C = 0, \qquad \lim_{x\to 0}\|A\|\,\|D\| = 0, \tag{4}$$

$$\sup_{P\in\Pi\setminus\Pi_0}\sqrt{\|A\|\,\|D\|} + \sqrt{\sup_{P\in\Pi\setminus\Pi_0}\|B\| \sup_{P\in\Pi\setminus\Pi_0}\|C\|} < 1, \tag{5}$$

then the map has a continuous invariant foliation of smooth leaves of the form $(x,\varphi) = h(z)\big|_{-1\leqslant z\leqslant 1}$, where the derivative $h'(z)$ is uniformly bounded. If, moreover,

$$\sup_{P\in\Pi\setminus\Pi_0}\|A\| + \sqrt{\sup_{P\in\Pi\setminus\Pi_0}\|B\| \sup_{P\in\Pi\setminus\Pi_0}\|C\|} < 1, \tag{6}$$

then the foliation is contracting, and under the additional condition that for some $\beta > 0$

$$\begin{gathered}\text{the functions } A|x|^{-\beta},\ D|x|^{\beta},\ B,\ C \text{ are uniformly bounded and Hölder,}\\ \frac{\partial \ln\det D}{\partial z},\ \frac{\partial \ln\det D}{\partial(x,\varphi)}D|x|^{\beta} \text{ are uniformly bounded,}\end{gathered} \tag{7}$$

it is absolutely continuous. If, furthermore,

$$\sup_{P\in\Pi\setminus\Pi_0}\sqrt{\det D} + \sqrt{\sup_{P\in\Pi\setminus\Pi_0}\|B\| \sup_{P\in\Pi\setminus\Pi_0}\|C\|} < 1, \tag{8}$$

then the quotient map $\widetilde{T}$ expands area.

We remark that, as follows from [8], in the case when the equilibrium state is a saddle-focus, the Poincaré map can be written in the form

$$(\overline{x}, \overline{\varphi}) = Q_\pm(Y, Z), \qquad \overline{z} = R_\pm(Y, Z) \tag{9}$$

near $\Pi_0 = \Pi \cap W^s$ for a suitable choice of coordinates. Here

$$\begin{aligned} Y &= |x|^\rho \begin{pmatrix} \cos(\Omega \ln |x| + \varphi) & \sin(\Omega \ln |x| + \varphi) \\ -\sin(\Omega \ln |x| + \varphi) & \cos(\Omega \ln |x| + \varphi) \end{pmatrix} + \Psi_1(x, \varphi, z), \\ Z &= \Psi_2(x, \varphi, z), \end{aligned} \tag{10}$$

where $\rho = \lambda/\gamma < \frac{1}{2}$ (see (1)), $\Omega = \omega/\gamma$, and for some $\eta > \rho$

$$\left\| \frac{\partial^{p+|q|} \Psi_i}{\partial x^p \partial(\varphi, z)^q} \right\| = O(|x|^{\eta - p}), \qquad 0 \leqslant p + |q| \leqslant r - 2; \tag{11}$$

the $Q_\pm$ and $R_\pm$ in (9) ("+" corresponds to $x > 0$, that is, to the map T_+, and "−" corresponds to $x < 0$, that is, to the map T_-) are smooth functions in a neighbourhood of $(Y, Z) = 0$ for which we can write the Taylor expansions

$$Q_\pm = (x_\pm, \varphi_\pm) + a_\pm Y + b_\pm Z + \cdots, \qquad R_\pm = z_\pm + c_\pm Y + d_\pm Z + \cdots. \tag{12}$$

It is clear from (9)–(12) that if O is a saddle-focus satisfying (1), then T satisfies the conditions (4) and (7) with $\beta \in (\rho, \eta)$ if $a_+ \neq 0$ and $a_- \neq 0$. Moreover, analogues of the conditions (3), (5), (6), and (8) hold, where the supremum is taken not over $|x| \leqslant 1$ but only over small x. It is not hard to extend the map (9), (10), (12) to the whole of Π in such a way that the conditions of the lemma hold in entirety. For example,

$$\begin{aligned} \overline{x} &= 0.9|x|^\rho \cos(\ln |x| + \varphi), \\ \overline{\varphi} &= 3|x|^\rho \sin(\ln |x| + \varphi), \\ \overline{z} &= (0.5 + 0.1 z |x|^\eta) \operatorname{sgn} x, \end{aligned} \tag{13}$$

where $0.4 = \rho < \eta$, is such a map.

As already noted, the expansion of volumes for the quotient semiflow imposes restrictions on the possible types of limit behaviour of the orbits. Namely, $\mathcal{D}$ *cannot contain stable periodic orbits.* What is more, *any orbit in* $\mathcal{D}$ *has a positive Lyapunov exponent.* Consequently, in this case we must speak of a strange attractor.

We first recall some definitions and simple facts from topological dynamics (see, for example, [13] and [14] and the literature cited there). Let $X_t P$ be the shift of a point P along the orbit of the flow X during the time t. For given $\varepsilon > 0$ and $\tau > 0$ we define an *(ε, τ)-orbit* to be a sequence of points $P_1, P_2, \dots, P_k$ such that P_{i+1} lies at a distance less than ε from $X_t P_i$ for some $t > \tau$. A point Q is said to be *(ε, τ)-accessible* from a point P if there exists an (ε, τ)-orbit joining P and Q, and *accessible* from P if for some $\tau > 0$ it is (ε, τ)-accessible from P for any ε (this definition obviously does not depend on the choice of τ). A set C is *accessible* from a point P if it contains a point that is accessible from P.

A point P is said to be *chain-recurrent* if it is accessible from X_tP for any t. A closed invariant set C is said to be *chain-transitive* if, for any points P and Q in C and for any $\varepsilon > 0$ and $\tau > 0$, C contains an (ε, τ)-orbit joining P and Q. Obviously, all points of a chain-transitive set are chain-recurrent.

A compact invariant set C is said to be *orbitally stable* if for any neighbourhood U of it there is a neighbourhood $V(C) \subseteq U$ such that the orbits starting in V do not leave U for $t \geqslant 0$. An orbitally stable set is said to be *completely stable* if for any neighbourhood $U(C)$ there exist $\varepsilon_0 > 0$ and $\tau > 0$ and a neighbourhood $V(C) \subseteq U$ such that the (ε_0, τ)-orbits starting in V do not leave U (it is clear that this is none other than stability with respect to constantly acting perturbations). It is known that a set C is orbitally stable if and only if $C = \bigcap_{j=1}^{\infty} U_j$, where the U_j form a system of nested invariant open sets, and it is completely stable if and only if the sets U_j not only are invariant but also are absorbing regions (that is, orbits starting on ∂U_j enter U_j in a time not exceeding some τ_j; here it is obvious that (ε, τ)-orbits starting on ∂U_j always remain inside U_j if ε is sufficiently small and $\tau \geqslant \tau_j$). Since a maximal invariant set (*a maximal attractor*) contained in each absorbing region is clearly *asymptotically stable*, every completely stable set either is asymptotically stable or is an intersection of countably many nested closed invariant asymptotically stable sets.

Returning to the original system X, we define the *attractor* of the system to be the set $\mathcal{A}$ of points accessible from the equilibrium state O (in particular, $\mathcal{A}$ contains the separatrices of O and their closure). The specific choice of the set $\mathcal{A}$ is due to the fact that, as shown in the next section, *$\mathcal{A}$ is chain-transitive, completely stable, and accessible from any point of the absorbing region $\mathfrak{D}$*; moreover, *$\mathcal{A}$ is the unique chain-transitive completely stable set in $\mathfrak{D}$*, and it is the intersection of all the completely stable invariant subsets of $\mathfrak{D}$ (Theorems 1 and 2).

These assertions follow mainly from the fact that *points asymptotic to O as $t \to +\infty$ are dense in $\mathfrak{D}$ (that is, the stable manifold of O is dense in $\mathfrak{D}$)* (Lemma 2), which follows in turn from the fact that the quotient of the Poincaré map T with respect to the leaves of the contracting foliation expands areas. We note that an analogous property holds for the Lorenz attractor, too. Also, by analogy with the Lorenz attractor, we show (Theorem 3) that the number of connected components of the intersection of the attractor $\mathcal{A}$ with the cross-section is bounded, and we give for the number of components an estimate that is analogous to the estimate for the number of lacunae in the Lorenz attractor [7].

This estimate is important for the proof of Theorem 4, where we construct a wild set contained in the attractor $\mathcal{A}$. That theorem is our main result.

We remark that the wild set in Theorem 4 is determined in a completely constructive way. For this we introduce the extra assumption that the system X *has a homoclinic loop of the saddle-focus O* (Fig. 2). More precisely, we consider a one-parameter family X_μ of the form described and we assume that for $\mu = 0$ there is a homoclinic loop, that is, one of the separatrices returns to the point O as $t \to +\infty$. The systems with a homoclinic loop form bifurcation surfaces of codimension 1 in the space of dynamical systems and it can be assumed without loss of generality that for $\mu = 0$ the family X_μ intersects the corresponding surface *transversally*. Theorem 4 asserts that *$\mu = 0$ is an accumulation point of a sequence of intervals Δ_i such that for $\mu \in \Delta_i$ the attractor $\mathcal{A}_\mu$ contains a wild set and, further, for any*

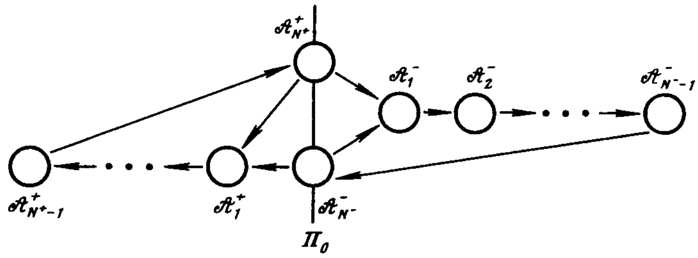

Figure 2

$\mu^* \in \Delta_i$ the attractor $\mathcal{A}$ of any system close to X_{μ^*} in the C^r-topology also contains a wild set.

The proof of the theorem is based on the following arguments. It is well known that a neighbourhood of a homoclinic loop of a saddle-focus contains a non-trivial transitive hyperbolic set Λ [15]. This is a non-closed set, and its closure contains the equilibrium state O. Further, it is clear that any point of Λ is accessible from O, and thus Λ lies in the attractor for $\mu = 0$. Since Λ is non-closed, it is not entirely preserved under small changes of μ, though its closed invariant subsets are preserved, of course. These are basic hyperbolic sets, they are unstable, and they are bounded away from the point O. Nevertheless, we explicitly single out a subset which we can prove (Lemmas 3 and 4) belongs to the attractor for all small $\mu > 0$ (here positive values of μ correspond to an inwards splitting of the loop) and for all systems close to X_μ for $\mu > 0$. In the spirit of [8] we next show that $\mu = 0$ is an accumulation point (from the positive side) of values $\mu = \mu_i$ corresponding to a homoclinic tangency of the invariant manifolds of a periodic orbit in the indicated subset (Lemma 5). On the basis of [16], this implies the existence now not of isolated values μ but of intervals on which the given subset is wild (that is, for each μ in such an interval its unstable manifold has points of tangency with the stable manifold, and this property is preserved under small perturbations of the system). This concludes the proof of the theorem. We remark that the use of results from [16] requires the verification of a number of non-degeneracy conditions for a homoclinic tangency when $\mu = \mu_i$; this is done in Lemma 5. Results analogous to [16] are proved also in [17] under the additional assumption that there is a sufficiently smooth linearization for the Poincaré map in a neighbourhood of a periodic orbit whose invariant manifolds have a homoclinic tangency. In our case, however, verifying this condition would be difficult and would possibly require additional restrictions on the spectrum of the non-leading characteristic exponents α_i of O, and this is not related to the essence of the matter.

With regard to the generic case when the separatrices of O do not form a loop, we remark that, as follows from the denseness of the stable manifold in $\mathcal{D}$, the separatrices of O are *non-wandering* orbits, and hence it is natural to suppose that they can be closed by a small perturbation. A similar problem arose already in the case of the Lorenz map, and it was overcome by using the specific nature of the Poincaré map. At the present time we have a very important lemma of Hayashi [18] that also permits this problem to be solved. Therefore, here we have the following result.

Assertion. *Systems with a homoclinic loop of the saddle-focus O are dense in the C^1-topology in the class of systems under consideration.*

As already mentioned, the existence of non-transversal homoclinic orbits in the attractor leads to very non-trivial dynamics. For instance, using results in [4] and [5], we deduce from Theorem 4 that *systems whose attractor contains non -transversal homoclinic orbits of arbitrarily high order of tangency are dense* in the regions constructed in the space of dynamical systems (Theorem 5), as are *systems whose attractor contains non-hyperbolic periodic orbits of arbitrarily high order of degeneracy* (Theorem 6). As a special case of Theorem 6 we can see that for the family X_μ the values of μ for which the attractor of the system contains a periodic orbit of saddle-saddle type together with its three-dimensional unstable manifold are dense in the intervals Δ_i constructed in Theorem 4 and, correspondingly, *for these values of μ the topological dimension of the attractor is equal to three* (Theorem 7). The latter means, in particular, that the given class of systems provides an example of so-called hyperchaos.

§ 2. Preliminary description of the attractor of a pseudohyperbolic flow

In this section we define the attractor of the system X and give an estimate of the number of connected components of its intersection with the cross-section Π (analogous to the estimate of the number of lacunae in the Lorenz attractor [7]).

Definition. We define the *attractor* of the system to be the set $\mathcal{A}$ of points accessible from the equilibrium state O.

This definition is justified by the following theorem.

Theorem 1. *The set $\mathcal{A}$ is chain-transitive, completely stable, and accessible from any point of the absorbing region $\mathcal{D}$.*

The stability of $\mathcal{A}$ follows immediately from the definition: it is known that for any initial point (and for the point O, in particular) the set of points accessible from it is completely stable (a system of absorbing regions is formed by the sets of points that are (ε_j, τ)-accessible from the initial point, with arbitrary $\varepsilon_j \to +0$ and $\tau > 0$).

To prove the remaining part of the theorem we note the following result.

Lemma 2. *The points asymptotic to O as $t \to +\infty$ are dense in $\mathcal{D}$ (in other words, the stable manifold of O is dense in $\mathcal{D}$).*

Indeed, we take an arbitrary point on Π and let U be an arbitrary neighbourhood of it. If U did not intersect the inverse images of the surface $\Pi_0 = W^s_{\mathrm{loc}} \cap \Pi$, then for all i the map $T^i|_U$ would be continuous, and for the sets T^iU the areas of the projections on $z = 0$ along the leaves of the invariant foliation would grow exponentially (in view of the property (B)), which contradicts the boundedness of Π. Thus, the inverse images of Π_0 (and this is the intersection of the stable manifold of O with the cross-section Π) are dense in Π, and hence the stable manifold of O is dense in $\mathcal{D}$.

It follows at once from Lemma 2 that the point O is accessible from any point in $\mathcal{D}$. In particular, for any points P and Q in $\mathcal{A}$ the point O is accessible from P, while Q is accessible from O by the definition of the set $\mathcal{A}$. Thus, Q is accessible from P, and for the proof of the chain transitivity of $\mathcal{A}$ it remains to show that the (ε, τ)-orbits joining P and Q can be selected to be contained in $\mathcal{A}$. The latter, however, follows from the complete stability of $\mathcal{A}$: as follows from the definition, for any $\delta > 0$ an (ε, τ)-orbit joining P and Q does not leave the δ-neighbourhood of $\mathcal{A}$ if ε is sufficiently small, and hence this orbit can be approximated by an (ε', τ)-orbit lying entirely in $\mathcal{A}$, where ε' can even be larger than ε, though clearly $\varepsilon' \to 0$ as $\varepsilon \to 0$ and $\delta \to 0$.

It follows from Theorem 1 that $\mathcal{A}$ is the *smallest* completely stable set in $\mathcal{D}$: since any completely stable set must contain all the points accessible from any one of its points, Theorem 1 implies that any completely stable set in $\mathcal{D}$ must contain O and together with it the set $\mathcal{A}$. Thus, $\mathcal{A}$ is the intersection of all the completely stable sets in $\mathcal{D}$.

The next theorem shows that $\mathcal{A}$ is the *unique* chain-transitive completely stable set in $\mathcal{D}$.

Theorem 2. *Any orbitally stable set in $\mathcal{D}$ containing points outside $\mathcal{A}$ contains points that are not chain-recurrent.*

Proof. Since the stable manifold of O is dense in $\mathcal{D}$, O obviously belongs to any orbitally stable subset of $\mathcal{D}$. A connected component of an orbitally stable set is itself orbitally stable, so each such set contains O and thus any orbitally stable subset of $\mathcal{D}$ is connected. Let C be such a set and suppose that $P \in C$ and $P \notin \mathcal{A}$. Because $\mathcal{A}$ is completely stable, there is an absorbing region U containing $\mathcal{A}$ and not containing P. Since O belongs to $\mathcal{A}$ and to C, we have obtained that C contains both interior points of U and points outside U. The set C is connected, so it contains at least one point on ∂U; but as we have already pointed out, the (ε, τ)-orbits starting on ∂U always remain inside U at a finite distance from ∂U for sufficiently small ε and sufficiently large τ, that is, there are no chain-recurrent points on ∂U.

The next theorem carries the result in part 1 of Theorem 1.5 in [7] over to the multidimensional case and gives a description of the partition into connected components of the intersection of the attractor $\mathcal{A}$ and the cross-section Π; we use this in the proof of Theorem 4. We denote by $q > 1$ the coefficient of area expansion by the quotient map $(\widetilde{T})$: if V is a connected region in Π disjoint from Π_0, then

$$S(TV) > qS(V), \tag{14}$$

where S denotes the area of the projection of the region on the surface $\{z = 0\}$ along leaves of the invariant foliation. According to (B*), $q > 1$. Moreover, since $S(\Pi) = 2S(\Pi_+)$ and $T\Pi_+ \subset \Pi$, it follows that $S(T\Pi_+)/S(\Pi_+) < 2$ and hence

$$1 < q < 2.$$

We denote the separatrices of O by Γ^+ and Γ^- and we let $\{P_i^\pm\}$ be the points of successive intersection of $\Gamma^\pm$ and Π. These sequences can be infinite or finite; the latter happens if the corresponding separatrix forms a loop (returns to O as $t \to +\infty$).

Theorem 3. *The number N of connected components of the set $\mathcal{A} \cap \Pi$ is finite and satisfies the estimate*

$$2 \leqslant N < 2 + \frac{|\ln(q-1)|}{\ln q}. \tag{15}$$

Each connected component contains at least one of the points $P_i^\pm$. Furthermore, for some integers $N_+ \geqslant 1$ and $N_- \geqslant 1$ such that $N_+ + N_- = N$ and

$$q^{-N_+} + q^{-N_-} > 1, \tag{16}$$

the set $\mathcal{A} \cap \Pi$ can be represented in the form

$$\mathcal{A} \cap \Pi = \mathcal{A}_1^+ \cup \cdots \cup \mathcal{A}_{N_+}^+ \cup \mathcal{A}_1^- \cup \cdots \cup \mathcal{A}_{N_-}^-, \tag{17}$$

where $\mathcal{A}_i^+$ and $\mathcal{A}_i^-$ are the components containing the respective points P_i^+ and P_i^-. In this formula all the components $\mathcal{A}_i^\pm$ are distinct,

$$\begin{aligned} \mathcal{A}_i^+ \cap \Pi_0 = \varnothing \quad \text{for } i < N^+, \qquad & \mathcal{A}_i^- \cap \Pi_0 = \varnothing \quad \text{for } i < N^-, \\ \mathcal{A}_{N_+}^+ \cap \Pi_0 \neq \varnothing, \qquad & \mathcal{A}_{N_-}^- \cap \Pi_0 \neq \varnothing, \end{aligned} \tag{18}$$

and

$$\begin{gathered} T_+\big((\mathcal{A}_{N_-}^- \cup \mathcal{A}_{N_+}^+) \cap (\Pi_+ \cup \Pi_0)\big) = \mathcal{A}_1^+, \\ T_-\big((\mathcal{A}_{N_-}^- \cup \mathcal{A}_{N_+}^+) \cap (\Pi_- \cup \Pi_0)\big) = \mathcal{A}_1^-, \\ \mathcal{A}_i^- = T^{i-1}\mathcal{A}_1^- \quad \text{for } i < N^-, \qquad \mathcal{A}_i^+ = T^{i-1}\mathcal{A}_1^+ \quad \text{for } i < N^+. \end{gathered} \tag{19}$$

Proof. Since the set $\mathcal{A}$ is orbitally stable, it can be represented as the intersection of a sequence of nested invariant regions. Correspondingly, the set $\mathcal{A} \cap \Pi$ is an intersection of nested neighbourhoods that are invariant with respect to the positive iterates of the map T. Let U be one of these regions: $\mathcal{A} \cap \Pi \subset U$, $TU \subseteq U$. We can remove from U the connected components not containing points in $\mathcal{A} \cap \Pi$, whereupon U remains an invariant set. And since $\mathcal{A} \cap \Pi$ is compact, only finitely many components remain in U. Let V_i^+ and V_i^- be the components of U containing P_i^+ and P_i^-, respectively. Since $P_1^+ \in T_+(\Pi_+ \cup \Pi_0)$ and $P_1^- \in T_-(\Pi_- \cup \Pi_0)$ and since the sets $T_+(\Pi_+ \cup \Pi_0)$ and $T_-(\Pi_- \cup \Pi_0)$ are bounded away from each other, V_1^+ and V_1^- are distinct.

We remark that since $TU \subseteq U$ and T is continuous on $\Pi \setminus \Pi_0$, it follows that if some component of U is disjoint from Π_0, then its image lies entirely inside some other component (or coincides with it). More precisely, if a point P_1 and the point $P_2 = TP_1$ belong to the respective components V_1 and V_2 and if $V_1 \cap \Pi_0 = \varnothing$, then $TV_1 \subseteq V_2$. But if V_1 intersects Π_0, then $V_2 = V_1^+$ or $V_2 = V_1^-$, depending on whether

$P_2 = T_+P_1$ or $P_2 = T_-P_1$ (this follows from the fact that $T_+(\Pi_0) = P_1^+ \in V_1^+$ and $T_-(\Pi_0) = P_1^- \in V_1^-$). Thus, if $\{P_j\}$ is some orbit of T, and the V_j are the components of U containing the P_j, then $T^k V_j \subseteq V_{j+k}$ for $k \geqslant 1$ if all the components $V_j, \dots, V_{j+k-1}$ are disjoint from Π_0. But if $V_j \cap \Pi_0 \neq \varnothing$ for some j, then each component with index greater than j coincides with one of the components $V_i^\pm$.

We now show that there are no connected components of U other than the sets $V_i^\pm$. Suppose that some component V does not contain any points $P_i^\pm$. By definition, V contains at least one point P in $\mathcal{A}$. Since $\mathcal{A} \cap \Pi$ is invariant, that is, it consists of entire orbits, it then contains the reverse semiorbit $P_0 \equiv P, P_{-1}, P_{-2}, \dots$ $(TP_{-(j+1)} = P_{-j})$ of P. Let $V_0 \equiv V, V_{-1}, V_{-2}, \dots$ be the sequence of connected components containing the points P_{-j}. By assumption, V does not coincide with any of the components $V_i^\pm$, so the V_j do not intersect Π_0, and $TV_{-(j+1)} \subseteq V_{-j}$ for $j = 0, \dots, \infty$. Because U only has finitely many components, the sequence V_{-j} must be periodic, that is, $V = V_{-k}$ for some $k > 0$. This would mean that T^k is continuous on V and $T^k V \subseteq V$, which is impossible, since $S(V) > q^k S(V)$ by virtue of area expansion by the quotient map (see (14)). Thus, the set U is exhausted by the components $V_i^\pm$.

Since $TP_{i-1}^+ = P_i^+$ and $TP_{i-1}^- = P_i^-$, it follows that $TV_{i-1}^\pm \subseteq V_i^\pm$ as long as $V_{i-1}^\pm \cap \Pi_0 = \varnothing$. As we pointed out (Lemma 2), the inverse images of the line Π_0 are dense in Π. In particular, they intersect V_1^+ and V_1^-. This means that some iterates of the regions V_1^+ and V_1^- intersect Π_0. Consequently, there are integers $N_+(U) \geqslant 1$ and $N_-(U) \geqslant 1$ such that $V_{N_+(U)}^+$ and $V_{N_-(U)}^-$ intersect Π_0 and the components $V_i^\pm$ with smaller indices do not intersect Π_0. Thus,

$$\begin{aligned} V_i^+ \cap \Pi_0 = \varnothing \quad \text{for} \quad i < N^+(U), \qquad V_i^- \cap \Pi_0 = \varnothing \quad \text{for} \quad i < N^-(U), \\ V_{N_+(U)}^+ \cap \Pi_0 \neq \varnothing, \qquad V_{N_-(U)}^- \cap \Pi_0 \neq \varnothing, \end{aligned} \tag{20}$$

and

$$\begin{gathered} T_+\big((V_{N_-(U)}^- \cup V_{N_+(U)}^+) \cap (\Pi_+ \cup \Pi_0)\big) \subseteq V_1^+, \\ T_-\big((V_{N_-(U)}^- \cup V_{N_+(U)}^+) \cap (\Pi_- \cup \Pi_0)\big) \subseteq V_1^-, \\ V_i^- \subseteq T^{i-1} V_1^- \quad \text{for} \quad i < N^-(U), \qquad V_i^+ \subseteq T^{i-1} V_1^+ \quad \text{for} \quad i < N^+(U). \end{gathered} \tag{21}$$

It follows from (21) that

$$U = V_1^+ \cup \dots \cup V_{N_+(U)}^+ \cup V_1^- \cup \dots \cup V_{N_-(U)}^-. \tag{22}$$

We show that $N_+(U)$ and $N_-(U)$ satisfy the inequality

$$q^{-N_+(U)} + q^{-N_-(U)} > 1. \tag{23}$$

Indeed, it is clear that

$$\begin{aligned} S\big((V_{N_-(U)}^- \cup V_{N_+(U)}^+) \cap \Pi_+\big) \geqslant S^+ \equiv \max\{S(V_{N_-(U)}^- \cap \Pi_+), S(V_{N_+(U)}^+ \cap \Pi_+)\}, \\ S\big((V_{N_-(U)}^- \cup V_{N_+(U)}^+) \cap \Pi_-\big) \geqslant S^- \equiv \max\{S(V_{N_-(U)}^- \cap \Pi_-), S(V_{N_+(U)}^+ \cap \Pi_-)\} \end{aligned}$$

(recall that S denotes the area of the projection on the surface $\{z = 0\}$ along the leaves of the invariant foliation). Hence, using (21) and (14),

$$S(V^-_{N_-(U)}) > q^{N_-(U)} S^-, \qquad S(V^+_{N_+(U)}) > q^{N_+(U)} S^+.$$

Since $S(V^-_{N_-(U)}) \leqslant S^- + S^+$ and $S(V^-_{N_+(U)}) \leqslant S^- + S^+$, we get that

$$q^{-N_-(U)} > \frac{S^-}{S^- + S^+}, \qquad q^{-N_+(U)} > \frac{S^+}{S^- + S^+},$$

which implies (23).

It clearly follows from (23) that the number of connected components of U is finite and bounded above by a constant independent of U:

$$N_+(U) + N_-(U) < 2 + \frac{|\ln(q-1)|}{\ln q}. \tag{24}$$

Since the neighbourhood U is arbitrarily close to $\mathcal{A} \cap \Pi$, this implies that the number of connected components of $\mathcal{A} \cap \Pi$ is also finite. Correspondingly, if U is sufficiently close to $\mathcal{A} \cap \Pi$, then each component of U contains exactly one component of $\mathcal{A} \cap \Pi$. The relations (15)–(18) now follow from (20)–(24), and, moreover,

$$\begin{gathered} T_+\big((\mathcal{A}^-_{N_-} \cup \mathcal{A}^+_{N_+}) \cap (\Pi_+ \cup \Pi_0)\big) \subseteq \mathcal{A}^+_1, \\ T_-\big((\mathcal{A}^-_{N_-} \cup \mathcal{A}^+_{N_+}) \cap (\Pi_- \cup \Pi_0)\big) \subseteq \mathcal{A}^-_1, \\ \mathcal{A}^-_i \subseteq T^{i-1}\mathcal{A}^-_1 \quad \text{for} \quad i < N^-, \qquad \mathcal{A}^+_i \subseteq T^{i-1}\mathcal{A}^+_1 \quad \text{for} \quad i < N^+. \end{gathered} \tag{25}$$

The set $\mathcal{A} \cap \Pi$ is invariant with respect to T: $T\mathcal{A} \cap \Pi = \mathcal{A} \cap \Pi$. From this and the fact that T is a homeomorphism of the set $\Pi \setminus \Pi_0$ onto its image $T_+\Pi_+ \cup T_-\Pi_-$ it follows that the image of any connected component of $\mathcal{A} \cap \Pi$ disjoint from Π_0 is also a connected component of $\mathcal{A} \cap \Pi$. For the components intersecting Π_0 the union of the images of their intersections with $\Pi_+ \cup \Pi_0$ makes up $\mathcal{A}^+_1$, while the union of the images of their intersections with $\Pi_- \cup \Pi_0$ makes up $\mathcal{A}^-_1$. Thus, in (25) we can replace the inclusion sign by equality and this yields (19). Since $\mathcal{A}^+_1$ and $\mathcal{A}^-_1$ are distinct, we now get that all the components in (17) are distinct. The proof of the theorem is complete.

It is clear from the proof that the theorem remains valid for an arbitrary invariant orbitally stable set in $\mathcal{D}$. We remark also that in proving Theorems 1–3 we did not use at all the fact that the equilibrium state O is a saddle-focus, that is, they remain true also under the condition that O is a saddle, provided only that there is an invariant foliation and that the quotient map expands areas. Here it is not essential that the foliation has codimension two: Theorems 1–3 remain true also in the general case when T has an invariant absolutely continuous contracting foliation of codimension $k \geqslant 1$ and the quotient map expands k-dimensional volumes. The Lorenz attractor ($k = 1$ and O a saddle) is thus included in this scheme.

§ 3. Construction of a wild attractor

In this section we prove our main result. Let us return to the case when O is a saddle-focus and the contracting foliation has codimension two. We show that in this case the attractor $\mathcal{A}$ can contain a wild hyperbolic set. In particular, this means that it is different from the Lorenz attractor (and from other well-known attractors).

We take a one-parameter family X_μ of systems of this form and assume that *for $\mu = 0$ there is a homoclinic loop of the saddle-focus O*, that is, one of the separatrices (say Γ_+) returns to O as $t \to +\infty$ (Fig. 2). In other words, we assume that for $\mu = 0$ the family X_μ intersects the bifurcation surface of systems with a homoclinic loop of a saddle-focus in the space of dynamical systems and we suppose that the intersection is *transversal.* Transversality means that as μ varies the loop splits and that if M is the index of the last point where the separatrix Γ_+ intersects the cross-section for $\mu = 0$ ($P_M^+ \in \Pi_0$ for $\mu = 0$), then the distance from P_M^+ to Π_0 varies 'at a non-zero rate' as μ varies. We choose the sign of μ so that $P_M^+ \in \Pi_+$ for $\mu > 0$ (respectively, $P_M^+ \in \Pi_-$ for $\mu < 0$).

Theorem 4. *There is a sequence of intervals Δ_i accumulating at $\mu = 0$ such that for $\mu \in \Delta_i$ the attractor $\mathcal{A}_\mu$ contains a wild set (a non-trivial transitive closed hyperbolic set whose unstable manifold has points of tangency with the stable manifold). Further, for any $\mu^* \in \Delta_i$ the attractor $\mathcal{A}$ of any system close to X_{μ^*} in the C^r-topology also contains a wild set.*

Proof. By our condition, for $\mu = 0$ the separatrix Γ_+ intersects Π at finitely many points P_i^+ and the last point P_M^+ belongs to Π_0. Without loss of generality it can be assumed that the number M is sufficiently large. Namely, for all small μ the quotient map $\widetilde{T}$ expands areas and it is clear that the expansion coefficient q is bounded below by a constant independent of μ. Correspondingly, by Theorem 3 the number of connected components of the attractor in the intersection with the cross-section Π is bounded above by a constant N independent of μ and we shall assume below that the number M of points of intersection is at least $N + 1$. This condition is not a restriction, because according to [19] a homoclinic loop of a saddle-focus can be split in such a way as to result in a loop of arbitrarily many circuits with respect to the original loop. More precisely, surfaces corresponding to homoclinic loops of arbitrarily many circuits accumulate at the bifurcation surface corresponding to a homoclinic loop of a saddle-focus. The family X_μ, which is transversal to the surface corresponding to the original loop, remains transversal to all C^1-close surfaces. Correspondingly, if for $\mu = 0$ the number M of points of intersection of the loop Γ_+ with Π is less than $N + 1$, then by an arbitrarily small change in μ it is possible to get a loop for which $M \geqslant N + 1$ (and the family X_μ remains transversal to the corresponding bifurcation surface). Therefore, after proving the theorem for such loops, we automatically obtain the statement of the theorem also in the general case.

As is known [15], in any small neighbourhood of a homoclinic loop of a saddle-focus there are countably many single-circuit (that is, homotopic to the loop in the given neighbourhood) saddle periodic orbits (in general this assertion holds only under certain additional conditions of general position, which, however, hold automatically in our case; see below). Each of these orbits intersects a small cross-section transversal to the loop at a single point (we take a small neighbourhood of $\Pi_0 \equiv W^s_{\mathrm{loc}} \cap \Pi$ in Π as such a cross-section). The given orbits $L_1, L_2, \ldots$ can be numbered in such a way that the intersection points $R_1, R_2, \ldots$ will lie at a distance $\mathrm{dist}(R_k, W^s_{\mathrm{loc}}) \sim C\exp(-\frac{\pi\gamma}{\omega}k)$ from W^s_{loc}, where C is some constant (recall that the characteristic exponents of the saddle-focus are $\gamma, -\lambda \pm i\omega, -\alpha_1, \ldots, -\alpha_{n-3}$, where $\gamma > 0$, $0 < \lambda < \mathrm{Re}\,\alpha_j$, and $\omega \neq 0$).

It is shown in [15] that on a cross-section it is possible to single out 'strips' accumulating at the intersection of the cross-section with W^s_{loc}: neighbourhoods $\sigma_1, \sigma_2, \ldots$ of the respective points $R_1, R_2, \ldots$ such that for $\mu = 0$ and for any $\rho' > \rho = \lambda/\gamma$ there exists a $\bar{k}$ such that for any $i \geqslant \bar{k}$ the image of the strip σ_i under the action of the Poincaré map (in our case this is the map $T' \equiv T^M$, where M is the number of points of intersection of the loop with Π) *regularly* intersects all the strips with indices $j \geqslant \rho' i$. Here "regularly" means that the intersection $T'\sigma_i \cap \sigma_j$ is connected and the map $T'|_{\sigma_i \cap T'^{-1}\sigma_j}$ is a saddle map in the sense of [20]. This means [15] that in a small neighbourhood of the loop there is a hyperbolic invariant set Λ that is in one-to-one correspondence with the set of all possible two-sided infinite sequences of integers $\{j_s\}$ satisfying the conditions $j_s \geqslant \bar{k}$ and $j_{s+1} \geqslant \rho' j_s$: corresponding to a sequence $\{j_s\}$ is an orbit whose successive points of intersection with the cross-section lie in the strips with indices $\{j_s\}$. In particular, the sequence $\{\ldots iiiiii \ldots\}$ corresponds to the single-circuit periodic orbit L_i.

It should be noted that the set Λ is not closed — its closure contains the equilibrium state O, the separatrix Γ^+, and orbits in the stable manifold. For an arbitrary integer $k \geqslant \bar{k}$ we single out in Λ the closed subset Λ_k corresponding to the sequences $\{j_s\}$ satisfying the condition $j_s \leqslant k$. Since for each fixed k the set Λ_k is bounded away from W^s_{loc}, this is a closed hyperbolic set, and hence it is preserved for all small μ (and depends continuously on μ). We have the following result.

Lemma 3. *There exists a C such that if $\bar{k}$ is sufficiently large and $k \geqslant \bar{k}$, then the set Λ_k is preserved for all μ with*

$$|\mu| < C\exp(-\tfrac{\pi\rho\gamma}{\omega}k). \tag{26}$$

To simplify the notation we denote by Λ_μ the set Λ_k corresponding to the smallest k for which (26) holds. The arguments used to prove Lemma 3 involve maps close to the homoclinic loop of the saddle-focus and are not connected with the specific nature of the given problem. By contrast, the following result makes essential use of Theorem 3 and the assumption that the homoclinic loop has sufficiently many circuits.

Lemma 4. *For all small $\mu \geqslant 0$ the set Λ_μ belongs to the attractor $\mathcal{A}_\mu$.*

We remark that for each fixed μ the set Λ_μ is a closed hyperbolic set and therefore it is preserved under small perturbations of the system. It becomes clear from the proof of Lemma 4 that this set remains in the attractor for all sufficiently close systems.

The unstable manifold of the hyperbolic set Λ_μ also belongs to the attractor (in view of its complete stability). We prove the theorem by showing that the intervals of values of μ (Newhouse regions) for which the set Λ_μ is *wild* accumulate at $\mu = 0$ (from the positive side), that is, its unstable manifold is tangent to its stable manifold. According to [16], for this it suffices to show that $\mu = 0$ is an accumulation point of a sequence of positive values of μ such that the unstable manifold of some periodic orbit in Λ_μ is tangent in a non-degenerate way to its stable manifold (we make more precise the non-degeneracy conditions below). As such a periodic orbit we choose the single-circuit orbit $L_j \in \Lambda_\mu$. The desired result follows directly from the next two lemmas.

Lemma 5. *There is a sequence $\mu_j \to +0$ such that for $\mu = \mu_j$ the family X_μ intersects transversally the bifurcation surface on which the separatrix Γ_+ falls on the stable manifold of the orbit $L_j \in \Lambda_\mu$ and further, the unstable manifold of L_j intersects $W^s(O)$ transversally.*

Lemma 6. *There are values of μ arbitrarily close to $\mu = \mu_j$ such that the stable and unstable manifolds of the saddle periodic orbit have a non-degenerate tangency.*

We proceed to proofs of Lemmas 3–6. Lemmas 3, 5, and 6 are proved by computations analogous to those in [8] and [15]; therefore we leave out the details and confine ourselves to the scheme. By our condition, the separatrix Γ_+ forms a homoclinic loop for $\mu = 0$: P_M^+ (the Mth point of intersection of Γ_+ with Π) lies on Π_0. We remark that this loop satisfies the general position conditions in [15]. One of these conditions is that the separatrix goes to O (as $t \to +\infty$) while being tangent to the leading plane, that is, it does not lie in the strongly stable invariant manifold of O. We remark that the strongly stable manifold is the *unique* $(n-3)$-dimensional smooth invariant manifold passing through O and tangent at O to the eigenspace corresponding to the non-leading characteristic exponents $-\alpha_i$. In our case the leaf of the foliation $\mathcal{N}^{ss}$ passing through O is such a manifold. By assumption, it has the form $(x, y) = h^{ss}(z)$ and does not intersect the cross-section Π, so Γ_+ cannot lie in it. Another general position condition is formulated in [15] as the condition that a certain quantity be non-zero, and is equivalent (see [21]) to the existence of a strongly contracting foliation in a neighbourhood of the loop; in our case this condition is satisfied by assumption. We note that, as follows from [21] and [22], the given general position conditions ensure that for all small μ the system has in a neighbourhood of the loop an invariant three-dimensional C^1-manifold $\mathcal{M}_\mu$ that is transversal to the foliation $\mathcal{N}^{ss}$ and that contains all the orbits that remain in a small neighbourhood of the loop as $t \to -\infty$.

According to [8], in suitable coordinates the Poincaré map $T' \equiv T^M\big|_{\Pi_+}$ can be written near Π_0 in the form

$$\begin{aligned}
\bar{x} &= x^* + a'x^\rho \sin(\Omega \ln x + \varphi - \varphi^*) + \psi_1(x, \varphi, z, \mu), \\
\bar{\varphi} &= \varphi^* + b'x^\rho \cos(\Omega \ln x + \varphi - \varphi^* + \theta) + \psi_2(x, \varphi, z, \mu), \\
\bar{z} &= z^* + \psi_3(x, \varphi, z, \mu),
\end{aligned} \tag{27}$$

where (x^*, φ^*, z^*) are the coordinates of P_M^+, a', b', and θ are certain quantities, and

$$\left\| \frac{\partial^{p+|q|}\psi_i}{\partial x^p \partial(\varphi, z, \mu)^q} \right\| = O(|x|^{\eta - p}), \qquad 0 \leqslant p + |q| \leqslant r - 2, \tag{28}$$

for some $\eta > \rho$. All the constants in (27) depend on μ. By our condition, $P_M^+ \in \Pi_0$ (that is, $x^* = 0$) for $\mu = 0$. The transversality of the family X_μ to the bifurcation surface corresponding to homoclinic loops means that $\left.\dfrac{\partial x^*}{\partial \mu}\right|_{\mu=0} \neq 0$. Without loss of generality we can let

$$x^* \equiv \mu. \tag{29}$$

We observe that coordinates in which the representation (27), (28) is valid can be introduced in such a way that in the restriction to W^s_{loc} the derivative $\dot{y}$ is independent of z (see [8]). Therefore, the leaves of the invariant foliation $\mathcal{N}^{ss}$ in the intersection with W^s_{loc} have the form $\{\varphi = \mathrm{const}\}$. From this it is obvious that the invariance of the foliation $\mathcal{N}^{ss} \cap \Pi$ with respect to the map $(T')^{-1}$ is equivalent to the condition

$$a' \neq 0, \qquad b' \neq 0, \qquad \cos\theta \neq 0. \tag{30}$$

The fixed points of T' are found from the condition ($\overline{x} = x$, $\overline{\varphi} = \varphi$, $\overline{z} = z$). Consequently, in this case the last two equations of the system (27) can be used to express φ and z as functions of x and μ for small x and μ (see (27)–(30)), after which the equation in the coordinate x takes the form

$$x = \mu + a'x^\rho \sin(\Omega \ln x) + o(x^\rho).$$

For $\mu = 0$ we have for the fixed point R_j that

$$x_j = \exp(-\tfrac{\pi j}{\Omega}) + o(\exp(-\tfrac{\pi j}{\Omega})) \tag{31}$$

and

$$\varphi_j = \varphi^* + o(\exp(-\tfrac{\pi j}{\Omega})), \qquad z_j = z^* + o(\exp(-\tfrac{\pi j}{\Omega})), \tag{32}$$

where (x_j, φ_j, z_j) are the coordinates of the fixed point and the index j runs through all natural numbers beginning with some sufficiently large $\overline{k}$. Obviously, the relations (31) and (32) remain valid also in the case $\mu \neq 0$ if $\mu \exp(\frac{\pi\rho j}{\Omega})$ is sufficiently small. The multipliers of the fixed point can be estimated as follows:

$$\begin{aligned}
&\nu_1 \sim (-1)^j a'\Omega x_j^{\rho-1},\\
&\nu_2 \sim -(-1)^j b' \frac{\rho\cos\theta}{\Omega} x_j^\rho,\\
&\nu_m = o(x_j^\rho), \qquad m = 3, \dots, n-1.
\end{aligned} \tag{33}$$

We note that ν_1 is the only multiplier greater than 1 in absolute value, and the others are interior to the unit disc. Therefore, the unstable manifold of the point R_j is one-dimensional and the stable manifold $(n-2)$-dimensional (correspondingly, the single-circuit periodic orbit L_j intersecting Π at R_j has a two-dimensional unstable manifold and an $(n-1)$-dimensional stable manifold).

To find the stable manifold W_j^s of R_j we proceed as follows. We take a small piece of the surface $x = x_j$ containing R_j and consider its inverse image with respect to the map T^{-1}. According to (27)–(32), it can be represented in the form

$$\sin(\Omega \ln x + \varphi - \varphi^*) = O\left(\frac{x_j}{x^\rho} + \frac{\mu}{x^\rho} + x^{\eta-\rho}\right)$$

if x is sufficiently small. We assume that

$$x < 2\exp(-\tfrac{\pi \bar{k}}{\Omega}),$$

where $\bar{k}$ is sufficiently large. Also, if x is not too small, namely, if x_j/x^ρ and μ/x^ρ are small, then the equation of the given surface takes the form

$$\Omega \ln x = -\pi j - \varphi + \varphi^* + O\left(\frac{x_j}{x^\rho} + \frac{\mu}{x^\rho} + x^{\eta-\rho}\right)$$

or

$$x = \exp(-\tfrac{\varphi-\varphi^*+\pi j}{\Omega})(1+\cdots), \tag{34}$$

where the dots stand for a function of φ and μ that is small together with its derivatives up to order $r-2$. As we noted, the representation is valid if x, x_j/x^ρ, and μ/x^ρ are sufficiently small, that is, on the region of values of φ

$$\pi\bar{k} < \varphi - \varphi^* + \pi j < \pi k, \tag{35}$$

where $\bar{k}$ is an arbitrary sufficiently large integer independent of j and μ, and k can be chosen to be much larger than j under the condition that

$$|\mu|\exp(\tfrac{\pi\rho}{\Omega}k) \text{ and } \exp(\tfrac{\pi}{\Omega}(\rho k - j)) \text{ are small} \tag{36}$$

(the last condition can be satisfied for k much greater than j because $\rho < 1$). Using the formulae (27)–(32), we can verify that the inverse image under T^{-1} of a small piece of this surface containing the fixed point R_j can be represented in the very same form on the very same interval of values of φ. The same remains valid for all successive iterates, and the quantity $o(1)$ in (34) stays uniformly small. Since the manifold W^s_j is the limit of a sequence of such iterates, we get as a result that on the region (35), (36) the stable manifold of the point R_j is given by the equation (34).

For the unstable manifold W^u_j of R_j it is possible to show (by taking the image of a small piece of the line $\varphi = \varphi_j$ and estimating the correction for the successive iterates) that it is C^{r-2}-close to the spiral determined by a parametric relation of the form

$$\begin{aligned} x &= \mu - a'\exp(-\tfrac{\rho}{\Omega}u)\sin u,\\ \varphi &= \varphi^* + b'\exp(-\tfrac{\rho}{\Omega}u)\cos(-u+\theta),\\ z &= z^* \end{aligned} \tag{37}$$

on the region corresponding to the parameter values $u > \pi\rho' j$, where $\rho' > \rho$ (and can be chosen arbitrarily close to ρ), under the condition that j is sufficiently large and $|\mu|\exp(\frac{\rho}{\Omega}j)$ is small. We remark that the spiral winds onto the point P^+_M and intersects Π_0 transversally. It is clear from the formulae (34)–(37) that W^u_j has points of transversal intersection with W^s_i for $i > \rho' j$. In particular, there are points of transversal intersection of W^u_j with W^s_j, and hence the points R_j are included in a non-trivial hyperbolic set.

We choose a small $\xi > 0$ and consider the neighbourhoods

$$\sigma_j:\ e^\xi < x\exp(\tfrac{\pi j}{\Omega})\exp(\tfrac{\varphi-\varphi^*}{\Omega})\exp(\tfrac{\arctan(\Omega/\rho)}{\Omega}) < e^{\pi-\xi}, \quad |\varphi-\varphi^*| < \pi$$

of the points R_j (informally speaking, we have excluded from consideration the values of x corresponding to extrema of the function $x^\rho \sin(\Omega \ln x)$). Using the formulae (27) and (28) we can verify that for $\mu = 0$ and for any $\rho' > \rho$ there exists a $\bar{k}$ such that for any $j \geqslant \bar{k}$ the image of the strip σ_j under the action of T' intersects regularly all the strips with indices $i \geqslant \rho' j$ in the sense that the intersection $T'\sigma_j \cap \sigma_i$ is connected and the map $T'\big|_{\sigma_j \cap T'^{-1}\sigma_i}$ is a saddle map in the sense of [20] (it is strongly expanding with respect to the coordinate x and strongly contracting with respect to the coordinates (φ, z)).

Let $\{P_s\}_{s=-\infty}^{s=+\infty}$ be an orbit of T' lying entirely in the union of strips $\bigcup_{j=\bar{k}}^{\infty} \sigma_j$: $P_s \in \sigma_{j_s}$. We call the sequence $\{j_s\}_{s=-\infty}^{s=+\infty}$ a *coding* of the given orbit. By the lemma on a sequence of saddle maps [20], to each coding satisfying the conditions $j_s \geqslant \bar{k}$ and $j_{s+1} \geqslant \rho' j_s$ there corresponds precisely one orbit with that coding. The union of all such orbits forms a hyperbolic invariant set Λ that contains, in particular, all the points R_j. It is not hard to verify that, as before, for $\mu \neq 0$ the image of the strip σ_j intersects the strips σ_i with $i > \rho' j$ regularly if $|\mu| \exp(\frac{\rho}{\Omega} j)$ is sufficiently small, and this gives us the statement of Lemma 3 by virtue of the lemma on a sequence of saddle maps [20].

To prove Lemma 5 it suffices to observe that the desired values $\mu_j \sim \exp(-\frac{\pi j}{\Omega})$ for which the point P_M^+ falls on the stable manifold of R_j can be found with the help of the formula (34), where in place of x and φ one should substitute the coordinates of P_M^+: μ and φ^*, respectively. Since $\rho < 1$, it follows that $\mu_j \exp(\frac{\pi \rho j}{\Omega})$ is small and the approximation (34) is indeed applicable for W_j^s.

To prove Lemma 6 we note that, because the unstable manifold W_j^u has the form of a spiral winding onto the point P_M^+, there are values of μ near μ_j for which W_j^u is tangent to W_j^s. We can see from (34) and (37) by a direct check that the tangency is *quadratic* and that this tangency splits in a generic way as μ varies. We must also verify the additional conditions of non-degeneracy that are necessary for using Theorem 2 in [16], namely, that near the values of μ corresponding to a homoclinic tangency there are intervals of values of μ for which the corresponding hyperbolic set is wild.

We remark that the fixed point R_j has (see (33)) exactly one multiplier (ν_2) closest to the unit circle from inside (that is, R_j is a *saddle of type* (1,1) in the terminology of [16]). Here only one multiplier lies outside the unit disc. In this case the non-degeneracy conditions reduce to two requirements: the orbit of the homoclinic tangency *must not lie in the non-leading manifold of* R_j, and everywhere at the points of that orbit *the extended unstable manifold* W_j^{ue} *of* R_j *must be transversal to the strongly stable foliation of the stable manifold.* Locally, the non-leading manifold W_j^{ss} is a smooth invariant manifold tangent at R_j to the eigenspace corresponding to the non-leading multipliers $\nu_3, \dots, \nu_n$. In our case the leaf of the foliation $\mathcal{N}^{ss} \cap \Pi$ passing through R_j is just such a manifold and therefore it coincides locally with W_j^{ss} (by the uniqueness of the latter). It is known that the non-leading manifold of the fixed point is included in the strongly stable invariant foliation and since this foliation is uniquely determined everywhere on the stable manifold, it coincides with $\mathcal{N}^{ss} \cap \Pi$ on the stable manifold of R_j.

As we remarked, the foliation is transversal to the three-dimensional manifold $\mathcal{M}_\mu$ which contains all the orbits that lie entirely in a small neighbourhood of the homoclinic loop of the saddle-focus. In particular, $\mathcal{M}_\mu \cap \Pi$ contains the point R_j and the orbit of the homoclinic tangency under consideration. By transversality, R_j is an isolated point of the intersection of W_j^{ss} and $\mathcal{M}_\mu \cap \Pi$, so W_j^{ss} cannot contain orbits that are asymptotic to R_j and lie in $\mathcal{M}_\mu \cap \Pi$. Consequently, the orbit of the homoclinic tangency under consideration does indeed not lie in W_j^{ss}.

The two-dimensional manifold $\mathcal{M}_\mu \cap \Pi$ is invariant and transversal to W_j^{ss} and, as a consequence, it is tangent at R_j to the two-dimensional eigenspace corresponding to the multipliers (ν_1, ν_2). Any invariant manifold tangent at the fixed point to the eigenplane corresponding to the unstable and to the leading stable multipliers can be regarded as the extended unstable manifold W_j^{ue}: any one of them contains the unstable manifold and any two of them are tangent to each other everywhere on the unstable manifold (see [21] for details). Since $\mathcal{M}_\mu \cap \Pi$ is transversal to the foliation $\mathcal{N}^{ss} \cap \Pi$, we get that at any point of the non-transversal homoclinic orbit under consideration the extended unstable manifold is indeed transversal to the strongly stable foliation. Thus, the non-degeneracy conditions are satisfied and Theorem 2 in [16] really is applicable in our case.

To conclude the proof of the theorem it remains to prove Lemma 4. We represent the solid torus Π as the strip $\widetilde{\Pi} = D^{n-2} \times \mathbb{R}^1$ (where (x, z) serve as coordinates in D^{n-2}, and φ serves as a coordinate in $\mathbb{R}^1$) in which the points (x, z, φ) and $(x, z, \varphi + 2\pi k)$ are identified for any integer k. We take the lifting of Π onto $\widetilde{\Pi}$ and let $\mathcal{P}$ be the image of the set $T_+(\Pi_+ \cup \Pi_0) \cup T_-(\Pi_- \cup \Pi_0)$ under this lifting. Since $T_+(\Pi_+ \cup \Pi_0) \cup T_-(\Pi_- \cup \Pi_0)$ is *contractible to a point* in Π, $\mathcal{P}$ lies in the bounded region $\widetilde{\Pi}$ for all μ, that is, for some integer s

$$\mathcal{P} \subset \{-\pi s < \varphi - \varphi^* < \pi s\}.$$

For the attractor $\mathcal{A}_\mu$ of the system X_μ the intersection $\mathcal{A}_\mu \cap \Pi$ is contained in $T_+(\Pi_+ \cup \Pi_0) \cup T_-(\Pi_- \cup \Pi_0)$, so the lifting to $\widetilde{\Pi}$ of the intersection of the attractor with the cross-section is contained in $\mathcal{P}$ and hence lies entirely in the bounded region $\{-\pi s < \varphi - \varphi^* < \pi s\}$.

As shown above, the stable manifold of R_j contains the piece given by the relations (34)–(36), where $\overline{k}$ is sufficiently large and is fixed, while k increases in proportion to j as j increases and as μ tends to zero. We can assume that $(k - \overline{k})$ is divisible by $2s$ and that $(j - \overline{k} - s)$ is even, and we can slice the lifting of W_j^s into pieces corresponding to the values of $(\varphi - \varphi^*)$ in the intervals $(\pi\overline{k} - \pi j + 2\pi s l, \pi\overline{k} - \pi j + 2\pi s(l+1))$, where $l = 0, \ldots, (k - \overline{k})/(2s - 1)$. Since we identify points whose φ-coordinates differ by a multiple of 2π, we can shift the piece with index l by $-2\pi(sl - (j - \overline{k} - s)/2)$. As a result we get a collection of surfaces $\mathcal{W}_l$ of the form

$$x = w_l(\varphi, z, \mu),$$

where the functions w_l are defined for $-\pi s < \varphi - \varphi^* < \pi s$ and

$$w_l \sim \exp\left(-\frac{\varphi - \varphi^* + \pi(\overline{k} + s) + 2\pi s l}{\Omega}\right)$$

(see (34)). In particular, for the upper surface w_0 we have that

$$x > \frac{1}{2} \exp\left(-\frac{\pi\overline{k} + 2\pi s}{\Omega}\right)$$

and for the lower surface $w_{\bar{l}}$, where $\bar{l} = (k - \bar{k})/(2s - 1)$, we have that

$$x < 2\exp(-\tfrac{\pi k - 2\pi s}{\Omega}).$$

It is obvious that the restrictions (36) do not hinder (since $\rho < 1$) choosing k and $\bar{k}$ in such a way that the lifting of the point P_M^+, whose x-coordinate is equal to μ, lies below the surface w_0 and above the surface $w_{\bar{l}}$. We remark that on both these surfaces the coordinate x is small. The point P_M^+ belongs to the attractor $\mathcal{A}_\mu$ (since this point is on the separatrix). Let V be the connected component of $\mathcal{A}_\mu \cap \Pi$ containing P_M^+, and let $\widetilde{V}$ be its lifting to $\widetilde{\Pi}$. As noted, the set $\widetilde{V}$ lies entirely in the region $\{-\pi s < \varphi - \varphi^* < \pi s\}$. By Theorem 3, V contains one of the points $P_1^+, \dots, P_N^+$ or $P_1^-, \dots, P_N^-$, where N is the number of connected components of the set $\mathcal{A}_\mu \cap \Pi$. Since $M > N$ by assumption, the points P_i^+, $i = 1, \dots, N$, lie at a finite distance from Π_0 for all small μ. If for $\mu = 0$ the separatrix Γ_- does not form a loop, then the same is true also for the points P_i^-, $i = 1, \dots, N$. Thus, in this case the component V together with the point P_M^+ contains a point lying at a finite distance from Π_0, that is, the coordinate x of this point is bounded away from zero. The coordinate x is small on the surfaces w_0 and $w_{\bar{l}}$, so the lifting of the given point to $\widetilde{P}$ lies outside the region bounded by the given surfaces (the region $w_{\bar{l}}(\varphi, z, \mu) < x < w_0(\varphi, z, \mu)$). At the same time, the lifting of P_M^+ lies interior to this region. Since $\widetilde{V}$ is connected, this set intersects at least one of the surfaces w_0 or $w_{\bar{l}}$.

We recall that the given surfaces are pieces of the lifting of the manifold W_j^s to $\widetilde{\Pi}$, that is, we have obtained that for all small μ the attractor contains at least one point on the stable manifold of the orbit L_j. This implies that the attractor contains L_j itself, its unstable manifold, and the closure of the latter. In particular, it contains the hyperbolic set Λ_μ.

If for $\mu = 0$ the separatrix Γ_- also forms a loop, then in a completely analogous way the hyperbolic set Λ_μ^- lies in a neighbourhood of this loop for all small μ. For the pieces of the stable and unstable manifolds of a single-circuit periodic orbit in Λ_μ^- we have representations of the form (34) and (37) with a change of the sign of x from positive to negative. If $N^+ < N < M$ is the number in Theorem 3, then the point $P_{N^+}^+$ belongs to a component V of $\mathcal{A}_\mu \cap \Pi$ intersecting Π_0. As we noted, $P_{N^+}^+$ lies at a finite distance from Π_0. Since V contains both $P_{N^+}^+$ and some point on Π_0, it follows just as above that V intersects the stable manifold of either one of the periodic points in Λ_μ^+ (if $P_{N^+}^+ \in \Pi_+$) or one of the periodic points in Λ_μ^- (if $P_{N^+}^+ \in \Pi_-$). Consequently, either Λ_μ^+ or Λ_μ^- lies in the attractor. But since the unstable manifolds of the periodic orbits in Λ_μ^- intersect Π_0 transversally (see (37)), while the stable manifolds of the periodic points in Λ_μ^+ get arbitrarily close to Π_0 for sufficiently small μ (see (34)–(36)), it follows that the unstable manifold of Λ_μ^- intersects the stable manifold of Λ_μ^+, that is, in either case Λ_μ^+ is in the attractor and this concludes the proof of the theorem.

We have determined regions in the space of dynamical systems in which the attractor contains a wild set along with its unstable manifold. These regions contain densely systems for which one of the periodic orbits of the wild set has the orbit of a homoclinic tangency of the stable and unstable manifolds. According to [4] any neighbourhood of such a system contains systems for which the stable and unstable manifolds of this periodic orbit have tangencies of arbitrarily high order. Since the homoclinic orbits belong to the unstable manifold and hence are in the attractor, we obtain the following result.

Theorem 5. *Systems whose attractors contain non-transversal homoclinic orbits of arbitrarily high orders of tangency are dense in the regions constructed.*

This theorem shows that the bifurcations of the attractor under consideration cannot be completely described in any finite-parameter family. Another reflection of this property is that arbitrarily degenerate periodic orbits can be in the attractor.

Theorem 6. *Systems whose attractors contain non-hyperbolic periodic orbits with one multiplier equal to* 1 *and an arbitrarily large number of Lyapunov coefficients equal to* 0 *are dense in the regions constructed.*

This theorem is also a corollary to the denseness in the regions constructed of systems with homoclinic tangencies: according to [4] any neighbourhood of a system with a homoclinic tangency contains systems with arbitrarily degenerate periodic orbits. The next lemma refines a result in [4] (in order not to make the exposition too cumbersome, we do not reproduce the proof here).

Lemma 7. *Suppose that some system has a saddle periodic orbit L with multipliers $\nu_1, \nu_2, \dots, \nu_n$, and assume that $|\nu_1| > 1$ and $1 > |\nu_2| > |\nu_m|$ for $m = 3, \dots, n$ (this means, in particular, that ν_1 and ν_2 are real). Also, let*

$$|\nu_1 \cdot \nu_2| > 1.$$

Suppose that the unstable and stable manifolds of L have a tangency of order k along some orbit. Then for any k-parameter family in general position containing the given system there are parameter values corresponding to the existence of a non-hyperbolic periodic orbit with one multiplier equal to 1, *one multiplier greater than* 1 *in absolute value, and the remaining multipliers interior to the unit disc, and for the restriction of the Poincaré map to the centre manifold (corresponding to the multiplier* 1) *the first $(k-1)$ Lyapunov coefficients are equal to* 0. *Further, the stable manifold of the given periodic orbit has orbits of transversal intersection with the unstable manifold of L.*

To apply this lemma to the proof of Theorem 6 we observe that saddle periodic orbits with non-transversal homoclinic orbits of arbitrary orders of tangency are in the attractor according to Theorem 5 (for a dense set of systems in the regions under consideration). As is clear from (33), the given saddle periodic orbits (the orbits L_j) satisfy the conditions of Lemma 7. Consequently, the indicated regions contain densely systems for which the unstable manifold of a saddle periodic orbit in the attractor intersects the stable manifold of some non-hyperbolic periodic orbit of arbitrarily high previously specified order of degeneracy. Since this orbit belongs to the closure of the unstable manifold of an orbit in the attractor, it is itself in the attractor.

Applying Lemma 7 to the one-parameter ($k = 1$) family X_μ in Theorem 4, we get that the values μ for which the attractor contains a non-hyperbolic periodic orbit of saddle-saddle type are dense in the intervals Δ_i. For such an orbit one multiplier is greater than 1 in absolute value, one multiplier equals 1 and the corresponding first Lyapunov coefficient is non-zero, and the remaining multipliers lie interior to the unit disk. The unstable manifold of such an orbit is three-dimensional. Since each orbit is in the attractor together with its unstable manifold, we get the following result.

Theorem 7. *The values of μ for which the topological dimension of the attractor $\mathcal{A}_\mu$ equals three are dense in the intervals Δ_i.*

Bibliography

[1] S. E. Newhouse, "The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms", *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 101–151.

[2] S. E. Newhouse, "Diffeomorphisms with infinitely many sinks", *Topology* **13** (1974), 9–18.

[3] S. V. Gonchenko, L. P. Shil'nikov, and D. V. Turaev, "On models with non-rough Poincaré homoclinic curves", *Phys. D* **62**:1–4 (1993), 1–14.

[4] S. V. Gonchenko, D. V. Turaev, and L. P. Shil'nikov, "Dynamical phenomena in multidimensional systems with a non-transversal homoclinic Poincaré curve", *Ross. Akad. Nauk Dokl.* **330** (1993), 144–147; English transl. in *Russian Acad. Sci. Dokl. Math.* **47** (1993).

[5] S. V. Gonchenko, L. P. Shil'nikov, and D. V. Turaev, "Dynamical phenomena in systems with structurally unstable Poincaré homoclinic orbits", *Chaos* **6** (1996), 15–31.

[6] V. S. Afraimovich, V. V. Bykov, and L. P. Shil'nikov, "On the appearance and structure of Lorenz attractors", *Dokl. Akad. Nauk SSSR* **234** (1977), 336–339; English transl. in *Soviet Math. Dokl.* **18** (1977).

[7] V. S. Afraimovich, V. V. Bykov, and L. P. Shil'nikov, "On structurally unstable attracting limit sets of the Lorenz attractor type", *Trudy Moskov. Mat. Obshch.* **44** (1982), 150–212; English transl. in *Trans. Moscow Math. Soc.* **1983**:2.

[8] I. M. Ovsyannikov and L. P. Shil'nikov, "On systems with a saddle-focus homoclinic curve", *Mat. Sb.* **130 (172)** (1986), 552–570; English transl. in *Math. USSR Sb.* **58** (1987).

[9] M. W. Hirsh, C. C. Pugh, and M. Shub, "Invariant manifolds", *Lecture Notes in Mathematics*, vol. 583, Springer–Verlag, Berlin 1977.

[10] J. Guckenheimer and R. F. Williams, "Structural stability of Lorenz attractors", *Astérisque* **50** (1979), 307–320.

[11] D. V. Anosov, "Geodesic flows on closed Riemannian manifolds of negative curvature", *Trudy Inst. Mat. Steklov.* **90** (1967); English transl. in *Proc. Steklov Inst. Math.* **90** (1969).

[12] Ya. B. Pesin, "Characteristic Lyapunov exponents and smooth ergodic theory", *Uspekhi Mat. Nauk* **32**:4 (1977), 55–111; English transl. in *Russian Math. Surveys* **32**:4 (1977).

[13] D. V. Anosov, S. Kh. Aranson, I. U. Bronshtein, and V. Z. Grines, "Smooth dynamical systems. II", *Itogi Nauki i Tekhniki : Sovrem. Probl. Mat.: Fund. Naprav.*, vol. 1, VINITI, Moscow 1985, pp. 151–242; English transl., *Encyclopaedia Math. Sci.*, vol. 1 [*Dynamical Systems*, I], Springer–Verlag, Berlin 1988.

[14] J. Auslander, "Generalized recurrence in dynamical systems", *Contributions to differential equations*, vol. 3, John Wiley and Sons, Inc., New York 1964, pp. 65–74.

[15] L. P. Shil'nikov, "On the question of the structure of an extended neighbourhood of a structurally stable equilibrium state of saddle-focus type", *Mat. Sb.* **81 (123)** (1970), 92–113; English transl. in *Math. USSR Sb.* **10** (1970).

[16] S. V. Gonchenko, D. V. Turaev, and L. P. Shil'nikov, "On the existence of Newhouse regions near systems with a non-transversal homoclinic Poincaré curve (the multidimensional case)", *Ross. Akad. Nauk Dokl.* **329** (1993), 404–407; English transl. in *Russian Acad. Sci. Dokl. Math.* **47** (1993).

[17] J. Palis and M. Viana, "High-dimension diffeomorphisms displaying infinitely many periodic attractors", *Ann. of Math.* (2) **140** (1994), 207–250.

[18] S. Hayashi, "On the solution of C^1-stability conjecture for flow", Preprint.

[19] J. A. Feroe, "Homoclinic orbits in a parametrized saddle-focus system", *Phys. D* **62**:1–4 (1993), 254–262.
[20] L. P. Shil'nikov, "On a problem of Poincaré and Birkhoff", *Mat. Sb.* **74 (116)** (1967), 378–397; English transl. in *Math. USSR Sb.* **3** (1967).
[21] D. V. Turaev, "On dimension of non-local bifurcational problems", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996), 919–948.
[22] B. Sandstede, "Center manifolds for homoclinic solutions", Preprint, 1994.

Weierstrass Institute for Applied Analysis and Stochastics,
Berlin
E-mail address: turaev@wias-berlin.de
Research Institute for Applied Mathematics and Cybernetics,
Nizhni Novgorod
E-mail address: shilnikov@focus.nnov.su

Received 20/JAN/97
Translated by H. H. McFADEN

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX