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BIFURCATIONS OF TWO-DIMENSIONAL DIFFEOMORPHISMS WITH NON-ROUGH HOMOCLINIC CONTOURS

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Two-dimensional diffeomorphisms having two saddle points for which one pair of stable and unstable manifolds intersect transversely and the other pair has a quadratic tangency are considered.

1. Introduction

Consider a C^r-smooth ($r \geq 3$) two-dimensional diffeomorphism f having two saddle fixed points O_1 and O_2 with multipliers λ_i, γ_i where $|\lambda_i| < 1$, $|\gamma_i| > 1$, $i = 1, 2$. Suppose $W^u(O_1)$ and $W^s(O_2)$ intersect each other transversely along a heteroclinic orbit Γ_{12} and $W^u(O_2)$ and $W^s(O_1)$ exhibit a quadratic tangency along a heteroclinic orbit Γ_{21} (Fig. 1). We will say that f has *a non-rough homoclinic contour* $C = O_1 \cup O_2 \cup \Gamma_{12} \cup \Gamma_{21}$.

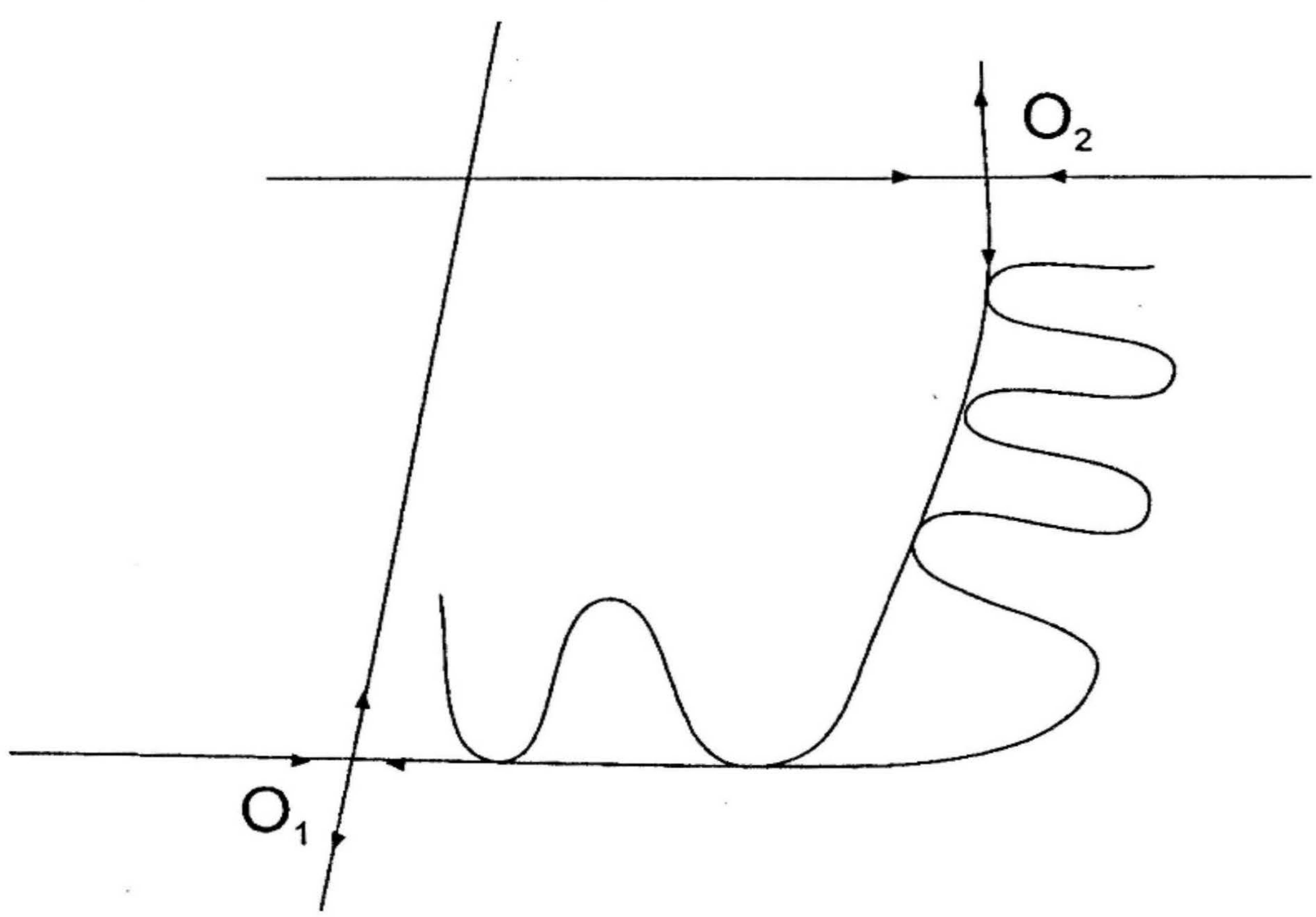

FIG. 1.

Denote by N the set of orbits of f which lie in a sufficiently small neighbourhood U of C. We show that such diffeomorphisms may be divided into three classes, depending on the character of tangency of $W^u(O_2)$ and $W^s(O_1)$ along Γ_{21} and give a description of the structure of the set N. For the first class the structure is trivial, for the second class it admits a complete description in terms of symbolic dynamics. For the third class the following results are obtained: the principal moduli of Ω-conjugacy are found; the density of systems possessing non-rough periodic orbits and the density of systems having infinitely many stable or (and) completely unstable periodic orbits are proved.

2. Three Classes of Diffeomorphisms

It is shown in [1, 2] that in a small neighbourhood U_i of the point O_i there exist C^{r-1}-coordinates (x_i, y_i) such that the map $f_{|U_i}$ has the form

$$\overline{x}_i = \lambda_i x_i + f_i(x_i, y_i) x_i y_i\,, \qquad \overline{y}_i = \gamma_i y_i + g_i(x_i, y_i) x_i) y_i\,, \tag{1}$$

where $f_i(0, y_i) \equiv 0$, $g_i(x_i, 0) \equiv 0$. Equations of the manifolds $W^s_{\rm loc}(O_i)$ and $W^u_{\rm loc}(O_i)$ are $y_i = 0$ and $x_i = 0$, respectively.

Choose two points of orbit Γ_{12}: the point $M^-(0, y^-) \in U_2$ and the point $M^+(x^+, 0) \in U_1$. Evidently, $f^n(M^-) = M^+$ for some integer n. The map $T \equiv f^n$ from some neighbourhood of M^- into some neighbourhood of M^+ may be written in the form

$$\begin{aligned} \overline{x}_1 - x^+ &= a x_2 + b_(y_2 - y^-) + \dots\,, \\ \overline{y}_1 &= c x_2 + d(y_2 - y^-)^2 + \dots\,. \end{aligned} \tag{2}$$

Note that T is a diffeomorphism, therefore, the Jacobian does not vanish at M^-: $bc \neq 0$. Also, $d \neq 0$ because the tangency is quadratic. The character of adjoining of $W^u(O_2)$ to $W^s(O_1)$ at M^+ is determined by signs of c and d (Fig. 2). For instance, $W^u(O_2)$ touches $W^s(O_1)$ from below if $d < 0$, and from above if $d > 0$.

The diffeomorphisms with non-rough homoclinic contours are divided into three classes, depending on signs of λ_1, γ_2, c and d. The combinations $\lambda_1 > 0$, $\gamma_2 > 0$, $c < 0$, $d < 0$ and $\lambda_1 > 0$, $\gamma_2 > 0$, $c < 0$, $d > 0$ correspond to the first and the second classes, respectively (Fig. 2 a and Fig. 2 b). The remaining cases (among them those with negative λ_1 and γ_2) correspond to the third class.

For the diffeomorphisms of the first class the set N is trivial: $N = C$. For the diffeomorphisms of the second class the structure of N is non-trivial. Here, all orbits of the set $N \backslash \Gamma_{21}$ are of saddle type and N admits a complete description in terms of symbolic dynamics.

Diffeomorphisms of the third class also have nontrivial hyperbolic subsets. These subsets, however, may not exhaust all the set $N \backslash \Gamma_{21}$. Moreover, the structure of N changes when the value of the invariant $\theta = -((\ln|\lambda_2|)/(\ln|\gamma_1|))$ changes. Namely, let H_3 be a codimension one bifurcation surface in the space of dynamical systems, composed by diffeomorphisms of the third class.

THEOREM 1. *If $f, f' \in H_3$ and f is Ω-conjugate to f', then $\theta = \theta'$.*

Theorem 1 means that θ is a modulus of Ω-conjugacy [1, 2] for the third class. Moreover, similarly to the case of homoclinic tangency of invariant manifolds of a single saddle point [3, 4], the following result is proved:

THEOREM 2. *Systems with a countable set of moduli of Ω-conjugacy are dense in H_3.*

As it is argued in [3, 4], if a system has Ω-moduli, then changing the values of the moduli causes bifurcations of nonwandering orbits (in particular, periodic and homoclinic orbits). Moreover, when a countable set of Ω-moduli exists, we deduce that those bifurcations may be very complex.

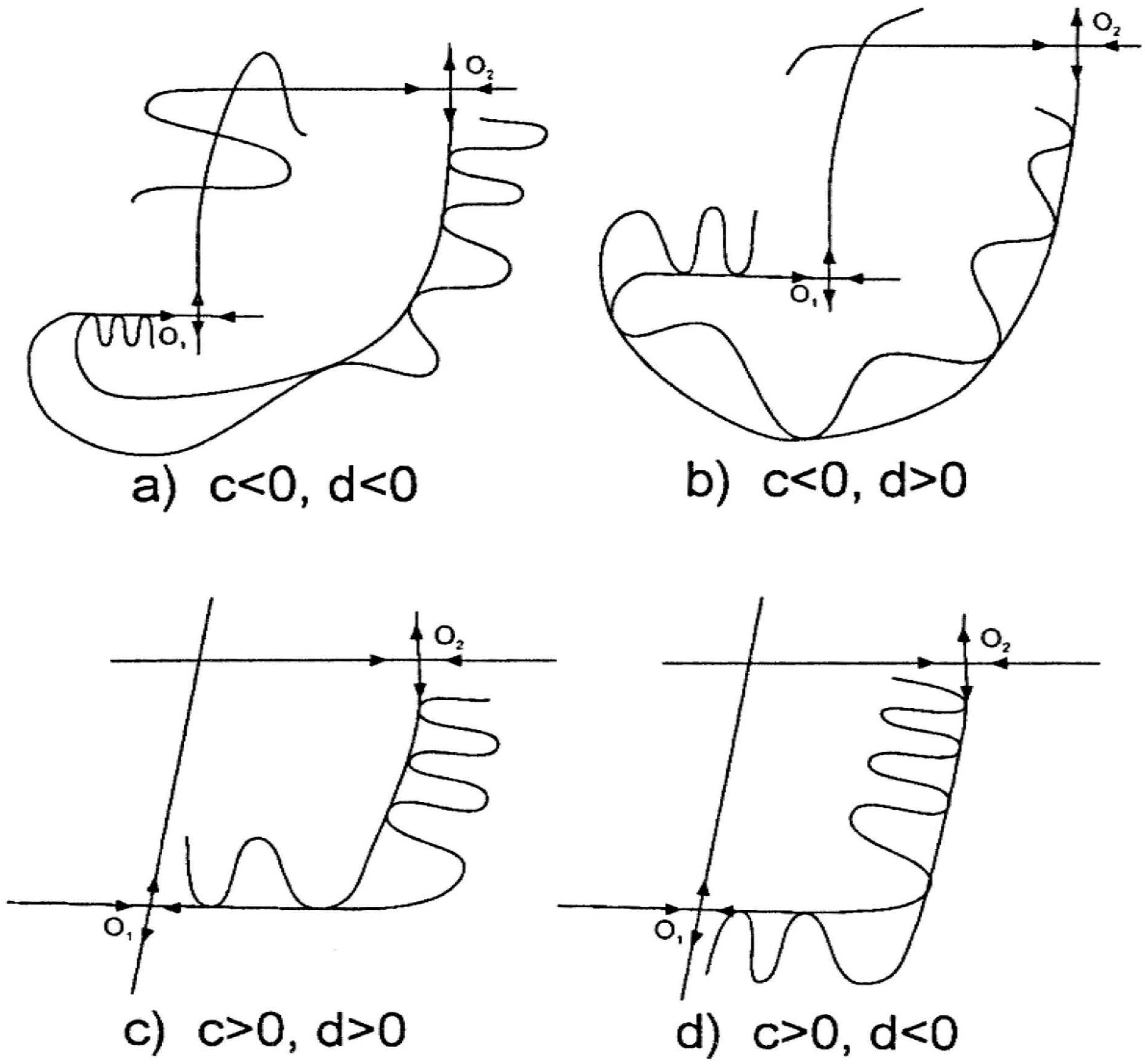

FIG. 2.

THEOREM 3. *Systems with non-rough periodic orbits of any order of degeneracy and systems with a homoclinic tangencies of any order are dense in* H_3.

3. Sinks and Sources of Diffeomorphisms of the Third Class

Note, that saddle-node periodic orbits having one unit multiplier and non-zero first Lyapunov value are the simplest form of non-rough periodic orbits mentioned in the Theorem 3. Stable (sinks) or completely unstable (sources) periodic orbits may appear when the saddle-nodes bifurcate. The type of stability of these orbits depends, first of all, on the saddle values $\sigma_i = |\lambda_s|\,|\gamma_i|$ of the points O_i.

THEOREM 4.

1. *Systems with a countable set of sinks (resp., sources) are dense in* H_3 *in the case* $\sigma_1 < 1$, $\sigma_2 < 1$ *(resp., in the case* $\sigma_1 > 1$, $\sigma_2 > 1$*).*

2. *If* $\sigma_1 < 1$, $\sigma_2 < 1$, *then neither* f *nor any nearby system has sources in* U. *If* $\sigma_1 > 1$, $\sigma_2 > 1$, *then neither* f *nor any nearby system has sinks in* U.

In the case where one of the saddle values is less than 1 and the other is greater than 1, systems in H_3 may have sinks and sources simultaneously. Here, an important quantity is also $\alpha = \sigma_1^{\theta}\sigma_2$. Divide the surface H_3 into two parts. Denote as H_s the part of H_3 where $\alpha < 1$, and as H_u the part of H_3 where $\alpha > 1$.

THEOREM 5. *Systems with a countable set of sinks (sources) are dense in* H_s *(in* H_u*).*

THEOREM 6. *In the cases where* λ_1 *or* γ_2 *are negative, systems having a countable set of sinks and sources simultaneously are dense in* $H_s \cup H_u$.

In the case where λ_1 and γ_2 are positive, let H_{s+} be the subset of H_s for which $d > 0$, $\sigma_1 > 1$, $\sigma_2 < 1$ or $d < 0$, $\sigma_1 < 1$, $\sigma_2 > 1$, let H_{u+} be the subset of H_u for which $d > 0$, $\sigma_1 < 1$, $\sigma_2 > 1$ or $d < 0$, $\sigma_1 > 1$, $\sigma_2 < 1$.

THEOREM 7.

1. *Systems in* H_{s+} *do not have sources and systems in* H_{u+} *do not have sinks.*
2. *Systems having a countable set of sinks and sources simultaneously are dense in* $(H_s \backslash H_{s+}) \cup (H_u \backslash H_{u+})$.

4. Newhouse Regions

The coexistence of periodic sinks and sources is not only the property of diffeomorphisms on H_3; in fact, it is a general situation in the following sense

THEOREM 8. *Let* f *be a diffeomorphism with non-rough homoclinic contour (*f *is not assumed here to belong to the third class) and let one of the saddle values be less than* 1 *and the other be greater than* 1. *Then, in any neighbourhood of* f *in the space of dynamical systems there exist open domains (Newhouse regions) where systems having a countable set of sinks and sources simultaneously are dense.*

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