Math. Ann.
DOI 10.1007/s00208-016-1468-0

Mathematische Annalen

Degenerate behavior in non-hyperbolic semigroup actions on the interval: fast growth of periodic points and universal dynamics

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Received: 18 September 2015 / Revised: 26 June 2016

Abstract We consider semigroup actions on the unit interval generated by strictly increasing C^r-maps. We assume that one of the generators has a pair of fixed points, one attracting and one repelling, and a heteroclinic orbit that connects the repeller and attractor. We also assume that the other generators form a robust blender, which can bring the points from a small neighborhood of the attractor to an arbitrarily small neighborhood of the repeller. This is a model setting for partially hyperbolic systems with one central direction. We show that, under additional conditions on $\frac{f''}{f'}$ and the Schwarzian derivative, the above semigroups exhibit, C^r-generically for any $r \geq 3$, arbitrarily fast growth of the number of periodic points as a function of the period. We also show that a C^r-generic semigroup from the class under consideration supports an ultimately complicated behavior called universal dynamics.

Mathematics Subject Classification 37C35 · 37D30 · 37E05 · 37H05 · 37D45

Communicated by Nalini Anantharaman.

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Published online: 03 September 2016

1 Introduction

One of the great mysteries of dynamical chaos is its extreme richness. Only in uniformly hyperbolic systems the variability of chaotic dynamics can be controlled: every basic set in Axiom A systems has a finite Markov partition [13], which implies that the chaotic behavior is self-similar in this case. However, once we leave uniformly hyperbolic systems, the self-similarity gets broken. It is typical for non-hyperbolic chaotic systems that going to longer time scales and finer phase space scales exhibits dynamics which is not present on the previous scales. Moreover, the diversity of dynamics that emerges in this process can be unlimited. As typical manifestations of such complexity, in this paper we investigate the phenomena of the *fast* (*super-exponential*) *growth of the number of periodic points* and the *universal dynamics* in a class of non-hyperbolic systems.

The super-exponential growth of periodic points does not seem to be natural. By Artin and Mazur [2] a generic polynomial map has only exponential growth of the number of periodic orbits as a function of period. The same holds true for hyperbolic systems: as we mentioned, their dynamics can be described by means of a finite Markov partitions and, consequently, the number of periodic orbits can grow at most exponentially. Notice also the result of Martens, de Melo, and van Strien [26] which implies the exponential growth of periodic points for C^r-endomorphisms ($2 \leq r \leq \infty$) of the unit interval with non-flat critical points. However, in other situations the super-exponential growth appears generically and is a characteristic feature of the wild behavior. The C^r-genericity of the super-exponential growth for systems in the Newhouse domain (the open region of diffeomorphisms with robust homoclinic tangencies [28]) was discovered by Kaloshin [25]. He noticed that the highly degenerate local bifurcations in the Newhouse domain, which were found in [17,18], create large numbers of periodic orbits of the same period, and this, generically, leads to the super-exponential growth. By a similar strategy, the C^1-genericity of the super-exponential growth was shown in [10] for systems with robust heterodimensional cycles.

Another important phenomenon observed in the non-hyperbolic setting is the generic occurrence of universal dynamics. This phenomenon, discovered in [17,18], can be formally described by the concept of a universal map which was introduced in [8,35]. A map is (d, r)-universal if the set of its iterations, each restricted to an appropriate region in the phase space and written in certain rescaled coordinates, approximates all possible maps from a unit ball to $\mathbb{R}^d$ with arbitrarily good precision in the C^r-topology.

By definition, a single universal map mimics all of the d-dimensional dynamics, i.e., it gives an example of chaos of ultimate complexity. The existence of universal maps may, at the first glance, seem improbable. Indeed, if one considers one dimensional dynamics, it is an easy consequence of Belitsky–Mather theory [5,27] that there exists no C^2-universal diffeomorphisms of an interval, see [37]. However, when the dimension of the phase space is 2 or higher, then, under the presence of robust non-hyperbolicity, universal maps do exist in abundance: they form a residual subset of certain open regions in the space of dynamical systems. Namely, C^1-universal maps are generic in the Bonatti–Díaz domain (the class of systems with robustly non-dominated heterodimensional cycles, see [8]), and (d, r)-universal maps with arbitrarily large $r \geq 2$ are generic in the Newhouse domain for $d = 2$ [20,35,37].

One may wonder if there is another non-uniformly hyperbolic setting which leads to the wild behavior as we saw above. According to the famous conjecture of Palis, the mechanism of non-hyperbolicity is ascribed to the existence of homoclinic tangency or heterodimensional cycles (see [12,30] for the precise statement and relevant discussions). Thus it is interesting to ask if such degenerate behavior is observable in systems exhibiting robust heterodimensional cycles.

By the pioneering work of Bonatti and Díaz [9], we certainly know that the occurrence of heterodimensional cycles can be C^1-robust (as a result, C^r-robust) through the presence of blenders (see [12]). In such situation, it is not hard to prove that C^1-generic dynamical systems exhibit super-exponential growth of the number of periodic points (see [10]) and has universal dynamics to the center direction [7]. However, it is not obvious if such phenomena occur under higher regularity settings.

In this article, aiming at the understanding of the behavior in generic C^r-systems with robust heterodimensional cycles, where $r \geq 2$, we investigate *semigroup actions on the interval*. As suggested in several papers (see for example [4,16,23]), they serve as simplified models of systems with robust heterodimensional cycles and, more generally, partially hyperbolic systems with one central direction. We prove that under certain mild non-hyperbolicity conditions such systems do exhibit the wild behavior.

Let us briefly see the statement of our main result; for basic definitions and precise statements, see Sect. 2. We investigate semigroup actions on the interval $I = [0, 1]$ generated by three maps $f_i : I \to I$ $(i = 0, 1, 2)$, which are smooth and strictly increasing. We assume that f_0 contains a repeller-attractor heteroclinic connection. We also assume that (f_1, f_2) is a persistent blender on an interval containing the repeller-attractor heteroclinic (see Sect. 2 for the definitions). The fact that (f_1, f_2) is a blender means that the action of (f_1, f_2) spreads points over the interval. This implies that there are orbits starting near the attracting fixed point of f_0 and ending near the repelling fixed point. Thus there are transient orbits going back and forward between the repelling point and the attracting point, which supports the non-hyperbolic behavior.

Let us introduce the C^r-topology in the space of such semigroups. Then, our result can be stated as follows:

Main Result: *For each $r \geq 1$, there exists a non-empty, C^r-open region $\mathcal{W}^r$ in the space of semigroups satisfying above conditions, in which C^r-generic elements exhibit super-exponential growth of periodic points* (*Theorem* 2.1) *and have C^r-universal dynamics* (*Theorem* 2.2).

We also prove that the itineraries along which we observe the super-exponential growth are quite abundant (see Theorem 2.3).

Local genericity of the super-exponential growth of periodic points for $r = 1$ is just an easy analog of a result in [10]. Our main contribution is the case $r \geq 2$. One interesting feature of our result is that the dynamics we describe is controlled by derivatives of order higher than 1. Usually, the conditions on the generic dynamics are formulated in terms of the first derivative only. It seems probable that no conditions involving higher order derivatives restrict the possible richness of chaotic dynamics in systems with homoclinic tangencies [21,24,34,36,37]. Surprisingly, as the results of the present paper suggest, this "the first derivative alone" principle should not be applicable to the description of dynamics near semigroup actions and, accordingly, robust heterodimensional cycles.

The class $\mathcal{W}^r$ is described in terms of the second order derivatives if $r \geq 2$ and the third order derivatives if $r \geq 3$ (see Sect. 2.1). A simple speculation reveals the necessity of such conditions. Consider the semigroups for which all the maps f_i have strictly positive second derivatives everywhere. Any composition of increasing convex functions is, obviously, convex. Therefore, any composition of the maps f_i cannot have more than 2 fixed points, i.e., it cannot approximate a map with a higher number of fixed points, so any such semigroup can not have universal dynamics. Neither can it have a super-exponential growth: a periodic orbit corresponds to a periodic itinerary; if for each such itinerary the corresponding period map cannot have more than 2 fixed points, then the number of periodic orbits is not more than twice the number of the periodic itineraries, and the latter grows exponentially with period. The same holds true if all the maps are concave. Similarly, if the Schwarzian derivative is negative for all the maps f_i (or positive for all of them), then every composition of f_i has negative (resp. positive) Schwarzian derivative too. This restricts the number of periodic points for every given itinerary by 3, so we do need some condition on the third derivatives, involving the Schwarzian derivatives, in order to have universal dynamics and/or the superexponential growth.

As we have already mentioned, the semigroups which we study here can serve as simplified models for the study of systems with heterodimensional cycles in partially hyperbolic systems. Therefore, our result suggests that the rate of the growth of the number of periodic orbits for a C^r-generic system having robust heterodimensional cycles can be determined by different factors for $r = 1$, $r = 2$ and $r \geq 3$.

Let us briefly explain the scheme of the proof of the Theorem. First, we construct an r-flat periodic point by an arbitrarily small perturbation of the maps (see Sect. 5). A periodic point is r-flat if it is neutral (i.e., the first derivative of the period map at that point is equal to 1) and the derivatives of orders from 2 to r vanish at this point, i.e., the period map is given by $x \mapsto x + o(x^r)$.

The construction of such periodic points is done by induction in the order of flatness: we show that if the semigroup with the persistent blender has a sufficiently large number of k-flat periodic points with $k \geq 3$, then a $(k+1)$-periodic point can be created by an arbitrarily small C^r-perturbation of the system. Moreover, the existence of a persistent blender allows to place this point within any given interval and also let its itinerary to follow the itinerary of any given orbit as long as we want. The perturbation can be localized in an arbitrarily small neighborhood of some finite set of points, so the process can be repeated without destroying any finite number of the flat points created at the previous steps.

A k-flat periodic point corresponds to a codimension-k bifurcation. A local unfolding of this bifurcation can only decrease the codimension. However, the presence of homoclinic and heteroclinic points due to the blender allows for creating bifurcations of codimension $(k+1)$. The fact that homoclinic bifurcations can lead to an unbounded increase in the codimension of accompanying bifurcations of periodic orbits was discovered in [17,18]. It was related to the presence of hidden bifurcation parameters (moduli of conjugacy) at typical homoclinic bifurcations [22]. The strategy of our proof here has the same flavor as the proof of a similar result for systems with homoclinic tangencies [19,20]. Indeed, the main argument is based on the calculation of a superposition of polynomial maps. However, the actual construction is quite different: in the case of a homoclinic tangency the argument unfolds in a neighborhood of a critical point, while the maps we consider here are diffeomorphisms of an interval, which results in a different algebraic structure.

As we have seen before, the information on the signature of the second and Schwarzian derivatives is important in our case. For example, the induction cannot start at $k \leq 2$. If we write the period map near a k-flat point as $x \mapsto x + a_{k+1}x^{k+1} + o(x^{k+1})$, then the sign of a_2 for a 1-flat point is the sign of the second derivative; the sign of a_3 for a 2-flat point is the sign of the Schwarzian derivative. Therefore, for a semigroup whose generators have second (resp. Schwarzian) derivatives with definite signature, we see that we cannot produce 2- (resp. 3-) flat points. In our proof, 2-flat periodic points are created by a perturbation over 1-flat periodic points with different signs of a_2. Such 1-flat points are obtained by a perturbation of a heteroclinic cycle which includes a pair of repeller-attractor heteroclinic connections of different characteristics in terms of the second derivative; the existence of such heteroclinics is a part of the conditions that describe the set $\mathcal{W}^2$. Similarly, when the same heteroclinics carry also an opposite sign of Schwarzian derivative (this condition is assumed in the definition of $\mathcal{W}^r$ with $r \geq 3$), we are able to make the resulting 2-flat points having opposite signs of a_3. Then, a perturbation including such 2-flat points creates a 3-flat point. After that the above described induction can start, as the sign of a_{k+1} does not play a role if $k \geq 3$. The difference between the case $k \leq 3$ and $k > 3$ will be elucidated through the proof of local algebraic lemmas given in Sect. 3.

Once the possibility to create an r-flat periodic point by an arbitrarily small perturbation is established, the proof of the generic super-exponential growth of periodic points is done by the Kaloshin argument (see Sect. 5). The creation of universal dynamics out of the abundant r-flat points is less straightforward. The idea of creating universal dynamics by perturbing a flat periodic point can be traced back to Ruelle and Takens work [31]. This task is not trivial in the C^r-topology setting with large r. While it was solved in [37] for maps of dimension 2 and higher, the methods of [37] are inapplicable in the one-dimensional setting. Therefore, we derive the genericity of the C^r-universal maps in $\mathcal{W}^r$ from the occurrence of r-flat points by employing a completely different technique; we also make a substantial use of the existence of a blender which gives us the freedom in choosing orbit itineraries (see Sect. 6).

Let us discuss one technical matter. Even though we use localized (hence, non-analytic) perturbations in our proof, it seems that the construction can be modified (in the spirit of [14,20]) in order to encompass the analytic case, and we believe that semigroups that have C^ω-universal semigroups are generic in $\mathcal{W}^\omega$. However, we do not know whether the generic super-exponential growth holds in the analytic case or not. Recently, the genericity of the super-exponential growth for real-analytic area-preserving diffeomorphisms was shown in [3]. The class of semigroups we consider in this paper can, possibly, serve as another sufficiently simple non-trivial class of systems for which this question can be investigated.

In conclusion, we also remark that the analysis of semigroup actions presented in this paper has similar flavor with the study of dynamics of cocycles. The reader will find similarities of arguments among papers such as [1,11,29]. Indeed, these two objects are tightly related: given a semigroup action, one can construct a diffeomorphism cocycle dynamics over shift spaces by taking a skew product. Consequently, all the results we obtain immediately give us corresponding results for the skew-product systems.

We also expect that the statement similar to our theorems should hold true for partially hyperbolic diffeomorphisms with heterodimensional cycles with one-dimensional central direction. However, compared to the semigroups case, the holonomies along the center foliation of a generic partially hyperbolic map have low regularity, which prevents a direct transfer of the results we obtain for one-dimensional semigroups to multi-dimensional partially hyperbolic diffeomorphisms. Meanwhile, we believe that the main techniques are transferrable, and should be useful for the further work in this direction.

In the next section, we start rigorous arguments: we give basic definitions and precise statement of our results. The organization of this paper is explained at the end of Sect. 2.

2 Main results

As explained in Sect. 1, the aim of this paper is to show that generic semigroup actions in a certain open subset of the space of actions has a wild behavior. The first theorem (Theorem 2.1) asserts that an arbitrarily fast growth of the number of attracting periodic points is generic in $\mathcal{W}^r$, an open region whose precise definition is explained below. The second theorem (Theorem 2.2) asserts that a generic triple in $\mathcal{W}^r$ generates ''universal dynamics''. The last theorem (Theorem 2.3) asserts that under certain additional conditions, the number of attracting periodic points grows, along a generic infinite word, faster than any given function of the period for a generic triple in $\mathcal{W}^r$.

2.1 Space of semigroup actions and its open subset $\mathcal{W}^r$

In this subsection, we prepare basic terminologies used throughout the paper.

For a finite set S, we denote the set $\bigsqcup_{n\geq 0} S^n$ by S^*. The set S^* is called the set of *words* of alphabets S. Under concatenation of words, S^* forms a semigroup. For $1 \leq r \leq \infty$, let $\mathcal{E}^r$ be the space of orientation-preserving C^r embeddings from $[0, 1]$ to $(0, 1)$ endowed with C^r-topology. By composition of maps, it is also a semigroup. We write $\mathcal{A}^r(S)$ for the set of families $(f_s)_{s\in S}$ of maps in $\mathcal{E}^r$ indexed by S. This set is endowed with the product topology of the C^r-topology of $\mathcal{E}^r$. For a family $\rho = (f_s)_{s\in S} \in \mathcal{A}^r(S)$ and a word $\omega = s_n \dots s_1 \in S^*$, we define a map ρ^ω in $\mathcal{E}^r$ by $\rho^\omega = f_{s_n} \circ \cdots \circ f_{s_1}$. Then, the map $\omega \mapsto \rho^\omega$ is a homomorphism between S^* and $\mathcal{E}^r$. This homomorphism is called the *semigroup action* generated by $\rho = (f_s)_{s\in S}$. It is easy to see that any homomorphism from S^* to $\mathcal{E}^r$ is generated by a family in $\mathcal{A}^r(S)$.

For $\rho \in \mathcal{A}^r(S)$ and $x \in [0, 1]$, let $\mathcal{O}_+(x, \rho)$ be the *forward orbit of x under ρ* $\{\rho^\omega(p) \mid \omega \in S^*\}$. We call an element $\rho \in \mathcal{A}^r(S)$ a *blender* on a closed (non-trivial) interval $J \subset (0,1)$ if the closure of $\mathcal{O}_+(x,\rho)$ contains J for any $x \in J$. We say that a blender ρ on J is *C^r-persistent* if any semigroup action which is C^r-close to ρ is a blender on J. It is known that a C^r-persistent blender exists. For example, suppose that $(f_1, f_2) \in \mathcal{A}^r(\{1,2\})$ satisfies that $f_1' < 1$ and $f_2' < 1$ on a closed interval $[a,b] \subset (0,1)$, $f_1(a) = a$, $f_2(b) = b$, and $f_1(b) > f_2(a)$. Then (f_1, f_2) is a C^r-persistent blender for any closed interval $J \subset (a,b)$ (see [32, Example 1]).

Let f be a map in $\mathcal{E}^r$ with $1 \le r \le \infty$. A pair (p,q) of points in $[0,1]$ is a *repeller-attractor pair* if p and q are fixed points of f such that $f'(p) > 1 > f'(q)$ and $W^u(p) \cap W^s(q) \neq \emptyset$. A point in $W^u(p) \cap W^s(q)$ is called a *heteroclinic point* of (p,q). We define two quantities $\tau_A(z_0, f), \tau_S(z_0, f) \in \{\pm 1, 0\}$ for a heteroclinic point z_0 as follows. For a map $g \in \mathcal{E}^r$ and $x \in [0,1]$, let $A(g)_x$ and $S(g)_x$ be the *non-linearity* and the *Schwarzian derivative* of g at x, defined as follows:

$$A(g)_x = \frac{g''(x)}{g'(x)}, \qquad S(g)_x = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2,$$

where $A(g)_x$ is defined only if $r \ge 2$ and $S(g)_x$ is defined only if $r \ge 3$. When $r \ge 2$, there exist normalized C^r-linearizations $\varphi : W^u(p) \to \mathbb{R}$ and $\psi : W^s(q) \to \mathbb{R}$ at p and q, i.e., orientation preserving diffeomorphisms satisfying $\varphi \circ f(x) = \lambda_p \varphi(x)$, $\psi \circ f(x) = \lambda_q \psi(x)$, and $\varphi'(p) = \phi'(q) = 1$, where $\lambda_p = f'(p)$ and $\lambda_q = f'(q)$. For a heteroclinic point $z_0 \in W^u(p) \cap W^s(q)$, set

$$\tau_A(z_0, f) = \operatorname{sgn}(A(\psi \circ \varphi^{-1})_{\varphi(z_0)}),$$
$$\tau_S(z_0, f) = \operatorname{sgn}(S(\psi \circ \varphi^{-1})_{\varphi(z_0)}),$$

where $\operatorname{sgn} : \mathbb{R} \to \{0, \pm 1\}$ is the sign function. We call the pair $(\tau_A(z_0, f), \tau_S(z_0, f))$ *the sign* of z_0 if f is of class C^3. If f is only C^2, then the sign of z_0 is just one number, $\tau_A(z_0, f)$.

The normalized C^r-linearizations φ and ψ are known to exist uniquely if $r \ge 2$ (see [33, Theorem 2]). Therefore, the map $\psi \circ \varphi^{-1}$ is uniquely defined and is an invariant (a functional modulus) of the smooth conjugacy of maps of the interval [5,27]. Hence, $\tau_A(z_0, f)$ is well-defined if $r \ge 2$, and $\tau_S(z_0, f)$ is well-defined if $r \ge 3$. Moreover, the sign of the heteroclinic point is invariant with respect to C^r-smooth coordinate transformations, and it is the same for every point of the orbit of z_0 by f.

Let $\mathcal{W}^1$ be the set of $\rho = (f_0, f_1, f_2) \in \mathcal{A}^1(\{0,1,2\})$ which satisfy the following conditions:

Existence of Blender $(f_1, f_2) \in \mathcal{A}^1(\{1,2\})$ is a C^1-persistent blender on a closed interval $J \subset [0,1]$.
Non-hyperbolicity f_0 admits a repeller-attractor pair (p,q) in Int J.
The set $\mathcal{W}^1$ is an open subset of $\mathcal{A}^1(\{0,1,2\})$. Let $\mathcal{W}^2$ be the set of $\rho = (f_0, f_1, f_2) \in \mathcal{W}^1 \cap \mathcal{A}^2(\{0,1,2\})$ which satisfy the following condition:
Sign condition I f_0 admits repeller-attractor pairs (p_1, q_1) and (p_2, q_2) in Int J (the case where $(p_1, q_1) = (p_2, q_2)$ is allowed) and there exist heteroclinic points $z_1 \in W^u(p_1) \cap W^s(q_1)$ and $z_2 \in W^u(p_2) \cap W^s(q_2)$ for f_0 such that $\tau_A(z_1, f_0) \cdot \tau_A(z_2, f_0) < 0$.

Finally, let $\mathcal{W}^3$ be the set of $\rho = (f_0, f_1, f_2) \in \mathcal{W}^2 \cap \mathcal{A}^3(\{0, 1, 2\})$ which satisfy the following condition:
Sign condition II f_0 admits repeller-attractor pairs (p_3, q_3) and (p_4, q_4) in Int J (the case where $(p_i, q_i) = (p_j, q_j)$ for some $1 \le i < j \le 4$ is allowed) and there exist heteroclinic points $z_3 \in W^u(p_3) \cap W^s(q_3)$ and $z_4 \in W^u(p_4) \cap W^s(q_4)$ for f_0 such that $\tau_S(z_3, f_0) \cdot \tau_S(z_4, f_0) < 0$.

Remark that $\mathcal{W}^3 \subset \mathcal{W}^2 \subset \mathcal{W}^1$. We also define $\mathcal{W}^r = \mathcal{W}^3 \cap \mathcal{A}^r(\{0, 1, 2\})$ for $r \ge 4$ (i.e. we do not apply any other conditions except for smoothness if $r \ge 4$). It is known that the normalized C^r linearization of a one-dimensional C^r map f at a hyperbolic fixed point depends continuously on the map f in the C^r topology (see Section 5.2 of [6] for example). This fact implies that $\tau_A(z_0, f_0)$ (*resp.* $\tau_S(z_0, f_0)$) does not change at C^2(*resp.* C^3)-small perturbations if it is non-zero. Hence, $\mathcal{W}^r$ is an open subset of $\mathcal{A}^r(\{0, 1, 2\})$ for any $r \ge 1$. In Sect. 8 we give simple sufficient criteria for the fulfillment of the sign conditions, which do not require the computation of the Belitsky–Mather invariant $\psi \circ \varphi^{-1}$. These criteria are formulated in terms of the first derivatives only, so we can conclude that $\mathcal{W}^1$ has a C^1 open subset where each C^r semigroup belongs to W^r.

2.2 Arbitrary growth of the number of periodic points

For a map $f \in \mathcal{E}^r$, set

$$\mathrm{Fix}(f) = \{x \in [0, 1] \mid f(x) = x\}, \quad \mathrm{Fix}_a(f) = \{x \in \mathrm{Fix}(f) \mid f'(x) < 1\}.$$

The following is our first result.

Theorem 2.1 *For any $1 \le r \le \infty$ and any sequence $\underline{a} = (a_n)_{n=1}^{\infty}$ of positive integers, a generic (in the sense of Baire) element ρ in $\mathcal{W}^r$ satisfies*

$$\limsup_{n\to\infty} \frac{\sum_{\omega\in\{1,2,3\}^n} \#\,\mathrm{Fix}_a(\rho^\omega)}{a_n} = \infty.$$

Thus, in $\mathcal{W}^r$, semigroups which exhibit arbitrarily fast growth of $\#\,\mathrm{Fix}_a(\rho^\omega)$ are quite abundant. Meanwhile, notice that every semigroup action can be C^r-approximated by the one generated by polynomial maps, and for these maps, by estimating the growth of the degree, we can easily see that $\#\,\mathrm{Fix}_a(\rho^\omega)$ grows at most at an exponential rate (this is analogous to the theorem by Artin and Mazur [2]).

Theorem 2.1 shows an interesting contrast between the average growth of the number of periodic points and the growth along almost every infinite word. Let μ be the uniform distribution on $\{0, 1, 2\}$. We denote the product probability on $\{0, 1, 2\}^\infty$ by μ^∞. For $\underline{\omega} = \cdots s_2 s_1 \in \{0, 1, 2\}^\infty$ and $n \ge 1$, set $\underline{\omega}|_n = s_n \ldots s_1$. As we will see in Sect. 8, the set

$$\mathcal{W}^r_{\mathrm{att}} = \{\rho = (f_0, f_1, f_2) \in \mathcal{W}^r \mid |f_0'|_\infty \cdot |f_1'|_\infty \cdot |f_2'|_\infty < 1\}$$

is non-empty, where $|h|_\infty = \sup_{x\in[0,1]} |h(x)|$. By the law of large numbers,

$$\lim_{n\to\infty} \frac{\#\{m \in \{1,\dots,n\} \mid s_m = s\}}{n} = \frac{1}{3}$$

for any $s \in \{0, 1, 2\}$ and μ_∞-almost every $\underline{\omega} \in \{0, 1, 2\}^\infty$. By using this, we can see that

$$\limsup_{n\to\infty} |(\rho^{\underline{\omega}|_n})'|_\infty^{1/n} \le (|f_0'|_\infty \cdot |f_1'|_\infty \cdot |f_2'|_\infty)^{1/3} < 1$$

holds for any $\rho \in \mathcal{W}^r_{\rm att}$ and μ_∞-almost every infinite word $\underline{\omega} \in \{0, 1, 2\}^\infty$. Hence,

$$\lim_{n\to\infty} \#\,\mathrm{Fix}(\rho^{\underline{\omega}|_n}) = 1 \quad \text{for } \mu^\infty\text{-almost every } \underline{\omega}.$$

On the other hand, Theorem 2.1 implies that semigroup actions ρ satisfying

$$\limsup_{n\to\infty} \frac{1}{a_n} \int_{\{0,1,2\}^\infty} \#\,\mathrm{Fix}_a(\rho^{\underline{\omega}|_n}) d\mu^\infty(\underline{\omega})$$
$$= \limsup_{n\to\infty} \frac{1}{a_n} \frac{\sum_{\omega\in\{0,1,2\}^n} \#\,\mathrm{Fix}_a(\rho^\omega)}{3^n} = \infty$$

is generic in $\mathcal{W}^r_{\rm att}$ for any sequence $\underline{a} = (a_n)_{n=1}^\infty$ of positive integers (by applying Theorem 2.1 replacing (a_n) by $(3^n a_n)$). Therefore, the μ^∞-averaged growth of $\#\,\mathrm{Fix}_a(\rho^\omega)$ and the growth of $\#\,\mathrm{Fix}_a(\rho^{\underline{\omega}|_n})$ along μ^∞-almost every infinite words $\underline{\omega}$ are completely different for generic ρ in $\mathcal{W}^r_{\rm att}$.

2.3 Universal dynamics

For finite sets S and S', and families $\rho = (f_s)_{s\in S} \in \mathcal{A}^r(S)$ and $\rho' = (g_{s'})_{s'\in S'} \in \mathcal{A}^r(S')$, we say that ρ *realizes* ρ' if there exists a closed interval $I \subset [0, 1]$, a diffeomorphism $\Phi : [0, 1]\to I$, and a family $(\omega_{s'})_{s'\in S'}$ of words in S^* such that $g_{s'} = \Phi^{-1} \circ (\rho^{\omega_{s'}}|_I) \circ \Phi$ holds for any $s' \in S'$. In other words, the semigroup actions generated by $\rho' = (g_{s'})_{s'\in S'}$ and $(\rho^{\omega_{s'}}|_I)_{s'\in S'}$ are conjugate by the diffeomorphism Φ. We say that $\rho \in \mathcal{A}^r(S)$ generates a *universal semigroup* if for each finite set S' there exists a dense subset $\mathcal{D}_{S'}$ of $\mathcal{A}^r(S')$ such that ρ realizes any $\rho' \in \mathcal{D}_{S'}$.

The following is our second result.

Theorem 2.2 *For any $1 \le r \le \infty$, a generic element in $\mathcal{W}^r$ generates a universal semigroup.*

2.4 Wild behavior along generic infinite words

Under an additional mild condition, the semigroup generated by a generic element of $\mathcal{W}^r$ exhibits wild behavior along generic infinite words. Let $\mathcal{W}^r_\#$ be the set consisting of elements $\rho = (f_0, f_1, f_2)$ of $\mathcal{W}^r$ such that $\mathcal{O}_+(x, \rho) \cap \operatorname{Int} J \neq \emptyset$ for each $x \in [0, 1]$, where J is the interval on which (f_1, f_2) is a persistent blender. The set $\mathcal{W}^r_\#$ is a non-empty open subset of $\mathcal{A}^r(\{0, 1, 2\})$. We furnish the product topology on $\{0, 1, 2\}^\infty$ induced by the discrete topology of the set $\{0, 1, 2\}$.

Theorem 2.3 *For any sequence $(a_n)_{n=1}^\infty$ of positive integers, a generic $\rho \in \mathcal{W}^r_\#$ satisfies*

$$\limsup_{n\to\infty} \frac{\# \operatorname{Fix}_a(\rho^{\underline{\omega}|_n})}{a_n} = \infty$$

for every generic infinite word $\underline{\omega} \in \{0, 1, 2\}^\infty$.

As we will see in Sect. 8, $\mathcal{W}^r_\# \cap \mathcal{W}^r_{\mathrm{att}}$ is non-empty. For any $\rho \in \mathcal{W}^r_\# \cap \mathcal{W}^r_{\mathrm{att}}$ and μ^∞-almost every $\underline{\omega}$, $\rho^{\underline{\omega}|_n}$ is a uniform contraction for any sufficiently large n. This implies that the generic infinite words in Theorem 2.3 form a null subset of $\{0, 1, 2\}^\infty$ with respect to the probability measure μ^∞.

2.5 Organization of this paper

The rest of this paper is organized as follows. In Sect. 3, we prepare several local algebraic results about the composition of germs. In Sect. 4, we prepare the notation for the perturbation of semigroups and give several lemmas, which produce orbits that realize desired germs. In Sect. 5, by using the techniques which we prepare in Sects. 3 and 4, we give the induction argument producing r-flat periodic orbits, and complete the proof of Theorem 2.1. In Sect. 6, we prove Theorem 2.2 by using the construction of r-flat periodic orbits (which is already obtained in Sect. 5) together with a lemma about the decomposition of diffeomorphisms on the interval (Lemma 3.3). In Sect. 7, we prove Theorem 2.3. The proof is done by a careful reiteration of the proof of Theorem 2.1 together with a genericity argument (Lemma 7.1). Finally, in Sect. 8, we give a simple sufficient condition for the fulfillment of the sign conditions. As an application, we give a simple polynomial example for a semigroup in $\mathcal{W}^\infty_\# \cap \mathcal{W}^\infty_{\mathrm{att}}$.

3 Cancellation of germs

As explained in the introduction, for the proof of Theorems 2.1 and 2.2, we first produce r-flat periodic points by an arbitrarily small perturbation. The construction of such periodic points will be done inductively. In this section, we derive local algebraic propositions needed for the inductive step. Namely, we show how to obtain an $(r+1)$-flat germ as a composition of iterations of r-flat germs.

Let $\mathcal{D}^r$ be the set of germs of an orientation-preserving local C^r-diffeomorphisms of $\mathbb{R}$ with a fixed point at the origin. We simply write $\mathcal{D}$ for $\mathcal{D}^\infty$. For $F \in \mathcal{D}$ and $s \geq 1$, we denote by $F^{(s)}$ (or F' for $s = 1$) the s-th derivative of F at 0. We define a pseudo-distance d on $\mathcal{D}$ by

$$d(F, G) = \sum_{r=1}^{\infty} 2^{-r} \left(\frac{|F^{(r)} - G^{(r)}|}{1 + |F^{(r)} - G^{(r)}|} \right).$$

The pseudo-distance d defines a topology on $\mathcal{D}$. This topology is non-Hausdorff. Indeed, a germ F with $F' = 1$ and $F^{(r)} = 0$ for all $r \geq 2$ is not separated from the identity germ I. We say that $F \in \mathcal{D}$ is *r-flat* if $F' = 1$ and $F^{(s)} = 0$ for all $s = 2, \ldots, r$. The term ∞-flat will mean r-flat for every $r \geq 1$. For $F \in \mathcal{D}$, let $A(F)$ and $S(F)$ be the non-linearity and the Schwarzian derivative of F at 0 respectively. The *sign* of a germ $F \in \mathcal{D}^r$ is the pair $(\mathrm{sgn}(A(F)), \mathrm{sgn}(S(F)))$ We say that two germs $F, G \in \mathcal{D}^r$ have the same or opposite signs if both $A(F) \cdot A(G)$ and $S(F) \cdot S(G)$ are positive or, resp., negative. For $f \in \mathrm{Diff}^r([0, 1])$ and $x \in (0, 1)$, we define a germ $[f]_x$ in $\mathcal{D}^r$ by $[f]_x(y) = f(x + y) - f(x)$.

We start with recalling the fact (Lemma 3.1) that any germ in $\mathcal{D}$ equals to a time-one map of a local flow up to order r, for each fixed $r \geq 1$. For $\alpha > 0$, let L_α be the element of $\mathcal{D}$ given by $L_\alpha(x) = \alpha x$.

Lemma 3.1 *For any $F \in \mathcal{D}$ and $r \in [1, +\infty)$, there exists a continuous family of germs $(F^t)_{t \in \mathbb{R}}$ in $\mathcal{D}$ such that F^0 is the identity map, $F^1 = F + o(x^r)$, and $F^t \circ F^{t'} = F^{t+t'} + o(x^r)$ for any $t, t' \in \mathbb{R}$.*

Proof Recall that F is orientation preserving. Put $\alpha = F' > 0$. If $\alpha \neq 1$, then F is smoothly linearizable at 0. This means that there exists $\Phi \in \mathcal{D}$ such that $\Phi \circ F \circ \Phi^{-1}(x) = \alpha x$. In this case, the family $(\Phi^{-1} \circ L_{\alpha^t} \circ \Phi)_{t \in \mathbb{R}}$ satisfies the required properties.

By $\mathcal{D}(r)$, we denote the subgroup of $\mathcal{D}$ consisting of r-flat elements, where the group operation is given by the composition of germs. Suppose that $F' = 1$. Then F belongs to $\mathcal{D}(1)$. The group $\mathcal{D}(1)/\mathcal{D}(r)$ is a finite-dimensional, connected, and simply connected nilpotent Lie group and it is well-known that the exponential map is a diffeomorphism for such Lie groups (see [15, Theorem 1.2.1] for example). Hence, there exists an element ξ in the Lie algebra of $\mathcal{D}(1)/\mathcal{D}(r)$ such that $\exp(\xi) = F + o(x^r)$. Then the family $(\exp(t\xi))_{t \in \mathbb{R}}$ satisfies the required properties.

Remark 3.2 There is an explicit inductive construction of the family $(F_t)_{t \in \mathbb{R}}$ for the case $F' = 1$: The constant family $(F_1^t \equiv I)_{t \in \mathbb{R}}$ satisfies the required condition for $r = 1$. Suppose that we have a family $(F_r^t)_{t \in \mathbb{R}}$ which satisfies the required condition for some $r \geq 1$. Put $a = [F^{(r+1)}(0) - F_r^{(r+1)}(0)]/(r + 1)!$ and let $(G^t)_{t \in \mathbb{R}}$ be the germ of local flow generated by the vector field $ax^{r+1}(\partial/\partial x)$ at 0. We set $F_{r+1}^t = F_r^t \circ G^t$. Then, $F_{r+1}^1(x) = F_r^1(x) + ax^{r+1} + o(x^{r+1}) = F(x) + o(x^{r+1})$. Since the germ G^t is r-flat, the maps F_r^t and G^t commute up to order x^{r+1} (see also Remark 3.4). Hence, we have $F_{r+1}^{t+t'} = F_{r+1}^t \circ F_{r+1}^{t'} + o(x^{r+1})$. Thus the family $(F_{r+1}^t)_{t \in \mathbb{R}}$ satisfies the required condition for $r + 1$.

For $h \in \mathrm{Diff}^r([0, 1])$ or $\mathrm{Diff}^r(\mathbb{R})$, the *support* of h, denoted by $\mathrm{supp}\, h$, is the closure of $\{h(x) \neq x\}$. For a family of diffeomorphism (h_t), its *support* means the closure of the union $\cup_t \mathrm{supp}\, h_t$. The following lemma plays a key role in our construction of universal semigroups.

Lemma 3.3 *Let $r \in [2, +\infty]$. Let I be a compact interval in $\mathbb{R}$ and F be an orientation-preserving C^r-diffeomorphism of $\mathbb{R}$ such that* $\mathrm{supp}(F) \subset I$. *Then, there exist one-parameter groups $(G^t)_{t\in\mathbb{R}}$ and $(H^t)_{t\in\mathbb{R}}$ of C^r-diffeomorphisms of $\mathbb{R}$ and a compact interval I' such that $F = G^1 \circ H^1$ on I and the support of (G^t) and (H^t) are both contained in I'.*

Proof Take $0 < \lambda < 1$ such that the map $F_\lambda(x) = \lambda F(x)$ is a uniform contraction on $\mathbb{R}$. The contraction property implies that the map F_λ has a unique fixed point p_*. This fixed point is exponentially stable. It follows that F_λ is C^r-linearizable on $\mathbb{R}$. More precisely, there exists a C^r-diffeomorphism φ of $\mathbb{R}$ and a constant $\mu > 0$ such that $F_\lambda \circ \varphi(x) = \varphi(\mu x)$ for any $x \in \mathbb{R}$.

Put $\bar{G}^t(x) = \lambda^{-t}x$ and $\bar{H}^t(x) = \varphi(\mu^t \cdot \varphi^{-1}(x))$ for $x \in \mathbb{R}$. These are one-parameter groups of diffeomorphisms of $\mathbb{R}$. Take a compact interval I' whose interior contains $\bar{G}^t \circ F_\lambda(I)$ and $\bar{H}^t(I)$ for any $t \in [0, 1]$. By cutting-off the vector fields generating $(\bar{G}^t)$ and $(\bar{H}^t)$ outside I', we obtain one-parameter groups $(G^t)_{t\in\mathbb{R}}$ and $(H^t)_{t\in\mathbb{R}}$ of diffeomorphisms of $\mathbb{R}$ such that $G^t(x) = \bar{G}^t(x)$ and $H^t(x) = \bar{H}^t(x)$ for any $x \in I \cup F_\lambda(I)$ and $t \in [0, 1]$, and the supports of G^t and H^t are both contained in I'. By construction, $G^1 \circ H^1(x) = \lambda^{-1} \cdot F_\lambda(x) = F(x)$ for any $x \in I$. □

In the following, we give three lemmas on the cancellation of germs, which are the main ingredient of the proofs of Theorems 2.1 and 2.2 for the case $r \geq 2$. Their proofs will be done by calculating compositions of polynomials. In the proofs, we will exploit the following elementary observations.

Remark 3.4
1. Non-linearities and Schwarzian derivatives satisfy the following *cocycle properties*: for $F, G \in \mathcal{D}$, we have

$$A(F \circ G) = A(F) \cdot (G') + A(G), \quad S(F \circ G) = S(F) \cdot (G')^2 + S(G).$$

   In particular, if the germs F and G are 1-flat, then

$$A(F \circ G) = A(F) + A(G), \quad S(F \circ G) = S(F) + S(G).$$

2. Suppose that $F \in \mathcal{D}$ is 1-flat and $G \in \mathcal{D}$ satisfies $G(x) = x + cx^{r+1} + o(x^{r+1})$ (*i.e*, G is r-flat). Then an easy computation shows that

$$F \circ G(x) \equiv G \circ F(x) \equiv F(x) + cx^{r+1} \mod o(x^{r+1}).$$

   In particular, $(F \circ G)^{(r+1)} = (G \circ F)^{(r+1)} = F^{(r+1)} + G^{(r+1)}$.

For a germ $F \in \mathcal{D}$ satisfying $A(F) \neq 0$, we put $(S/A)(F) = S(F)/A(F)$.

Lemma 3.5 *Let F_1 and F_2 be 1-flat germs in $\mathcal{D}$ with the opposite signs and satisfying $|(S/A)(F_1)| > |(S/A)(F_2)|$. Then, for any neighborhood $\mathcal{V}$ of the identity germ in $\mathcal{D}$ and any $\alpha, \beta \in \mathbb{R}$, there exist 1-flat germ $H \in \mathcal{V}$ and $m, n \geq 1$ such that the following holds:*

$$A(F_2^n) + A((H \circ F_1)^m) + \alpha = 0,$$
$$S(F_1) \cdot \left\{S(F_2^n) + S((H \circ F_1)^m) + \beta\right\} > 0.$$

Proof Since $A(F_1) \cdot A(F_2) < 0$, there exist sequences $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} m_k = +\infty$ and $|m_k A(F_1) + n_k A(F_2)| < 1$ for any $k \geq 1$. Put $c_k = (m_k A(F_1) + n_k A(F_2) + \alpha)/(2m_k)$ and let H_k be the germ in $\mathcal{D}$ given by $H_k(x) = x - c_k x^2 + c_k^2 x^3$. Then, $A(H_k) = -2c_k$, $S(H_k) = 0$, and H_k converges to the identity germ in $\mathcal{D}$. Notice that, by Remark 3.4, we have

$$A(F_2^{n_k}) + A((H_k \circ F_1)^{m_k}) = m_k\,(A(F_1) + A(H_k)) + n_k A(F_2) = -\alpha$$

and

$$\begin{aligned}
\frac{S(F_2^{n_k}) + S((H_k \circ F_1)^{m_k})}{m_k S(F_1)} &= \frac{n_k S(F_2) + m_k[S(H_k) + S(F_1)]}{m_k S(F_1)} \\
&= \frac{n_k S(F_2)}{m_k S(F_1)} + (0+1) \\
&= -\frac{A(F_1) + A(H_k) + (\alpha/m_k)}{A(F_2)} \cdot \frac{S(F_2)}{S(F_1)} + 1 \\
&\xrightarrow{k\to\infty} -\frac{A(F_1)S(F_2)}{A(F_2)S(F_1)} + 1.
\end{aligned}$$

Since $|S(F_1)/A(F_1)| > |S(F_2)/A(F_2)|$, the last term is positive. This implies that

$$\lim_{k\to\infty} S(F_1) \cdot \{S(F_2^{n_k}) + S((H \circ F_1)^{m_k})\} = +\infty.$$

This shows that H_k, m_k and n_k satisfy the desired properties for sufficiently large k. □

Lemma 3.6 *Suppose $r \geq 2$. Let F_1 and F_2 be r-flat germs in $\mathcal{D}$ such that $F_1^{(r+1)} \cdot F_2^{(r+1)} < 0$. Then, for any neighborhood $\mathcal{V}$ of the identity and $\alpha \in \mathbb{R}$, there exist $H \in \mathcal{D}^r$ and $m, n \geq 1$ such that*

$$F_2^m \circ (H \circ F_1)^n(x) = x + \alpha x^{r+1} + o(x^{r+1}).$$

Proof Proof is similar to Lemma 3.5. Put $\alpha_i = F_i^{(r+1)}/(r+1)!$ for $i = 1, 2$. Since $\alpha_1 \cdot \alpha_2 < 0$, there exists sequences $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ such that $m_k \to \infty$ as $k \to \infty$ and $|m_k\alpha_2 + n_k\alpha_1| < 1$ holds. Put

$$c_k = \frac{m_k\alpha_2 + n_k\alpha_1 - \alpha}{n_k}$$

and $H_k(x) = x - c_k x^{r+1}$. Then, H_k converges to the identity in $\mathcal{D}$. Since r-flat germs are commutative up to $(r+1)$-st order, we have

$$\begin{aligned}
F_2^{m_k} \circ (H_k \circ F_1)^{n_k} &= x + (m_k\alpha_2 + n_k(\alpha_1 - c_k))x^{r+1} + o(x^{r+1}) \\
&= x + \alpha x^{r+1} + o(x^{r+1}).
\end{aligned}$$

This shows that H_k, m_k and n_k satisfy the desired property if k is sufficiently large. □

Lemma 3.7 *Suppose $r \geq 3$. Let $F_1, \ldots, F_4$ be r-flat germs in $\mathcal{D}$. Then, for any neighborhood $\mathcal{V}$ of the identity map in $\mathcal{D}$ and $\alpha \in \mathbb{R}$, there exist $H_1, \ldots, H_4 \in \mathcal{V}$ and $n \geq 1$ such that*

$$(H_4 \circ F_4)^n \circ \cdots \circ (H_1 \circ F_1)^n(x) = x + \alpha x^{r+1} + o(x^{r+1}).$$

Proof Set $c = (\sum_{i=1}^4 F_i^{(r+1)})/(r+1)!$. Let $(G^t)_{t \in \mathbb{R}}$ and $(H_\mu^t)_{t \in \mathbb{R}}$ be the germs of local flows generated by vector fields $x^2 \frac{\partial}{\partial x}$ and $\mu x^r \frac{\partial}{\partial x}$, respectively. They satisfy the following:

$$G^t(x) = \frac{x}{1 - tx} = x + tx^2 + \cdots + t^r x^{r+1} + o(x^{r+1}),$$
$$H_\mu^t(x) = x + \mu t x^r + o(x^{r+1}).$$

Since

$$\begin{aligned} G^t \circ H_\mu^t(x) &= (x + \mu t x^r) + t(x + \mu t x^r)^2 + \cdots + o(x^{r+1}) \\ &= G^t(x) + \mu t x^r + 2\mu t^2 x^{r+1} + o(x^{r+1}), \\ H_\mu^t \circ G^t(x) &= G^t(x) + \mu t(x + tx^2 + \cdots)^r + o(x^{r+1}) \\ &= G^t(x) + \mu t x^r + r\mu t^2 x^{r+1} + o(x^{r+1}), \end{aligned}$$

we have

$$G^t \circ H_\mu^t(x) = H_\mu^t \circ G^t(x) - (r-2)\mu t^2 x^{r+1} + o(x^{r+1}).$$

This implies that

$$G^t \circ H_\mu^t \circ G^{-t} \circ H_\mu^{-t}(x) = x - (r-2)\mu t^2 x^{r+1} + o(x^{r+1}).$$

Set $\mu_n = c - (\alpha/n)$ and $t_n = 1/\sqrt{n(r-2)}$ for $n \geq 1$. The germs G^{t_n} and $H_{\mu_n}^{t_n}$ converge to the identity in $\mathcal{D}$. Since r-flat germs commute with any germs in $\mathcal{D}$ up to $(r+1)$-st order, we have

$$\begin{aligned} &(G^{t_n} \circ F_4)^n \circ (H_{\mu_n}^{t_n} \circ F_3)^n \circ (G^{-t_n} \circ F_2)^n \circ (H_{\mu_n}^{-t_n} \circ F_1)^n \\ &= G^{nt_n} \circ H_{\mu_n}^{nt_n} \circ G^{-nt_n} \circ H_{\mu_n}^{-nt_n}(x) + cnx^{r+1} + o(x^{r+1}) \\ &= x + \left[-(r-2)\mu_n(nt_n)^2 + cn\right] x^{r+1} + o(x^{r+1}) \\ &= x + (c - \mu_n)n \cdot x^{r+1} + o(x^{r+1}) \\ &= x + \alpha x^{r+1} + o(x^{r+1}). \end{aligned}$$

Thus, letting n large, $H_1 = H_{\mu_n}^{-t_n}$, $H_2 = G^{-t_n}$, $H_3 = H_{\mu_n}^{t_n}$ and $H_4 = G^{t_n}$, we complete the proof. □

Remark that the above proof needs the assumption $r \geq 3$ because of the expression $t_n = 1/\sqrt{n(r-2)}$.

These lemmas allow us to construct flat germs. For the practical use, we need to take their realizations as close to identity diffeomorphisms. The following statement shows that such realization are always possible (we omit the proof since it is well-known):

Remark 3.8 For any neighborhood $\mathcal{N}$ of the identity map in $\mathrm{Diff}^{\infty}([-1, 1])$, any $x \in (0, 1)$, and any neighborhood $V \subset [-1, 1]$ of x, there exists a neighborhood $\mathcal{M}$ of the identity germ in $\mathcal{D}$ such that for every $F \in \mathcal{M}$ there exists a diffeomorphism $\tilde{F} \in \mathcal{N}$ such that $\tilde{F}(x) = x$, $[\tilde{F}]_x = F$, and $\mathrm{supp}\tilde{F} \subset V$.

4 Connecting lemmas

In this section, we show that, in the presence of a blender, we can create an orbit connecting any two prescribed points by a small perturbation. We consider semigroup actions $\rho = (f_0, f_1, f_2) \in \mathcal{A}^r(\{0, 1, 2\})$ (where $r \geq 1$) which satisfy the following conditions:

1. $(f_1, f_2) \in \mathcal{A}^r(\{1, 2\})$ is a blender on a closed interval $J \subset [0, 1]$.
2. $\mathrm{Int}(J) \cap f_0(\mathrm{Int}(J)) \neq \emptyset$.

Notice that these conditions hold for $\rho \in \mathcal{W}^r$.

We prepare several definitions. For $x \in [0, 1]$, we put

$$\mathcal{O}_-(x, (f_1, f_2)) = \{y \in [0, 1] \mid \rho^{\omega}(y) = x \text{ for some } \omega \in \{1, 2\}^*\}.$$

We say that a point $x \in J$ is (f_1, f_2)-*generic* if the closure of $\mathcal{O}_-(x, (f_1, f_2))$ contains J. An (f_1, f_2)-generic point is a generic point in the sense of Baire as well. Indeed, the set $J(U) = J \cap \bigcup_{\omega \in \{1,2\}^*} \rho^{\omega}(U)$ is an open and dense subset of J for any non-empty open subset U of J. Take a countable open basis $(U_n)_{n \geq 1}$ of J. Since (f_1, f_2) is a blender on J, every point in the residual subset $\bigcap_{n \geq 1} J(U_n)$ of J is (f_1, f_2)-generic.

For $h \in \mathrm{Diff}^r([0, 1])$, we define an element ρ_h of $\mathcal{A}^r(\{0, 1, 2\})$ by

$$\rho_h = (h \circ f_0, f_1, f_2).$$

For a point $x \in [0, 1]$ and a word $\omega = s_n \cdots s_1 \in \{0, 1, 2\}^*$, set

$$\Sigma_h^{\omega}(x) = \{\rho_h^{s_k \dots s_1}(x) \mid s_k = 0,\ k = 1, \dots, n\}.$$

When h is the identity map, we simply write ρ^{ω} and Σ^{ω} for ρ_h^{ω} and Σ_h^{ω}. If $h \in \mathrm{Diff}^r([0, 1])$ satisfies $\mathrm{supp}(h) \cap \Sigma^{\omega}(x) = \emptyset$, then $[\rho_h^{\omega}]_x = [\rho^{\omega}]_x$, hence $[\rho_h^{0\omega}]_x = [h]_{\rho^{0\omega}(x)} \circ [\rho^{0\omega}]_x$ (remember that by $[\,\cdot\,]_x$ we denote the germ of a diffeomorphism at x, see Sect. 3). For $\omega \in \{0, 1, 2\}^*$, a point $x \in [0, 1]$ is ω-*periodic* for ρ if $\rho^{\omega}(x) = x$. For $r \in [1, +\infty]$, we say that an ω-periodic point x is r-*flat* if the germ $[\rho^{\omega}]_x$ is r-flat. For a word $\omega \in S^*$ with an alphabet S, we denote the length of ω by $|\omega|$, i.e. $|\omega| = n$ if $\omega = s_n \dots s_1$. For $1 \leq k \leq n$, we set $\omega|_k = s_k \cdots s_1$.

Lemma 4.1 *Let p and p' be points in J such that p' is (f_1, f_2)-generic. For any neighborhood $\mathcal{N} \subset \mathrm{Diff}^\infty([0,1])$ of the identity map, any non-empty open subset U of $J \cap f_0(J)$ and any $l \geq 1$, there exist $h \in \mathcal{N}$ and $\omega \in \{0,1,2\}^*$ such that $\mathrm{supp}(h) \cup \Sigma_h^\omega(p) \subset U$, $\rho_h^\omega(p) = p'$, and $|\omega| \geq l$.*

Proof Since p' is (f_1, f_2)-generic, $\mathcal{O}_-(p', \rho)$ intersects with $U \subset J \cap f_0(J)$. Take $\eta \in \{1,2\}^*$ such that $|\eta| \geq l$ and $q = (\rho^\eta)^{-1}(p')$ belongs to U. Since (f_1, f_2) is a blender on J, the point $f_0^{-1}(q)$ is contained in the closure of $\mathcal{O}_+(p, \rho)$. Hence, there exist $h \in \mathcal{N}$ and $\eta' \in \{1,2\}^*$ such that $\mathrm{supp}(h) \subset U$ and $h \circ f_0 \circ \rho^{\eta'}(p) = q$. Put $\omega = \eta 0 \eta'$. Then, $\rho_h^\omega(p) = p'$, $\Sigma_h^\omega(p) = \{q\} \subset U$, and $|\omega| \geq l$. □

Remark 4.2 Notice that in the word ω obtained in the above proof the letter 0 appears only once.

The next lemma shows that when there is an r-flat periodic point somewhere in J, then the connecting orbit can be constructed in such a way that the germ $[\rho_h^\omega]_p$ will coincide with any prescribed one up to order r. In particular, we can construct the connecting orbit for which the corresponding germ will be r-flat.

Lemma 4.3 *Let p and p' be points in J such that p' is (f_1, f_2)-generic. Suppose that there exist an (f_1, f_2)-generic point $\hat{p} \in J$, a word $\gamma \in \{0,1,2\}^*$, and $r_0 \in [1, r]$ such that $\hat{p}$ is an r_0-flat 0γ-periodic point of ρ and $\hat{p} \notin \Sigma^\gamma(\hat{p})$. Then, for any neighborhood $\mathcal{N} \subset \mathrm{Diff}^\infty([0,1])$ of the identity map, any neighborhood V of $\hat{p}$, any non-empty open subset U of $J \cap f(J)$, any germ $F \in \mathcal{D}$ and any $l \geq 1$, there exist $\omega \in \{0,1,2\}^*$ and $h \in \mathcal{N}$ such that $|\omega| \geq l$, $\mathrm{supp}(h) \subset U \cup V$, $\rho_h^\omega(p) = p'$, $[\rho_h^\omega(t)]_p = F(t) + o(t^{r_0})$, and $\Sigma_h^\omega(p) \subset \Sigma^\gamma(\hat{p}) \cup U \cup V$.*

Proof Without loss of generality, we may assume that the sets U, V, and $\Sigma^\gamma(\hat{p})$ are mutually disjoint. Applying Lemma 4.1 for pairs $(p, \hat{p})$, $(\hat{p}, p')$ and the open set U, we obtain $\omega_1, \omega_2 \in \{1,2\}^*$ and $\bar{h} \in \mathcal{N}$ such that $|\omega_1| \geq l$, $\mathrm{supp}(\bar{h}) \subset U$, $\rho_{\bar{h}}^{\omega_1}(p) = \hat{p}$, $\rho_{\bar{h}}^{\omega_2}(\hat{p}) = p'$, and $\Sigma_{\bar{h}}^{\omega_1}(p) \cup \Sigma_{\bar{h}}^{\omega_2}(\hat{p}) \subset U$. More precisely, we first take $U_1, U_2 \subset U$ satisfying $U_1 \cap U_2 = \emptyset$. Then we apply Lemma 4.1 for $(p, \hat{p})$ and U_1, and $(\hat{p}, p')$ and U_2 to obtain two diffeomorphisms h_1 and h_2 respectively. Then, since they have disjoint support, their composition $\bar{h} = h_1 \circ h_2$ gives us the desired $\bar{h}$.

Put $F_1 = [\rho_{\bar{h}}^{\omega_1}]_p$ and $F_2 = [\rho_{\bar{h}}^{\omega_2}]_{\hat{p}}$. By Lemma 3.1, there exists a one-parameter family of germs $(\phi^t)_{t \in \mathbb{R}}$ in $\mathcal{D}$ (which is a one-parameter group up to order r_0) such that

$$\phi^1(x) = (F_2)^{-1} \circ F \circ F_1^{-1}(x) + o(x^{r_0}).$$

Notice that, by Remark 3.8, for each $\phi^{1/N}$, we can choose a diffeomorphism $\varphi^{1/N}$: $[0,1] \to [0,1]$, such that $\varphi^{1/N}(\hat{p}) = \hat{p}$ and $[\varphi^{1/N}]_{\hat{p}} = \phi^{1/N}$. Furthermore, by choosing N sufficiently large, we can assume that $h = \varphi^{1/N} \circ \bar{h}$ is contained in $\mathcal{N}$, being the support of $\varphi^{1/N}$ arbitrarily close to the point $\{\hat{p}\}$.

Then, the support of h is contained in $U \cup V$, hence, it does not intersect $\Sigma^\gamma(\hat{p})$. Put $\omega = \omega_2 (0\gamma)^N \omega_1$. Since $\hat{p}$ is an r_0-flat 0γ-periodic point, we have

$$
\begin{aligned}
[\rho_h^\omega(x)]_p &= F_2 \circ ([\varphi^{1/N}]_{\hat{p}} \circ [\rho^\gamma]_{\hat{p}})^N \circ F_1(x) \\
&= F_2 \circ [\varphi^1]_{\hat{p}} \circ F_1(x) + o(x^{r_0}) \\
&= F(x) + o(x^{r_0}).
\end{aligned}
$$

We can also see that $|\omega| \geq l$ and $\Sigma_h^\omega(p) = \Sigma^\gamma(\hat{p}) \cup \{\hat{p}\} \cup \Sigma_{h_1}^{\omega_1}(p) \cup \Sigma_{h_2}^{\omega_2}(\hat{p})$. The latter implies that $\Sigma_h^\omega(p) \subset \Sigma^\gamma(\hat{p}) \cup U \cup V$. □

Remark 4.4 In this lemma, because of Remark 4.2, we can assume that there exists a point $y \in \Sigma_h^\omega(p) \cap U$ and a unique integer $k \geq 1$ such that $\rho_h^{\omega|_k}(p) = y$ and $\omega_k = 0$, where $\omega = \omega_{|\omega|} \cdots \omega_1$. Indeed, the point in $U_1 \cap \Sigma_h^\omega(p)$ is such a point. Roughly speaking, y is a point which appears in $\Sigma_h^\omega(p)$ only once.

Below we will use the following perturbation result whose proof we omit.

Remark 4.5 Let $f \in \mathrm{Diff}^r([0, 1])$ where $r \in [1, +\infty]$. If f has an r-flat fixed point x, then C^r-arbitrarily close to the identity there exists $g \in \mathrm{Diff}^\infty([0, 1])$ whose support is contained in an arbitrarily small neighborhood of $\{x\}$ such that $g \circ f$ coincides with the identity map near x.

For the case $r = \infty$, we have the following

Lemma 4.6 *Under the assumptions of Lemma 4.3, we furthermore assume that $\rho \in \mathcal{A}^\infty(\{0, 1, 2\})$ and $\hat{p}$ is ∞-flat 0γ-periodic point of ρ. Then, for any neighborhood $\mathcal{N} \subset \mathrm{Diff}^\infty([0, 1])$ of the identity map, any neighborhood V of $\hat{p}$, any non-empty open subset U of $J \cap f(J)$, there exist $\omega \in \{0, 1, 2\}^*$ and $h \in \mathcal{N}$ such that $\mathrm{supp}(h) \subset U \cup V$, $\rho_h^\omega(p) = p'$, $[\rho_h^\omega(t)]_p = [t]$, and $\Sigma_h^\omega(p) \subset \Sigma^\gamma(\hat{p}) \cup V \cup U$.*

Proof For a given $\mathcal{N} \subset \mathrm{Diff}^\infty([0, 1])$, since the C^∞-topology coincides with the projective limit of C^r-topology, there exists $s \geq 1$ and $\mathcal{N}^s \subset \mathrm{Diff}^s([0, 1])$ such that $\mathcal{N}^s \cap \mathrm{Diff}^\infty([0, 1]) \subset \mathcal{N}$. We fix such s and $\mathcal{N}^s$.

Then, for given ρ, U and V, by applying Lemma 4.3 where we let F be the identity germ and $r_0 = s$, we take h which is arbitrarily C^∞-close to the identity map and $\omega \in \{0, 1, 2\}^*$ such that $\rho_h^\omega(p) = p'$ and $[\rho_h^\omega(t)]_p = t + o(t^s)$.

Now, by Remark 4.5, we choose $\tilde{h} \in \mathrm{Diff}^\infty([0, 1])$ supported in an arbitrarily small neighborhood of p' such that it is arbitrarily C^s close to the identity and $[\tilde{h} \circ \rho_h^\omega(t)] = [t]$. We take a point $y \in \Sigma_h^\omega(p) \cap U$ given by Remark 4.4 and choose ω_y, ω'_y such that $\rho_h^{\omega_y}(p) = y$ and $\omega = \omega'_y \omega_y$.

Now, we take $\bar{h} = (\rho_h^{\omega'_y})^{-1} \circ \tilde{h} \circ \rho_h^{\omega'_y}$. Then, by taking $\tilde{h}$ arbitrarily C^s-close to the identity, we can assume that $\bar{h} \in \mathcal{N}^s \cap \mathrm{Diff}^\infty([0, 1])$. Furthermore, by shrinking the support of $\tilde{h}$, we can check that $\rho_{\bar{h} \circ h}^\omega(p) = p'$ keeping $[\rho_{\bar{h} \circ h}^\omega(t)]_p = [t]$ and all the other conditions. Thus, $\bar{h} \circ h$ gives us the desired map g. □

5 Creation of r-flat periodic orbits

In this section, we prove the following proposition which implies Theorem 2.1.

Proposition 5.1 *For any $\rho = (f_0, f_1, f_2) \in \mathcal{W}^\infty$, any open neighborhood $\mathcal{N}$ of the identity map in* $\mathrm{Diff}^\infty([0,1])$*, any $N, l \geq 1$, and $r \in [1, \infty]$, there exist $h \in \mathcal{N}$, (f_1, f_2)-generic distinct points $\hat{p}_1, \ldots, \hat{p}_N$ in J, and $\gamma_1, \ldots, \gamma_N \in \{0,1,2\}^*$ such that $|\gamma_i| \geq l$, $\hat{p}_i$ is an r-flat $0\gamma_i$-periodic point for ρ_h for any $i = 1, \ldots, N$, and the sets $\{\hat{p}_1, \ldots, \hat{p}_N\}$, $\Sigma^{\gamma_1}(\hat{p}_1), \ldots, \Sigma^{\gamma_N}(\hat{p}_N)$ are mutually disjoint.*

Let us, first, see how we derive Theorem 2.1 from Proposition 5.1. In the proof, we use perturbations given by the following construction.

Remark 5.2 Let $f \in \mathrm{Diff}^r([0,1])$ where $1 \leq r \leq +\infty$. If f coincides with the identity map on some non-empty open interval U, then for every $\ell > 0$, there exists $g \in \mathrm{Diff}^\infty([0,1])$ which is arbitrarily C^∞-close to the identity and is supported in an arbitrarily small interval in U, such that $g \circ f$ has more than ℓ attracting fixed points in U. For instance, one can build such g as follows: let g be a map which has the form $x + a\sin(kx)$ (a and k are some constants). Collapse it to the identity map outside U by some bump function. Then, by choosing a and k appropriately, one can see that g will be arbitrarily C^∞-close to the identity and have an arbitrarily large number of attracting periodic points in U. The details are left to the reader.

Another important remark is that $\mathcal{W}^\infty$ is dense in $\mathcal{W}^r$ for any $r \geq 1$. For $r \geq 3$, it is trivial, since $\mathcal{A}^\infty(\{0,1,2\})$ is dense in $\mathcal{A}^r(\{0,1,2\})$. So, let us consider the case $r = 2$. For any $\rho = (f_0, f_1, f_2) \in \mathcal{W}^2$ with heteroclinic points z_1 and z_2 satisfying Sign Condition I, a C^2-small (but C^3-large) perturbation of f_0 at z_1 and z_2 creates a C^∞ map f such that $\tau_S(z_1, f) \cdot \tau_S(z_2, f) < 0$. This implies that $\mathcal{W}^\infty$ is C^2-dense in $\mathcal{W}^2$. Similarly, we can see that $\mathcal{W}^\infty$ is C^1-dense in $\mathcal{W}^1$.

Proof of Theorem 2.1 from Proposition 5.1 Fix $1 \leq r \leq \infty$ and a sequence $(a_n)_{n=1}^\infty$ of integers. Put

$$\mathcal{U}(\omega) = \{\rho \in \mathcal{W}^r \mid \#\mathrm{Fix}_a(\rho^\omega) \geq |\omega| \cdot a_{|\omega|}\}$$

for $\omega \in \{0,1,2\}^*$ and $\mathcal{U}_n = \bigcup_{|\omega| \geq n} \mathcal{U}(\omega)$ for $n \geq 1$. By the persistence of attracting periodic points, $\mathcal{U}(\omega)$ is open for every ω and, accordingly, $\mathcal{U}_n$ is open as well. Notice that every $\rho \in \bigcap_{n \geq 1} \mathcal{U}_n$ satisfies

$$\limsup_{n \to \infty} \frac{\sum_{\omega \in \{0,1,2\}^n} \mathrm{Fix}_a(\rho^\omega)}{a_n} = \infty.$$

Hence, it is sufficient to show that $\mathcal{U}_n$ is a dense subset of $\mathcal{W}^r$ for every $n \geq 1$.

Fix a non-empty open subset $\mathcal{U}$ of $\mathcal{W}^r$, $\rho = (f_0, f_1, f_2) \in \mathcal{U} \cap \mathcal{W}^\infty$, and $n \geq 1$. Take a neighborhood $\mathcal{N}^{1/3}$ of the identity map in $\mathrm{Diff}^\infty([0,1])$ such that $((h_3 \circ h_2 \circ h_1) \circ f_0, f_1, f_2) \in \mathcal{U}$ for any $h_1, h_2, h_3 \in \mathcal{N}^{1/3}$. By Proposition 5.1, there exist $h_1 \in \mathcal{N}^{1/3}$, $p \in J$, and $\gamma \in \{0,1,2\}^*$ such that $|\gamma| \geq n$ and p is an r-flat 0γ-periodic point of ρ_{h_1} satisfying $p \notin \Sigma^\gamma_{h_1}(p)$. Take $h_2 \in \mathcal{N}^{1/3}$ such that $\mathrm{supp}(h_2) \cap \Sigma^\gamma_{h_1}(p) = \emptyset$, and $\rho^{0\gamma}_{h_2 \circ h_1} = h_2 \rho^{0\gamma}_{h_1}$ is the identity map on a small neighborhood V of p (see Remark 4.5). We also take $h_3 \in \mathcal{N}^{1/3}$ such that $\mathrm{supp}(h_3) \cap \Sigma^\gamma(p) = \emptyset$ and $\rho^{0\gamma}_{h_3 \circ h_2 \circ h_1}$ admits more than $a_{|0\gamma|}|0\gamma|$ attracting fixed points in V (see Remark 5.2).

Now $((h_3 \circ h_2 \circ h_1) \circ f_0, f_1, f_2)$ is contained in $\mathcal{U} \cap \mathcal{U}_n$. Since the choice of $\mathcal{U}$ is arbitrary, the set $\mathcal{U}_n$ is dense in $\mathcal{W}^r$. □

Let us prove Proposition 5.1. The proof is done by several inductive steps. The following notation will be used throughout this section. Let $\rho = (f_0, f_1, f_2) \in \mathcal{A}^\infty(\{0, 1, 2\})$ and $J \subset [0, 1]$ be a closed interval such that (f_1, f_2) is a blender on J. We assume that f_0 has a repeller-attractor pair (p, q) in Int J.

In the following first step, we create 1-flat periodic points which satisfy certain estimates on A and S. Recall that for a germ $F \in \mathcal{D}$, we denote $(S/A)(F) = S(F)/A(F)$.

Lemma 5.3 *In the above setting, assume that for a repeller-attractor pair (p, q) the repeller p is (f_1, f_2)-generic, $(\log f_0'(p))/(\log f_0'(q))$ is irrational, and the pair (p, q) has a heteroclinic point z_* with $\tau_A(z_*, f_0) \neq 0$ and $\tau_S(z_*, f_0) \neq 0$. Then, for any neighborhood $\mathcal{N}$ of the identity map in* $\mathrm{Diff}^\infty([0, 1])$ *and any finite subset Λ of $[0, 1]$ and $\nu > 0$, there exist $h \in \mathcal{N}$, an (f_1, f_2)-generic point $\hat{p} \in J \setminus \Lambda$, and $\gamma \in \{0, 1, 2\}^*$ such that $\hat{p}$ is a 1-flat 0γ-periodic point for ρ_h, $(\mathrm{supp}(h) \cup \Sigma_h^\gamma(\hat{p})) \cap \Lambda = \emptyset$, the sign of the germ $[\rho_h^{0\gamma}]_{\hat{p}}$ is $(\tau_A(z_*, f_0), \tau_S(z_*, f_0))$, and $|(S/A)([\rho_h^{0\gamma}]_{\hat{p}})| > \nu$.*

We remark that the finite set Λ in the lemma can contain p and q.

Proof We may assume that $\Lambda \cap \{f_0^i(z_*) \mid i \in \mathbb{Z}\} = \emptyset$ by replacing z_* with a nearby point if necessary.

We denote

$$X = f_0^{-1}(\Lambda) \cup \Lambda \cup f_0(\Lambda) \cup [p, q],$$

where $[p, q]$ denotes the closed interval whose end points are p and q (note that we do not assume p is to the left of q). Notice that $\{f^i(z_*) \mid i \in \mathbb{Z}\} \subset [p, q]$. Since Int J contains a repelling fixed point p of f_0, we see that $[\mathrm{Int}(J) \cap f_0(\mathrm{Int}(J) \cap f_0^{-1}(\mathrm{Int}(J))] \setminus X \neq \emptyset$. Take a point $\hat{p}$ in this set such that $\hat{p}$ is (f_1, f_2)-generic and $f_0(\hat{p}) \neq \hat{p}$. Remark that the three points $\hat{p}$, $f_0(\hat{p})$, and $f_0^{-1}(\hat{p})$ are mutually distinct (since f_0 is orientation-preserving). Also, notice that by construction $\{\hat{p}, f_0(\hat{p}), f_0^{-1}(\hat{p})\} \cap (\Lambda \cup [p, q]) = \emptyset$. We take an open neighborhood U of $\{\hat{p}, f_0(\hat{p}), f_0^{-1}(\hat{p})\}$ in $(\mathrm{Int}(J) \cap f_0(\mathrm{Int}(J)) \cap f_0^{-1}(\mathrm{Int}(J))) \setminus (\Lambda \cup [p, q])$. Then we take a neighborhood V of $(f_0(\hat{p}), \hat{p})$ in $U \times U$, a real number $\epsilon > 0$, and a continuous family $(h_{s,v})_{s \in (-\epsilon, \epsilon), v \in V}$ in $\mathcal{N}$ such that

1. $\mathrm{supp}(h_{s,(x,y)}) \subset U$,
2. $h_{s,(x,y)}(f_0(\hat{p})) = x$, $(h_{s,(x,y)})'(f_0(\hat{p})) = 1$,
3. $h_{s,(x,y)}(y) = \hat{p}$, and $(h_{s,(x,y)})'(y) = e^s$,

for any $s \in (-\epsilon, \epsilon)$ and $(x, y) \in V$. Since, by construction, $f_0(\hat{p})$, $f_0^{-1}(\hat{p}) \in J$ and p is (f_1, f_2)-generic, there exist $\eta, \eta' \in \{1, 2\}^*$ such that $\bar{v} = ((\rho^\eta)^{-1}(p), f_0 \circ \rho^{\eta'}(q))$ is contained in V. Let $\gamma_{m,n} = 0\eta' 0^{m+n} \eta 0$ and

$$v_{m,n} = ((\rho^\eta)^{-1}(f_0^{-m}(z_*)), f_0 \circ \rho^{\eta'}(f_0^n(z_*)))$$

for $m, n \geq 1$. Since $f_0^n(z_*)$ converges to q and $f_0^{-n}(z_*)$ converges to p as $n \to \infty$, $v_{m,n}$ converges to $\bar{v}$ as $m, n \to \infty$. Fix $N \geq 1$ such that $v_{m,n} \in V$ for any $m, n \geq N$. For any $m, n \geq N$, we have

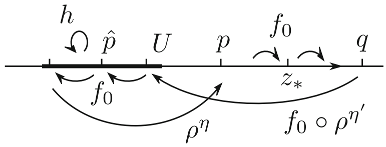

Fig. 1 Proof of Lemma 5.3

$$\rho^{\gamma_{m,n}}_{h_{s,v_{m,n}}} = (h_{s,v_{m,n}} \circ f_0) \circ \rho^{\eta'} \circ f_0^{m+n} \circ \rho^\eta \circ (h_{s,v_{m,n}} \circ f_0) \tag{1}$$

in a small neighborhood of $\hat{p}$. In particular, $\rho^{\gamma_{m,n}}_{h_{s,v_{m,n}}}(\hat{p}) = \hat{p}$; see Fig. 1.

We will show that $\gamma = \gamma_{m,n}$ and $h = h_{s,v_{m,n}}$ satisfy the required properties at $\hat{p}$ for suitably chosen m, n and s. Let $\varphi : W^u(p) \to \mathbb{R}$ and $\psi : W^s(q) \to \mathbb{R}$ be linearizations of f_0 at p and q respectively. Notice that since $\operatorname{supp}(h_{s,(x,y)}) \subset U$, the perturbation by $h_{s,(x,y)}$ does not change the behavior of φ and ψ on (p, q). Let $\lambda_p = f_0'(p)$, $\lambda_q = f_0'(q)$, and define a continuous function

$$c(x, y) = (f_0 \circ \rho^{\eta'} \circ \psi^{-1})'((f_0 \circ \rho^{\eta'} \circ \psi^{-1})^{-1}(y)) \cdot (\varphi \circ \rho^\eta)'(x) \cdot f_0'(\hat{p})$$

on V. By equality (1), notice that the following holds:

$$\left(\rho^{\gamma_{m,n}}_{h_{s,v_{m,n}}}\right)'(\hat{p}) = e^s \cdot \lambda_p^m \cdot \lambda_q^n \cdot c(v_{m,n}) \cdot (\psi \circ \varphi^{-1})'(\varphi(z_*)).$$

for any $m, n \geq N$. Since $\lambda_p > 1 > \lambda_q$ and the ratio $\log \lambda_p / \log \lambda_q$ is irrational, there exist increasing sequences $(m_k)_{k \geq 1}$ and $(n_k)_{k \geq 1}$ such that

$$\lim_{k \to \infty} \lambda_p^{m_k} \cdot \lambda_q^{n_k} \cdot c(\bar{v}) \cdot (\psi \circ \varphi^{-1})'(\varphi(z_*)) = 1.$$

By the continuity of the function c, the sequence $c(v_{m_k,n_k})$ converges to $c(\bar{v})$ as $k \to \infty$. Hence, we can choose a converging to zero sequence of real numbers $(s_k)_{k \geq 1}$ such that $|s_k| < \epsilon$ and

$$\left(\rho^{\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}\right)'(\hat{p}) = e^{s_k} \cdot \lambda_p^{m_k} \cdot \lambda_q^{n_k} \cdot c(v_{m_k,n_k}) \cdot (\psi \circ \varphi^{-1})'(\varphi(z_*)) = 1$$

for all large k.

Let us estimate $A([\rho_h^{0\gamma}]_{\hat{p}})$ and $S([\rho_h^{0\gamma}]_{\hat{p}})$ for $(\gamma, h) = (\gamma_{m_k,n_k}, h_{s_k,v_{m_k,n_k}})$. Let

$$F_k = [\varphi \circ \rho^\eta \circ (h_{s_k,v_{m_k,n_k}} \circ f_0)]_{\hat{p}},$$
$$G_k = [(h_{s_k,v_{m_k,n_k}} \circ f_0) \circ \rho^{\eta'} \circ \psi^{-1}]_{\psi \circ f_0^{n_k}(z_*)}.$$

Then, F_k and G_k converge to $[\varphi \circ \rho^\eta \circ (h_{0,\bar{v}} \circ f_0)]_{\hat{p}}$ and $[(h_{0,\bar{v}} \circ f_0) \circ \rho^{\eta'} \circ \psi^{-1}]_{\psi(q)}$ respectively, and

$$\left[\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}\right]_{\hat{p}} = G_k \circ L^{n_k}_{\lambda_q} \circ [\psi \circ \varphi^{-1}]_{\varphi(z_*)} \circ L^{m_k}_{\lambda_p} \circ F_k,$$

where L_λ is the germ of the map $x \mapsto \lambda x$ at 0. By the cocycle property of $A(\cdot)$ and the equality $(\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}})'(\hat{p}) = 1$, together with the obvious relation $A(L_\lambda) = 0$, we have

$$A\left(\left[\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}\right]_{\hat{p}}\right) = A(G_k) \cdot ((G_k)')^{-1} + A([\psi \circ \varphi^{-1}]_{\varphi(z_*)}) \cdot \lambda_p^{m_k} (F_k)' + A(F_k).$$

This implies that

$$\lim_{k\to\infty} \lambda_p^{-m_k} \cdot A\left(\left[\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}\right]_{\hat{p}}\right) = A([\psi \circ \varphi^{-1}]_{\varphi(z_*)}).$$

Similarly, we have

$$\lim_{k\to\infty} \lambda_p^{-2m_k} \cdot S\left(\left[\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}\right]_{\hat{p}}\right) = S([\psi \circ \varphi^{-1}]_{\varphi(z_*)}).$$

Therefore, the germ $[\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}]_{\hat{p}}$ has the same sign as $[\psi \circ \varphi^{-1}]_{\varphi(z_*)}$ and $|(S/A)[\rho^{0\gamma_{m_k,n_k}}_{h_{s_k,v_{m_k,n_k}}}]_{\hat{p}}| > \nu$ for all large k.

Thus we have completed the construction of $h = h_{s_k,v_{m_k,n_k}}$ and $\hat{p}$ having all the desired properties. □

Remark 5.4 In the proof of Lemma 5.3, we can assume that the length of γ is arbitrarily long. Indeed, we only need to choose one which corresponds to a large k.

On the second step, we construct a 2-flat periodic point from five 1-flat periodic points.

Lemma 5.5 *Suppose that there exist mutually distinct (f_1, f_2)-generic points $p_1, \ldots, p_5$ in J and $\gamma_1, \ldots, \gamma_5 \in \{0, 1, 2\}^*$ such that*

1. *each p_i is a 1-flat $0\gamma_i$-periodic point,*
2. *$\Sigma^{\gamma_1}(p_1), \ldots, \Sigma^{\gamma_5}(p_5)$ and $\{p_1, \ldots, p_5\}$ are mutually disjoint,*
3. *$[\rho^{0\gamma_4}]_{p_4}$ and $[\rho^{0\gamma_5}]_{p_5}$ have opposite signs, and*

$$|(S/A)([\rho^{0\gamma_4}]_{p_4})| > |(S/A)([\rho^{0\gamma_5}]_{p_5})|.$$

Then, for any neighborhood V of $\{p_1, \ldots, p_5\}$, any neighborhood $\mathcal{N}$ of the identity map in $\mathrm{Diff}^\infty([0, 1])$*, and any non-empty open subset U of $J \cap f_0(J)$, there exist an (f_1, f_2)-generic point $\hat{p} \in U$, $\hat{\gamma} \in \{0, 1, 2\}^*$, and $h \in \mathcal{N}$ such that* $\mathrm{supp}(h) \subset U \cup V$*, $\hat{p}$ is a 2-flat $0\hat{\gamma}$-periodic point for ρ_h with $\hat{p} \notin \Sigma_h^{\hat{\gamma}}(\hat{p})$, $\Sigma_h^{\hat{\gamma}}(\hat{p}) \subset \bigcup_{i=1}^5 \Sigma^{\gamma_i}(p_i) \cup U \cup V$, and* $\mathrm{sgn}(S([\rho_h^{0\hat{\gamma}}]_{\hat{p}})) = \mathrm{sgn}(S([\rho^{0\gamma_4}]_{p_4}))$.

Proof Without loss of generality, we may assume that U, V and $\bigcup_{i=1}^{5} \Sigma^{\gamma_i}(p_i)$ are mutually disjoint.

We prepare several regions for the construction of the periodic orbit. Take a neighborhood $\mathcal{N}^{1/4}$ of the identity in $\mathrm{Diff}^{\infty}([0, 1])$ such that any composition of four diffeomorphisms in $\mathcal{N}^{1/4}$ belongs to $\mathcal{N}$. We take a neighborhood V_i of p_i for each $i = 1, \ldots, 5$ such that $V_1, \ldots, V_5$ are mutually disjoint subsets of V. Then, we take non-empty, mutually disjoint open sets U_i $(i = 0, 1, 2, 3)$ in U such that $f_0^{-1}(U_0)$ is also disjoint from U_i $(i = 1, \ldots, 3)$ and V_i $(i = 1, \ldots, 5)$. It is not difficult to choose such U_i and V_i, so we leave it to the reader. Finally, we take an (f_1, f_2)-generic point $\bar{p}$ in U_0 such that $f_0^{-1}(\bar{p})$ is also (f_1, f_2)-generic. One can check that such a choice is indeed possible by using the fact that being an (f_1, f_2)-generic point is a generic property in J.

Now we construct the periodic orbit by using our connecting lemmas. First, by Lemma 4.3, with $\bar{p}$, p_4, p_1, U_1 and V_1 corresponding to p, p', $\hat{p}$, U and V respectively and F being the identity germ, we obtain $\omega_1 \in \{0, 1, 2\}^*$ and $h_1 \in \mathcal{N}^{1/4}$ such that $\mathrm{supp}(h_1) \subset (U_1 \cup V_1)$, $\rho_{h_1}^{\omega_1}(\bar{p}) = p_4$, and $H_1 = [\rho_{h_1}^{\omega_1}]_{\bar{p}}$ is 1-flat. Similarly, we apply Lemma 4.3 to $(p_4, p_5, p_2, U_2, V_2)$ and, again, letting F be the identity germ, and to $(p_5, f_0^{-1}(\bar{p}), p_3, U_3, V_3)$ and $F = ([f_0]_{f_0^{-1}(\bar{p})})^{-1}$, and obtain that there exist $\omega_2, \omega_3 \in \{0, 1, 2\}^*$ and $h_2, h_3 \in \mathcal{N}^{1/4}$ such that

1. $\mathrm{supp}(h_i) \subset (U_i \cup V_i)$ for $i = 2, 3$,
2. $\rho_{h_2}^{\omega_2}(p_4) = p_5$, $H_2 = [\rho_{h_2}^{\omega_2}]_{p_4}$ is 1-flat, and
3. $\rho_{h_3}^{\omega_3}(p_5) = f_0^{-1}(\bar{p})$, $H_3 = [f_0]_{f_0^{-1}(\bar{p})} \circ [\rho_{h_3}^{\omega_3}]_{p_5}$ is 1-flat.

Let $h_\flat = h_1 \circ h_2 \circ h_3$ and $\gamma_{m,n} = \omega_3(0\gamma_5)^m \omega_2(0\gamma_4)^n \omega_1$ for $m, n \geq 1$. Notice that, by construction, $\bar{p}$ is a 1-flat $0\gamma_{m,n}$-periodic point of $\rho_{h_\flat}$. We show that by adding a further perturbation and choosing m, n appropriately, we can obtain the 2-flatness and the condition on the Schwarzian derivative.

Let $\alpha = \sum_{i=1}^{3} A(H_i)$ and $\beta = \sum_{i=1}^{3} S(H_i)$. By Lemma 3.5 (where we let $F_1 = [\rho^{0\gamma_4}]_{p_4}$ and $F_2 = [\rho^{0\gamma_5}]_{p_5}$), together with Remark 3.8, we take $h_4 \in \mathcal{N}^{1/4}$ and $m, n \geq 1$ such that $\mathrm{supp}(h_4) \subset V_4$, $h_4(p_4) = p_4$, $[h_4]_{p_4}$ is 1-flat, and the following holds:

$$A([\rho^{0\gamma_5}]_{p_5}^m) + A(([h_4]_{p_4} \circ [\rho^{0\gamma_4}]_{p_4})^n) + \alpha = 0.$$
$$S([\rho^{0\gamma_4}]_{p_4}) \cdot \left\{ S([\rho^{0\gamma_5}]_{p_5}^m) + S(([h_4]_{p_4} \circ [\rho^{0\gamma_4}]_{p_4})^n) + \beta \right\} > 0.$$

Let $h = h_1 \circ h_2 \circ h_3 \circ h_4 = h_\flat \circ h_4$. This is a map in $\mathcal{N}$ such that $\mathrm{supp}(h) \subset (\bigcup_{i=1}^{3} U_i \cup \bigcup_{i=1}^{4} V_i)$. Since $U_1, U_2, U_3, V_1, \ldots, V_5$ are mutually disjoint and they do not intersects with $\{\bar{p}, f_0^{-1}(\bar{p})\} \cup \bigcup_{i=1}^{5} \Sigma^{\gamma_i}(p_i)$, we have

$$\begin{aligned}[\rho_h^{0\gamma_{m,n}}]_{\bar{p}} &= [\rho_{h_3}^{0\omega_3}]_{p_5} \circ [\rho^{0\gamma_5}]_{p_5}^m \circ [\rho_{h_2}^{\omega_2}]_{p_4} \circ [\rho_{h_4}^{0\gamma_4}]_{p_4}^n \circ [\rho_{h_1}^{\omega_1}]_{\hat{p}} \\ &= H_3 \circ [\rho^{0\gamma_5}]_{p_5}^m \circ H_2 \circ [\rho_{h_4}^{0\gamma_4}]_{p_4}^n \circ H_1.\end{aligned}$$

This, together with the 1-flatness of each germ, implies that

$$A([\rho_h^{0\gamma_{m,n}}]_{\bar{p}}) = A([\rho^{0\gamma_5}]_{p_5}^m) + A(([h_4]_{p_4} \circ [\rho^{0\gamma_4}]_{p_4})^n) + \alpha,$$
$$S([\rho_h^{0\gamma_{m,n}}]_{\bar{p}}) = S([\rho^{0\gamma_5}]_{p_5}^m) + S(([h_4]_{p_4} \circ [\rho^{0\gamma_4}]_{p_4})^n) + \beta.$$

Therefore, $A([\rho_h^{0\gamma_{m,n}}]_{\bar{p}}) = 0$ and $S([\rho_h^{0\gamma_{m,n}}]_{\bar{p}}) \cdot S([\rho^{0\gamma_4}]_{p_4}) > 0$. By construction, one can see that $\Sigma_h^{\gamma_{m,n}}(\bar{p}) \subset (\bigcup_{i=1}^3 U_i \cup \bigcup_{i=1}^5 (\Sigma^{\gamma_i}(p_i) \cup V_i))$. In particular, $\bar{p} \notin \Sigma_h^{\gamma_{m,n}}(\bar{p})$ and $\Sigma_h^{\gamma_{m,n}}(\bar{p}) \subset \bigcup_{i=1}^5 \Sigma^{\gamma_i}(p_i) \cup U \cup V$.

Thus we have constructed the desired $\hat{p} = \bar{p}$, $\hat{\gamma} = \gamma_{m,n}$ and h. □

The following lemma is the third step.

Lemma 5.6 *Suppose that there exist mutually distinct (f_1, f_2)-generic points $p_1, \dots, p_5$ in J and $\gamma_1, \dots, \gamma_5 \in \{0, 1, 2\}^*$ such that*

1. *each p_i is a 2-flat $0\gamma_i$-periodic point,*
2. *$\Sigma^{\gamma_1}(p_1), \dots, \Sigma^{\gamma_5}(p_5)$ and $\{p_1, \dots, p_5\}$ are mutually disjoint,*
3. *$S([\rho^{0\gamma_4}]_{p_4}) \cdot S([\rho^{0\gamma_5}]_{p_5}) < 0$.*

Then, for any neighborhood V of $\{p_1, \dots, p_5\}$, any neighborhood $\mathcal{N}$ of the identity map in $\mathrm{Diff}^\infty([0, 1])$*, and any non-empty open subset U of $J \cap f_0(J)$, there exist an (f_1, f_2)-generic point $\hat{p} \in U$, $\gamma \in \{0, 1, 2\}^*$, and $h \in \mathcal{N}$ such that* $\mathrm{supp}(h) \subset U \cup V$*, $\hat{p}$ is a 3-flat 0γ-periodic point for ρ_h with $\hat{p} \notin \Sigma^\gamma(\hat{p})$, and $\Sigma^\gamma(\hat{p}) \subset \bigcup_{i=1}^5 \Sigma^{\gamma_i}(p_i) \cup U \cup V$.*

Proof The proof is done similarly to the proof of Lemma 5.5, with the use of Lemma 3.6 instead of Lemma 3.5.

Without loss of generality, we may assume that U, V and $\bigcup_{i=1}^5 \Sigma^{\gamma_i}(p_i)$ are mutually disjoint. Take a neighborhood $\mathcal{N}^{1/4}$ of the identity in $\mathrm{Diff}^\infty([0, 1])$ such that any composition of four diffeomorphisms in $\mathcal{N}^{1/4}$ belongs to $\mathcal{N}$. We fix open neighborhoods V_i of p_i for each $i = 1, \dots, 5$ such that $V_1, \dots, V_5$ are mutually disjoint subsets of V. We also take mutually disjoint non-empty open sets U_i $(i = 0, 1, 2, 3)$, such that $f_0^{-1}(U_0)$ is also disjoint from U_i $(i = 0, 1, 2, 3)$ and V_i $(i = 1, \dots 5)$. Finally, we take an (f_1, f_2)-generic point $\bar{p} \in U_0$ such that $f_0^{-1}(\bar{p})$ is also (f_1, f_2)-generic.

Then, by applying Lemma 4.3 with $r_0 = 2$ in the same way as Lemma 5.5, we obtain $\omega_1, \omega_2, \omega_3 \in \{0, 1, 2\}^*$, $h_1, h_2, h_3 \in \mathcal{N}^{1/4}$ such that

1. $\mathrm{supp}(h_i) \subset (U_i \cup V_i)$ for $i = 1, 2, 3$,
2. $\rho_{h_1}^{\omega_1}(\bar{p}) = p_4$, $H_1 = [\rho_{h_1}^{\omega_1}]_{\bar{p}}$ is 2-flat,
3. $\rho_{h_2}^{\omega_2}(p_4) = p_5$, $H_2 = [\rho_{h_2}^{\omega_2}]_{p_2}$ is 2-flat, and
4. $\rho_{h_3}^{\omega_3}(p_5) = f_0^{-1}(\bar{p})$, $H_3 = [f_0]_{f_0^{-1}(\bar{p})} \circ [\rho_{h_3}^{\omega_3}]_{p_2}$ is 2-flat.

Since each H_i is 2-flat, there exists $\alpha_0 \in \mathbb{R}$ such that

$$H_3 \circ H_2 \circ H_1(x) = x + \alpha_0 x^3 + o(x^3).$$

By Lemma 3.6 for $r = 2$ and $\alpha = \alpha_0$ (notice that since p_i is 2-flat, our third assumption implies $[\rho^{0\gamma_4}]_{p_4}^{(3)} \cdot [\rho^{0\gamma_5}]_{p_5}^{(3)} < 0$), there exist $h_4 \in \mathcal{N}^{1/4}$ and $m, n \geq 1$ such that $\mathrm{supp}(h_4) \subset V_4$, $h_4(p_4) = p_4$, $[h_4]_{p_4}$ is 2-flat, and

$$[\rho^{0\gamma_5}]_{p_5}^m \circ ([h_4]_{p_4} \circ [\rho^{0\gamma_4}]_{p_4})^n(x) = x - \alpha_0 x^3 + o(x^3).$$

Let $h = h_4 \circ h_3 \circ h_2 \circ h_1$. This is a map in $\mathcal{N}$ such that $\mathrm{supp}(h) \subset (\bigcup_{i=1}^3 U_i \cup \bigcup_{i=1}^4 V_i)$. We also put $\gamma_{m,n} = \omega_3(0\gamma_5)^m \omega_2(0\gamma_4)^n \omega_1$. Since $V_1, \dots, V_5$ are mutually disjoint and they do not intersects with $\{\bar{p}, f_0^{-1}(\bar{p})\} \cup \bigcup_{i=1}^5 \Sigma^{\gamma_i}(p_i)$, we have

$$\begin{aligned}[\rho_h^{0\gamma_{m,n}}]_{\bar{p}} &= [\rho_{h_3}^{0\omega_3}]_{p_5} \circ [\rho^{0\gamma_5}]_{p_5}^m \circ [\rho_{h_2}^{\omega_2}]_{p_4} \circ [\rho_{h_4}^{0\gamma_4}]_{p_4}^n \circ [\rho_{h_1}^{\omega_1}]_{\bar{p}} \\ &= H_3 \circ [\rho^{0\gamma_5}]_{p_5}^m \circ H_2 \circ ([h]_{p_4} \circ [\rho^{0\gamma_4}]_{p_4})^n \circ H_1.\end{aligned}$$

Since any 2-flat germs commute with each other modulo $o(x^3)$, this implies that the germ $[\rho_h^{0\gamma_{m,n}}]_{\bar{p}}$ is 3-flat. As in the proof of Lemma 5.5, by construction we can check $\bar{p} \notin \Sigma_h^{\gamma_{m,n}}(\bar{p})$ and $\Sigma_h^{\gamma_{m,n}}(\bar{p}) \subset \bigcup_{i=1}^5 \Sigma^{\gamma_i}(p_i) \cup U \cup V$.

Thus $\hat{p} = \bar{p}$ is the desired 3-flat $(0\gamma_{m,n})$-periodic point for ρ_h. □

The fourth step is the following

Lemma 5.7 *Suppose that $3 \le r < \infty$ and there exist mutually distinct (f_1, f_2)-generic points $p_1, \dots, p_9$ in J and $\gamma_1, \dots, \gamma_9 \in \{0, 1, 2\}^*$ such that*

1. *each p_i is an r-flat $0\gamma_i$-periodic point, and*
2. *$\Sigma^{\gamma_1}(p_1), \dots, \Sigma^{\gamma_9}(p_9)$, and $\{p_1, \dots, p_9\}$ are mutually disjoint.*

Then, for any neighborhood V of $\{p_1, \dots, p_9\}$, any neighborhood $\mathcal{N}$ of the identity map in $\mathrm{Diff}^\infty([0, 1])$, and any non-empty open subset U of $J \cap f_0(J)$, there exist an (f_1, f_2)-generic point $\hat{p} \in U$, $\gamma \in \{0, 1, 2\}^*$, and $h \in \mathcal{N}$ such that $\mathrm{supp}(h) \subset U \cup V$, $\hat{p}$ is an $(r+1)$-flat 0γ-periodic point for ρ_h with $\hat{p} \notin \Sigma^\gamma(\hat{p})$, $\Sigma^\gamma(\hat{p}) \subset \bigcup_{i=1}^9 \Sigma^{\gamma_i}(p_i) \cup U \cup V$.

Proof One proves this in the same way as Lemma 5.6, using Lemma 3.7 instead of Lemma 3.6. Hence we omit the details. □

Now we finish the proof of Proposition 5.1. The case $r = \infty$ is reduced to the case $r < \infty$. So, first we consider the case $r < \infty$.

Proof of Proposition 5.1 for $r < \infty$ Take $\rho = (f_0, f_1, f_2) \in \mathcal{W}^\infty$. Let J be the closed subinterval of $[0, 1]$ on which (f_1, f_2) is a blender, $(\bar{p}_1, \bar{q}_1), \dots, (\bar{p}_4, \bar{q}_4)$ be repeller-attractor pairs in $\mathrm{Int}\, J$, $z_i \in W^u(\bar{p}_i) \cap W^s(\bar{q}_i)$ $(i = 1, 2, 3, 4)$ be heteroclinic points such that $\tau_A(z_1, f_0) > 0 > \tau_A(z_2, f_0)$ and $\tau_S(z_3, f_0) > 0 > \tau_S(z_4, f_0)$. By perturbing f_0 if it is necessary, we may assume that $\bar{p}_i$ is (f_1, f_2)-generic, $(\log f_0'(\bar{p}_i))/(\log f_0'(\bar{q}_i))$ are irrational, $\tau_A(z_i, f_0) \neq 0$, and $\tau_S(z_i, f_0) \neq 0$ for $i = 1, \dots, 4$. By an elementary combinatorial argument,[1] there exists a pair (i_1, i_2) in $\{1, 2, 3, 4\}$ such that the signs of z_{i_1} and z_{i_2} are opposite.

First, we show how we construct a 3-flat periodic point. Fix a neighborhood $\mathcal{N}$ of the identity map in $\mathrm{Diff}^\infty([0, 1])$. Take a smaller neighborhood $\mathcal{N}^{1/3}$ of the identity such that any composition of three maps in $\mathcal{N}^{1/3}$ belongs to $\mathcal{N}$. Applying Lemma

[1] If $\tau_S(z_1, f_0) \neq \tau_S(z_2)$, then choose $(i_1, i_2) = (1, 2)$. If $\tau_S(z_1, f_0) = \tau_S(z_2) = +1$ and $\tau_A(z_4, f_0) = +1$, then choose $(i_1, i_2) = (2, 4)$. Other cases are similar.

5.3 repeatedly to the heteroclinic points z_{i_1} and z_{i_2}, we obtain a diffeomorphism $h_1 \in \mathcal{N}^{1/3}$, (f_1, f_2)-generic points $p_1, \ldots, p_{25} \in J$, and $\gamma_1, \ldots, \gamma_{25} \in \{0, 1, 2\}^*$ such that p_i is a 1-flat $0\gamma_i$-periodic point satisfying the following:

- $\Sigma_{h_1}^{\gamma_1}(p_1), \ldots, \Sigma_{h_1}^{\gamma_{25}}(p_{25})$, and $\{p_1, \ldots, p_{25}\}$ are mutually disjoint,
- $[\rho_{h_1}^{0\gamma_{5j-1}}]_{p_{19}}$ and $[\rho_{h_1}^{0\gamma_{5j}}]_{p_{24}}$ have opposite signs and $|(S/A)([\rho_{h_1}^{0\gamma_{5j-1}}]_{p_{5j-1}})| > |(S/A)([\rho_{h_1}^{0\gamma_{5j}}]_{p_{5j}})|$ for any $j = 1, \ldots, 5$,
- $S([\rho_{h_1}^{0\gamma_{19}}]_{p_{19}}) > 0 > S([\rho_{h_1}^{0\gamma_{24}}]_{p_{24}})$.

Notice that we may apply Lemma 5.3 repeatedly in such a way that the perturbations do not interfere with each other, since Lemma 5.3 allows us to localize the support of perturbations away from a given finite set.

Then, by applying Lemma 5.5 to each quintuple of 1-flat periodic points $(p_{5j-4}, \ldots, p_{5j})$ of ρ_{h_1} $(j = 1, \ldots, 5)$, we take $h_2 \in \mathcal{N}^{1/3}$, $p'_j \in J$, and $\gamma'_j \in \{0, 1, 2\}^*$ such that p'_j is an (f_1, f_2)-generic, 2-flat $0\gamma'_j$-periodic point of $\rho_{h_2 \circ h_1}$ for $j = 1, \ldots 5$, $\Sigma_{h_2 \circ h_1}^{\gamma'_1}(p'_1), \ldots, \Sigma_{h_2 \circ h_1}^{\gamma'_5}(p'_5)$, and $\{p'_1, \ldots, p'_5\}$ are mutually disjoint, and

$$S\left(\left[\rho_{h_2 \circ h_1}^{0\gamma'_4}\right]_{p'_4}\right) > 0 > S\left(\left[\rho_{h_2 \circ h_1}^{0\gamma'_5}\right]_{p'_5}\right).$$

Then, by applying Lemma 5.6 to $(p'_1, \ldots, p'_5)$, we take a diffeomorphism $h_3 \in \mathcal{N}^{1/3}$, an (f_1, f_2)-generic point $\hat{p}$ in J, and $\hat{\gamma} \in \{0, 1, 2\}^*$ such that $\hat{p}$ is a 3-flat $0\hat{\gamma}$-periodic point of $\rho_{h_3 \circ h_2 \circ h_1}$ and $\hat{p} \notin \Sigma_h^{\hat{\gamma}}(\hat{p})$. Thus we have constructed a 3-flat 0γ-periodic point $\hat{p}$ of ρ_h, where $h = h_3 \circ h_2 \circ h_1$ is a diffeomorphism in $\mathcal{N}$. Notice that the length of γ can be taken arbitrarily large, since by Remark 5.4 we can assume that the lengths of the 1-flat periodic points produced in the first step are arbitrarily large.

In a similar way, for $r \geq 3$, by Lemma 5.7 we can construct an $(r + 1)$-flat periodic point with an arbitrary large period from nine r-flat periodic points by a small perturbation. Hence, we obtain N of r-flat periodic points with an arbitrary large period starting with $25 \cdot 9^{r-3} N$ of 1-flat periodic points. □

Finally, let us consider the case $r = \infty$.

Proof of Proposition 5.1 for $r = \infty$ For simplicity we only consider the case $N = 1$ (the proof for the general case is done similarly). First, as in the proof of Lemma 4.6, given a neighborhood $\mathcal{N} \subset \mathrm{Diff}^\infty([0, 1])$ of the identity map, there exist $s > 1$ and a C^s-neighborhood $\mathcal{N}^s \subset \mathrm{Diff}^s([0, 1])$ of the identity map such that $\mathcal{N}^s \cap \mathrm{Diff}^\infty([0, 1]) \subset \mathcal{N}$.

For $\rho \in \mathcal{W}^\infty$ in the assumption of the proposition, we apply the already proven result for finite r with $r = s$. This gives us $h \in \mathcal{N}$, $\hat{p} \in J$, and $\gamma \in \{0, 1, 2\}^*$ such that $\hat{p}$ is an s-flat 0γ-periodic point for ρ_h such that $\hat{p} \notin \Sigma^\gamma(\hat{p})$. Now, by Remark 4.5, we take a C^∞-diffeomorphism g which is C^s-close to the identity such that $[g]_{\hat{p}} \circ [\rho_h^{0\gamma}]_{\hat{p}}$ is the identity germ and $\mathrm{supp}(g) \cap \Sigma_h^\gamma(\hat{p}) = \emptyset$. By choosing g sufficiently C^s-close to the identity, we can assume that $g \circ h \in \mathcal{N}^s$. Accordingly, we have $g \circ h \in \mathcal{N}$. Now, $\hat{p}$ is an ∞-flat 0γ-periodic point for $\rho_{g \circ h}$. Thus the proof is completed. □

6 Universal semigroups

In this section, we prove the following proposition which, together with Proposition 5.1, implies Theorem 2.2.

Proposition 6.1 *Let* $\rho = (f_0, f_1, f_2)$ *be an element of* $\mathcal{A}^\infty(\{0, 1, 2\})$ *and* $(\theta_1, \dots, \theta_N)$ *be in* $(\mathcal{E}^\infty)^N$. *Suppose that there exist distinct* (f_1, f_2)*-generic points* $p_1, \dots, p_{4N}$ *and words* $\gamma_1, \dots, \gamma_{4N} \in \{0, 1, 2\}^*$ *such that* p_i *is an* ∞*-flat* $0\gamma_i$*-periodic point for each* $i = 1, \dots, 4N$, *and the sets* $\{p_1, \dots, p_{4N}\}$, $\Sigma^{\gamma_1}(p_1)$, ..., $\Sigma^{\gamma_{4N}}(p_{4N})$ *are mutually disjoint. Then, for any neighborhood* $\mathcal{N}$ *of the identity in* $\mathrm{Diff}^\infty([0, 1])$, *there exist a map* $h \in \mathcal{N}$, *a closed interval* $I \subset [0, 1]$, *an affine diffeomorphism* $\Phi : [0, 1] \to I$, *and words* $\omega_1, \dots, \omega_N \in \{0, 1, 2\}^*$ *such that* $\Phi \circ \theta_k = \rho_h^{\omega_k} \circ \Phi$ *for every* $k = 1, \dots, N$.

First, let us derive Theorem 2.2 from Proposition 6.1.

Proof of Theorem 2.2 from Proposition 6.1 Fix $1 \le r \le \infty$. The space $(\mathcal{E}^r)^N$ admits a countable open basis $(\mathcal{O}_n)_{n=1}^\infty$. Let $\mathcal{U}_n$ be the set consisting of $\rho \in \mathcal{W}^r$ which realize the semigroup action generated by some elements of $\mathcal{O}_n$. The set $\mathcal{U}_n$ is an open subset of $\mathcal{W}^r$ and any element in $\bigcap_{n \ge 1} \mathcal{U}_n$ generates a universal semigroup. Hence, it is sufficient to show that every $\mathcal{U}_n$ is a dense subset of $\mathcal{W}^r$.

To see this, we fix a non-empty open subset $\mathcal{U}$ of $\mathcal{W}^r$ and an element $(\theta_1, \dots, \theta_N)$ of $\mathcal{O}_n \cap (\mathcal{E}^\infty)^N$ (notice that $\mathcal{O}_n \cap (\mathcal{E}^\infty)^N$ is dense in $\mathcal{O}_n$). By Proposition 5.1 and the density of $\mathcal{W}^\infty$ in $\mathcal{W}^r$, we take $\tilde{\rho} \in \mathcal{U} \cap \mathcal{A}^\infty(\{0, 1, 2\})$ which satisfies the hypothesis of Proposition 6.1. Then, by Proposition 6.1 there exist $\rho = (f_0, f_1, f_2) \in \mathcal{U} \cap \mathcal{A}^\infty(\{0, 1, 2\})$, a closed interval $I \subset [0, 1]$, an affine diffeomorphism $\Phi : [0, 1] \to I$, and words $\omega_1, \dots, \omega_N \in \{0, 1, 2\}^*$ such that $\Phi \circ \theta_k = (\rho^{\omega_k}|_I) \circ \Phi$ for every $k = 1, \dots, N$. This implies that ρ realizes the semigroup action generated by $(\theta_1, \dots, \theta_N)$. Therefore $\mathcal{U}$ intersects with $\mathcal{U}_n$. Since the choice of $\mathcal{U}$ is arbitrary, $\mathcal{U}_n$ is a dense subset of $\mathcal{W}^r$. □

Now we prove Proposition 6.1.

Proof of Proposition 6.1 Let $\mathcal{N}^\#$ be a neighborhood of the identity in $\mathrm{Diff}^\infty([0, 1])$ such that any composition of $6N$ maps in $\mathcal{N}^\#$ belongs to $\mathcal{N}$. Take a neighborhood V_i of p_i for each $i = 1, \dots, 4N$, non-empty open subsets $U_1, \dots, U_{4N}$, $W_1, \dots, W_{2N}$ of $J \cap f(J)$, and an (f_1, f_2)-generic point $\bar{p} \in J$ such that the sets

$$\{\bar{p}\},\ V_1, \dots, V_{4N},\ U_1, \dots, U_{4N},\ W_1, \dots, W_{2N},$$
$$f_0^{-1}(W_1), \dots, f_0^{-1}(W_{2N}),\ \bigcup_{i=1}^{4N} \Sigma^{\gamma_i}(p_i)$$

are mutually disjoint. It is not difficult to check that such choice is possible (the detail is left to the reader). Then choose $q_i \in W_i$ $(i = 1, \dots, 2N)$ so that q_i and $f_0^{-1}(q_i)$ are (f_1, f_2)-generic. For each $j = 1, \dots, 2N$, by applying Lemma 4.6 (for the neighborhood V_j, the open subset U_j, and the triple $(\bar{p}, f_0^{-1}(q_j), p_j)$ taken as $(p, p', \hat{p})$) we obtain $h_j \in \mathcal{N}^\#$ and $\eta_j \in \{0, 1, 2\}^*$ such that

- $\mathrm{supp}(h_j) \subset U_j \cup V_j$, $\Sigma_{h_j}^{\eta_j}(\bar{p}) \subset \Sigma^{\gamma_j}(p_j) \cup U_j \cup V_j$,

- $\rho_{h_j}^{\eta_j}(\bar{p}) = f_0^{-1}(q_j)$, and $[f_0]_{f_0^{-1}(q_j)} \circ [\rho_{h_j}^{\eta_j}]_{\bar{p}}$ is ∞-flat.

Similarly, for each $j = 1, \ldots, 2N$, by viewing $(q_j, \bar{p}, p_{2N+j})$ as $(p, p', \hat{p})$ in Lemma 4.6, we take $h_{2N+j} \in \mathcal{N}^{\#}$ and $\eta_{2N+j} \in \{0, 1, 2\}^*$ (for $j = 1, \ldots, 2N$) such that

- $\mathrm{supp}(h_{2N+j}) \subset U_{2N+j} \cup V_{2N+j}$, $\Sigma_{h_{2N+j}}^{\eta_{2N+j}}(q_j) \subset \Sigma^{\gamma_{2N+j}}(p_{2N+j}) \cup U_{2N+j} \cup V_{2N+j}$,
- $\rho_{h_{2N+j}}^{\eta_{2N+j}}(q_j) = \bar{p}$, and $[\rho_{h_{2N+j}}^{\eta_{2N+j}}]_{q_j}$ is ∞-flat.

Put $\bar{\omega}_j = \eta_{2N+j} 0 \eta_j$ for $j = 1, \ldots, 2N$. Since the germ $[\rho_{h_{2N+j}}^{\eta_{2N+j}}]_{q_j} \circ [\rho_{h_j}^{0\eta_j}]_{\bar{p}}$ is ∞-flat and $q_j \notin \Sigma_{h_{2N+j}}^{\eta_{2N+j}}(q_j) \cup \Sigma_{h_j}^{\eta_j}(\bar{p})$, there exists $\bar{h}_j \in \mathcal{N}^{\#}$ such that $\mathrm{supp}\, \bar{h}_j \subset W_j$, $\bar{h}_j(q_j) = q_j$, and

$$\left[\rho_{\hat{h}_j \circ h_{2N+j} \circ h_j}^{\bar{\omega}_j}\right]_{\bar{p}} = \left[\rho_{h_{2N+j}}^{\eta_{2N+j}}\right]_{q_j} \circ [\bar{h}_j]_{q_j} \circ \left[\rho_{h_j}^{0\eta_j}\right]_{\bar{p}}$$

is equal to the identity map as a germ. In particular, there exists an open interval I_0 containing $\bar{p}$ such that $\rho_{\bar{h}_j \circ h_{2N+j} \circ h_j}^{\bar{\omega}_j}(x) = x$ for all $x \in I_0$. Put $h_{\#} = h_1 \circ \cdots \circ h_{4N} \circ \bar{h}_1 \circ \cdots \circ \bar{h}_{2N}$. Remark that $h_{\#}$ is a composition of $6N$ maps in $\mathcal{N}^{\#}$, and hence, it belongs to $\mathcal{N}$. Now, $\rho_{h_{\#}}^{\bar{\omega}_j}$ is the identity map on I_0. Notice that this is also true for smaller intervals in I_0 containing $\bar{p}$. We shrink I_0 so that $\rho_{h_{\#}}^{0\eta_j}(I_0) \subset W_j$ for every $j = 1, \ldots, 2N$.

For maps $\theta_1, \ldots, \theta_N$ in the assumption of Proposition 6.1, we take their extensions $\bar{\theta}_1, \ldots, \bar{\theta}_N$ over $\mathbb{R}$ in such a way that each $\bar{\theta}_i$ has compact support. Now, for each $\bar{\theta}_i$ we apply Lemma 3.3 for $I = [0, 1]$ to obtain one-parameter groups $(\tilde{G}_1^t)_{t\in\mathbb{R}}, \ldots, (\tilde{G}_N^t)_{t\in\mathbb{R}}, (\tilde{H}_1^t)_{t\in\mathbb{R}}, \ldots, (\tilde{H}_N^t)_{t\in\mathbb{R}}$ of C^∞-diffeomorphisms of $\mathbb{R}$ having compact support and satisfying $\tilde{G}_i^1 \circ \tilde{H}_i^1|_{[0,1]} = \theta_i$ for every i. Let I_1 be a compact interval in $\mathbb{R}$ which contains $\mathrm{supp}(\tilde{G}_i^t)$ and $\mathrm{supp}(\tilde{H}_i^t)$ for every i.

Now we take an affine map $\Phi : \mathbb{R} \to \mathbb{R}$ satisfying $\Phi(I_1) = I_0$ and define one-parameter groups (G_i^t) and (H_i^t) $(i = 1, \ldots, N)$ of $\mathrm{Diff}^\infty(\mathbb{R})$ by $G_i^t = \Phi \circ \tilde{G}_i^t \circ \Phi^{-1}$ and $H_i^t = \Phi \circ \tilde{H}_i^t \circ \Phi^{-1}$. Notice that $\mathrm{supp}(G_i^t)$ and $\mathrm{supp}(H_i^t)$ are contained in I_0, hence their restrictions on $[0, 1]$ give diffeomorphisms of $[0, 1]$. Also, by definition we have $(G_i^1 \circ H_i^1) \circ \Phi = \Phi \circ \bar{\theta}_i = \Phi \circ \theta_i$ on $[0, 1]$ for every $i = 1, \ldots, N$. Take a continuous family $(\bar{h}^t)_{t\in\mathbb{R}}$ of diffeomorphism of $[0, 1]$ defined as follows: for $k = 1, \ldots, N$, we put

$$\bar{h}^t(x) = \begin{cases} \rho_{h_{\#}}^{0\eta_{2k-1}} \circ H_k^t \circ (\rho_{h_{\#}}^{0\eta_{2k-1}})^{-1}(x) & (x \in W_{2k-1}), \\ \rho_{h_{\#}}^{0\eta_{2k}} \circ G_k^t \circ (\rho_{h_{\#}}^{0\eta_{2k}})^{-1}(x) & (x \in W_{2k}), \end{cases}$$

and then extend them as the identity map outside W_j. Notice that this is a well-defined procedure because of the choice of I_0. Then, we have

$$\rho_{\bar{h}^t \circ h_{\#}}^{\bar{\omega}_{2k-1}} = \rho_{h\#}^{\eta_{2N+2k-1}} \circ \bar{h}^t \circ \rho_{h_{\#}}^{0\eta_{2k-1}} = H_k^t,$$

$$\rho^{\bar{\omega}_{2k}}_{\bar{h}^t \circ h_\#} = \rho^{\eta_{2N+2k}}_{h_\#} \circ \bar{h}^t \circ \rho^{0\eta_{2k}}_{h_\#} = G^t_k$$

on I_0. Hence, for any $m \geq 1$,

$$\rho^{\bar{\omega}^m_{2k}\bar{\omega}^m_{2k-1}}_{\bar{h}^{1/m} \circ h_\#} = G^1_k \circ H^1_k = \Phi \circ \theta_k \circ \Phi^{-1}$$

on I_0. If m is sufficiently large, $\bar{h}^{1/m} \circ h_\#$ is contained in $\mathcal{N}$. Therefore, $h = \bar{h}^{1/m} \circ h_\#$, $(\omega_k = \bar{\omega}^m_{2k}\bar{\omega}^m_{2k-1})^N_{k=1}$, and the map Φ satisfies the statement of the Proposition for sufficiently large m.

7 Wild behavior along generic infinite words

In this section we prove Theorem 2.3. We start with a general lemma on generic infinite words.

Lemma 7.1 *Let X be a Baire space, k be a positive integer, and $(X(\omega))_{\omega \in \{1,\ldots,k\}^*}$ be a family of open subsets of X. Suppose that $\bigcup_{\eta \in \{1,\ldots,k\}^*} X(\eta\omega)$ is dense in X for any $\omega \in \{1,\ldots,k\}^*$. Then, for generic $x \in X$, the set*

$$\{\underline{\omega} \in \{1,\ldots,k\}^\infty \mid x \in X(\underline{\omega}|_n) \textit{ for infinitely many} n' \text{s}\}$$

is a residual subset of $\{1,\ldots,k\}^\infty$.

Proof By the assumption, the set

$$\bigcap_{\omega \in \{1,\ldots,k\}^*} \left(\bigcup_{\eta \in \{1,\ldots,k\}^*} X(\eta\omega) \right),$$

is a residual subset of X. Fix a point x in this subset. For $n \geq 1$, put

$$\Omega_n(x) = \{\underline{\omega} \in \{1,\ldots,k\}^\infty \mid x \in X(\underline{\omega}|_n)\}.$$

This is an open subset of $\{1,\ldots,k\}^\infty$. By the choice of x, for each $\underline{\omega} \in \{1,\ldots,k\}^\infty$ and $m \geq 1$, there exists $\eta_m \in \{1,\ldots,k\}^*$ such that $x \in X(\eta_m(\underline{\omega}|_m))$. Hence, the open set $\{\underline{\omega}' \in \{1,\ldots,k\}^* \mid \underline{\omega}'|_m = \underline{\omega}|_m\}$ intersects with $\Omega_{m+|\eta_m|}(x)$ for any m. This implies that $\bigcup_{n \geq N} \Omega_n(x)$ is a dense subset of $\{1,\ldots,k\}^\infty$ for any $N \geq 1$. Hence, $\bigcap_{N \geq 1} \bigcup_{n \geq N} \Omega_n(x)$ is a residual subset of $\{1,\ldots,k\}^\infty$. Any infinite word $\underline{\omega}$ in this residual subset satisfies $x \in X(\underline{\omega}|_n)$ for infinitely many n's. □

Now let us finish Theorem 2.3.

Proof of Theorem 2.3 Fix a sequence $(a_n)_{n \geq 1}$ of positive integers. For $\omega \in \{0,1,2\}^*$, put

$$\mathcal{W}(\omega) = \{\rho \in \mathcal{W}^r_\# \mid \#\,\mathrm{Fix}_a(\rho^\omega) \geq |\omega| \cdot a_{|\omega|}\}.$$

Since attracting periodic points are persistent under small perturbations, $\mathcal{W}(\omega)$ is an open subset of $\mathcal{W}^r_{\#}$. By Lemma 7.1, it is sufficient to show that $\bigcup_{\eta\in\{0,1,2\}^*}\mathcal{W}(\eta\omega)$ is a dense subset of $\mathcal{W}^r_{\#}$. In other words, our goal is to show that for any given $\rho=(f_0,f_1,f_2)\in\mathcal{W}^r_{\#}$, $\omega_0\in\{0,1,2\}^*$, and any neighborhood $\mathcal{U}\subset \mathrm{Diff}^r([0,1])$ of the identity map, there exist $\eta\in\{0,1,2\}^*$ and $h\in\mathcal{U}$ such that $\#\,\mathrm{Fix}_a(\rho_h^{\eta\omega_0})\geq n\cdot a_n$, where $n=|\eta\omega_0|$.

First, by Proposition 5.1, after a small perturbation of f_0 if necessary, we may assume that there exist $p_*\in \mathrm{Int}\,J$ and $\gamma\in\{0,1,2\}^*$ such that p_* is an r-flat 0γ-periodic point with $p_*\notin\Sigma^{\gamma}(p_*)$. Now we choose $x_0\in J\cap f_0^{-1}(J)$ such that x_0 is (f_1,f_2)-generic and $x_0\notin\mathcal{O}_-(p_*,\rho)$. Such x_0 exists since these are generic conditions in $J\cap f_0^{-1}(J)$. We put $x_1=\rho^{\omega_0}(x_0)$. Since $\rho\in\mathcal{W}^r_{\#}$, we can take $\omega_1\in\{0,1,2\}^*$ such that $x_2=\rho^{\omega_1\omega_0}(x_0)\in\mathrm{Int}(J)$.

We choose two disjoint non-empty open intervals U and V in $\mathrm{Int}(J)\cap\mathrm{Int}(f_0(J))$ such that

- V is a neighborhood of p_*, and
- U, V are so small that both are disjoint from $\Sigma^{\omega_1\omega_0}(x_0)\cup\Sigma^{\gamma}(p_*)$.

Now we apply Lemma 4.3 (viewing (x_2,x_0,p_*) as $(p,p',\hat{p})$) to obtain ω_2 and $h_1\in\mathcal{U}$ such that $\mathrm{supp}(h_1)\subset U\cup V$, $\rho_{h_1}^{\omega_2}(x_2)=x_0$, and $[\rho_{h_1}^{\omega_2}(t)]_{x_2}=[\rho^{\omega_1\omega_0}(t)]^{-1}+o(t^r)$.

By Remark 4.4, we can choose $y\in U$ which appears in $\Sigma_{h_1}^{\omega_2}(x_2)$ only once. We take the word ω_2' of the form $\omega_2|_k$ for some k such that $y=\rho_{h_1}^{\omega_2'}(x_2)\in U$. Since U is disjoint from $\Sigma^{\omega_1\omega_0}(x_0)\cup\Sigma^{\gamma}(p_*)$, we have $y\notin\Sigma_{h_1}^{\omega_2'\omega_1\omega_0\omega_2''}(y)$, where ω_2'' is the (unique) word which satisfies $\omega_2=\omega_2''\omega_2'$. Notice that, by construction, we also know that y is an r-flat $(\omega_2'\omega_1\omega_0\omega_2'')$-periodic point of ρ_{h_1}.

Then, as in the proof of Theorem 2.1 in Sect. 5 (see also Remark 5.2), we can find $h_2\in\mathrm{Diff}^\infty([0,1])$ which is sufficiently close to the identity in C^r and supported in an arbitrarily small neighborhood of y such that $\mathrm{Fix}_a(\rho_{h_2\circ h_1}^{\omega_2'\omega_1\omega_0\omega_2''})\geq|\omega_2'\omega_1\omega_0\omega_2''|\cdot a_{|\omega_2'\omega_1\omega_0\omega_2''|}$.

Since each generator of $\rho_{h_2\circ h_1}$ is a diffeomorphisms on its image, we have the same estimate for $\mathrm{Fix}_a(\rho_{h_2\circ h_1}^{\omega_2''\omega_2'\omega_1\omega_0})$. Thus letting $\eta=\omega_2''\omega_2'\omega_1=\omega_2\omega_1$ and $h=h_2\circ h_1$, we complete the proof. $\square$

8 A criterion for sign conditions

In this section, we give the following simple criterion for sign conditions, which is given only in terms of positions and multipliers of repellers and attractors.

Proposition 8.1 *If $f\in\mathcal{E}^3$ has three fixed points $q_0<p<q_1$ such that $f'(q_0)<f'(p)^{-1}<f'(q_1)<1<f'(p)$ and $W^u(p,f)=(q_0,q_1)$, then f satisfies Sign conditions I and II.*

The following criterion for the existence of a persistent blender is shown in [32, Example 1].

Proposition 8.2 *Suppose that* $(f_1, f_2) \in \mathcal{A}^1(\{1,2\})$ *satisfies that* $f_1' < 1$ *and* $f_2' < 1$ *on an closed interval* $[a,b] \subset (0,1)$, $f_1(a) = a$, $f_2(b) = b$, *and* $f_1(b) > f_2(a)$. *Then* (f_1, f_2) *is a* C^1*-persistent blender for any closed interval* $J \subset (a,b)$.

Let us give a simple example of semigroups in $\mathcal{W}^\infty$ by using these propositions. Fix real numbers p, q_0, q_1, r, δ such that $0 < q_0 < p < q_1 < 1 < r, r - p < q_1 - q_0$, and $0 < 2\delta < \min\{q_0, 1 - q_1\}$. Let $\rho = (f_0, f_1, f_2) \in \mathcal{A}^\infty(\{0,1,2\})$ be a triple given by

$$\begin{aligned} f_0(x) &= x + \epsilon(x - q_0)(x - p)(x - q_1)(x - r), \\ f_1(x) &= (1-\delta)x + \delta^2, \\ f_2(x) &= (1-\delta)x + \delta(1-\delta). \end{aligned}$$

with $\epsilon > 0$. A direct computation shows that (p, q_0), (p, q_1) are repeller-attractor pairs of f_0, the point δ is the unique fixed point of f_1, and $1 - \delta$ is the unique fixed point of f_2. Set $J = [2\delta, 1 - 2\delta]$. The interval Int J contains the fixed points p, q_0, q_1 of f_0 and all heteroclinic points between them. Since $0 < 2\delta < 1$, we have $f_1(1-\delta) > f_2(\delta)$. By Proposition 8.2, the pair (f_1, f_2) is a C^1-persistent blender on J. Since $q_0 < p < q_1 < r, r - p < q_1 - q_0$, and

$$f_0'(p) \cdot f_0'(q_i) = 1 - \epsilon(p - q_i)^2\{r - p + (-1)^i(q_1 - q_0)\} + O(\epsilon^2)$$

for each $i = 0, 1$, we have $f_0'(p) \cdot f_0'(q_0) < 1 < f_0'(p) \cdot f_0'(q_1)$ if $\epsilon > 0$ is sufficiently small. By Proposition 8.1, f_0 satisfies Sign conditions I and II. Therefore, $\rho = (f_0, f_1, f_2)$ is an element of $\mathcal{W}^\infty$ when ϵ is sufficiently small. We also see that the set $\mathcal{W}^\infty_\# \cap \mathcal{W}^\infty_{\mathrm{att}}$ defined in Sect. 2 is non-empty. Indeed, if $\delta > 0$ is sufficiently small, then $f_1^N([0,\delta]) \cup f_2^N([1-\delta, 1]) \subset J$ for some large N. This implies that ρ is an element of $\mathcal{W}^\infty_\#$. It is easy to check that ρ is an element of $\mathcal{W}^\infty_{\mathrm{att}}$ if ϵ is sufficiently smaller than δ.

Proposition 8.1 is a direct consequence of the following

Proposition 8.3 *The following hold for* $f \in \mathcal{E}^r$ *with* $r \geq 2$ *and its repeller-attractor pair* (p, q)*:*

1. *There exists* $z_* \in (p, q)$ *such that* $\tau_A(z_*, f) = \mathrm{sgn}(p - q)$.
2. *If* $r \geq 3$ *and* $f'(p)f'(q) \neq 1$, *then there exists* $z_\# \in (p, q)$ *such that*

$$\tau_S(z_\#, f) = \mathrm{sgn}(f'(p)f'(q) - 1).$$

We reduce the proposition to the following

Lemma 8.4 *Let* F *be a* C^r *map from* $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ *to* $\mathbb{R}_- = \{x \in \mathbb{R} \mid x < 0\}$ *with* $r \geq 2$. *Suppose that* $F' > 0$ *and there exist positive real numbers* λ *and* μ *such that* $\mu < 1 < \lambda$ *and* $F(\lambda^n) = -\mu^n$ *for all* $n \in \mathbb{Z}$. *Then, the following hold:*

1. *There exists* $x_* \in \mathbb{R}_+$ *such that* $A(F)_{x_*} < 0$.
2. *If* $r \geq 3$ *and* $\lambda\mu \neq 1$, *then there exists* $x_\# \in \mathbb{R}^+$ *such that* $\mathrm{sgn}(S(F)_{x_\#}) = \mathrm{sgn}(\lambda\mu - 1)$.

Proof of Proposition 8.3 from Lemma 8.4 Let $\varphi : W^u(p, f) \to \mathbb{R}$ and $\psi : W^s(q, f) \to \mathbb{R}$ be the linearizations of f at p and q such that $\varphi'(p) = \psi'(q) = 1$. Set $I_p = \varphi(W^u(p, f) \cap W^s(q, f))$, $I_q = \psi(W^u(p, f) \cap W^s(q, f))$, and $H = \psi \circ \varphi^{-1}$.

First, we suppose that $p < q$. Then, $I_p = \mathbb{R}_+$, $I_q = \mathbb{R}_-$, and

$$H(\lambda^n) = \psi \circ f^n \circ \varphi^{-1}(1) = \mu^n \cdot H(1)$$

for any $n \in \mathbb{Z}$. Set $F(x) = H(x)/|H(1)|$. Then, the map F satisfies the assumption of Lemma 8.4. Since $A(H)_x = A(F)_x$ and $S(H)_x = S(F)_x$, by applying Lemma 8.4, we obtain the proposition for this case. The proof for the case $q > p$ can be obtained in a similar way with $F(x) = -H(x)/H(-1)$. □

Proof of Lemma 8.4 By the mean value theorem, there exists $x_n \in (\lambda^n, \lambda^{n+1})$ such that

$$F'(x_n) = \frac{F(\lambda^{n+1}) - F(\lambda^n)}{\lambda^{n+1} - \lambda^n} = \left(\frac{\mu}{\lambda}\right)^n \frac{1-\mu}{\lambda - 1}$$

for any $n \in \mathbb{Z}$. Since $\lambda > 1 > \mu > 0$, we have $F'(x_{n+1}) < F'(x_n)$ for any n. This implies that there exists $x_* \in (x_0, x_1)$ such that $F''(x_*) < 0$.

Set $G = (F')^{-\frac{1}{2}}$. By a direct calculation, we have

$$G'' = -\frac{1}{2}(F')^{-\frac{1}{2}} \cdot S(F). \tag{2}$$

Since $\lambda^n < x_n < \lambda^{n+1}$, we have

$$\lambda^2 - \lambda < \frac{x_{n+2} - x_n}{\lambda^n} < \lambda^3 - 1.$$

We also have

$$\frac{G(x_{n+2}) - G(x_n)}{x_{n+2} - x_n} = \frac{\left(\left(\frac{\lambda}{\mu}\right)^{\frac{n+2}{2}} - \left(\frac{\lambda}{\mu}\right)^{\frac{n}{2}}\right) \cdot \sqrt{\frac{\lambda-1}{1-\mu}}}{x_{n+2} - x_n}$$
$$= (\lambda\mu)^{-\frac{n}{2}} \cdot \left(\frac{\lambda}{\mu} - 1\right) \cdot \sqrt{\frac{\lambda - 1}{1-\mu}} \cdot \frac{\lambda^n}{x_{n+2} - x_n}.$$

Hence, we can choose a constant $C > 0$ such that

$$C^{-1}(\lambda\mu)^{-\frac{n}{2}} < \frac{G(x_{n+2}) - G(x_n)}{x_{n+2} - x_n} < C(\lambda\mu)^{-\frac{n}{2}}$$

for any $n \in \mathbb{Z}$.

By the mean value theorem, there exists $y_n \in (x_n, x_{n+2})$ such that $G'(y_n) = (G(x_{n+2}) - G(x_n))/(x_{n+2} - x_n)$. Notice that $\lim_{n\to+\infty} y_n = +\infty$ and

$\lim_{n\to-\infty} y_n = 0$. If $\lambda\mu > 1$, then $\lim_{n\to+\infty} G'(y_n) = 0$ and $\lim_{n\to-\infty} G'(y_n) = +\infty$. This implies that there exists $x_\# \in \mathbb{R}_+$ such that $G''(x_\#) < 0$, and hence, $S(F)_{x_\#} > 0$ by Equation (2). If $\lambda\mu < 1$, then $\lim_{n\to+\infty} G'(y_n) = +\infty$ and $\lim_{n\to-\infty} G'(y_n) = 0$. This implies that there exists $x'_\# \in \mathbb{R}_+$ such that $G''(x_\#) > 0$, and hence, $S(F)_{x'_\#} < 0$. □

Acknowledgments The authors thank to Pierre Berger for useful dicsussions. KS and DT are grateful for the hospitality of Department of Mathematics of Kyoto University. This paper is supported by GCOE program of Kyoto University, JSPS KAKENHI Grant-in-Aid for Young Scientists (A) (22684003), Scientific Research (C) (26400085), and JSPS Fellows (26·1121), by Grant No. 14-41-00044 of RSF (Russia), and by the Royal Society Grant IE141468.

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