On Smooth Hamiltonian Flows Limiting to Ergodic Billiards

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AMS No.: 58F15, 82C05, 34C37, 58F05, 58F13, 58F14.

Abstract. Sufficient conditions are found so that a family of smooth Hamiltonian flows limits to a billiard flow as a parameter $\epsilon \to 0$. This limit is proved to be C^1 near non-singular orbits and C^0 near orbits tangent to the billiard boundary. These results are used to prove that scattering (thus ergodic) billiards with tangent periodic orbits or tangent homoclinic orbits produce nearby Hamiltonian flows with elliptic islands. This implies that ergodicity may be lost for smooth potentials which are arbitrarily close to ergodic billiards. Thus, in some cases, anomoulous transport associated with stickiness to stability islands is expected

1 Introduction

The billiard model is concerned with the motion of a point particle traveling with a constant speed in a region and undergoing elastic collisions at the region's boundary. This motion is very much like in that of a real billiard table - the main difference is that there is no friction in the model (so the ball never stops nor rolls). In the two-dimensional setting of our model, the ball is actually a small disk (a two-dimensional ball). Different shapes of the billiard table, and the number of balls that one considers influence the type of motion a ball may execute. Ergodic billiards are billiard tables in which the balls execute a uniformly disordered motion: all possible positions and velocities are realized by the traveling billiard balls (for almost all initial positions).

The billiard problem has been extensively studied both in its classical and quantized formulation. Numerous applications lead to study such a model problem; First, there exist direct mechanical realizations of this model (e.g. the motion of N rigid d-dimensional spheres in a d-dimensional box may be reduced to a billiard problem, possibly in higher dimensions [21, 22, 7]. See also [6] for the inelastic case.). Second, it serves as an idealized model for the motion of charged particles in a potential, a model which enables the examination of the relation between classical and quantized systems, see [14] and references therein. Finally, and most important, this model has been suggested [21] as a first step for substantiating the basic assumption of statistical mechanics - the ergodic hypothesis of Boltzmann (see especially the discussion and references in [22, 24]).

In all the applications of this model, of special interest are so-called *scattering* billiards, i.e., billiards in a complement to the union of a finite number of closed convex regions. For example - the two-dimensional idealization of a gas in the form of a lattice of rigid disks produces a scattering billiard ("the Sinai billiard"). The motion in a scattering billiard is highly unstable thus produces strong mixing in the phase space. More precisely, it has been shown [21, 11, 1] that the corresponding dynamical system is (non-uniformly) hyperbolic, it is ergodic with respect to the natural invariant measure and it possesses K-property. Based on this theory, statistical properties of various scattering systems have been analyzed (see [5, 4]).

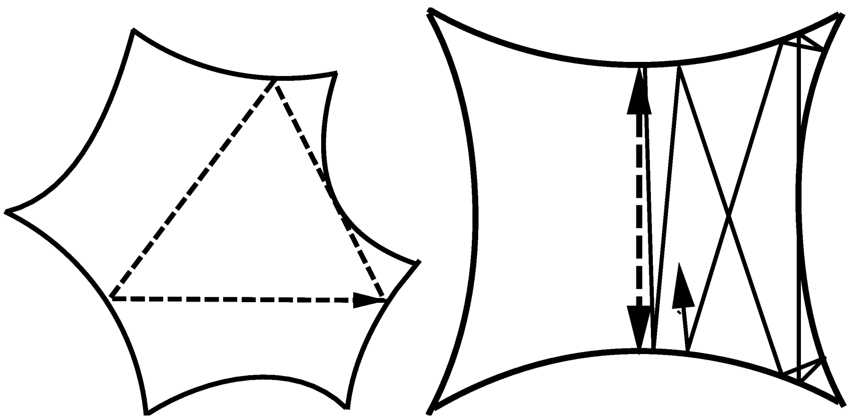

Fig. 1.1. Tangent trajectories

a) Singular (tangent) periodic trajectory
b) – – – – –non-singular periodic trajectory,
———Tangent homoclinic trajectory to the periodic orbit.

Do small perturbations ruin the ergodicity property of a scattering billiard? In this paper we consider the perturbation caused by the "natural" smoothening of a billiard flow, by which the step-function potential at the billiard boundary is replaced by a family of smooth potentials approaching the step function, preserving the correct reflection law near the boundary. We stress that the billiard reflection rule ("angle of reflection equals angle of incidence") appears as a limit only, and the billiard itself is, of course, an idealized model to the real motion. Therefore, the problem of relating the statistics manifested by the billiard dynamical systems to actual physical applications must inevitably include the study of the smoothening of the billiard potential.

The influence of such smoothening is a non-trivial question, since the dynamical system associated with the billiard we consider (in the simplest setting, this is a two-dimensional area-preserving mapping [21]) is singular. In particular, as explained more precisely in section 2.1, singularities appear near trajectories which are tangent to the billiard's boundary - like the ones shown in figure 1.1. Thus, even though the scattering billiard is hyperbolic almost everywhere, theoretically, there exists a possibility that the singular set (e.g. singular periodic orbits) will produce stability islands under small perturbation. While such a phenomenon seems to be quite common, general theory does not exist. Indeed, it is clear that the results are not straightforward - namely it is not true that all smooth systems approaching a singular hyperbolic and mixing system have stable periodic orbits nor is the converse - that they have the same ergodic properties as the singular system. (As an example, consider an analogous problem for one dimensional maps; For a family of tent maps of an interval which are known to be ergodic and mixing, the ergodicity property may be easily destroyed in an arbitrarily close smooth family: if the maximum of the interval image produces a periodic orbit, it is clearly stable. However, the smooth one-dimensional map does not always possess stable periodic orbits: there may be a positive measure set of parameter values for which the smooth maps are ergodic and mixing [16]).

In this paper we prove that, indeed, *a perturbation of a scattering billiard to a smooth Hamiltonian flow may create stability islands near singular periodic and homoclinic orbits of the billiard.* An important ingredient of the proof is the established connection between the limiting smooth Hamiltonian flows and the singular billiard flow. This connection, which seems to be fundamental for understanding the applicability and limitations of the billiards to more realistic models of particle motion has not been previously formalized (to the best of our knowledge), and has received surprisingly little attention.

In the physics community it has been assumed to exist; For example, in [15] the qualitative behavior of orbits of the diamagnetic Kepler problem has been analyzed by studying the four-disk billiard system which has similar spatial structure. Furthermore, in that paper, the correspondence between elliptic periodic orbits of the smooth Hamiltonian system and singular periodic orbits of the modeling billiard was noticed. Nevertheless, our analysis reveals non-trivial requirements on smooth potentials approaching the billiard potential, which are essential for the dynamics of the corresponding Hamiltonian system to follow the dynamics of the billiard flow. Therefore, a rigorous proof of a correspondence between billiard and "smooth" orbits can not be immediate.

Mathematically, Marsden [19] has studied a more general question of the behavior of the symplectic structure when a family of smooth Hamiltonians approaches a singular limit, and related these problems to the general study of distributions on manifolds. In this setting, he showed that some properties of the smooth Hamiltonians are preserved by the singular one. For example, he proved that if the families of Hamiltonians are uniformly mixing then the mixing property carries to the singular system as well. Here we investigate the other direction of the above result - namely given a singular system which is mixing - what can be said on the natural family of smooth Hamiltonian which approaches this limiting system.

Recently, an example of another kind of smooth analogue of a scattering billiard with elliptic islands was constructed [9]; namely, for the motion of a point-wise particle in a finite-range smooth potential, where the potential's support consists of a finite number of non-overlapping disks on a plane torus. It was shown that in this geometry the smooth potential effect is to create a finite-length-travel along the scattering disks, and this produces focusing shifts near tangent trajectories even in the limit of high energies. Thus, it was proved that for any given energy level, there exists an arrangement of the disks for which elliptic islands exist. Here, a completely different approach is taken, which in particular, does not assume any specific geometry of the scatterers nore that the potential is of a finite-range.

Another type of natural perturbation of a billiard is achieved by a deformation of the billiard's boundary (in a non-smooth fashion for scattering billiards with a piece-wise smooth boundary). While such deformations have been extensively studied numerically, we are not aware of theoretical approaches for studying the near-ergodic regime. On the other end, perturbations of near-integrable billiards may be studies using Melnikov technique [8].

Traditionally, transport properties of the extended Sinai billiard were studied in terms of the decay of the correlation function [5]. More recently (see [27] and references therein), Poincaré recurrences and stickiness in phase space of both Sinai billiards and Casini billiards were numerically studied. It has been demonstrated that the appearance of sticky islands for some parameter values causes anomoulous transport - specifically power-law decay for the Poincaré recurrences distribution. To produce the anomoulos transport a parameter controlling the shape of the billiard was carefully tuned to produce self-similar sticky island structure. Moreover, it has been observed that such a tuning is possible near any parameter value for which islands exist. Here, we prove that islands may be produced by smoothening of the billiard boundary. Combining these results implies that by tuning the smoothening one can obtain sticky islands and thus *anomoulous transport for the Lorenz gas model with arbitrarily sharp smooth potentials.*

The general scheme of the paper is as follows: In 2.1 we introduce the billiard flow in a general domain, and describe its nature near regular and tangent collision points and its relation to the standard billiard map. Then, in 2.2, we introduce a class of one-parameter families of Hamiltonians and formulate sufficient conditions on this class so that as the parameter $\epsilon \to 0$

they approach the billiard flows. In section 2.3 some examples of families of smooth Hamiltonians satisfying our assumptions are presented. In section 2.4 we formulate the main theorems which establish in which sense the Hamiltonian flows approach the billiard flow. In section 3 we utilize these theorems to prove the existence of elliptic islands in Hamiltonian flows which approximate scattering (Sinai) billiards; First, we study the phase space structure of the billiard map near singular periodic orbits and near singular homoclinic orbits. We prove that existence of such orbits implies the appearance of a non-smooth analogue of the Smale horseshoe, similar to the horseshoe in the Hénon map. Then, using the closeness results of section 2.4 we establish that if a singular periodic orbit/homoclinic orbit exists for the billiard map, then necessarily there exist nearby Hamiltonians with elliptic periodic orbits. The appearance of persistent singular homoclinics and singular (tangent) periodic orbits for scattering billiards is conjectured and the former is numerically demonstrated. Section 4 is devoted to a discussion on the implication of these results. In appendix A examples showing the necessity of some of the conditions imposed on the family of Hamiltonians are presented.

2 Closeness of plane billiards and smooth Hamiltonian flows

2.1 Billiard flow.

Consider an open bounded region D on a plane with a piecewise smooth (C^{r+1}, $r \geq 2$) boundary S. On S there is a finite set C of so-called *corner points* $c_1, c_2, \ldots$ such that the arc of the boundary that connects two neighboring corner points is C^{r+1}-smooth. Let us call these arcs *the boundary arcs* and denote them by $S_1, S_2, \ldots$. The set C includes all the points where the boundary loses smoothness and all the points where the curvature of the boundary vanishes. Thus, the curvature has a constant sign on each of the arcs S_i. Being equipped with the field of inward normals, the arc is called *convex* if its curvature is negative (with respect to the chosen equipment) and it is called *concave* if its curvature is positive (see figure 2.1).

Consider *the billiard flow* on $\bar{D}$ which describes the motion of a point mass moving with a constant velocity between consecutive elastic collisions with S. The phase space of the flow is co-ordinatized by (x, y, p_x, p_y) where (x, y) is the position of the particle in $\bar{D}$ and (p_x, p_y) is the velocity vector:

$$\dot{x} = p_x \quad \dot{y} = p_y. \tag{2.1}$$

Henceforth, to distinguish between the phase space and the configuration space $\bar{D}$ we reserve the term "orbit" for the orbits in the phase space and the term "trajectory" for the projection of an orbit to the (x, y)-plane.

The flow is defined by the condition that the velocity vector (p_x, p_y) is constant in the interior, and at the boundary it changes by the elastic reflection rule so $p_x^2 + p_y^2 = const$ and the angle of reflection equals the angle of incidence with the opposite sign. Taking the point of reflection as the origin of the coordinate frame and the boundary's normal at that point as the y-axis, the reflection rule is simply

$$p_x \rightarrow p_x, p_y \rightarrow -p_y; \tag{2.2}$$

namely, the angle of incidence ϕ is $arctan p_y/p_x$. This law is well defined only when the normal can be well defined: it is invalid at the corners where the boundary looses its smoothness.

Generally, the incidence angle ϕ belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, but if the boundary is convex, $|\phi| < \frac{\pi}{2}$. If the boundary arc is concave, it is possible to have $\phi = \pm\frac{\pi}{2}$ (figure 2.1) which corresponds to a trajectory tangent to S.

A special case is a tangent trajectory ($\phi = \pm\frac{\pi}{2}$) which reaches the boundary at an inflection point. One can easily see that any close trajectory undergoes an unboundedly large number of collisions before leaving a small neighborhood of the inflection point, and for the trajectory tangent to the boundary at the inflection point itself there is no reflection at all (figure 2.1). The trajectory is terminated at the moment of such tangency and the corresponding orbit of the flow is not defined for greater times. That is the reason for excluding the inflection points from consideration by putting them into the corner set.

Denote points in the phase space of the billiard flow as $q \equiv (x, y, p_x, p_y)$ and the time t map of the flow as $b_t : q_0(x_0, y_0, p_{x0}, p_{y0}) \mapsto q_t(x_t, y_t, p_{xt}, p_{yt})$. Recall that the reflection law is not defined at the corner points; thus, by writing $q_t = b_t q_0$, we mean, in particular, that the piece of trajectory that connects (x_0, y_0) and (x_t, y_t) is on a finite distance of the corner set C. At the same time we allow the trajectory to have one or more points of tangency with concave components of S.

A point $q(x, y, p_x, p_y)$ in the phase space is called *an inner point* if $(x, y) \notin S$, and *a collision point* if $(x, y) \in (S \backslash C)$. Obviously, if q_0 and $q_t = b_t q_0$ are inner points, then q_t depends continuously on q_0 and t. Otherwise, if q_t is a (non-tangent) collision point, the velocity vector undergoes a jump: denoting by $q_{t-0} = b_{t-0} q_0$ and $q_{t+0} = b_{t+0} q_0$ the points just before and just after the collision, it follows that (p_{xt+0}, p_{yt+0}) and (p_{xt-0}, p_{yt-0}) are related by the elastic reflection law. To avoid ambiguity we assume that at a collision point the velocity vector is oriented inside D; thus, we put $b_t \equiv b_{t+0}$.

Further, if q_t is an inner point and if the piece of trajectory that connects (x_0, y_0) and (x_t, y_t) does not have tangencies with the boundary, then q_t depends C^r-smoothly on q_0 and t. On the other hand, it is well known [21] that the map b_t loses smoothness at any point q_0 whose trajectory is tangent to the boundary at least once on the interval $[0, t]$. Indeed, choosing coordinates so that the origin is a point on a concave boundary arc S_i, the y-axis is the normal to S_i and the x-axis is tangent to S_i, the arc is locally given by the equation

$$y = -x^2 + \dots$$

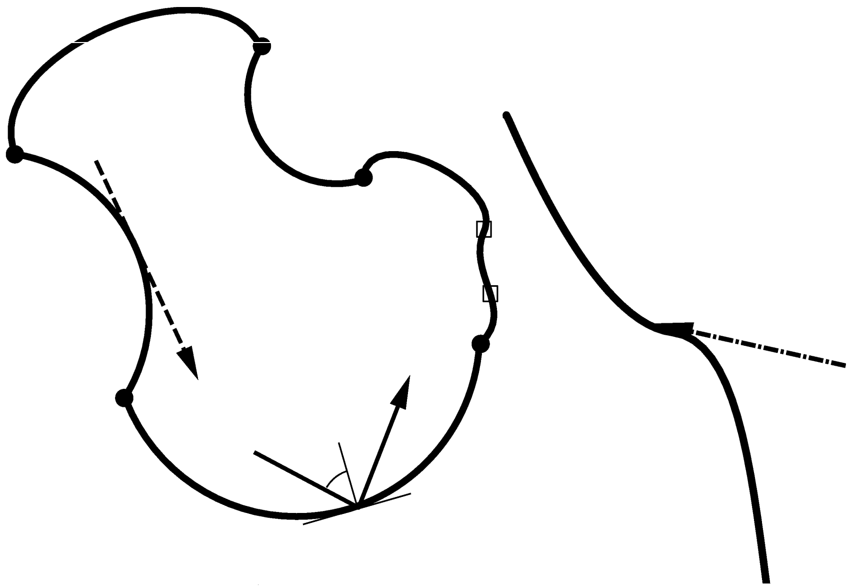

Fig. 2.1. Billiard flow

a) • - standard corner points, ▫ - inflection corner points
$S_{1,3,5}$ - concave boundary arcs, $S_{2,4,6,7}$ - convex arcs
—— Regular reflection, - - - Tangent trajectory
b) $- \cdot -\cdot$ Tangent trajectory terminated at an inflection point

It follows that for small $\delta > 0$ the time $t = \delta$ map of the slanted line ($x_0 = -\delta/2 + ay_0, p_{x0} = 1, p_{y0} = 0$) has a square root singularity in the limit $y_0 \to -0$ which corresponds to the tangent trajectory (see figure 2.2; $a \neq 0$ for graphical purposes):

$$(x_\delta, y_\delta, p_{x\delta}, p_{y\delta}) = \begin{cases} (\frac{1}{2}\delta + ay_0, y_0, 1, 0) \quad \text{at } \ y_0 \geq 0 \\ (\frac{1}{2}\delta + ay_0 + O(\delta y_0), 2\sqrt{-y_0}\delta + O(\delta y_0), 1 + O(y_0) \\ \qquad , 2\sqrt{-y_0} + O(y_0)) \quad \text{at } \ y_0 \leq 0 \end{cases}$$

If q_0 and $q_t = b_t q_0$ are inner points, then for arbitrary two small cross-sections in the phase space, one through q_0 and the other through q_t, *the local Poincaré map* is defined by the orbits of the billiard flow. If no tangency to the boundary arcs is encountered between q_0 and q_t, then *the Poincaré map is locally a C^r-diffeomorphism.*

One can easily prove that the same remains valid if q_0, or q_t, or both of them are collision points, provided the corresponding cross-sections are composed of the nearby collision points. In fact, the collision set (the surface

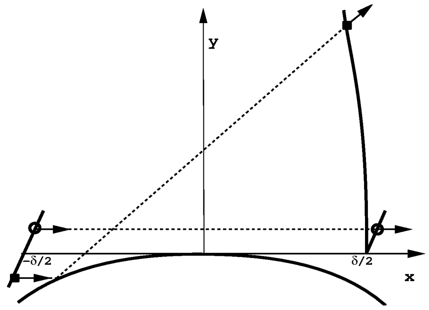

Fig. 2.2. Singularity near a tangent trajectory

$(x, y) \in S$ in the phase space) provides a *global* cross-section for the billiard flow. The corresponding Poincaré map relating consecutive collision points is called *the billiard map*. A point on the surface is determined by the position s on the boundary S and by the reflection angle ϕ which yields the direction of the outgoing velocity vector (the absolute value of the velocity does not matter because $p_x^2 + p_y^2$ is a conserved quantity - the energy - and it may be taken arbitrary by rescaling the time). The initial conditions, corresponding to a trajectory directed to a corner or tangent to a boundary arc at the moment of the next collision, form *the singular set* on the (s, ϕ)-surface. Generically, the singularity set is a collection of lines which may be glued at some points. The billiard map is a C^r-diffeomorphism outside the singular set; it may be discontinuous at the singular points. Near a singular point corresponding to the tangent trajectory the continuity of the map can be restored locally by taking two iterations of the map on a half of the neighborhood of the singular point (see figure 2.2). The obtained map will, nevertheless, be non-smooth at the singular point, having the square root singularity described above.

2.2 Class of smooth Hamiltonians.

Formally, the billiard flow may be considered as a Hamiltonian system of the form

$$H_b = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V_b(x, y) \tag{2.3}$$

where the potential vanishes inside the billiard region D and equals to infinity outside:

$$V_b(x, y) = \begin{cases} 0 & (x, y) \in D \\ +\infty & (x, y) \notin D \end{cases} \tag{2.4}$$

Clearly, this is an approximate model of the motion of a pointwise particle in a *smooth* potential which stays nearly constant in the interior region and grows very fast near the boundary. However, it is not obvious immediately when (and in which sense) this motion is indeed close to the billiard motion. We examine this question in this section and describe a class of potentials for which the billiard approximation (2.4) is correct in some reasonable sense.

Consider a Hamiltonian system associated with

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y; \epsilon) \tag{2.5}$$

where the potential $V(x, y; \epsilon)$ tends to zero inside the region D as $\epsilon \longrightarrow 0$ and it tends to infinity outside. Specifically, we require that

I. *For any compact region $K \subset D$ the potential $V(x, y; \epsilon)$ diminishes along with all its derivatives as $\epsilon \longrightarrow 0$:*

$$\lim_{\epsilon \rightarrow +0} ||V(x, y; \epsilon)|_{\{(x,y)\in K\}}||_{C^{r+1}} = 0. \tag{2.6}$$

The growth of the potential to infinity across the boundary is a more delicate issue. The crucial construction here is that V is evaluated along the level sets of some *finite* function near the boundary. Namely, putting the set C of corner points c_i out of consideration, we suppose that in a neighborhood of the set $(D\backslash C)$ there exists a function $Q(x, y; \epsilon)$ which is C^{r+1} with respect to (x, y) and it depends continuously on ϵ (in C^{r+1}-topology) at $\epsilon \geq 0$. Specifically, $Q(x, y; \epsilon)$ along with its derivatives have a proper limit as $\epsilon \longrightarrow 0$. Assume that

IIa *On the boundary, the function $Q(x, y; 0)$ is constant between any two neighboring corner points:*

$$Q(x, y; \epsilon = 0)|_{(x,y)\in S_i} \equiv Q_i \tag{2.7}$$

We call *Q a pattern function.* For each boundary component S_i, for Q close to Q_i, let us define *a barrier function* $W_i(Q; \epsilon)$ which does not depend explicitly on (x, y) and assume that:

IIb *There exists a small neighborhood N_i of the arc S_i on which the potential V is given by W_i evaluated along the level sets of the pattern function Q:*

$$V(x, y; \epsilon)|_{(x,y)\in N_i} \equiv W_i(Q(x, y; \epsilon); \epsilon) \tag{2.8}$$

IIc *The gradient of V does not vanish in a finite neighborhood of the boundary arcs:*

$$\nabla V|_{(x,y)\in N_i} \neq 0 \tag{2.9}$$

which is equivalent to the following conditions

$$\nabla Q|_{(x,y)\in N_i} \neq 0 \tag{2.10}$$

and

$$\frac{d}{dQ} W_i(Q; \epsilon) \neq 0. \tag{2.11}$$

Conditions **IIa,b,c** formalize the requirement that the direction of the gradient of the potential must be normal to the boundary as $\epsilon \to +0$. Obviously, this is necessary for having a proper reflection law in the limit: if the reflecting force has a component tangent to the wall, then the tangent component p_x of the momentum will not be preserved during the collision (see (2.2)).

Now we may describe the rapid growth of the potential across the boundary in terms of the barrier functions W_i only. Choose any of the arcs S_i and henceforth suppress the index i. Without loss of generality assume $Q = 0$ on S. By (2.10), the pattern function Q is monotonically increasing across S and we assume Q is positive inside D near S and negative outside (otherwise, change inequalities in (2.12) to the opposite ones). Assume

III *As $\epsilon \to +0$ the barrier function increases from zero to infinity across the boundary S_i:*

$$\lim_{\epsilon \to +0} W(Q;\epsilon) = \begin{cases} +\infty & Q < 0 \\ 0 & Q > 0 \end{cases} \tag{2.12}$$

Note that according to **I.** and **IIb.**, for any $Q_0 > 0$

$$\lim_{\epsilon \to +0} ||W(Q,\epsilon)|_{Q \geq Q_0}||_{C^{r+1}} = 0. \tag{2.13}$$

Clearly, it will cause no troubles if one allows W to take infinite values: by (2.11), the function W is monotonic and if it is infinite at some Q, it is infinite for all smaller Q; on the other hand, trajectories always stay in the region where W is bounded: since the energy given by (2.5) is conserved, the value of the potential is bounded by the initial value of H. We will study limiting behavior (as $\epsilon \to +0$) of the smooth Hamiltonian system (2.5) in a given, *fixed* energy level, $H = H^*$. This implies that all trajectories stay in the region $W \leq H^*$ for any ϵ. It follows that the symbol $+\infty$ in (2.12) may be replaced by any value greater than H^*.

It is immediately evident that the particle in the potential V satisfying condition **I** moves in the interior of D with essentially constant velocity along a straight line until it reaches a thin layer near the boundary S where the potential runs from small to very large values (the smaller the value of ϵ, the thinner the boundary layer). By virtue of condition **III**, if the particle enters the layer near an interior point of some boundary arc (corner points are not considered in this paper), it can not penetrate the layer and go outside - because fixing the value of the energy bounds the potential from above. Thus, the particle is either reflected, exiting the boundary layer near the point where it entered, or it might, in principle, stick into the layer, traveling along the boundary far away from the entrance point. As simple arguments show (see the proof of theorem 1 below), condition **II** guarantees that when a reflection does occur it will be of the right character, approximately preserving the tangential component (p_x) of the momentum and changing sign of the normal component (p_y). However, as argued below, and shown by an example in Appendix A, conditions **I-III** are insufficient for preventing the existence of non-reflecting trajectories. Since such finite length travels along the boundary layer must be forbidden in the limit $\epsilon \to 0$, we impose an additional restriction on the shape of the potential near the boundary. Denote the normal force function by $F(Q;\epsilon) = \frac{d}{dQ}W(Q,\epsilon)$ and require the following:

IV *The normal force is a monotonic function of Q:*

$$W''(Q) \equiv F'(Q) \geq 0. \tag{2.14}$$

(According to condition **III**, since W decays rapidly across $Q = 0$, it follows that its derivative $F(Q)$ is close to $-\infty$ at small Q. Then, as Q grows, $F(Q) \to 0$ by (2.13). Thus, $F(Q)$ can not be strictly decreasing function and the monotonicity of $F(Q)$ is indeed equivalent to the positiveness of $F'(Q)$.)

To see how a violation of the monotonicity condition can lead to the appearance of non-reflecting trajectories suppose that for arbitrarily small ϵ there is an interval of values of Q, arbitrarily close to the boundary, on which the graph of absolute value of $F(Q)$ is as shown in figure 2.3: it grows from zero to very large values, then decays back to nearly zero at a value $Q_\epsilon$which approaches zero as $\epsilon \to 0$, and only after that it grows to infinity. Since the force is the gradient of the potential and, according to condition **II**, it is proportional to $F(Q)$ whereas the distance to the wall is proportional to Q, it follows that the graph of the normal component of the reflecting force *vs* the distance to the wall has the same shape as in figure 2.3. Thus, the initial velocity of the particle can be taken such that the normal component of the velocity is completely damped when moving through the region of the first peak of $F(Q)$, leading to the trapping of the particle in the zone where the reflecting force is nearly zero with the normal component of velocity close to zero too. In this case the distance to the wall will change very slowly and the particle may stay at a small distance to the wall for a long time, traveling along the boundary instead of making reflection. An explicit example of such trapping in a circular billiard is presented in Appendix A. In fact, the geometry of the boundary plays a crucial role here: one can show that the finite length travels along a *concave* boundary arc are forbidden even for the non-monotonic $F(Q)$ (though the reflection time may still be unboundedly large in this case).

Conditions **I-IV** guarantee, as is precisely formulated in section 2.4, a correct reflection law only in the C^0-topology and not in the C^1-topology. As this issue is very important for the sequel, we explain its intuitive implication now. Let us take a point (x_0, y_0) and momentum (p_{x0}, p_{y0}) as initial conditions for an orbit of the Hamiltonian system (2.5) and let us take the same initial conditions for the billiard orbit. Consider a time interval t for which the billiard orbit collides with the boundary S only once, at some point (x_c, y_c) (see figure 2.4). Here, the incidence angle ϕ^{in} is the angle between the vector $(x_0 - x_c, y_0 - y_c)$ and the inward normal to S at the point (x_c, y_c); the reflection angle ϕ^{out} is the angle between the vector $(x_t - x_c, y_t - y_c)$ and the normal, where (x_t, y_t) is the point reached by the billiard trajectory at the

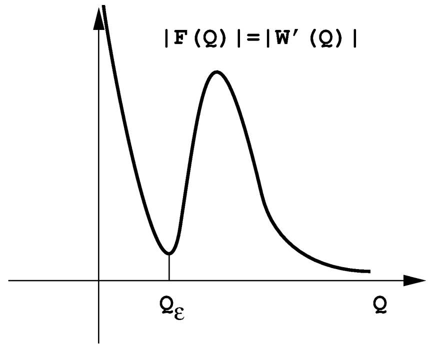

Fig. 2.3. Non-monotonic normal force

time t. In the same way one may define the incidence and reflection angles for the trajectory of the Hamiltonian system where (x_0, y_0) and (x_c, y_c) are taken the same as for the billiard trajectory and $(x_t(\epsilon), y_t(\epsilon))$ is now defined by the Hamiltonian flow (see figure 2.4). We expect the trajectory of the Hamiltonian system to be close to the billiard trajectory; in particular, it should demonstrate a correct reflection law

$$\phi^{in}(\epsilon) + \phi^{out}(\epsilon) \approx 0$$

for sufficiently small ϵ. Note, however, that $(\phi^{in} + \phi^{out})$ is a function of the initial conditions. Conditions **I-IV** give only C^0-closeness of these functions to zero and to ensure a C^1-correct reflection law we need the following additional condition on $W(Q)$:
V *There exists an $\alpha \in (0,1)$ such that the following holds for any interval $[Q_1(\epsilon), Q_2(\epsilon)]$ on which $W(Q)$ is bounded away from zero and infinity for all ϵ:*

$$\lim_{\epsilon\to 0} \frac{W''(Q)}{|W'(Q)|^{3+\alpha}} = 0, \tag{2.15}$$

uniformly on the interval $[Q_1, Q_2]$.

This condition is used directly in the proof of theorem 1 (see [25]). To give the reader a feeling of how the smoothness may be lost, consider a *one-dimensional* reflection described by the equation $\ddot{Q} + W'(Q;\epsilon) = 0$ where $Q \geq 0$, $W(0;\epsilon) = +\infty$, $\lim_{\epsilon\to 0} W(Q;\epsilon) = 0$ at $Q > 0$. Here, Q is the position of a particle moving inertially until a collision with the wall at $Q = 0$, after which the particle reflects elastically and moves back. The time of collision is given by $\tau = \displaystyle\int_{Q^*}^{1} \frac{\sqrt{2}dQ}{\sqrt{H - W(Q)}}$ where H is the value of energy and $Q^*(\epsilon)$ is such that $W(Q^*;\epsilon) = H$. Differentiation with respect to H gives

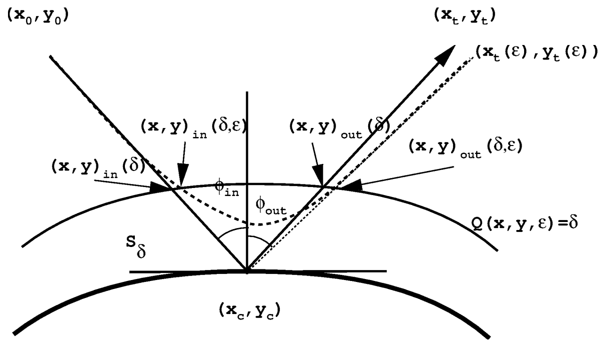

Fig. 2.4. Reflection by Hamiltonian flow

$$\frac{d\tau}{dH} = \frac{\sqrt{2}}{W'(Q^*)}(H - W(1))^{-1/2}$$
$$-\frac{1}{\sqrt{2}W'(Q^*)}\int_{Q^*}^{1}\frac{W'(Q^*) - W'(Q)}{W(Q^*) - W(Q)}(H - W(Q))^{-1/2}dQ. \quad (2.16)$$

Note that $(W'(Q^*) - W'(Q))/(W(Q^*) - W(Q)) \approx W''/W'$, therefore restrictions should be imposed on W'', like in condition **V**, to have $d\tau/dH$ bounded.

2.3 Examples for smooth Hamiltonians limiting to billiards.

Conditions **I-V** are in fact quite general, and they are fulfilled by many reasonable choices of the pattern and barrier functions. For the pattern function, consider any smooth function Q depending on two variables (x, y). Corners are created at the singularities of the level sets and at the points of inflection.

For the barrier function conditions **I-V** need to be fulfilled. For example, the following barrier functions $W(Q, \epsilon)$ satisfy them (for $\beta > 0$):

$$\frac{\epsilon}{Q^\beta}, \quad (1 - Q^\beta)^{\frac{1}{\epsilon}}, \quad \epsilon e^{-\frac{1}{Q^\beta}}, \quad \epsilon|\ln Q|^\beta, \quad \epsilon \ln \ldots |\ln Q|.$$

One may easily produce more examples because there is no restriction on the growth rate: given any potential V satisfying conditions **I-V** the potential $\psi(V)$ also satisfies these conditions provided ψ is a smooth monotonic function of V such that $\psi(0) = 0$, $\psi(\infty) = \infty$.

In section 3 we consider the billiard corresponding to the following family of pattern functions:

$$Q(x, y; \gamma) = \gamma\Big(\frac{1}{x^2 + (y - \frac{1}{\gamma})^2 - R^2} + \frac{1}{x^2 + (y + \frac{1}{\gamma})^2 - R^2}$$
$$+\frac{1}{(x - \frac{1}{\gamma})^2 + y^2 - R^2} + \frac{1}{(x + \frac{1}{\gamma})^2 + y^2 - R^2}\Big)^{-1} \quad (2.17)$$

where $R^2 = 1 + (1 - \frac{1}{\gamma})^2$ and γ is a parameter (not necessary small). The billiard domain is bounded by the level set $Q(x, y) = 0$. For $\gamma \to 0$ this defines a square whereas for $\gamma > 0$ it defines a concave shape bounded by the four circles of radius R which intersect at the four corner points $(x, y) = (\pm 1, \pm 1)$.

Taking the barrier function in the simplest form $W(Q, \epsilon) = \frac{\epsilon}{Q}$ produces the following Hamiltonian system:

$$H_{\gamma,\epsilon}(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \epsilon\Big(\frac{1}{\gamma(x^2+y^2-2)+2(1-y)} + \frac{1}{\gamma(x^2+y^2-2)+2(1+y)} + \frac{1}{\gamma(x^2+y^2-2)+2(1-x)} + \frac{1}{\gamma(x^2+y^2-2)+2(1+x)}\Big)$$

Notice that for $\gamma \to 0$, the square geometry produces separable - hence integrable - Hamiltonian flow. This is, of course, a very interesting limit, which is not studied in this paper. Notice also that here the limit $\epsilon \to 0$ is equivalent to the limit $H \to \infty$ with ϵ held fixed.

2.4 Closeness theorems.

Denote the Hamiltonian flow of (2.5) by $h_t(\epsilon)$. Given t and ϵ, the flow maps a phase point $q_0 \equiv (x_0, y_0, p_{x0}, p_{y0})$ to $q_t(\epsilon) \equiv (x_t(\epsilon), y_t(\epsilon), p_{xt}(\epsilon), p_{yt}(\epsilon))$. We will call $q_t(\epsilon)$ *the smooth orbit* of q_0 and will examine how close is it to *the billiard orbit* $b_t q_0 \equiv q_t(0)$. The corresponding trajectories $(x_t(\epsilon), y_t(\epsilon))$ and $(x_t(0), y_t(0))$ on the (x, y)-plane will be called *the smooth* and, respectively, *the billiard* trajectories.

Let (x_c, y_c) be the first point of collision of the billiard trajectory with the boundary S; by definition, $(x_c, y_c) = (x_0, y_0) + (p_{x0}, p_{y0})t_c$, where $t = t_c$ is the moment of collision. Since the potential V is nearly zero in the interior of the billiard domain D, the smooth orbit of q_0 is arbitrarily close (as $\epsilon \to 0$) to the billiard orbit before the collision: namely, the point $(x_t(\epsilon), y_t(\epsilon))$ moves with essentially constant velocity until reaching a small neighborhood of (x_c, y_c). Take a small $\delta > 0$ and consider *the boundary layer* $S_\delta \equiv \{|Q(x, y; \epsilon) - Q(x_c, y_c; \epsilon)| \le \delta\}$, where Q is the pattern function. For any small δ, if ϵ is sufficiently small, the smooth trajectory enters the boundary layer at some time $t_{in}(\epsilon)$. Denote $q_{in}(\epsilon) = q_{t_{in}}(\epsilon)$; by definition, $|Q(x_{in}(\epsilon), y_{in}(\epsilon); \epsilon) - Q(x_c, y_c; \epsilon)| = \delta$. The closeness of the billiard and the smooth orbits (before the collision) implies the existence of the limits (see figure 2.4)

$$t_{in} \equiv \lim_{\epsilon \to 0} t_{in}(\epsilon), \quad q_{in} \equiv \lim_{\epsilon \to 0} q_{in}(\epsilon);$$

moreover,

$$\lim_{\delta \to 0} t_{in} = t_c, \ \lim_{\delta \to 0}(x_{in}, y_{in}) = (x_c, y_c), \ \lim_{\delta \to 0}(p_{x,in}, p_{y,in}) = (p_{x0}, p_{y0}).$$

Analogously, denote the moment when the smooth trajectory exits the boundary layer as $t_{out}(\epsilon)$ (we will prove that such a moment exists) and denote the corresponding value of $q_t(\epsilon)$ as $q_{out}(\epsilon)$. The time interval $(t_{out}(\epsilon) - t_{in}(\epsilon))$ will be called *the collision time.* For fixed δ, the limiting values of the introduced quantities as $\epsilon \to 0$ will be denoted as t_{out}, q_{out} (the existence of the limits is given by Theorem 1 below).

It is natural to call the relation between the limits q_{out} and q_{in} *the reflection law.* By definition, $q_{out}(\epsilon)$ and $t_{out}(\epsilon)$ are functions of q_{in}. If the convergence of $\lim_{\epsilon\to 0}(q_{out}, t_{out})(\epsilon)$ is uniform in some neighborhood of a given q_{in}, then the reflection law is C^0. If, moreover, there is a uniform convergence for the derivatives with respect to q_{in}, then these limit to $\dfrac{\partial(q_{out}, t_{out})}{\partial q_{in}}$, so the reflection law is C^1.

Note that the relation between the reflection laws corresponding to different values of δ is found trivially for the billiard flow, and it is absolutely the same for the Hamiltonian flow because it limits to the billiard flow out of any fixed boundary layer. Therefore, no information is lost if one considers the limit of the reflection law as $\delta \to 0$, as it is done in the following theorem.

Theorem 1. *For the Hamiltonian system (2.5) where the potential $V(x, y; \epsilon)$ satisfies conditions* **I-IV**, *if initial conditions q_0 are such that for the billiard orbit $b_t q_0$ the point of reflection is not a corner: $(x_c, y_c) \in S\backslash C$, then for any sufficiently small δ the limits (as $\epsilon \to 0$) q_{out} and t_{out} are well defined. As $\delta \to 0$, the collision time tends to zero:*

$$\lim_{\delta\to 0}(t_{out} - t_{in}) = 0 \tag{2.18}$$

and the limiting C^0 reflection law is:

$$\begin{aligned} &(x_{out}, y_{out}) = (x_{in}, y_{in}) \\ &(p_{x,out}, p_{y,out}) + (p_{x,in}, p_{y,in}) = 2(p_{x,in}e_x + p_{y,in}e_y)(e_x, e_y) \end{aligned} \tag{2.19}$$

where $\bar{e} = (e_x, e_y)$ is the unit vector tangent to the boundary at the point (x_c, y_c).

If, additionally, condition **V** *is fulfilled and the ingoing velocity vector $(p_{x,in}, p_{y,in})$ is not tangent to the boundary at the point (x_c, y_c), then the reflection law is C^1.*

One may check that the above reflection law is exactly the reflection law associated with the billiard flow. In other words, theorem 1 says that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}||(q_{out}(\epsilon), t_{out}(\epsilon)) - (q_{out}(0), t_{out}(0))|| = 0 \tag{2.20}$$

where the norm is C^0- or C^1-norm in a small neighborhood of q_{in}. Since out of the boundary layer the Hamiltonian flow limits to the billiard flow as $\epsilon \to 0$, this local result implies immediately the following global version.

Theorem 2: *If q_0 and $q_t = b_t q_0$ are inner phase points, then, as $\epsilon \to 0$, the time t map $h_t(\epsilon)$ of the flow defined by Hamiltonian (2.5) where $V(x, y; \epsilon)$ satisfies assumptions* **I-IV** *limits to the map b_t in the C^0-topology in a small neighborhood of q_0. If, additionally, condition* **V** *is fulfilled and if the billiard trajectory of q_0 has no tangencies to the boundary for the time interval $[0, t]$, then $h_t(\epsilon) \to b_t$ in the C^1 sense.*

Theorem 2 follows from theorem 1, and vice versa. The proof of the theorems (in fact, a C^r -convergence proof) is given in [25]. Namely, the following is proved there

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} ||(q_{out}(\epsilon), t_{out}(\epsilon)) - (q_{out}(0), t_{out}(0))|| = 0 \qquad (2.21)$$

which is formally weaker than (2.20), but it is, obviously, also sufficient for the validity of theorem 2.

The general idea of the proof is as follows (see details in [25]). By condition **II**, the gradient of the potential is close to normal to the boundary near the point of reflection. This implies, almost immediately, that the tangential component p_x of the momentum is approximately preserved during the collision. Essentially, this means that the motion described by the Hamiltonian system (2.5) can be thought as a sum of two almost independent motions: inertial motion parallel to the boundary and reflection in the normal direction. In the limit $\epsilon \to 0$, the parallel motion prevails in some sense for the nearly tangent trajectories, whereas for the non-tangent trajectories its contribution can be neglected. Thus, in both cases the consideration is essentially one-dimensional and this makes the proof of the C^0 part of theorem 1 pretty simple. The proof of the C^1 version is more involved and it requires estimates of some integrals along the orbit of the Hamiltonian system, necessary for the evaluation of the solution of the linearized equations.

A more specified way to formulate closeness of the Hamiltonian system under consideration to the billiard approximation is to use the Poincaré sections. Let q_0 and $q_t = h_t(\epsilon) q_0$ ($\epsilon \geq 0$) be inner phase points and ω_0 and ω_1 be small surfaces transverse to the flow near q_0 and q_t. Then the flow defines the local Poincaré map $h_{t_f}(\epsilon) : \omega_0 \to \omega_1$ where $t_f(\epsilon)$ is *the flight time* from ω_0 to ω_1. The Poincaré map preserves the foliation of the cross-sections by the levels of equal energy. Therefore, reduced Poincaré maps are defined taking fixed energy levels on ω_0 onto the levels of the same energy on S_1. For $\epsilon > 0$ (respectively $\epsilon = 0$) the reduced Poincaré map is a two-dimensional area-preserving C^r-diffeomorphism (respectively - almost everywhere C^r-diffeomprphism). Obviously, the flow is recovered by the set of reduced Poincaré maps along with the corresponding flight times, and vice versa. Thus, theorem 2 admits the following reformulation.

Theorem 3. *If q_0 and $q_t = b_t q_0$ are inner phase points and ω_0 and ω_1 are small cross-sections through q_0 and q_t respectively, then at all small ϵ the Hamiltonian flow (2.5) satisfying conditions* **I-IV** *defines the reduced Poincaré map of the the energy level of q_0 in ω_0 into ω_1. As $\epsilon \to 0$ this map* *limits (in C^0) to the reduced Poincaré map of the billiard flow as does the flight time. In addition, if condition* **V** *is satisfied and the segment of billiard trajectory between q_0 and q_t does not have tangencies to the boundary of the billiard domain, then the convergence is C^1.*

The last theorem allows one to utilize persistence theorems regarding two-dimensional area preserving diffeomorphisms in order to establish relations between periodic orbits of the billiard flow and of the Hamiltonian flows under consideration.

Recall that an orbit (e.g., a periodic orbit) of the billiard flow is called *non-singular* if its trajectory in the (x, y)-plane does not have tangencies with the boundary of the billiard domain (and by definition the trajectory cannot hit a corner either). For a non-singular periodic orbit, for a cross-section through an inner point on it, the reduced Poincaré map of the billiard flow is locally a diffeomorphism and the intersection of the periodic orbit with the cross-section in the phase space is a fixed point of the diffeomorphism. Generally, the fixed point is either hyperbolic or elliptic. Fixed points of both types are preserved under small smooth perturbations in the class of area preserving diffeomorphisms. Thus, theorem 3 implies the following statement.

Corollary 1 - persistence of periodic orbits: *If a non-singular periodic orbit L_0 of the billiard flow is hyperbolic or elliptic, then at ϵ sufficiently small the Hamiltonian flow $h_t(\epsilon)$ has a unique continuous family of hyperbolic or, respectively, elliptic periodic orbits L_ϵ in the fixed energy level of L_0 which limit to L_0 as $\epsilon \to 0$.*

If L_0 is hyperbolic, the local stable ($W^s_{\mathrm{loc}}(L_\epsilon)$) and unstable ($W^u_{\mathrm{loc}}(L_\epsilon)$) manifolds of L_ϵ depend continuously on ϵ (as smooth manifolds) and limit to $W^s_{\mathrm{loc}}(L_0)$ and $W^u_{\mathrm{loc}}(L_0)$ respectively. The global stable and unstable manifolds - $W^u(L_\epsilon)$ and $W^s(L_\epsilon)$ - are obtained as the continuation of $W^s_{\mathrm{loc}}(L_\epsilon)$ and $W^u_{\mathrm{loc}}(L_\epsilon)$ by the orbits of the flow. Note that for the billiard flow, by applying the continuation process tangencies to the boundary and corner points are bound to be encountered by some points belonging to the manifolds. Using local cross-sections as above, it is easy to see that the following result holds.

Corollary 2 - extensions of stable and unstable manifolds: *Any piece K_0 of $W^u(L_0)$ or $W^s(L_0)$ obtained as a time $t > 0$ shift of some region in $W^u_{\mathrm{loc}}(L_0)$ (respectively, a time $t < 0$ shift of some region in $W^s_{\mathrm{loc}}(L_0)$) is a C^0- or, if no tangencies to the boundary are encountered in the continuation process, C^1-limit of a family of surfaces $K_\epsilon \subset W^u(L_\epsilon)$ (resp. $K_\epsilon \subset W^s(L_\epsilon)$).*

The above persistence results apply only to non-singular periodic orbits; near the singular periodic orbits the billiard flow is non-smooth and the standard theory is not valid. However, it is of interest to study the behavior near a singular periodic orbit for $\epsilon > 0$. We consider this problem in the next section for the case of so-called *scattering* billiards. Here, the billiard flow is hyperbolic whence all non-singular periodic orbits are hyperbolic. We, nevertheless, show that the singular periodic orbits give rise to stable (elliptic) periodic orbits in the Hamiltonian systems (2.5) limiting to the scattering billiards.

3 Appearance of elliptic islands in the smooth Hamiltonian approximation of scattering billiards

Consider scattering billiards - namely billiards which are composed of concave arcs with the curvature bounded away from zero, and non-zero angles between the arcs at the corner points. The corresponding billiard flows are hyperbolic and exhibit strong ergodic properties (they are K-systems) [21, 1, 11]. In particular, almost every orbit covers the whole phase space densely. In this section we examine how these properties may be lost by the approximating smooth Hamiltonian flows for arbitrarily small positive ϵ values. We propose two mechanisms for the appearance of elliptic islands which destroy these properties: one mechanism is controlled by the existence, in the billiard flow, of a singular periodic orbit and another mechanism is controlled by the existence of a singular homoclinic orbit. To be specific, from here on, we consider only *simple singular* orbits; i.e., those for which the corresponding trajectories in the billiard domain have exactly one tangency to the billiard boundary and do not approach corner points.

First, we study the phase space structure of the local Poincaré map near such orbits, showing that locally these create a "sharp" horseshoe which, embedded in a one parameter family of billiard maps, unravels as the parameter γ varies (see figure 3.3). Then, using theorem 3, we establish that the two parameter family of Hamiltonian flows $h_t(\epsilon;\gamma)$ which approach the family of billiards as $\epsilon \to 0$ undergoes, for sufficiently small ϵ, a series of bifurcations associated with the disappearance of a Smale's horseshoe. It is well established that in this process elliptic islands are created. Thus, it follows that for each sufficiently small ϵ there exist intervals of γ values for which elliptic islands exist.

We end the section with some conjectures on the genericity of the phenomena mentioned above: we expect that singular homoclinic and periodic orbits are, in fact, unavoidable in scattering billiards. Apparently, systems possessing simple singular homoclinic and periodic orbits are dense among all scattering billiards. We provide a numerical example which supports such a conjecture regarding the density of billiards with singular homoclinic orbits. A proof of this conjecture combined with the results presented here would imply that *for any given scattering billiard on a plane, there exists a nearby Hamiltonian flow possessing elliptic islands.*

3.1 Singular periodic orbits.

The hyperbolic structure of the scattering billiards plays a crucial role in the understanding of the behavior near a singular periodic orbit. For the billiard map B (the map relating two consecutive collision points; see section 2.1), the presence of hyperbolic structure implies that for almost every point $P(s,\phi)$ in the phase space there exist stable and unstable directions E_P^u and E_P^s, depending continuously on P. The system of stable and unstable directions is invariant with respect to the linearized map: $d_P B E^{s(u)} = E_{BP}^{s(u)}$, which is uniformly expanding along the unstable direction and uniformly contracting along the stable direction: if $v \in E^u$ ($v \in E^s$), then $||d_P B v|| \geq e^{\lambda\tau}||v||$ (resp., $||d_P B v|| \leq e^{-\lambda\tau}||v||$) in a suitable norm; here, τ is the flight time from P to PB, the uniformity means that the value $\lambda > 0$ is independent of P (see details in [3]).

Equivalently, there is an invariant family of stable and unstable cones: the unstable cone at a point P is taken by the linearized map $d_P B$ into the unstable cone at the point BP; the image is stretched in the unstable direction and shrinks in the stable direction. Similar behavior appears for the stable cone under backward iterations. There is an explicit geometrical description of these cones for scattering billiards [26]. Consider a point (s,ϕ) in the phase space and a small curve passing through this point. Taking two points on this curve defines two inward directed rays emanating from the billiard boundary near s (see figure 3.1). If these rays intersect, then the tangent direction to this curve belongs to the stable cone of (s,ϕ); otherwise, it belongs to the unstable cone (in other words, the unstable cones are given by $ds \cdot d\phi > 0$ and the stable cones by $ds \cdot d\phi < 0$). Moreover, it can also be shown that if the intersection of the rays with each other occurs before the first intersection of the rays with the billiard boundary, then the tangent direction to the forward image of the small curve under consideration belongs to the unstable cone of the image of (s,ϕ).

It follows from the simple geometry above that the tangents to a line of singularity at any point lies in the stable cone, and the tangent to any iteration of the singularity line by the billiard map lies in the corresponding unstable cone. In particular, this implies that intersections of the singularity lines with their images are always transverse.

Next, we find the normal form of the first return map of the billiard map near a simple singular periodic orbit (a periodic orbit with only one tangency). More precisely, consider a periodic orbit L with the corresponding sequence of collision points $P_i(s_i,\phi_i)$ ($i = 0,\dots n-1$): $P_{i+1} = BP_i$ where $P_n = P_0$. Since L is a simple singular periodic orbit, assume that $P \equiv P_0$ belongs to the singular set (so $|\phi_1| = \frac{\pi}{2}$). Take a small neighborhood U of P and denote as Σ the line of singular points in U (it is the line composed of the points whose trajectories are tangent to the billiard boundary near s_1). Then, we prove the following proposition:

Proposition 3.1 *Given a simple singular periodic orbit L as above, the local return map near P_0 may be reduced to the form:*

$$\begin{cases} \bar{u} = v \\ \bar{v} = \xi(v - \sqrt{\max(v,0)}) - u + \dots \end{cases} \tag{3.1}$$

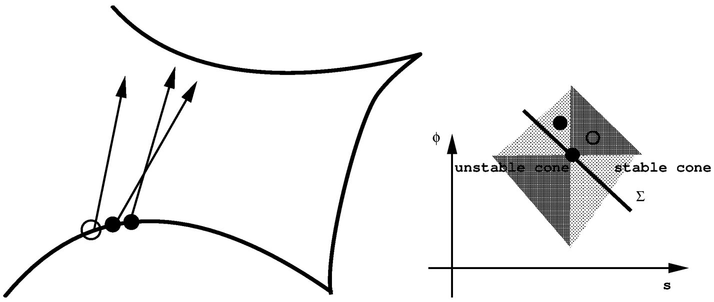

Fig. 3.1. Hyperbolic structure - the stable and unstable cones

a) Geometrical interpretation of stable/unstable directions
b) Phase space structure

where $v=0$ *gives the singularity line,* $u=0$ *is its image, and* $|\xi|>2$*.*

As will be apparent by the proof, it is useful to define an auxiliary billiard $B^{(r)}$, for which the boundary arc by which the tangency of the periodic orbit occurs (i.e. near s_0) is pushed slightly backwards so that the singular periodic orbit becomes a regular orbit for the auxiliary system. The quantity ξ in (3.1) is simply the trace of the linearization matrix of the first return map of the auxiliary billiard about the periodic orbit. Since the auxiliary billiard is scattering, its regular periodic orbits are hyperbolic, hence $|\xi|>2$.

Proof of Proposition 3.1 Consider the local structure in U, near the singularity line Σ. The line Σ divides U into two parts, U_r and U_s; the orbits starting on U_r (e.g. P_0'' in figure 3.2) do not hit the boundary near s_1 and approach it near the point s_2, the orbits starting on U_s (e.g. P_0' in figure 3.2) have a nearly tangent collision with the boundary in a neighborhood of s_1. Without loss of generality we assume that Σ is locally a straight line $(s-s_0)+k(\phi-\phi_0)=0$, where $k>0$ because Σ must lie in the stable cone $(s-s_0)(\phi-\phi_0)<0$, and that U_r is given by $(s-s_0)+k(\phi-\phi_0)<0$ and U_s by $(s-s_0)+k(\phi-\phi_0)\geq 0$.

Consider *the first return map* $\bar{B}$ defined on U. The map $\bar{B}$ equals $B_{n-1}\dots B_2B_1B_0$ on U_s and $B_{n-1}\dots B_2B_0$ on U_r where B_i is a restriction of the billiard map on a small neighborhood of P_i. According to section 2.1.1, $\bar{B}$ is a continuous map but it loses smoothness on Σ. Namely, the restriction B_{0s} of B_0 on U_s exhibits the square root singularity described in section 2.1.1 whereas the map $B|_{U_r}$ is regular and it can be continued onto the whole U as a smooth map B_{0r}: erasing a small piece of the boundary containing the tangency point s_1, B_{0r} will simply be the billiard map from U to a small neighborhood of P_2 (see the action of B_{0r} on P_0' in figure 3.2). Obviously, $B_{0r}\Sigma=B_1B_{0s}\Sigma$, therefore the first return map $\bar{B}$ is continuous. One may represent the map $\bar{B}$ as a superposition of regular and singular maps:

$$\bar{B}=B^{(r)}\cdot B^{(s)}$$

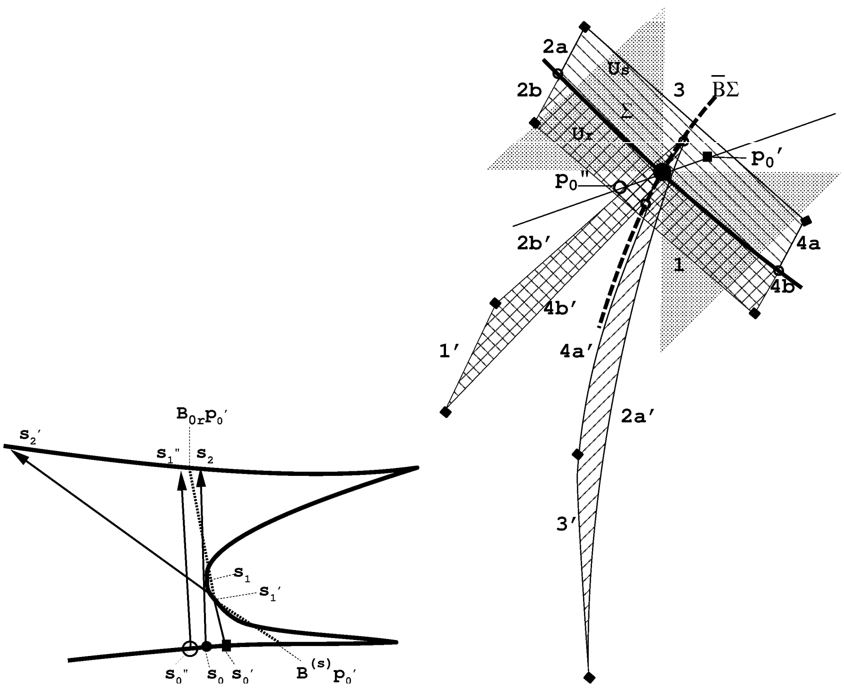

Fig. 3.2. Structure near singular periodic orbit

a) Action of billiard map near a singular segment of trajectory
b) Phase space structure near singular periodic orbit: 1234 is mapped onto 1′2′3′4′

where

$$B^{(r)} = B_{n-1} \dots B_2 B_{0r}$$

and

$$B^{(s)} = \begin{cases} id & \text{on } U_r \\ B_{0r}^{-1} B_1 B_{0s} & \text{on } U_s \end{cases}$$

The singular part $B^{(s)} : U \to U$ may be obtained by inverted reflection near the tangency point s_1 (see the action of $B^{(s)}$ on P_0' in figure 3.2). It is not hard to calculate that $B^{(s)}$ is given by

$$\begin{cases} S' = S + k\sqrt{\max(S + k\Phi, 0)} + \dots \\ \Phi' = \Phi - \sqrt{\max(S + k\Phi, 0)} + \dots \end{cases}$$

where $S = s - s_0$, $\Phi = \phi - \phi_0$ are coordinates in U, and the dots stand for the quantities infinitely small in comparison with S, Φ or $\sqrt{\max(S + k\Phi, 0)}$ as $S, \Phi \to 0$.

The regular part $B^{(r)}$ is, by definition, the first return map for the auxiliary billiard obtained by pushing the boundary near the tangency point s_1

slightly aside the trajectory of L. The point P is a fixed point for $B^{(r)}$ (as well as for the map $\bar{B}$). Since the auxiliary billiard is still scattering, the point P is a hyperbolic fixed point for $B^{(r)}$. Moreover, the unstable cone $S \cdot \Phi \geq 0$ must be mapped inside itself by the linearization of $B^{(r)}$ at P. If $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is the corresponding linearization matrix, the last condition is equivalent to the requirement that all b_{ij} are of same sign. Recall that $B^{(r)}$ is an area-preserving diffeomorphism, so

$$b_{11}b_{22} - b_{12}b_{21} = 1.$$

Superposition of $B^{(r)}$ and $B^{(s)}$ gives, to leading order in S, Φ and $\sqrt{\max(S + k\Phi, 0)}$, the following formula for the map $\bar{B}$:

$$\begin{cases} \bar{S} = b_{11}S + b_{12}\Phi - (b_{12} - b_{11}k)\sqrt{\max(S + k\Phi, 0)} + \dots \\ \bar{\Phi} = b_{21}S + b_{22}\Phi - (b_{22} - b_{21}k)\sqrt{\max(S + k\Phi, 0)} + \dots \end{cases} \tag{3.2}$$

Provided inequalities 3.5 are satisfied, as proved in the lemma below, the normal form 3.1 is obtained from the above expression by changing to the new coordinates u, v where u is aligned with the singularity line ($v \propto S + k\Phi$) and v is aligned with its image. From the calculation, it follows that the quantity ξ is $(b_{11} + b_{22})$, namely the sum of eigenvalues of the linearization of the regular part $B^{(r)}$ of $\bar{B}$ at P. Since the product of the eigenvalues equals to 1 and since they do not lie on the unit circle, it follows that

$$|\xi| > 2, \tag{3.3}$$

as indicated in the Proposition. □.

Lemma 3.1: *The coefficients b_{ij} in (3.2) obey the inequalities:*

$$(b_{12} - b_{11}k)(b_{22} - b_{21}k) \quad > 0 \tag{3.4}$$

$$|b_{12}| < |b_{11}|k \tag{3.5}$$

$$|b_{22}| < |b_{21}|k. \tag{3.6}$$

Proof: Since the image $\bar{B}\Sigma$ of the singularity line $S + k\Phi = 0$ must lie in the unstable cone $\bar{S} \cdot \bar{\Phi} > 0$, it follows from 3.2 that the first inequality $(b_{12} - b_{11}k)(b_{22} - b_{21}k) > 0$ holds. Moreover, it is geometrically evident that for a small piece l of a straight line through P which lies in the unstable cone, i.e., for which the increase of s is followed with the increase of ϕ (see figure 3.2 - imagine a line going through P_0'', P_0, P_0') the image of $l \cap U_r$ by B_0 and the image of $l \cap U_s$ by $B_1 B_0$ lie both to one side of the point P_2 (or s_2 when projected to the configuration plane). In other words, these images belong both to the same half of the unstable cone of P_2 corresponding to a definite sign of $(s - s_2)$. Since the linearization of each of the maps B_i preserves the decomposition into the stable and unstable cones, it follows that the image of l by $\bar{B}$ is a folded line with the vertex at P which divides $\bar{B}l$ in two parts belonging both to the same half of the unstable cone of P; i.e., $\bar{S}$ and $\bar{\Phi}$ have the same sign on $\bar{B}(l \cap U_r)$ and $\bar{B}(l \cap U_s)$. By (3.2), it is equivalent to the condition that the sign of $(b_{12} - b_{11}k)$ is opposite to the sign of b_{12} and b_{11} and the sign of $(b_{22} - b_{21}k)$ is opposite to the sign of b_{22} and b_{21} (recall that all b_{ij} are of same sign). Thus, the second and third inequalities $|b_{12}| < |b_{11}|k$ and $|b_{22}| < |b_{21}|k$ hold. □.

Now, embed the billiard under consideration in a one parameter family of scattering billiards $b_t(\cdot;\gamma)$ for which all arcs depend smoothly on the parameter γ, while the corner points are held fixed; we suppose that the billiard with the simple singular periodic orbit L is realized at $\gamma = 0$. The regular part $B^{(r)}$ of the first return map of U depends smoothly on γ, hence its hyperbolic fixed point $P_\gamma^{(r)}$ is also a smooth function of γ. The same is valid for the position of the singularity line Σ_γ. For a *general* family of billiards, the parameterization by γ may be chosen so that the distance between $P_\gamma^{(r)}$ and Σ_γ is proportional to γ (it is true if, for instance, one changes the billiard boundary locally, near the tangency point s_1 only: such a perturbation moves the singularity line but the map $B^{(r)}$ and the position of its fixed point remain unchanged). Assume, with no loss of generality, that $P_\gamma^{(r)} \in U_r$ for $\gamma > 0$ and that $P_\gamma^{(r)} \in U_s$ for $\gamma < 0$. Therefore, by the definition of $B^{(r)}$, its fixed point is a fixed point of $\bar{B}$ for $\gamma > 0$, and its fixed point is imaginary when $\gamma < 0$.

Thus, for such a family of billiards, the normal form (3.1) of the first return map $\bar{B}$ is now rewritten as

$$\begin{cases} \bar{u} = v \\ \bar{v} = \xi(\gamma + v - \sqrt{\max(v,0)}) - u + \dots \end{cases} \tag{3.7}$$

In this form, the map $\bar{B}_\gamma$ looks similar to the well-known Hénon map but it has another type of nonlinearity. In fact we show below: **Proposition 3.2** *Consider the map (3.7). For a small fixed neighborhood U of the origin, let Ω_γ be the set of all orbits of $\bar{B}_\gamma$ which never leave U. Then there exist $\gamma^\pm$ values such that $\Omega_\gamma = \emptyset$ for $\gamma < \gamma^- < 0$, and if $\gamma > \gamma^+ > 0$ and small, then Ω_γ is in one-to-one correspondence with the set of all sequences composed of two symbols (r, s): "r" corresponds to entering U_r and "s" corresponds to entering U_s.*

Proof: Indeed, take a small $\delta > 0$ and let the neighborhood U be a rectangle $\{-\delta < u < \kappa\delta,\ -\delta < v < \kappa\delta\}$ where $\kappa = \frac{1}{2}(\frac{1}{2}|\xi| - 1) > 0$ (recall that $|\xi| > 2$). Let $\gamma^+ = (\frac{1}{2} - \frac{1}{\xi})\delta > 0$ and $\gamma^- = -\frac{2}{|\xi|}\delta$. Then, for sufficiently small δ, one may check that for the given choice of U the map (3.7) takes the horizontal boundaries of U (marked 1 and 3 in figure 3.3) on a finite distance of U for all $\gamma \in [\gamma^-, \gamma^+]$. The images of the vertical boundaries 2 and 4 which intersect the singularity line, fold as indicated in figure 3.3: the segments 2a,4a are mapped to 2a',4a' and the segments 2b,4b are mapped to 2b',4b'. The folded lines 2',4' may intersect U but they lie on a finite distance of their preimages (the boundaries 2 and 4) for all $\gamma \in [\gamma^-, \gamma^+]$. Thus, the image of U by $\bar{B}_\gamma$ has a specific shape of a sharp horseshoe. Changing γ shifts the horseshoe along the v-axis, so at $\gamma = \gamma^+$ the intersection of the horseshoe with U consists of two distinct connected components (figure 3.3b). On each component the map $\bar{B}_\gamma$ is smooth and hyperbolic. The statement regarding the one-to-one correspondence to Bernoulli shift on two symbols follows as in the standard construction of the horseshoe map [23, 18]. *In particular, it implies that each of the two components has a hyperbolic fixed point.* On the other hand, at $\gamma = \gamma^-$ the intersection of $\bar{B}_\gamma U$ with U is empty (figure 3.3c) and no fixed points may exist in U. □.

Notice the following three important conclusions from the proof of the above proposition: first that there exist $\gamma^\pm$ values such that for γ^+ two hyperbolic fixed points exist and for γ^- no fixed points exist in the square region U near the intersection of the singularity line with its image. Second that $\gamma^\pm$ may be chosen arbitrarily small (by taking smaller U). Third, no fixed points can pass through the boundary of U as γ varies from γ^- to γ^+ because the image of the horizontal boundaries of U never intersects the boundary of U and the image of the vertical boundaries U may intersect only the horizontal parts of the boundary.

Now, take a *two-parameter* family of Hamiltonians $H(\cdot;\epsilon,\gamma)$ which approach the family of billiard flows $b_t(\cdot;\gamma)$ as $\epsilon \to 0$, in the sense that conditions **I-V** are satisfied uniformly with respect to γ. Note that for the billiard flow, the structure of the Poincaré map of an arbitrary small cross-section ω through an *inner* point on the simple singular periodic orbit L is absolutely the same as described above (because the map $\bar{B}$ is a particular case of the Poincaré map, corresponding to the cross-section made of collision points, and different Poincaré maps are smoothly conjugate near L; see section 2.1.1). Due to the C^0-closeness result of theorem 3, it follows that for ϵ sufficiently small the corresponding Poincaré map $\Pi_{\epsilon\gamma}$ for the Hamiltonian system transforms a rectangle $U' \subset \omega$ (analogous to the rectangle U) to a horseshoe shape (which is now smooth because the Hamiltonian system is smooth at all $\epsilon > 0$). At $\gamma = \gamma^-$ the intersection $\Pi_{\epsilon\gamma}U' \cap U'$ is empty for small ϵ whence $\Pi_{\epsilon\gamma^-}$ has no fixed points in U'. Moreover, no fixed points can pass through the boundary of U' as γ varies from γ^- to γ^+ because the fixed points of the first return billiard map stay on a finite distance from the boundary of U' for all $\gamma \in [\gamma^-, \gamma^+]$.

The two fixed points of the Poincaré map of the billiard flow which exist at $\gamma = \gamma^+$ are hyperbolic and do not belong to the singularity line. Thus, by the corollary 1 to theorem 3, each of these hyperbolic fixed points exists for the map $\Pi_{\epsilon\gamma^+}$ at all sufficiently small ϵ. Now, fixing any ϵ small enough, a fixed point of $\Pi_{\epsilon\gamma^+}$ changes continuously as γ decreases, until it merges with some other fixed point (as we argued, the fixed point must disappear before

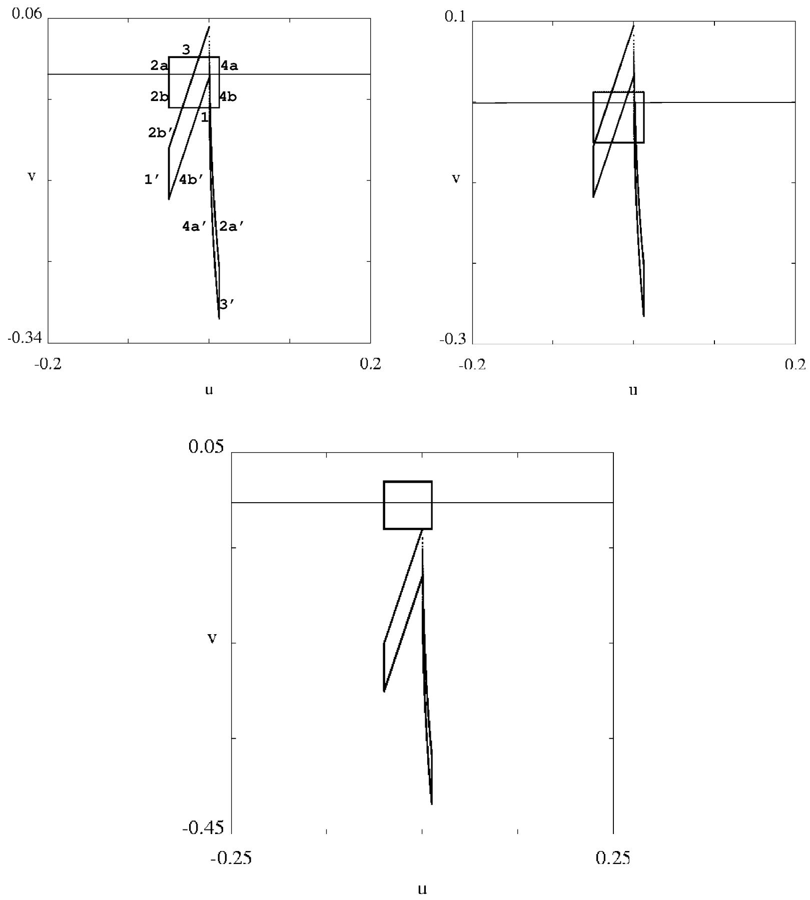

Fig. 3.3. Sharp horseshoe bifurcation near singular periodic orbit

One iterate of the indicated box by the truncation
$(\bar{u} = v,\ \bar{v} = \xi(\gamma + v - \sqrt{\max(v,0)}) - u)$ of the normal form (3.7)
in all figures $\xi = 3,\ \delta = 0.05$.
a) $\gamma = 0$ b) $\gamma = 0.015 > \gamma^+ = 1/160$ c) $\gamma = \gamma^- = -1/30$.
$\bullet$ - period n point.

$\gamma = \gamma^-$ and it can not leave U' via crossing the boundary). In a *general* family of sufficiently smooth Hamiltonian systems, one of the merging fixed points is necessary saddle and another is elliptic. Thus, we have established that

generically, for each ϵ small enough, there exists an interval of values of γ for which the smooth Hamiltonian system possesses an elliptic periodic orbit.

Without genericity assumptions, we may conclude the following. **Theorem**

4: *If a scattering billiard has a simple singular periodic orbit L, then there exists a one-parameter family of smooth Hamiltonian flows $h_t(\epsilon)$ limiting to the billiard flow as $\epsilon \to 0$ (i.e. satisfying conditions* **I-V***) and for which there exists a sequence of intervals of ϵ values converging to 0 on which elliptic periodic orbits L_ϵ exist in the energy level of L. These elliptic periodic orbits limit to the singular periodic orbit as $\epsilon \to 0$.*

3.2 Singular homoclinic orbits.

Consider a *non-singular* hyperbolic periodic orbit L_0 of the billiard flow. Suppose, its stable and unstable manifolds intersect along some orbit Γ. This is a *homoclinic* orbit; i.e., it asymptotes L_0 exponentially as $t \to \pm\infty$. Assume that Γ is *simple singular* which means that its trajectory has one point of tangency with the billiard's boundary (see figure 1.1 b).

Let $P(s, \phi)$ and $\bar{P}(\bar{s}, \bar{\phi})$ be collision points on Γ: P is the last before the tangency and P is the first after the tangency. By definition, $P = B^2 P$ where B is the billiard map. Consider, in the (s, ϕ) plane, the local segment W^u of the unstable manifold of L_0 to which P belongs. Since the tangent to W^u at P belongs to the unstable cone, it must intersect the singularity line transversely at P. Thus, as explained in the proof of lemma 3.1, the image of W^u in a neighborhood of $\bar{P}$ under the billiard map folds with a sharp square root singularity at $\bar{P}$, see figure 3.4. Now, the point $\bar{P}$ belongs to the stable manifold as well. Since the tangent to W^s belongs to the stable cone, it follows that the folded image of W^u lies to one side of W^s, so a sharp homoclinic tangency is created at $\bar{P}$, as shown in figure 3.4.

In a general family of scattering billiards (as in section 3.1), two transverse homoclinic intersections appear at $\gamma > 0$ and none at $\gamma < 0$. For the corresponding two-parameter Hamiltonian family, arguments analogous to those in the proof of theorem 4 show that
generically, for any ϵ sufficiently small there exists $\gamma^*(\epsilon)$ for which a quadratic homoclinic tangency occurs.

Recall that the occurrence of homoclinic tangencies is a well-known mechanism for the creation of elliptic islands [20]. Thus we have established:

Theorem 5: *If a scattering billiard has a simple singular homoclinic orbit Γ, then there exists a one-parameter family of smooth Hamiltonian flows $h_t(\epsilon)$ satisfying conditions* **I-V***, which limits to the billiard flow as $\epsilon \to 0$ and for which there exist a sequence of intervals of ϵ values converging to zero for which elliptic periodic orbits exist in the energy level of Γ.*

The period of the elliptic periodic orbits mentioned in Theorem 5 goes to infinity as $\epsilon \to 0$. In fact, in the two-parameter family of smooth Hamiltonians elliptic periodic orbits of bounded period limit, as $\epsilon \to 0$, to singular periodic orbits corresponding to $\gamma \neq 0$. Thus Theorems 5 and 4 are very much related. Indeed, like the appearance of stable periodic orbits near a homoclinic tangency is proved in smooth situation (see [12, 20, 13]), one may

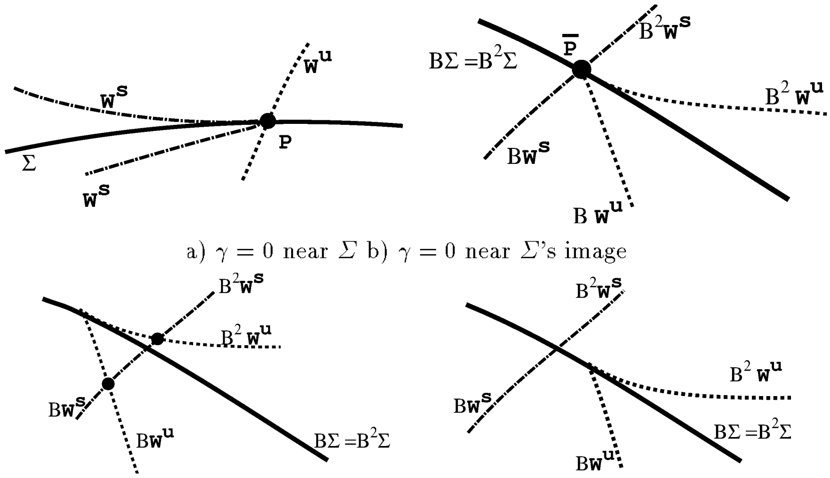

Fig. 3.4. Bifurcation of singular homoclinic orbit

c) $\gamma > 0$ d) $\gamma < 0$, near Σ's image
• - homoclinic points.

show that
in a general family of scattering billiards having a sharp homoclinic tangency at $\gamma = 0$ there is a sequence of values of γ accumulating at $\gamma = 0$ for which singular periodic orbits exist.
Now the reference to theorem 4 gives another proof of theorem 5.

3.3 On the genericity of the elliptic islands creation.

It is well known [17, 2, 3] that for scattering billiards the hyperbolic non-singular periodic orbits are dense in the phase space. The stable/unstable manifolds of such orbits cover the phase space densely and the orbits of their homoclinic intersections also form a dense set.

It follows that the periodic orbits and the homoclinic orbits get arbitrarily close to the singularity set. It seems thus intuitively clear that for any scattering billiard very small smooth perturbations may be applied to place a specific periodic orbit or a specific homoclinic orbit exactly on the singularity line, so that Theorem 4 and 5 may be applied. Proving these intuitive statements turns out to be quite a delicate issue, thus we formulate these as conjectures:

Conjecture 1: *Any scattering billiard may be slightly perturbed to a scattering billiard for which a singular (tangent) periodic orbit exists.*

Conjecture 2: *Any scattering billiard may be slightly perturbed to a scattering billiard for which there exists a non-singular hyperbolic periodic orbit which has a singular homoclinic orbit.*

3.4 Numerically produced singular homoclinic orbits

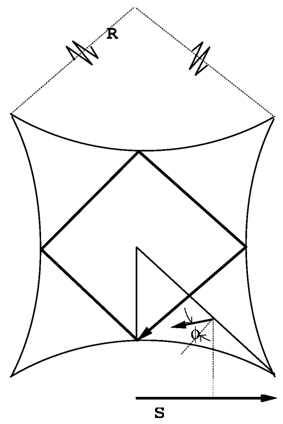

Fig. 3.5. Billiard between four disks

To examine the appearance of singular homoclinic orbits we consider the billiard in a domain bounded by four symmetrical circles

$$x^2 + (y \pm \frac{1}{\gamma})^2 = R^2;\ (x \pm \frac{1}{\gamma})^2 + y^2 = R^2$$

where $R^2 = 1 + (1 - \frac{1}{\gamma})^2$. The quantity γ (which is, approximately, the curvature of the circles) serves as the free parameter for unfolding the singularity. We find explicitly the corresponding billiard map, and using DSTOOL package[10], we find numerically hyperbolic periodic orbits of this mapping and their stable and unstable manifolds. The billiard map is found on the fundamental domain of the billiard - a triangular region cut by an arc as shown in the figure 3.5. We find the return map to the slanted side of the triangle, which is parameterized by s, the horizontal coordinate, and by ϕ, the outgoing angle to the normal vector $(-1, -1)$, see figure 3.5. We choose an arbitrary value of γ and the simplest hyperbolic non-singular periodic orbit, as shown in the figure (the fixed point of the return map to the slanted side of the reduced domain). Then, we construct the stable and unstable manifolds for this periodic orbit. We examine how these manifolds vary by small variation of γ, until we find a value of γ for which singular homoclinic orbit appears. The success (see figure 3.6 and figure 3.7) of the very crude search for such a delicate phenomena, near every γ value we have chosen, supports conjecture 2 regarding the density of systems for which such orbits exist. In fact we have found, by such a search near $\gamma_i = i * 0.05, i = 1, \ldots, 10$, eleven sharp homoclinics to this specific periodic orbit (at $\gamma = 0.0837, 0.10165, 0.1018, 0.153, 0.2077, 0.2552, 0.29245, 0.3329, 0.3832, 0.4143, 0.4692$).

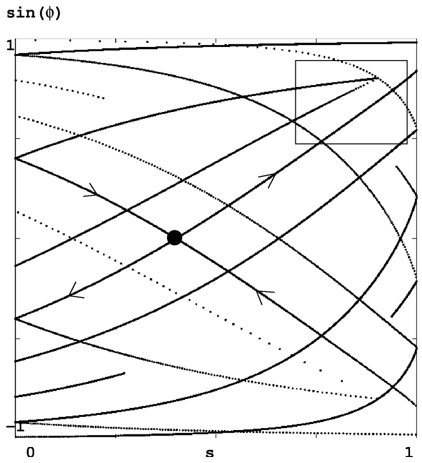

Fig. 3.6. Numerically produced sharp homoclinics

4 Conclusions

There are two main results in this paper; First, we have found sufficient conditions for establishing that a family of smooth Hamiltonian flows limits to the singular billiard flow (see Theorem 1, 2 and 3). These conditions are fulfilled by smooth Hamiltonians with potentials approaching a step function in almost arbitrary way (see section 2.3); they fail, nevertheless, when the potentials are highly oscillatory (i.e., condition IV or V fails).

Second, we have established that if a scattering billiard (we use the particular hyperbolic structure associated with such billiards) has a singular periodic orbit or a singular homoclinic orbit, then there exist arbitrarily close to it smooth Hamiltonian flows which possess elliptic islands, hence these are not ergodic (Theorem 4 and 5). Finally, we have conjectured, and have provided numerical support to these conjectures, that in fact scattering billiards with singular periodic orbits and singular homoclinic orbits are dense among scattering billiards (conjectures 1 and 2 of section 3.3). If these conjectures are correct, then theorems 4 and 5 will imply that arbitrarily close to any scattering billiard there exists a family of non-ergodic smooth Hamiltonian flows.

Such statements imply that *ergodicity and mixing results concerning two-dimensional non-smooth systems cannot be directly applied to the smooth dy-*

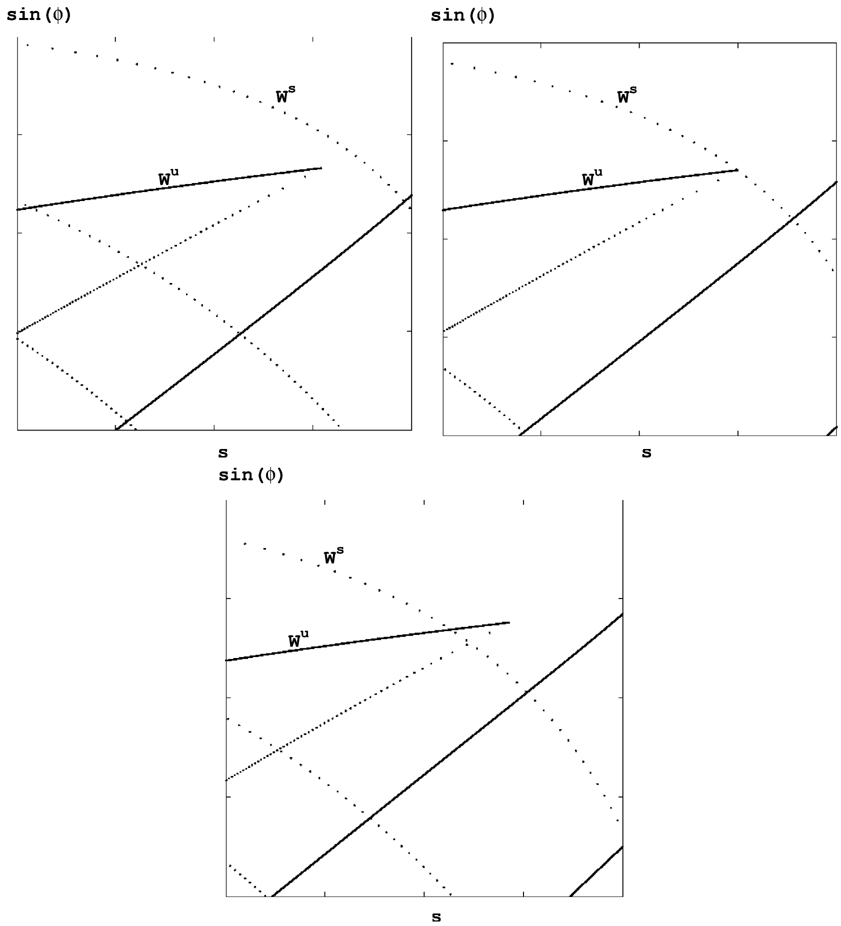

Fig. 3.7. Magnification near numerically produced sharp homoclinics
a) $\gamma = 0.28$ b) $\gamma = 0.29245$ c) $\gamma = 0.31$

namics they model. Whether the same holds for higher dimensional systems, e.g. three-dimensional billiards or multi-particle billiards, is yet to be studied.

On the other hand, eventhough stability islands may appear in smooth billiard-like problems, the size of an individual island is expected to be very small. Thus,with no doubt, while the smooth flow may be non-ergodic, it will "seem" to be ergodic for a very long time; Statistics (e.g. correlation function) which are based upon finite time realizations may appear to behave as in the scattering billiards (e.g. fall off quasi-exponentially [5]). Whether longer realizations will reveal very different statistical properties, depends on the number of elliptic islands, the total area they cover in the phase space and on the "typical" period of the islands. Thus, estimates of the islands sizes, their periods, and of the real potential steepness (the "physical ϵ") are necessary to supply estimates on the time scale for which the mixing property will appear to hold.

We may try to estimate the periodicity of the elliptic periodic orbits of smooth flows approaching generic scattering billiards, by very naive arguments. Indeed, since stable periodic orbits are generated from singular periodic orbits of the billiard, one may expect (if conjecture 1 is correct) that the least period of stable periodic orbits of a smooth Hamiltonian system which is ϵ-close to the billiard is of the order of the Poincaré return time to an ϵ-neighborhood of the singularity surface for the billiard flow. Notice that the billiard flow is a hyperbolic system; therefore, the return time in the billiard and, correspondingly, the typical period of the stable periodic motions in its smooth approximation must, essentially, be *logarithmic* in ϵ and not of a power-law type. Namely, very small ϵ values, corresponding to very steep potentials, may still produce stability islands which are observable on physical time-scales.

Acknowledgement

This work has been supported by MINERVA Foundation, Munich/Germany. We greatly acknowledge discussions with L. Bunimovich, E. Gutkin, A. Kirillov, J. Marsden, L.P. Shilnikov and U. Smilansky.

A An example of smooth Hamiltonian approximation of the circular billiard with non-reflecting trajectories

Consider the Hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \epsilon V(1 - x^2 - y^2) \tag{A.1}$$

where the potential V is given by

$$V(Q) = \frac{1}{Q}\exp\left(-AQ + \int_{1/Q}^{\infty}\frac{\sin z}{z}dz\right),\ Q > 0, \tag{A.2}$$

with some positive constant A. The potential is of the form $\epsilon V(Q)$ where the pattern function is defined by $Q(x,y;\epsilon) = 1 - x^2 - y^2$ for all ϵ. As $\epsilon \to 0$, the above Hamiltonian satisfies conditions **I - III**, which garuantee that near the boundary, $x^2 + y^2 = r^2 = 1$, the correct elastic reflection rules are approached. Thus one may expect that the motion described by (A.1,A.2) limits to the billiard in the unit circle. We show that this is not the case; there exist initial conditions inside the unit circle for which the orbits of the Hamiltonian system (A.1,A.2) stick to the circle boundary for infinitely long time at arbitrarily small ϵ. Notice that condition **IV** is violated.

The specific choice of V is not too important. Essentially, we use that

$$\liminf_{u\to 0}|\frac{V'}{V}| = A < \infty \quad \text{whereas} \quad \limsup_{u\to 0}|\frac{V'}{V}| = \infty. \tag{A.3}$$

Hamiltonian (A.1) is rotationally invariant, thus the particle's angular momentum Ω:

$$\Omega = r^2\dot\theta \tag{A.4}$$

is preserved. It follows that the system is integrable and may be easily analyzed. In polar coordinates ($x = r\cos\theta, y = r\sin\theta$) the equations of motion are of the form

$$\begin{aligned} \ddot r &= r(\dot\theta^2 + 2\epsilon V'(1-r^2)) = r(\tfrac{\Omega^2}{r^4} + 2\epsilon V'(1-r^2)) \\ \ddot\theta &= -2\tfrac{\dot r\dot\theta}{r}. \end{aligned} \tag{A.5}$$

The radial motion decouples, and is governed by the Hamiltonian:

$$H = \frac{1}{2}\dot r^2 + \frac{1}{2}\frac{\Omega^2}{r^2} + \epsilon V(1-r^2) = \frac{1}{2}\dot r^2 + V_{eff}(r;\Omega,\epsilon) \tag{A.6}$$

The maximal polar radius, r^*, reached by an initial condition $(r_0, \dot r_0)$ with $\Omega \neq 0$ is found from:

$$V_{eff}(r^*;\Omega,\epsilon) = \epsilon V(1-{r^*}^2) + \frac{1}{2}\frac{\Omega^2}{{r^*}^2} = \frac{1}{2}\dot r_0^2 + \frac{1}{2}\frac{\Omega^2}{{r_0}^2} + \epsilon V(1-r_0^2) = h \tag{A.7}$$

As $\epsilon \to 0$, the value of r^* tends to 1. The time spent by the orbit near $r = r^*$ is given by

$$2\int_{r^*}^{r} \frac{ds}{\sqrt{h - V_{eff}(s;\Omega,\epsilon)}}, \tag{A.8}$$

thus it is infinite if:

$$V'_{eff}(r^*;\Omega,\epsilon) = -r^*(\frac{\Omega^2}{{r^*}^4} + 2\epsilon V'(1-{r^*}^2)) = 0 \tag{A.9}$$

(i.e. if $\ddot r = 0$ at $r = r^*$). It follows, that if there exist $(r^*(r_0, \dot r_0, \Omega;\epsilon) > r_0, \epsilon)$ solving (A.7) and (A.9) simultaneously, then, the phase point will move for infinitely long time close to the unit circle with non-zero angular velocity $(\lim_{t\to\infty}\dot\theta = (r_0/r^*)^2\dot\theta_0)$.

Next, we show that such a solution exist for many initial condition and for a sequence of $\epsilon \to 0$ values. First, since $V(Q)$ is a monotonic function, for any $r^* > r_0$ one may find ϵ such that (A.7) is satisfied; moreover, $\epsilon \to 0$ as $r^* \to 1$. Resolving (A.7) with respect to ϵ and plugging the result in (A.9) we get

$${r^*}^2\frac{|V'(1-{r^*}^2)|}{V(1-{r^*}^2) - V(1-r_0^2)} = \frac{1}{(r^*/r_0)^2(\frac{\dot r_0}{r_0\dot\theta_0})^2 + (r^*/r_0)^2 - 1}. \tag{A.10}$$

According to (A.3), this equation is solved by an infinite number of values of r^* (with their corresponding $\epsilon(r^*; r_0, \dot{r}_0, \dot{\theta}_0)$) limiting to $r^* = 1$, provided

$$(\frac{\dot{r}_0}{r_0\dot{\theta}_0})^2 + 1 < r_0^2(1 + \frac{1}{A}). \tag{A.11}$$

Clearly, for any given $A > 0$, and for any $r_0 < \sqrt{\frac{A}{1+A}} < 1$ such initial conditions exist. Summarizing: if the initial conditions satisfy (A.11), then there exist an infinite number of values of ϵ, approaching $\epsilon = 0$, for which the orbit sticks to the boundary for infinitly long time.

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