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ON DIMENSION REDUCTION NEAR HOMOCLINIC AND HETEROCLINIC CYCLES

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An analogue of the center manifold theory is proposed for bifurcations of homo/heteroclinic cycles. In contrast to local bifurcations, the dimension of the non-local problems is shown to be determined by *three different* integers: *geometrical* dimension d_g (the dimension of the center manifold), *critical* dimension d_c (the codimension of the sum of strong-stable and strong-unstable invariant subspaces of the linearized flow), and *Lyapunov* dimension d_L. For the orbits arising at the bifurcation, effective restrictions on the maximal and minimal numbers of positive and negative Lyapunov exponents are obtained, involving the values d_c and d_L.

In this paper we consider most general properties of dynamical systems possessing *homo-* and *heteroclinic cycles.* A cycle is a union of a finite number of periodic orbits and (or) equilibrium states and a finite number of orbits asymptotic to them. We denote such a cycle as C, the periodic orbits and equilibrium states belonging to C are denoted by $L_1, L_2, \ldots$, and the orbits asymptotic to L_i are denoted by $\Gamma_1, \Gamma_2, \ldots$: $C = L_1 \cup L_2 \cup \ldots \cup \Gamma_1 \cup \Gamma_2 \cup \ldots$. Each orbit Γ_s is an intersection of the unstable $W^u(L_i)$ and stable $W^s(L_j)$ manifolds of some orbits L_i and L_j. If $i = j$, then Γ_s is called *homoclinic*, and it is called *heteroclinic* if $i \neq j$.

Let U be a small neighbourhood of C and let Ω be the set of orbits staying in U for all times. We will find effective restrictions on the possible numbers of positive and negative Lyapunov exponents (in other words, on possible dimensions of stable and unstable manifolds) for orbits in Ω both for the system itself and for any nearby system.

Such restrictions are well known in the local bifurcation theory. Namely, if some system has such a periodic orbit or an equilibrium state with k characteristic exponents on the imaginary axis, n characteristic exponents to the right and m characteristic exponents to the left of the imaginary axis, then: *any orbit $\mathcal{L}$ in a small neighbourhood of the bifurcating equilibrium state (the periodic orbit) can not have more than $(n + k)$ and less than n positive or more than $(m + k)$ and less than m negative Lyapunov exponents.*

This assertion follows from the "center manifold theorem" according to which the orbit $\mathcal{L}$ lies in a k-dimensional invariant center manifold which is an intersection of $(n + k)$-dimensional center-unstable and $(m + k)$-dimensional center-stable invariant manifolds. On the center-stable (center-unstable) manifold there exists a *contracting* strong-stable (respectively, *expanding* strong-unstable) invariant foliation with m-dimensional (resp., n-dimensional) leaves transverse to the center manifold.

The number k is called *the dimension of the bifurcational problem* and its determining is a standard preliminary step when studying any local bifurcation. Surprisingly, we show that the dimension of non-local bifurcational problems is determined not by one constant but by three different integers: $d_g \geq d_c \geq d_L$. Here, *the geometrical dimension* d_g is equal to the dimension of a non-local analogue of the center manifold; *the critical* *dimension* d_c is connected with the strong-stable and strong-unstable foliations: it is equal to the difference between the dimension of the phase space and the sum of dimensions of strong-stable and strong-unstable leaves; *the Lyapunov dimension* d_L relates to the estimates on possible dimensions of stable and unstable manifolds of the orbits that may be born at the bifurcation: it is equal to the maximal number of zero Lyapunov exponents possible for these orbits.

For any cycle C there can be found integers m, n and k ($m+n+k$ = the dimension of the phase space) such that the "trichotomy property" holds:

For each periodic orbit or equilibrium state $L_i \subseteq C$, for some positive β_i^u and β_i^s, exactly k characteristic exponents λ of L_i lie in the strip $-\beta_i^s < \Re e\lambda < \beta_i^u$ (the critical part of the spectrum), n characteristic exponents lie to the right of this strip (the strong-unstable part) and m characteristic exponents lie to the left (the strong-stable part).

The separating values β_i can be different for different orbits L_i, but the numbers k, m, n must not depend on L_i. We restrict the freedom in the choice of the trichotomy decomposition by the additional assumption that for each homo/heteroclinic orbit $\Gamma_s \subseteq C$ *the transversality conditions* are fulfilled.

Let $\Gamma_s \subseteq W^u(L_i) \cap W^s(L_j)$. It is known that $W^u(L_i)$ lies in an extended unstable invariant $(n+k)$-dimensional manifold $W^{ue}(L_i)$ which is tangent at L_i to the eigenspace corresponding to the center and strong-unstable characteristic exponents. Also, L_i lies in a strong-unstable invariant n-dimensional manifold $W^{uu}(L_i) \subset W^u(L_i)$ which is tangent to the eigenspace corresponding to the strong-unstable characteristic exponents. The manifold $W^{uu}(L_i)$ is embedded into invariant smooth foliation $F^{uu}(L_i)$ of the manifold $W^u(L_i)$. Since $\Gamma_s \subset W^u(L_i)$, an $(n+k)$-dimensional plane N^{cu} and n-dimensional plane $N^{uu} \subset N^{cu}$ can be constructed at each point of Γ_s such that the families of planes N^{cu} and N^{uu} are invariant with respect to the flow linearized along Γ_s: N^{cu} is tangent to $W^{ue}(L_i)$ and N^{uu} is tangent to the leaf of the foliation $F^{uu}(L_i)$ that passes through the point under consideration. Likewise, at each point of Γ_s there exist invariant families of $(m+k)$-dimensional and m-dimensional planes N^{cs} and $N^{ss} \subset N^{cs}$: N^{cs} is tangent to the *extended stable* manifold of L_j and N^{ss} is tangent to the leaf of the *strong-stable* foliation of $W^s(L_j)$. The planes N^{ss}, N^{cs}, N^{uu} and N^{cu} are defined uniquely. The transversality conditions are:

At each point of each orbit $\Gamma_s \subset C$ the space N^{cs} is transverse to N^{uu} and the space N^{cu} is transverse to N^{ss}.

Due to the invariance of the families of the planes with respect to the linearized flow, the transversality is to be verified in one point on each orbit Γ_s. Since N^{cs} and N^{uu} (N^{cu} and N^{ss}) have complementary dimensions ($(m+k)$ and n and $(n+k)$ and m, respectively), the transversality conditions are well posed.

THEOREM 1. *If the cycle C satisfies the trichotomy property and the transversality conditions for some k, m, n, then for any nearby system, for any orbit $\mathcal{L} \in \Omega$, the flow linearized along $\mathcal{L}$ admits an exponential trichotomy; i.e., there exists an m-dimensional strong-stable subspace for which the linearized flow is exponentially contracting, an n-dimensional strong-unstable subspace for which the linearized flow is expanding, and a k-dimensional center subspace for which contraction or expansion are weaker than those on the strong-stable and strong-unstable subspaces. The system of these subspaces is invariant with respect to the linearized flow, and they depend continuously on the point in Ω and on the system.*

The numbers k, m and n are not defined uniquely. We call the lowest possible value of k for which the trichotomy property and the transversality conditions remains valid (and, therefore, theorem 1 remains valid) *the critical dimension* d_c.

Theorem 1 shows that, for any orbit staying in a neighbourhood U of C for all times, *the number of negative Lyapunov exponents can not be greater than* $(m + d_c)$ *and less than* m *and the number of positive Lyapunov exponents can not be greater than* $(n + d_c)$ *and less than* n.

To improve this estimate we introduce the notion of *the Lyapunov dimension of the problem*, d_L. Let $k = d_c$ and let $\lambda_1^i, \ldots, \lambda_k^i$ be the critical exponents of an orbit $L_i \subset C$ (a periodic orbit or an equilibrium state). Suppose the critical exponents are ordered so that $\Re e\lambda_1^i \geq \Re e\lambda_2^i \geq \ldots \geq \Re e\lambda_k^i$. Let the sequential divergence S_i be less than zero: $S_i \equiv \Re e\lambda_1^i + \ldots + \Re e\lambda_k^i < 0$. Let k_i' be such that $\Re e\lambda_1^i + \ldots + \Re e\lambda_{k_i'}^i \geq 0$ and $\Re e\lambda_1^i + \ldots + \Re e\lambda_{k_i'+1}^i < 0$. If $S_i < 0$ for all orbits L_i, then we assume $d_L = \max_{L_i \subset C} k_i'$.

If $S_i > 0$ for each L_i, there are defined integers $k_i' < k$ such that $\Re e\lambda_{k_i'}^i + \ldots + \Re e\lambda_k^i \leq 0$ and $\Re e\lambda_{k_{i-1}'}^i + \ldots + \Re e\lambda_k^i > 0$. Here, $d_L = \max_{L_i \subset C}(k + 1 - k_i')$.

THEOREM 2. *If for each orbit* L_i *the sequential divergence* S_i *is negative, then any orbit* $\mathcal{L} \subseteq \Omega$ *can not have more than* $(n + d_L)$ *non-negative Lyapunov exponents. If all* S_i *are positive, then any orbit* $\mathcal{L} \subseteq \Omega$ *can not have more than* $(m + d_L)$ *non-positive Lyapunov exponents.*

The estimates given by theorems 1 and 2 seem to be final; namely, we propose the following "realization conjecture" (which is valid in all cases known to the author):

CONJECTURE 1. *Let all orbits* L_i *in an indecomposable cycle* C *be structurally stable. Then, by a small perturbation of the system, a periodic orbit can be born in* U*, having* d_L *zero characteristic exponents.*

Suppose now that the trichotomy property holds and the transversality conditions are fulfilled for some k, m and n (we do not assume here that k is taken as low as possible). We say that *the non-coincidence conditions* are fulfilled if

None of the orbits $\Gamma_s \subset C$ *tends to its* α*- or* ω*-limit orbit* L_i *lying in* $W^{uu}(L_i)$ *or, respectively, in* $W^{ss}(L_i)$ *and if there is more than one of orbits* Γ_s *tending to an orbit* L_i *as* $t \to +\infty$ *or as* $t \to -\infty$*, then different orbits* Γ_s *intersect different leaves of the foliations* $F^{ss}(L_i)$ *or, respectively,* $F^{uu}(L_i)$.

THEOREM 3. *The non-coincidence conditions are necessary and sufficient for the presence of a smooth* k*-dimensional invariant manifold* $\mathcal{W}^c$ *containing the cycle* C *and tangent at each point of* C *to the center subspace. The manifold* $\mathcal{W}^c$ *is the intersection of two invariant manifolds:* $\mathcal{W}^{cu}$ *and* $\mathcal{W}^{cs}$ *of dimensions* $(n + k)$ *and* $(m + k)$*, respectively. The invariant manifolds* $\mathcal{W}^{cu}$ *and* $\mathcal{W}^{cs}$ *exist for all nearby systems and depend on the system continuously. For any system close to* X *the manifold* $\mathcal{W}^{cu}$ *contains all orbits staying in the neighbourhood* U *for all negative times and the manifold* $\mathcal{W}^{cs}$ *contains all orbits staying in* U *for all positive times; the manifold* $\mathcal{W}^c = \mathcal{W}^{cu} \cap \mathcal{W}^{cs}$ *contains the set* Ω *of the orbits lying in* U *entirely. At each point of* Ω *the manifold* $\mathcal{W}^c$ *is tangent to the center subspace, the manifold* $\mathcal{W}^{cu}$ *to the sum of the center and strong-unstable subspaces, the manifold* $\mathcal{W}^{cs}$ *to the sum of the center and strong-stable subspaces.*

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