J. Nonlinear Sci. Vol. 9: pp. 525–573 (1999)

Journal of Nonlinear Science

An Existence Theorem of Smooth Nonlocal Center Manifolds for Systems Close to a System with a Homoclinic Loop

M. V. Shashkov[1] and D. V. Turaev[2]

[1] Department of Differential Equations, Institute for Applied Mathematics and Cybernetics, 10 Uljanova Street, Nizhny Novgorod 603005, Russia

[2] The Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, Berlin 10117, Germany

Received June 4, 1997; final revision received April 24, 1998
Communicated by Stephen Wiggins

Summary. In this paper we give a proof of the existence of smooth nonlocal center manifolds for systems close to a system with a homoclinic orbit to a saddle-type equilibrium point. Our proof is based on a consideration of some class of the boundary value problems (see Section 3). We obtain estimates for solutions of the boundary value problems that allow us to prove the theorem on the center manifolds at the C^1-assumptions for the smoothness of systems.

1. Introduction

It is well known that in neighborhoods of equilibrium points and periodic orbits of C^k-smooth dynamical systems there exist C^k-smooth invariant center manifolds. This result goes back to Pliss [1964], Kelley [1967], and then to Fenichel [1971], Hirsch et al. [1977], and Shoshitaishvili [1975]. For equilibria, the dimension of such manifolds is determined by the number of roots of the characteristic equation with zero real parts, and for periodic orbits it is determined by the number of multipliers that lie on the unit circle. Due to the existence of such manifolds, the investigation of local bifurcations (bifurcations of equilibrium points and periodic orbits) can be reduced to the study of the systems on the center manifolds.

However, a large number of the models of multidimensional dynamical systems is provided by nonlocal bifurcations and, in particular, by bifurcations of homoclinic and heteroclinic contours consisting of equilibria and orbits asymptotic to them. The investigations of such bifurcations were pioneered by Shilnikov. He studied the principal nonlocal bifurcations (bifurcations of homoclinic orbits to a saddle and to a saddle-node) of multidimensional systems (see Shilnikov [1963], [1968]) and discovered the complex structure near a homoclinic loop to a saddle-focus and near a homoclinic bunch to a saddle-saddle (see Shilnikov [1965], [1967], [1969], [1970]). By now bifurcations of homoclinic and heteroclinic contours have been developed intensively. For instance, codimension-two bifurcations of heteroclinic contours with two equilibrium states were studied in Bykov [1978], [1980], [1993]; Chow et al. [1990b]; Deng [1989], [1991]; Shashkov [1991b], [1992], [1994]; Shashkov and Turaev [1996]; bifurcations of a pair of homoclinic orbits to a saddle, in Turaev [1984], [1991]; Turaev and Shilnikov [1986]; Gambaudo et al. [1988]; bifurcations of two separatrices of a saddle, one of which forms a homoclinic loop and the other one tends to the loop, in Homburg, [1993], [1996]; and bifurcations of a homoclinic loop to a saddle at the resonant eigenvalues, in Chow et al. [1990a].

In connection with the study of the global bifurcations of multidimensional systems, a number of papers have been devoted, in recent years, to the extension of center-manifold theory to homoclinic and heteroclinic contours. First, the existence of a Lipschitz two-dimensional nonlocal invariant manifold was pointed out in [Turaev, 1984], where bifurcations of a system with two homoclinic trajectories to a saddle were studied. A proof of the existence of a C^1-smooth center manifold for C^4-smooth systems was done in [Turaev, 1991]. Note that in [Turaev, 1996] the center-manifold theory was developed up to more complex contours that consist of finite number of equilibria, limit cycles, homoclinic and heteroclinic orbits and, moreover, ω- and α-limit trajectories. The analogous theorems on the existence of smooth center manifolds for C^4-smooth systems, close to a system with a heteroclinic contour, have been proved in Shashkov [1991a], [1994]. Independently, Homburg [1993], [1996] has shown the existence of a smooth two-dimensional center manifold for sufficiently smooth systems with a homoclinic loop at the nonresonant eigenvalues. By now, the theorem on existence of a smooth center manifold near a homoclinic loop has been proved at $C^{k+\varepsilon}$ assumptions ($k \geq 1,\ \varepsilon > 0$) for the smoothness of vector fields Sandstede [1994]. Note that Sandstede [1994] has extended his results to an infinitely dimensional case.

In this paper we develop a tool that permits us to prove the presence of nonlocal center manifolds at minimal restrictions to the smoothness of the system. This tool involves solutions of some class of boundary value problems (see Section 3). Using the estimates for the solutions of one such problem near an equilibria, we prove the existence of the smooth center manifold for C^1-systems close to a system with a homoclinic orbit to a saddle equilibrium state. Finally (in Section 9) we give a C^1 example of a vector field with a smooth global center manifold.

2. Main Theorems

Let us consider a family of vector fields X_μ on an $(n+m)$-dimensional manifold,

$$\dot{X} = F(X, \mu), \qquad X \in R^{n+m}, \qquad n \geq 1, \quad m \geq 1, \quad \mu \in R^l, \quad l \geq 0.$$

We assume the function $F(X, \mu)$ to be C^k-smooth ($k \geq 1$) with respect to the phase variables X and the parameter μ.

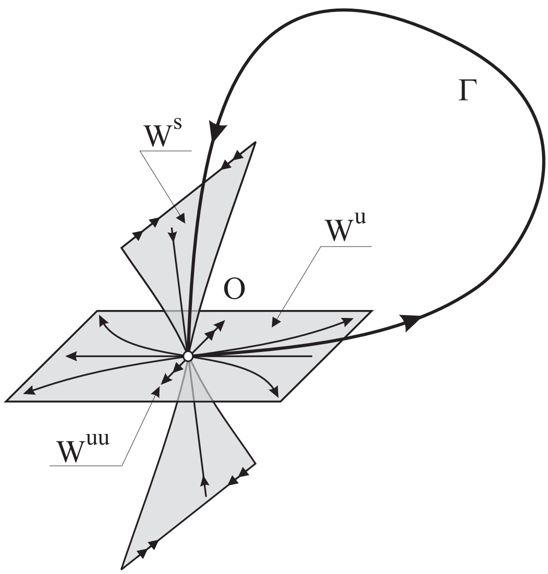

Fig. 1. The stable manifold W^s intersects the unstable manifold W^u along the orbit Γ, forming a homoclinic loop $\mathcal{L} = O \cup \Gamma$. The orbit Γ does not belong to the strong-unstable manifold W^{uu}.

We make the following assumptions:

(**A**) *at $\mu = 0$, the system X_0 has a saddle equilibrium point O and the roots $\lambda_n, \ldots, \lambda_1, \gamma_1, \ldots, \gamma_m$ of the characteristic equation of the linearized system at the point O satisfy the following inequalities:*

$$\operatorname{Re}\lambda_n \leq \cdots \leq \operatorname{Re}\lambda_1 < 0 < \gamma_1 < \operatorname{Re}\gamma_2 \leq \cdots \leq \operatorname{Re}\gamma_m.$$

In this case, the dimension of the stable manifold W^s of the equilibrium point O is equal to n and the unstable manifold W^u of O is an m-dimensional surface. Since $\gamma_1 < \operatorname{Re}\gamma_i, (i = 2, \ldots, m)$, there exists an $(m-1)$-dimensional strong-unstable invariant submanifold W^{uu} in W^u. The main feature characterizing W^{uu} is that all its orbits tend to O, as $t \to -\infty$, being tangent to the subspace corresponding to the eigenvalues $\gamma_2, \ldots, \gamma_n$, whereas all orbits of $W^u \backslash W^{uu}$ are tangent, as $t \to -\infty$, to the eigendirection corresponding to the *principle* eigenvalue γ_1. We shall assume that

(**B**) *at $\mu = 0$, the system X_0 has an orbit Γ that is doubly asymptotic to the equilibrium point O, i.e., $(W^s \cap W^u)/O \supset \Gamma$,*

and

(**C**) *the orbit Γ does not belong to the strong-unstable manifold W^{uu} (see Figure 1).*

Denote by $E^{s+} \subset R^{n+1}$ the invariant subspace of the linearization matrix of the system X_0 at the point O, which corresponds to the eigenvalues $\lambda_n, \ldots, \lambda_1, \gamma_1$. It is well known (Fenichel [1971]; Hirsch et al. [1977]) that, under the assumption (**A**), there exists

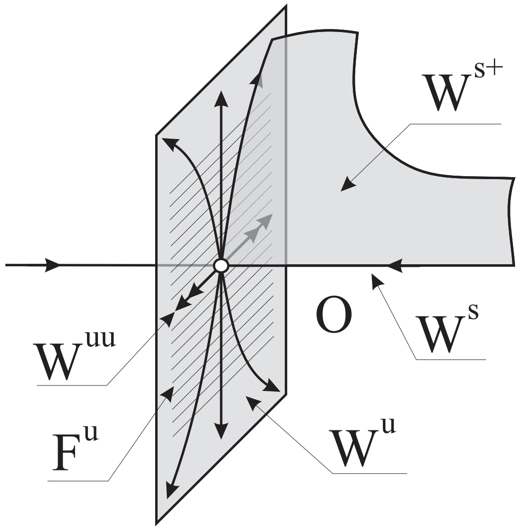

Fig. 2. There exists a C^1-smooth invariant manifold W^{s+}, containing W^s, that is tangent at O to the subspace corresponding to the eigenvalues $\lambda^n, \dots, \lambda^1, \gamma^1$. The manifold W^{s+} is not uniquely defined, but any two of such manifolds have the common tangent everywhere on W^s. The strong-unstable manifold W^{uu} is uniquely embedded into a smooth invariant codimension-one foliation F^u on W^u.

an invariant C^1-smooth manifold W^{s+} tangential to E^{s+} at the point O (see Figure 2). The manifold W^{s+} contains entirely the stable manifold W^s. It is not uniquely defined, but any two of them are tangent at all points of W^s. It is known that the manifold W^{uu} is uniquely included in a smooth invariant foliation F^u on the manifold W^u. We require the following condition to be fulfilled.

(**D**) *The manifold W^{s+} is transverse to the leaves of the foliation F^u at each point of the homoclinic orbit Γ (see Figure 3).*

Notice that condition (**D**) must be verified only at one point on the trajectory Γ, because the manifold W^{s+} and the foliation F^u are invariant with respect to the flow defined by the system X_0. It should also be noted that the dimension of the manifold W^{s+} and the dimension of the leaves of the foliation F^u complement each other; therefore condition (**D**), as well as conditions (**A**) and (**C**), are conditions of the general position.

Theorem 2.1. *If conditions* (**A**), (**B**), (**C**), *and* (**D**) *are fulfilled, then there exists a small neighborhood U of the loop $\mathcal{L} = O \cup \Gamma$ such that, for all μ small enough, the system X_μ has an $(n+1)$-dimensional invariant C^1-smooth manifold $\mathcal{M}^{cs}$ that depends smoothly on μ and such that any orbit not lying in $\mathcal{M}^{cs}$ leaves U as t tends to $+\infty$*

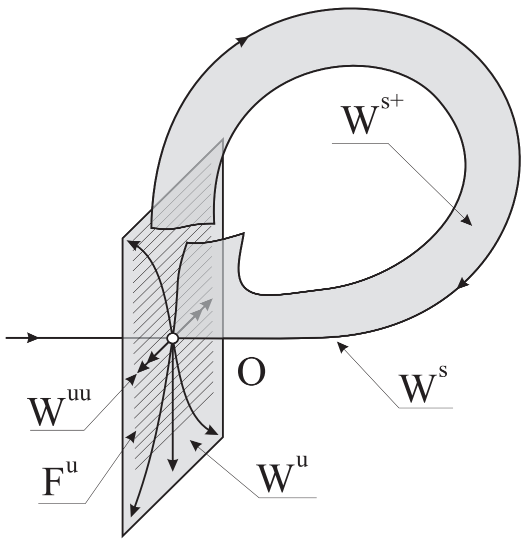

Fig. 3. The manifold W^{s+} transversely intersects the leaves of the foliation F^u.

(see Figure 4).[1] *The manifold $\mathcal{M}^{cs}$ is tangent at the point O to the invariant subspace corresponding to the eigenvalues $\lambda_n, \ldots, \lambda_1, \gamma_1$.*

Reversing the time, Theorem 2.1 immediately implies the following corollary. Let

(A′). *the eigenvalues at the point O satisfy the following conditions:*

$$\operatorname{Re}\lambda_n \leq \cdots \leq \operatorname{Re}\lambda_2 < \lambda_1 < 0 < \operatorname{Re}\gamma_1 \leq \cdots \leq \operatorname{Re}\gamma_m.$$

In this case, since $\lambda_1 > \operatorname{Re}\lambda_i$, $(i = 2, \ldots, n)$, there exists an $(n-1)$-dimensional strong-stable invariant submanifold W^{ss} lying entirely in the stable manifold W^s. We also modify the conditions (**C**) and (**D**). Namely,

(C′). *the homoclinic orbit Γ, which exists at $\mu = 0$, does not lie in the strong-stable manifold W^{ss} (see Figure 5).*

Denote by $E^{u+} \subset R^{m+1}$ the invariant subspace of the linearization matrix of the system X_0 at the point O, which corresponds to the eigenvalues $\gamma_m, \ldots, \gamma_1, \lambda_1$. Under the condition (**A′**), there exists an invariant C^1-smooth manifold W^{u+}, tangential to E^{u+} at the point O (see Figure 6). The manifold W^{u+} is not uniquely defined, but any two of them contain W^u entirely and are tangent at all points of W^u. The strong-unstable manifold W^{ss} is uniquely included in a smooth invariant foliation F^s on W^s. We require the following condition to be fulfilled.

[1] In fact, $\mathcal{M}^{cs} \in C^{k+\varepsilon}$ if $F(X, \mu) \in C^{k+\varepsilon}$ and $\operatorname{Re}\gamma_2/\gamma_1 > k + \varepsilon$.

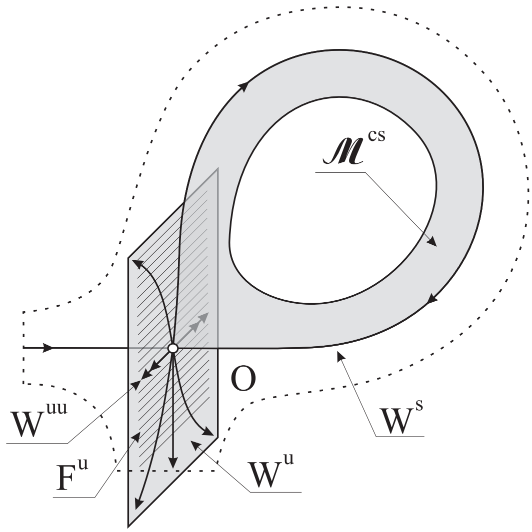

Fig. 4. The system X_μ has an $(n+1)$-dimensional C^1-smooth invariant manifold $\mathcal{M}^{cs}$ for all μ small enough. Any orbit not lying in $\mathcal{M}^{cs}$ escapes the neighborhood of the loop $\mathcal{L} = O \cup \Gamma$, which exists at $\mu = 0$, as $t + \infty$.

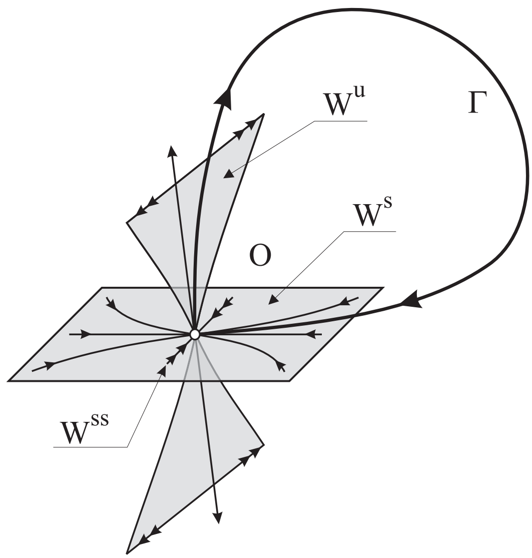

Fig. 5. The homoclinic orbit Γ, which exists at $\mu = 0$, does not belong to W^{ss}, i.e., Γ tends to the equilibrium O, if $t \to +\infty$, along the *principle* direction corresponding to λ_1.

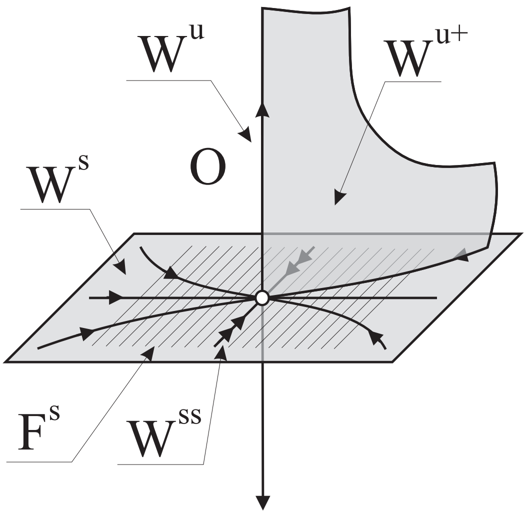

Fig. 6. There exists an invariant C^1-manifold W^{u+}, containing W^u, that is tangent at O to the subspace corresponding to the eigenvalues $\lambda^1, \gamma^1, \dots, \gamma^m$. The manifold W^{u+} is not unique, but any two of them have the common tangent everywhere on W^u. The strong-stable manifold W^{ss} is uniquely embedded into a smooth invariant codimension-one foliation F^s on W^s.

(D′). *At each point of* Γ*, the manifold* W^{u+} *is transverse to the leaves of the foliation* F^s *(see Figure 7).*

Theorem 2.2. *If the conditions* $(\mathbf{A}')$, $(\mathbf{B})$, $(\mathbf{C}')$*, and* $(\mathbf{D}')$ *are fulfilled, then there exists a small neighborhood* U *of the homoclinic loop* $\mathcal{L} = O \cup \Gamma$ *such that, for all* μ *small enough, the system* X_μ *has an* $(m+1)$*-dimensional invariant* C^1*-smooth manifold* $\mathcal{M}^{cu}$ *that depends smoothly on* μ *and such that any orbit, not lying in* $\mathcal{M}^{cu}$*, leaves* U *as* t *tends to* $-\infty$ *(see Figure 8). The manifold* $\mathcal{M}^{cu}$ *is tangent at the point* O *to the subspace corresponding to the eigenvalues* $\gamma_m, \dots, \gamma_1, \lambda_1$.

If the conditions of both Theorem 2.1 and Theorem 2.2 are fulfilled, then we have the following result.

Theorem 2.3. *The manifolds* $\mathcal{M}^{cu}$ *and* $\mathcal{M}^{cs}$ *intersect each other transversally along a two-dimensional invariant* C^1*-manifold* $\mathcal{M}^c$ *that depends smoothly on* μ*. The manifold* $\mathcal{M}^c$ *contains all orbits of* X_μ *lying in* U *entirely for all* $t \in (-\infty, +\infty)$ *and it is tangent at the point* O *to the subspace corresponding to the principle eigenvalues* γ_1, λ_1.

By definition, in the situation of Theorem 2.1, the n-dimensional stable manifold of the point O belongs to $\mathcal{M}^{cs}$. The invariant manifold $\mathcal{M}^{cs}$ is $(n+1)$-dimensional and,

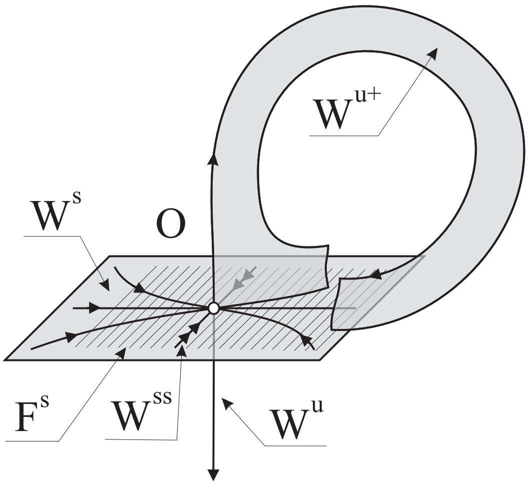

Fig. 7. The manifold W^{u+} is transverse to the leaves of the foliation F^s.

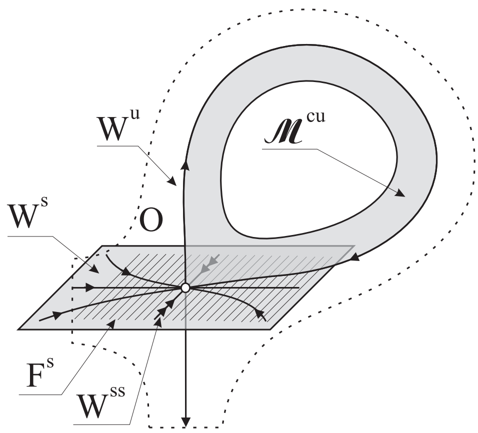

Fig. 8. The system X_μ has an $(m+1)$-dimensional C^1-smooth invariant manifold $\mathcal{M}^{cu}$ for all μ small enough. Any orbit not lying in $\mathcal{M}^{cu}$ escapes the neighborhood of the loop $\mathcal{L} = O \cup \Gamma$, which exists at $\mu = 0$, as $t \to -\infty$.

hence, the unstable manifold of O is one-dimensional for the restriction of the system X_μ onto $\mathcal{M}^{cs}$. Thus, the restriction onto the center-stable manifold $\mathcal{M}^{cs}$ reduces the dimension of the unstable manifold. Analogously, in the situation of Theorem 2.2, the restriction onto the center-unstable manifold $\mathcal{M}^{cu}$ reduces the dimension of the stable manifold.

Notice that the manifolds $\mathcal{M}^{cs}$, $\mathcal{M}^{cu}$, and $\mathcal{M}^{c}$ are in general only C^1-smooth.[2] Therefore, in contrast to the local bifurcation theory, we cannot apply the reduction to the center manifold directly for studying subtle bifurcational problems requiring higher smoothness of the system. Theorems 2.1–2.3 are, thus, qualitative results that allow one to evaluate possible dynamical behavior in a neighborhood of the homoclinic loop. For instance, they provide restrictions on the possible dimensions of the stable and unstable manifolds of the orbits in U or, which is the same, on the number of positive and negative Lyapunov exponents. Moreover, it is possible to use the restriction onto the invariant manifold for the study of a bifurcation problem that actually requires only C^1-smoothness. For instance, we believe that the bifurcation problems considered by Shilnikov [1963], [1965], [1968], [1970]; Turaev [1984], [1991]; Turaev and Shilnikov [1986]; Chow et al. [1990b]; Deng [1989], [1991]; Shashkov [1991b], [1992], [1994]; Shashkov and Turaev [1996]; Homburg [1993]; and Gambaudo [1988] actually require only C^1-smoothness.

Since both Theorems 2.2 and 2.3 follow from main Theorem 2.1, all of our considerations below will be focused on the proof of Theorem 2.1. We prove Theorem 2.1, reducing the problem on a Poincaré map (see Sections 4 and 5) and applying the standard arguments used in proving the contractibility for the graph transformations (see Section 6). In order to construct the Poincaré map, we use an appropriate boundary value problem that gives the proper estimates for the solutions of the system near the equilibrium point (see Section 3). A proof of the smoothness of the invariant manifold $\mathcal{M}^{cs}$ is carried out in Sections 7 and 8. We give an example of a C^1-smooth vector field with the nonlocal center manifold in Section 9.

3. On a Class of Boundary Value Problems

In order to prove Theorem 2.1, we need proper estimates for the orbits of the system near the homoclinic loop $\mathcal{L}$. Notice that the study of solutions near the equilibrium point is the most complicated because the *flight time* of orbits near O is unbounded and, therefore, we need estimates that are fulfilled for unbounded times. If the system can be linearized in the neighborhood of the equilibrium point, the question about the estimates does not arise. However, the smooth linearization requires that the superfluous additional resonance restrictions be satisfied. Moreover, the system should be extra smooth. Therefore, the general way to find the suitable estimates near the equilibrium point is to use Shilnikov's method, which is based on a consideration of some boundary value problem (see Shilnikov [1967]). This method gives the proper estimates (see Ovsyannikov and Shilnikov [1986], [1991], Turaev [1991]) for the solutions and their derivatives up to order k if the initial system is C^{k+3}-smooth. There are many nonlocal bifurcational problems that were solved by this method (see, for instance, Chow et al. [1990a], [1990b]; Deng [1989], [1991]; Fiedler and Turaev [1996]; Shashkov [1991a], [1991b], [1994]; Turaev [1984], [1991]; Turaev and Shilnikov [1986]).

Since we consider here vector fields that are only C^1-smooth, we cannot apply the results mentioned above and, therefore, we develop Shilnikov's method. Namely, we

[2] For instance, smoothness of $\mathcal{M}^{cs}$ is not higher than the integer part of $\operatorname{Re} \gamma_2/\gamma_1 > 1$.

consider the boundary value problem at the other assumptions for the boundary conditions (formulas (3.61)–(3.64) below). Moreover, since the boundary value problems are of interest in its own, in this section we consider a sufficiently wide class of these problems (see (3.1), (3.2), (3.5), and (3.6)). For C^k-smooth ($k \geq 1$) vector fields, we obtain estimates for the derivatives of the solutions up to order k. We also investigate the convergence of these derivatives when the flight time of the solutions tends to infinity.

3.1. The Existence Theorem

Consider a system of ordinary differential equations,

$$\begin{cases} \dot{u} = Au + f(u, v, \mu, t), \\ \dot{v} = Bv + g(u, v, \mu, t), \end{cases} \tag{3.1}$$

where $u \in R^n$, $v \in R^m$, t is time, and μ is a vector of parameters from some compact set $\mathcal{D} \subset R^l$, ($l \geq 0$). We assume that the functions f and g are C^k-smooth ($k \geq 1$) with respect to all variables (u, v, μ, t). Let the following conditions hold for the matrices A and B:

$$\begin{gathered} \operatorname{Spectr} A = \{\alpha_1, \dots, \alpha_n\}, \qquad \operatorname{Spectr} B = \{\beta_1, \dots, \beta_m\}, \\ \max_{i=1,\dots,n} \operatorname{Re} \alpha_i < \alpha < \beta < \min_{i=1,\dots,m} \operatorname{Re} \beta_i . \end{gathered} \tag{3.2}$$

In this case, it is possible to choose the norms of the vectors u and v in such a way that for $s \geq 0$,

$$\begin{gathered} \left\| e^{A\,s} \right\| \leq e^{\tilde{\alpha} s} \leq e^{\alpha s}, \\ \left\| e^{-B\,s} \right\| \leq e^{-\tilde{\beta} s} \leq e^{-\beta s}, \end{gathered} \tag{3.3}$$

where constants $\tilde{\alpha}$, $\tilde{\beta}$ satisfy the conditions

$$\max_{i=1,\dots,n} \operatorname{Re} \alpha_i < \tilde{\alpha} < \alpha \qquad \text{and} \qquad \beta < \tilde{\beta} < \min_{i=1,\dots,m} \operatorname{Re} \beta_i . \tag{3.4}$$ [3]

We also require that, for any $(u, v) \in R^{n+m}$ and $\mu \in \mathcal{D}$,

$$\left\| \frac{\partial (f, g)}{\partial (u, v)} \right\| < \xi, \tag{3.5}$$

where ξ is a small enough constant.[4]

We are interested in solutions of the system (3.1) that satisfy the following boundary conditions:

$$u(0) = u^0, \qquad v(\tau) = v^1, \qquad \tau > 0. \tag{3.6}$$

Notice that the difference between our boundary value problem (3.1), (3.6) and Shilnikov's is that we do not require the conditions $\alpha < 0$ and $\beta > 0$.

---

[3] Here and below, the symbol $\| \cdot \|$ denotes a norm in R^p if it is applied to a vector $x = (x_1, \dots, x_p)$, and denotes the compatible operator norm if it is applied to an operator.

[4] The exact value of the constant ξ can be extracted from the proofs of the theorems below.

Theorem 3.1. *A solution of the boundary value problem (3.1), (3.6),*

$$u(t) = u(t; u^0, v^1, \tau, \mu), \qquad v(t) = v(t; u^0, v^1, \tau, \mu) \tag{3.7}$$

exists. It is uniquely defined and depends C^k-smoothly on $(t; u^0, v^1, \tau, \mu)$. Moreover, the following estimates hold:

$$\left\| \frac{\partial(u, v)}{\partial u^0} \right\| \le Ce^{\alpha t}, \qquad \left\| \frac{\partial(u, v)}{\partial v^1} \right\| \le Ce^{\beta(t-\tau)}, \tag{3.8}$$

where C is some constant.

Before the proof of Theorem 3.1, we notice that vector fields in neighborhoods of equilibrium points give us one of the main examples of systems of the kind (3.1). Indeed, let us consider a family of C^k-smooth dynamical systems that depends on a parameter μ and is given on an $(n+m)$-dimensional manifold:

$$\begin{gathered} \dot{X} = F(X, \mu), \qquad X \in R^{n+m} \; (n \ge 1, \; m \ge 1), \\ \mu \in R^l \; (l \ge 0), \qquad F(X, \mu) \in C^k \; (k \ge 1). \end{gathered} \tag{3.9}$$

Assume that at $\mu = 0$ the system (3.9) has an equilibrium point O in the origin of the coordinates and the spectrum of the linear part of the system (3.9) in the point O can be divided onto two parts, i.e.,

$$\text{Spectr} \left(\frac{\partial F}{\partial X} \right) \bigg|_{(X,\mu)=0} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}, \qquad \max_{i=1,\ldots,n} \operatorname{Re} \alpha_i < \min_{j=1,\ldots,m} \operatorname{Re} \beta_j.$$

In this case, it is possible to introduce coordinates $u \in R^n$ and $v \in R^m$ such that, in a neighborhood of the point O for μ small enough, the system (3.9) is given by

$$\begin{cases} \dot{u} = Au + f(u, v, \mu), \\ \dot{v} = Bv + g(u, v, \mu), \end{cases} \tag{3.10}$$

where the matrices A and B satisfy conditions (3.2), (3.3), and the functions f, g are C^k-smooth and satisfy the following conditions:

$$f(0, 0, 0) = 0, \qquad g(0, 0, 0) = 0, \qquad \frac{\partial(f, g)}{\partial(u, v)} \bigg|_{(u,v,\mu)=0} = 0. \tag{3.11}$$

We are interested in solutions $(u(t), v(t))$, at $t \in [0, \tau]$, lying entirely ($t \in [0, \tau]$) in a small neighborhood of the point O. Therefore, the transition to a new system is justified

$$\begin{cases} \dot{u} = Au + \tilde{f}(u, v, \mu), \\ \dot{v} = Bv + \tilde{g}(u, v, \mu), \end{cases} \tag{3.12}$$

where the functions $\tilde{f}, \tilde{g} \in C^k$ are given by the following formulas:

$$\begin{aligned} \tilde{f}(u, v, \mu) &= f\left(\vartheta(\|(u, v)\|/\rho)u, \vartheta(\|(u, v)\|/\rho)v, \mu\right), \\ \tilde{g}(u, v, \mu) &= g\left(\vartheta(\|(u, v)\|/\rho)u, \vartheta(\|(u, v)\|/\rho)v, \mu\right). \end{aligned} \tag{3.13}$$

Here ρ is a small positive constant and the function $\vartheta(t) \in C^\infty$ has the following properties:

$$\vartheta(t) = \begin{cases} 1, & \text{if } t \leq 1, \\ 0, & \text{if } t \geq 2, \end{cases} \qquad \left| \frac{\partial \vartheta(t)}{\partial t} \right| < 2. \tag{3.14}$$

Notice that the functions $\tilde{f}$ and $\tilde{g}$ satisfy inequality (3.5) for any $(u, v) \in R^{n+m}$ and small μ. By (3.11), (3.13), and (3.14), and by choosing ρ small, the constant ξ can be made arbitrarily small. Therefore, the system (3.12) is a system of the kind (3.1). The functions $\tilde{f}$ and $\tilde{g}$ coincide with f and g correspondingly for $\|(u, v)\| \leq \rho$. So, if the boundary value problem (3.12), (3.6) has a solution $(u(t), v(t))$ lying entirely ($t \in [0, \tau]$) in the neighborhood $\|(u, v)\| \leq \rho$, then $(u(t), v(t))$ is also the solution of the boundary value problem (3.6), (3.10).[5]

Let us pass to a proof of Theorem 3.1. Consider the space H of continuous functions $(u(t), v(t))$, which are given on the segment $t \in [0, \tau]$. Define a γ-norm by the following formula:

$$\|(u, v)\|_\gamma = \sup_{t \in [0, \tau]} \left(\|(u(t), v(t))\| e^{-\gamma t} \right), \tag{3.15}$$

where

$$\alpha < \gamma < \beta. \tag{3.16}$$

Obviously, H with the γ-norm is a complete metric space.[6]

Let us introduce an integral operator T, which maps any function $(u(t), v(t)) \in H$ into the function $(\overline{u}(t), \overline{v}(t)) \in H$, by the following rule:

$$\begin{cases} \overline{u}(t) = e^{At} u^0 + \displaystyle\int_0^t e^{A(t-s)} f(u(s), v(s), \mu, t) ds, \\ \overline{v}(t) = e^{B(t-\tau)} v^1 + \displaystyle\int_\tau^t e^{B(t-s)} g(u(s), v(s), \mu, t) ds. \end{cases} \tag{3.17}$$

It is easy to check that any solution of the boundary value problem (3.1), (3.6) is a fixed point of the integral operator (3.17). It is also true that any fixed point of T is a solution of the boundary value problem (3.1), (3.6). Therefore, the question of the existence and uniqueness of the solution of the problem (3.1), (3.6) is reduced to the question of the existence and uniqueness of the fixed point of the operator T.

In order to show that T has a unique fixed point, we shall establish that T is a contraction operator in the space H. Let us check it. Consider any functions $(u_1, v_1) \in H$

---

[5] Notice the solution of the boundary value problem (3.12), (3.6), generally speaking, can fall outside the limits of the neighborhood of O. Therefore, in this case, the solution is not the solution of the boundary value problem (3.10), (3.6).

[6] Note that the γ-norm is equivalent to the usual uniform norm.

and $(u_2, v_2) \in H$. Let $T(u_1, v_1) = (\overline{u}_1, \overline{v}_1)$ and $T(u_2, v_2) = (\overline{u}_2, \overline{v}_2)$; then, by (3.2)–(3.5) and (3.15)–(3.17), we have the following relations:

$$\begin{aligned}
\|\overline{u}_1 - \overline{u}_2\| &= \left\| \int_0^t e^{A(t-s)}(f(u_1(s), v_1(s), \mu, t) - f(u_2(s), v_2(s), \mu, t))ds \right\| \\
&\le \int_0^t e^{\alpha(t-s)} \xi \|(u_1 - u_2, v_1 - v_2)\|_\gamma e^{\gamma s} ds \\
&= \frac{\xi}{\gamma - \alpha} e^{\alpha t} \left(e^{(\gamma-\alpha)t} - 1\right) \|(u_1 - u_2, v_1 - v_2)\|_\gamma \\
&\le \frac{\xi}{\gamma - \alpha} e^{\gamma t} \|(u_1, v_1) - (u_2, v_2)\|_\gamma, \qquad (3.18)
\end{aligned}$$

$$\begin{aligned}
\|\overline{v}_1 - \overline{v}_2\| &= \left\| \int_\tau^t e^{B(t-s)}(g(u_1(s), v_1(s), \mu, t) - g(u_2(s), v_2(s), \mu, t))ds \right\| \\
&\le \int_t^\tau e^{\beta(t-s)} \xi \|(u_1 - u_2, v_1 - v_2)\|_\gamma e^{\gamma s} ds \\
&= \frac{\xi}{\beta - \gamma} e^{\beta t} \left(e^{(\gamma-\beta)t} - e^{(\gamma-\beta)\tau}\right) \|(u_1 - u_2, v_1 - v_2)\|_\gamma \\
&\le \frac{\xi}{\beta - \gamma} e^{\gamma t} \|(u_1, v_1) - (u_2, v_2)\|_\gamma. \qquad (3.19)
\end{aligned}$$

By (3.15)–(3.19), we obtain

$$\begin{aligned}
\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_\gamma &\le \sup_{t\in[0,\tau]} \left(\|\overline{u}_1 - \overline{u}_2\| e^{-\gamma t} + \|\overline{v}_1 - \overline{v}_2\| e^{-\gamma t}\right) \\
&\le \xi \left(\frac{1}{\gamma - \alpha} + \frac{1}{\beta - \gamma}\right) \|(u_1, v_1) - (u_2, v_2)\|_\gamma. \qquad (3.20)
\end{aligned}$$

We assume that the constant ξ is so small that $\xi\,(1/(\gamma - \alpha) + 1/(\beta - \gamma))$ is less than 1. In this case $T : H \to H$ is a contraction operator and, therefore, T has a unique fixed point $(u(t), v(t)) \in H$. Moreover, any sequence $(u_0, v_0), (u_1, v_1), (u_2, v_2), \ldots,$ obtained by the iterations

$$(u_{n+1}(t), v_{n+1}(t)) = T\,(u_n(t), v_n(t))\,, \qquad (3.21)$$

with any initial function $(u_0(t), v_0(t)) \in H$, converges to the fixed point.

Thus, for any fixed values (u^0, v^1, τ, μ), the boundary value problem (3.1), (3.6) has a unique solution. Depending on the boundary conditions (u^0, v^1, τ) and μ, we obtain different solutions. Below we show the existence of a constant C such that the solution (3.7) satisfies the following Lipschitz conditions with respect to u^0 and v^1:

$$\begin{aligned}
&\left\|\left(u(t; u_1^0, v^1, \tau, \mu), v(t; u_1^0, v^1, \tau, \mu)\right) - \left(u(t; u_2^0, v^1, \tau, \mu), v(t; u_2^0, v^1, \tau, \mu)\right)\right\| \\
&\quad \le Ce^{\alpha t} \left\|u_1^0 - u_2^0\right\|, \qquad (3.22)
\end{aligned}$$

$$\left\|\left(u(t;u^0,v_1^1,\tau,\mu),v(t;u^0,v_1^1,\tau,\mu)\right)-\left(u(t;u^0,v_2^1,\tau,\mu),v(t;u^0,v_2^1,\tau,\mu)\right)\right\| \le Ce^{\beta(t-\tau)}\left\|v_1^1-v_2^1\right\|. \tag{3.23}$$

Consider the space $\tilde{H}$ of continuous functions $(u(t;u^0,v^1,\tau,\mu),v(t;u^0,v^1,\tau,\mu))$ that satisfy Lipschitz conditions (3.22) and (3.23). Here $\mu\in\mathcal{D}$, $\tau\ge 0$, $0\le t\le\tau$, $u^0\in R^n$ and $v^1\in R^m$. Let us define an operator $\tilde{T}$ in the space $\tilde{H}$ by the relations (3.17).[7] Below, we show that $\tilde{H}$ is an invariant space with respect to $\tilde{T}$. Consider any function $(u(t;u^0,v^1,\tau,\mu),v(t;u^0,v^1,\tau,\mu))\in\tilde{H}$. Now, we shall check that the function

$$\left(\overline{u}(t;u^0,v^1,\tau,\mu),\overline{v}(t;u^0,v^1,\tau,\mu)\right)=\tilde{T}\left(u(t;u^0,v^1,\tau,\mu),v(t;u^0,v^1,\tau,\mu)\right)$$

satisfies inequality (3.23). Fix any (u^0,τ,μ) and take any two values v^1: $v^1=v_1^1$ and $v^1=v_2^1$. Below, we use the following denotations:

$$\begin{aligned}
\left(u(t;u^0,v_1^1,\tau,\mu),v(t;u^0,v_1^1,\tau,\mu)\right)&\equiv(u_1(t),v_1(t)),\\
\left(u(t;u^0,v_2^1,\tau,\mu),v(t;u^0,v_2^1,\tau,\mu)\right)&\equiv(u_2(t),v_2(t)),\\
\left(\overline{u}(t;u^0,v_1^1,\tau,\mu),\overline{v}(t;u^0,v_1^1,\tau,\mu)\right)&\equiv(\overline{u}_1(t),\overline{v}_1(t)),\\
\left(\overline{u}(t;u^0,v_2^1,\tau,\mu),\overline{v}(t;u^0,v_2^1,\tau,\mu)\right)&\equiv(\overline{u}_2(t),\overline{v}_2(t)).
\end{aligned} \tag{3.24}$$

By (3.2)–(3.5), (3.17), (3.23), and (3.24), we have the following relations:

$$\begin{aligned}
\|\overline{u}_1-\overline{u}_2\| &= \left\|\int_0^t e^{A(t-s)}(f(u_1,v_1,\mu,t)-f(u_2,v_2,\mu,t))ds\right\|\\
&\le \int_0^t e^{\alpha(t-s)}\xi Ce^{\beta(s-\tau)}\|v_1^1-v_2^1\|ds\\
&= \frac{\xi}{\beta-\alpha}\left(1-e^{(\alpha-\beta)t}\right)Ce^{\beta(t-\tau)}\|v_1^1-v_2^1\|\\
&\le \frac{\xi}{\beta-\alpha}Ce^{\beta(t-\tau)}\|v_1^1-v_2^1\|,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\|\overline{v}_1-\overline{v}_2\| &\le e^{B(t-\tau)}\|v_1^1-v_2^1\|\\
&\quad+\left\|\int_\tau^t e^{B(t-s)}(g(u_1,v_1,\mu,t)-g(u_2,v_2,\mu,t))ds\right\|\\
&\le e^{\beta(t-\tau)}\|v_1^1-v_2^1\|+\int_t^\tau e^{\tilde{\beta}(t-s)}\xi Ce^{\beta(s-\tau)}\|v_1^1-v_2^1\|ds\\
&= \left(1+\frac{\xi}{\tilde{\beta}-\beta}\left(1-e^{(\tilde{\beta}-\beta)(t-\tau)}\right)C\right)e^{\beta(t-\tau)}\|v_1^1-v_2^1\|\\
&\le \left(1+\frac{\xi}{\tilde{\beta}-\beta}C\right)e^{\beta(t-\tau)}\|v_1^1-v_2^1\|.
\end{aligned} \tag{3.26}$$

[7] The difference between $\tilde{T}$ and T is that the operators are defined on different spaces of functions.

Therefore, by inequalities (3.25) and (3.26),

$$\begin{aligned}\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\| &\leq \|\overline{u}_1 - \overline{u}_2\| + \|\overline{v}_1 - \overline{v}_2\| \\ &\leq \left(1 + \xi\left(\frac{1}{\beta - \alpha} + \frac{1}{\tilde{\beta} - \beta}\right) C\right) e^{\beta(t-\tau)} \|v_1^1 - v_2^1\|.\end{aligned}$$

We assume that ξ is so small that $\xi(1/(\beta - \alpha) + 1/(\tilde{\beta} - \beta)) < 1$. In this case, if $C \geq 1/(1 - \xi(1/(\beta - \alpha) + 1/(\tilde{\beta} - \beta)))$, we have

$$\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\| \leq Ce^{\beta(t-\tau)} \|v_1^1 - v_2^1\|,$$

i.e., the function $(\overline{u}(t; u^0, v^1, \tau, \mu), \overline{v}(t; u^0, v^1, \tau, \mu))$ satisfies estimate (3.23). In the same way, it is possible to check that the function $(\overline{u}(t; u^0, v^1, \tau, \mu), \overline{v}(t; u^0, v^1, \tau, \mu))$ satisfies inequality (3.22). This means that $\tilde{H}$ is an invariant space with respect to the operator $\tilde{T}$.

Notice, if u^0, v^1, τ, and μ are fixed, the sequence of functions

$$\left(u_n(t; u^0, v^1, \tau, \mu), v_n(t; u^0, v^1, \tau, \mu)\right) = \tilde{T}^n(0, 0)$$

coincides with the sequence (3.21) with the initial element $(u_0(t), v_0(t)) = (0, 0) \in H$ and, therefore, converges to the solution (3.7) of the boundary value problem (3.1), (3.6) in H. Due to the theorem *on the passage to the limit in inequalities*, the solution also satisfies relations (3.22) and (3.23).

Below we shall show that the solution $\left(u(t; u^0, v^1, \tau, \mu), v(t; u^0, v^1, \tau, \mu)\right)$ is a C^k-smooth function with respect to all variables. To establish this, we consider Cauchy's initial value problem for the system (3.1) with the following initial conditions:

$$u(0) = u^0, \qquad v(0) = v^0. \tag{3.27}$$

Smoothness of the functions f and g in the right-hand side of the system (3.1) guarantees the existence and uniqueness of the solution

$$\left(u^*(t; u^0, v^0, \mu), v^*(t; u^0, v^0, \mu)\right), \tag{3.28}$$

which is C^k-smooth with respect to $(t; u^0, v^0, \mu)$. The phase trajectories of the system (3.1) determine the one-to-one correspondence between the initial (3.27) and the boundary (3.6) conditions. This correspondence can be specified by the following formulas:

$$\left(u^0, v^0\right) = \left(u(0; u^0, v^1, \tau, \mu), v(0; u^0, v^1, \tau, \mu)\right), \tag{3.29}$$

$$\left(u^0, v^1\right) = \left(u^*(0; u^0, v^0, \mu), v^*(\tau; u^0, v^0, \mu)\right). \tag{3.30}$$

Further, we use the following corollary of *the implicit function theorem*.

Lemma 3.1. *Let a function $F(x, y)$ satisfy the Lipschitz condition with respect to x, i.e., for any x_1, x_2, and y from the domain of definition*

$$\|F(x_1, y) - F(x_2, y)\| \leq L\|x_1 - x_2\|.$$

Let the equation $z = F(x, y)$ be resolved with respect to x, i.e., $x = \Phi(y, z)$, and, moreover, let $\Phi(y, z)$ be a C^k-smooth function with respect to (y, z). Then $F(x, y)$ is also a C^k-smooth function with respect to (x, y) and

$$\left\| \frac{\partial F}{\partial x} \right\| \leq L.$$

The functions in (3.30) depend C^k-smoothly on the variables because the functions in (3.28) are smooth. The functions in (3.29) satisfy the Lipschitz condition (3.23) with respect to v^1. Applying Lemma 3.1, we obtain that the functions in equality (3.29) are C^k-smooth. Now, C^k-smoothness of the solution (3.7) follows from the fact that it can be represented as a superposition of smooth functions:

$$\begin{aligned} u\left(t; u^0, v^1, \tau, \mu\right) &= u^*\left(t; u^0, v(0, u^0, v^1, \tau, \mu), \mu\right), \\ v\left(t; u^0, v^1, \tau, \mu\right) &= v^*\left(t; u^0, v(0, u^0, v^1, \tau, \mu), \mu\right). \end{aligned}$$

The smoothness of the solution (3.7) and the Lipschitz conditions (3.22), (3.23) imply estimates (3.8). The theorem is proved.

Notice that our boundary value problem (3.1), (3.6) makes sense with the following conditions:

$$v^1 = 0, \qquad \tau = \infty.$$

In this case, we define the solution as a fixed point of the following integral operator:

$$\begin{cases} \overline{u}(t) = e^{At}u^0 + \int\limits_0^t e^{A(t-s)} f(u(s), v(s), \mu, t)\, ds, \\ \overline{v}(t) = \int\limits_\infty^t e^{B(t-s)} g(u(s), v(s), \mu, t) ds. \end{cases}$$

This operator, as well as the operator T (see (3.17)), has a unique fixed point in the space H

$$u(t) = u(t; u^0, \mu), \qquad v(t) = v(t; u^0, \mu).^8$$

Since the function $\left(u(t; u^0, \mu), v(t; u^0, \mu)\right)$ is bounded in the γ-norm, the orbit $(u(t), v(t))$ belongs to an invariant manifold that corresponds to the eigenvalues $\alpha_1, \ldots, \alpha_n$ and, therefore, the function $v^0 = V(u^0, \mu) \equiv v(0; u^0, \mu)$ specifies that

1. the extended stable manifold if $0 < \alpha < \beta$;
2. the stable manifold if $\alpha < 0 < \beta$;
3. the strong stable manifold if $\alpha < \beta < 0$.

---

[8] The functions $u(t; u^0, \mu)$ and $v(t; u^0, \mu)$ depend C^k-smoothly on $(t; u^0, \mu)$ if $\alpha \leq 0$, and the smoothness is bounded by the integer part of β/α if $\alpha > 0$.

By virtue of the symmetry with respect to the changes

$$t \to \tau - t, \qquad u \to v, \qquad v \to u, \qquad u^0 \to v^1, \qquad v^1 \to u^0,$$

$$A \to -B, \qquad B \to -A, \qquad \alpha \to -\beta, \qquad \beta \to -\alpha,$$

there exists an invariant manifold[9] that corresponds to the eigenvalues $\beta_1, \dots, \beta_m$, i.e.,

1. the strong unstable manifold if $0 < \alpha < \beta$;
2. the unstable manifold if $\alpha < 0 < \beta$;
3. the extended unstable manifold if $\alpha < \beta < 0$.

3.2. Estimates for the Derivatives

Theorem 3.1 gives us the estimates (3.8) for the derivatives of the first order with respect to u^0 and v^1. In this subsection we estimate any derivatives for the solution (3.7) of the boundary value problem (3.1), (3.6) up to order k.

We use the following denotations for the derivatives of a vector function $\phi = (\phi_1, \dots, \phi_q) \in R^q$ with respect to a vector argument $x = (x_1, \dots, x_p) \in R^p$:

$$\frac{\partial^{|s|}\phi}{\partial x^s} \equiv \left(\frac{\partial^{s_1+\cdots+s_p}\phi_1}{\partial x_1^{s_1}\cdots\partial x_p^{s_p}}, \dots, \frac{\partial^{s_1+\cdots+s_p}\phi_q}{\partial x_1^{s_1}\cdots\partial x_p^{s_p}} \right).$$

Here, the vector $s = (s_1, \dots, s_p)$ consists of nonnegative integer-valued components and $|s| = s_1 + \cdots + s_p$.

Theorem 3.2. *Let the solution (3.7) of the boundary value problem (3.1), (3.6) lie in a bounded domain; then the following estimates hold.*

1. *Let* $0 < \alpha < \beta$; *then*

$$\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|}(u, v)}{\partial \left(u^0, \mu\right)^{k_1} \partial \left(v^1, \tau\right)^{k_2} \partial t^{k_3}} \right\| \le \begin{cases} C & \text{if } |k_1| = |k_2| = 0, \\ C\, e^{|k_1|\alpha t} & \text{if } |k_2| = 0 \text{ and } |k_1|\alpha - \beta < 0, \\ C\, e^{\beta(t-\tau)+|k_1|\alpha\tau} & \text{if } |k_2| \ne 0 \text{ or } |k_1|\alpha - \beta > 0. \end{cases} \tag{3.31}$$

2. *Let* $\alpha < 0 < \beta$; *then*

$$\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|}(u, v)}{\partial \left(u^0\right)^{k_1} \partial \left(v^1, \tau\right)^{k_2} \partial (t, \mu)^{k_3}} \right\| \le \begin{cases} C & \text{if } |k_1| = |k_2| = 0, \\ C\, e^{\alpha t} & \text{if } |k_2| = 0 \text{ and } |k_1| \ne 0, \\ C\, e^{\beta(t-\tau)} & \text{if } |k_1| = 0 \text{ and } |k_2| \ne 0, \\ C\, e^{\alpha t+\beta(t-\tau)} & \text{if } |k_1| \ne 0 \text{ and } |k_2| \ne 0. \end{cases} \tag{3.32}$$

[9] The smoothness of this manifold is bounded by the integer part of α/β if $\beta < 0$.

3. *Let* $\alpha < \beta < 0$*; then*

$$\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|}(u,v)}{\partial\left(u^0\right)^{k_1}\partial\left(v^1,\tau,\mu\right)^{k_2}\partial t^{k_3}} \right\| \le \begin{cases} C & \text{if } |k_1|=|k_2|=0, \\ C\,e^{|k_2|\beta(t-\tau)} & \text{if } |k_1|=0 \text{ and } \alpha-|k_2|\beta<0, \\ C\,e^{\alpha t-|k_2|\beta\tau} & \text{if } |k_1|\ne 0 \text{ or } \alpha-|k_2|\beta>0. \end{cases} \tag{3.33}$$

Here C *is some positive constant.*

Notice that our boundary value problem is symmetric with respect to the following change:

$$\begin{array}{lllll} t\to\tau-t, & \alpha\to-\beta, & \beta\to-\alpha, & u\to v, & \\ v\to u, & u^0\to v^1, & v^1\to u^0, & k_1\to k_2, & k_2\to k_1. \end{array} \tag{3.34}$$

Therefore, by Theorem 3.2, we have

Theorem 3.3. *Let the solution (3.7) of the boundary value problem (3.1), (3.6) lie in a bounded domain, then the following estimates hold.*

1. *Let* $0 < \alpha < \beta$*; then*

$$\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|}(u,v)}{\partial\left(u^0,\tau,\mu\right)^{k_1}\partial\left(v^1\right)^{k_2}\partial(\tau-t)^{k_3}} \right\| \le \begin{cases} C & \text{if } |k_1|=|k_2|=0, \\ C\,e^{|k_1|\alpha t} & \text{if } |k_2|=0 \text{ and } |k_1|\alpha-\beta<0, \\ C\,e^{\beta(t-\tau)+|k_1|\alpha\tau} & \text{if } |k_2|\ne 0 \text{ or } |k_1|\alpha-\beta>0. \end{cases} \tag{3.35}$$

2. *Let* $\alpha < 0 < \beta$*; then*

$$\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|}(u,v)}{\partial\left(u^0,\tau\right)^{k_1}\partial\left(v^1\right)^{k_2}\partial(\tau-t,\mu)^{k_3}} \right\| \le \begin{cases} C & \text{if } |k_1|=|k_2|=0, \\ C\,e^{\alpha t} & \text{if } |k_2|=0 \text{ and } |k_1|\ne 0, \\ C\,e^{\beta(t-\tau)} & \text{if } |k_1|=0 \text{ and } |k_2|\ne 0, \\ C\,e^{\alpha t+\beta(t-\tau)} & \text{if } |k_1|\ne 0 \text{ and } |k_2|\ne 0. \end{cases} \tag{3.36}$$

3. *Let* $\alpha < \beta < 0$*; then*

$$\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|}(u,v)}{\partial\left(u^0,\tau\right)^{k_1}\partial\left(v^1,\mu\right)^{k_2}\partial(\tau-t)^{k_3}} \right\| \le \begin{cases} C & \text{if } |k_1|=|k_2|=0, \\ C\,e^{|k_2|\beta(t-\tau)} & \text{if } |k_1|=0 \text{ and } \alpha-|k_2|\beta<0, \\ C\,e^{\alpha t-|k_2|\beta\tau} & \text{if } |k_1|\ne 0 \text{ or } \alpha-|k_2|\beta>0. \end{cases} \tag{3.37}$$

Here C *is some positive constant.*

Let us prove Theorem 3.2 for the first case, i.e. under the conditions $0 < \alpha < \beta$. Note that the estimates for the derivatives with respect to μ can be reduced to the estimates for the derivatives with respect to u^0. In order to see that it is sufficient to add the equation $\dot{\mu} = 0$ to the initial system (3.1) and the condition $\mu(0) = \mu$ to the boundary conditions (3.6).

Note also that the differentiation with respect to t does not change the estimates. Indeed, by (3.1), we have the following recurrence relations:

$$\begin{aligned}\frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|}u}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}\partial t^{k_5}} &= \frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5-1|}(A\,u + f(u,v,\mu,t))}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}\partial t^{k_5-1}},\\ \frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|}v}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}\partial t^{k_5}} &= \frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5-1|}(B\,v + g(u,v,\mu,t))}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}\partial t^{k_5-1}}.\end{aligned} \tag{3.38}$$

So, by (3.38),

$$\frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|}(u,v)}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}\partial t^{k_5}} = O\left(\frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|}(u,v)}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}}\right).$$

In order to find the estimates for the derivatives with respect to τ we use the following trick. By the definition of the solution of the boundary value problem we have the following identities:

$$\begin{aligned} u(t;u^0,v^1,\tau,\mu) &\equiv u(t;u^0,v(\tau+\delta;u^0,v^1,\tau,\mu),\tau+\delta,\mu)\,,\\ v(t;u^0,v^1,\tau,\mu) &\equiv v(t;u^0,v(\tau+\delta;u^0,v^1,\tau,\mu),\tau+\delta,\mu)\,,\end{aligned} \tag{3.39}$$

The differentiation of the identities (3.39) with respect to δ gives us the following equality:

$$\frac{\partial(u,v)}{\partial v^1}\left.\frac{\partial v}{\partial t}\right|_{t=\tau} + \frac{\partial(u,v)}{\partial\tau} \equiv 0. \tag{3.40}$$

This formula implies that the differentiation with respect to τ is “analogous” to the differentiation with respect to v^1, i.e.

$$\frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|}(u,v)}{\partial(u^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}\partial\tau^{k_4}} = O\left(\frac{\partial^{|k_1|+|k_6|+|k_3|}(u,v)}{\partial(u^0)^{k_1}\partial(v^1)^{k_6}\partial\mu^{k_3}}\right), \tag{3.41}$$

where $|k_6| = |k_2| + |k_4|$.

So, to prove Theorem 3.2, under the condition $0 < \alpha < \beta$, we need to check the following estimates:

$$\left\|\frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial\left(u^0\right)^{k_1}\partial\left(v^1\right)^{k_2}}\right\| \le \begin{cases} C\,e^{|k_1|\alpha t} & \textit{if } |k_2| = 0 \text{ and } |k_1|\alpha-\beta<0,\\ C\,e^{\beta(t-\tau)+|k_1|\alpha\tau} & \textit{if } |k_2| \ne 0 \text{ or } |k_1|\alpha-\beta>0.\end{cases} \tag{3.42}$$

We prove the validity of these estimates by induction on $|k_1| + |k_2|$. By virtue of Theorem 3.1 (see (3.8)), the estimates are fulfilled for $|k_1| + |k_2| = 1$. Assume that the estimates (3.42) are fulfilled for any k_1 and k_2 such that $|k_1| + |k_2| \le q$. Let us show the validity of the estimates for any k_1 and k_2 such that $|k_1| + |k_2| = q + 1$.

Take any k_1 and k_2 such that $2 \le |k_1| + |k_2| = q + 1 \le k$. Since the solution of the boundary value problem is a fixed point of the operator T (see (3.17)), the derivative $\frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}}$ satisfies the following equation:

$$\begin{cases} \dfrac{\partial^{|k_1|+|k_2|} u}{\partial \left(u^0\right)^{k_1} \partial \left(v^1\right)^{k_2}} = \displaystyle\int_0^t e^{A(t-s)} \dfrac{\partial^{|k_1|+|k_2|} f(u, v, \mu, s)}{\partial \left(u^0\right)^{k_1} \partial \left(v^1\right)^{k_2}} ds, \\ \dfrac{\partial^{|k_1|+|k_2|} v}{\partial \left(u^0\right)^{k_1} \partial \left(v^1\right)^{k_2}} = \displaystyle\int_\tau^t e^{B(t-s)} \dfrac{\partial^{|k_1|+|k_2|} g(u, v, \mu, s)}{\partial \left(u^0\right)^{k_1} \partial \left(v^1\right)^{k_2}} ds, \end{cases} \tag{3.43}$$

where the derivatives of the composite functions $f(u(s; u^0, v^1, \tau, \mu), v(s; u^0, v^1, \tau, \mu), \mu, s)$ and $g(u(s; u^0, v^1, \tau, \mu), v(s; u^0, v^1, \tau, \mu), \mu, s)$ are calculated by the following formula:

$$\begin{aligned} \frac{\partial^{|k_1|+|k_2|}(f, g)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}} &= \frac{\partial (f, g)}{\partial (u, v)} \frac{\partial^{|k_1|+|k_2|}(u, v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}} \\ &+ \sum_{|i|=2}^{|k_1|+|k_2|} \left(\frac{\partial^{|i|}(f, g)}{\partial (u, v)^i} \sum_{\substack{|l_1|+\cdots+|l_{|i|}|=|k_1| \\ |p_1|+\cdots+|p_{|i|}|=|k_2| \\ |l_m|+|p_m|\ge 1}} \left(C_{\substack{l_1 \ldots l_{|i|} \\ p_1 \ldots p_{|i|}}} \prod_{j=1}^{|i|} \frac{\partial^{|l_j|+|p_j|}(u, v)}{\partial (u^0)^{l_j} \partial (v^1)^{p_j}} \right) \right). \end{aligned} \tag{3.44}$$

Here $C_{\substack{l_1 \ldots l_{|i|} \\ p_1 \ldots p_{|i|}}}$ are some constants. Notice, by the proposition of induction, the derivatives $\frac{\partial^{|l_j|+|p_j|}(u,v)}{\partial (u^0)^{l_j} \partial (v^1)^{p_j}}$ satisfy the estimates (3.42) because $|l_j| + |p_j| \le q$.

Let us consider the space N of continuous functions $(\eta_1(t), \eta_2(t))$ which are defined on the segment $t \in [0; \tau]$. The space N with the γ-norm (see (3.15), (3.16)) is a complete metric space. Formulas (3.43) and (3.44) associate a map $P : N \to N$. Namely, we define $(\bar{\eta}_1, \bar{\eta}_2) = P[(\eta_1, \eta_2)]$ by the following relations:

$$\begin{cases} \bar{\eta}_1 = \displaystyle\int_0^t e^{A(t-s)} \frac{\partial f}{\partial (u, v)} (\eta_1, \eta_2)\, ds + \int_0^t e^{A(t-s)} \sum_{|i|=2}^{|k_1|+|k_2|} \left(\frac{\partial^{|i|} f}{\partial (u, v)^i} \cdots \right) ds, \\ \bar{\eta}_2 = \displaystyle\int_\tau^t e^{B(t-s)} \frac{\partial g}{\partial (u, v)} (\eta_1, \eta_2)\, ds + \int_\tau^t e^{B(t-s)} \sum_{|i|=2}^{|k_1|+|k_2|} \left(\frac{\partial^{|i|} g}{\partial (u, v)^i} \cdots \right) ds. \end{cases} \tag{3.45}$$

In order to obtain the formulas (3.45), we substituted (η_1, η_2) for $\frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}}$ in the right-hand side of the equation (3.43) and $(\bar{\eta}_1, \bar{\eta}_2)$ for $\frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}}$ in the left-hand side. Thus, the derivative $\frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}}$ is a fixed point of P.

Notice that the map P is a linear operator. It can be represented in the following form:

$$(\bar{\eta}_1, \bar{\eta}_2) = \mathcal{A}(\eta_1, \eta_2) + \mathcal{B}, \tag{3.46}$$

where

$$\mathcal{A}(\eta_1,\eta_2) \equiv \left(\int_0^t e^{A(t-s)} \frac{\partial f}{\partial (u,v)}(\eta_1,\eta_2)\, ds, \int_\tau^t e^{B(t-s)} \frac{\partial g}{\partial (u,v)}(\eta_1,\eta_2)\, ds \right), \quad (3.47)$$

and

$$\mathcal{B} \equiv \left(\int_0^t e^{A(t-s)} \sum_{|i|=2}^{|k_1|+|k_2|} \left(\frac{\partial^{|i|} f}{\partial (u,v)^i} \cdots \right) ds\, , \int_\tau^t e^{B(t-s)} \sum_{|i|=2}^{|k_1|+|k_2|} \left(\frac{\partial^{|i|} g}{\partial (u,v)^i} \cdots \right) ds \right). \quad (3.48)$$

By (3.2)–(3.5) and (3.47), there exists a constant $\rho < 1$ such that $\|\mathcal{A}\| \le \rho$, i.e., the operator P satisfies the contraction property

$$\operatorname{dist}(P(\eta_1,\eta_2)) \le \rho \operatorname{dist}(\eta_1,\eta_2)\,.$$

It means, in particular, that the derivative $\frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}}$ is a unique fixed point of the operator P and, moreover,

$$\left\| \frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}} \right\| \le (1-\rho)^{-1} \|\mathcal{B}\|\,. \quad (3.49)$$

Therefore, by (3.49) and (3.48), there exists a constant D such that

$$\left\| \frac{\partial^{|k_1|+|k_2|}(u,v)}{\partial (u^0)^{k_1} \partial (v^1)^{k_2}} \right\|$$
$$\le D \max_{\substack{i=2\ldots|k_1|+|k_2| \\ |l_1|+\cdots+|l_i|=|k_1| \\ |p_1|+\cdots+|p_i|=|k_2| \\ |l_m|+|p_m|\ge 1}} \left(\int_0^t \prod_{j=1}^i \left\| \frac{\partial^{|l_j|+|p_j|}(u,v)}{\partial (u^0)^{l_j} \partial (v^1)^{p_j}} \right\| ds;\ \int_t^\tau e^{\beta(t-s)} \prod_{j=1}^i \left\| \frac{\partial^{|l_j|+|p_j|}(u,v)}{\partial (u^0)^{l_j} \partial (v^1)^{p_j}} \right\| ds \right). \quad (3.50)$$

By the proposition of induction, the derivatives $\frac{\partial^{|l_j|+|p_j|}(u,v)}{\partial (u^0)^{l_j} \partial (v^1)^{p_j}}$ satisfy the estimates (3.42) and, therefore, the right-hand side of inequality (3.50) can be estimated indeed. Thus, by the relations (3.50) and (3.42), we must estimate the maximum of the integrals

$$\int_0^t e^{\alpha(t-s)}\ e^{m_1 \alpha s}\ e^{m_2 \beta (s-\tau) + m_3 \alpha \tau} ds \qquad \text{and} \qquad \int_t^\tau e^{\beta(t-s)}\ e^{m_1 \alpha s}\ e^{m_2 \beta (s-\tau) + m_3 \alpha \tau} ds,$$

where $m_1 + m_3 = |k_1|$ and, moreover, $m_2 = m_3 = 0$ if $|k_2| = 0$ and $(|k_1|-1)\alpha - \beta < 0$. Obviously, the integrals satisfy the estimates (3.42).

So, Theorem 3.2 is proved for the case $0 < \alpha < \beta$. The statements of the theorem for the other cases can be proved in the same way.

3.3. *Behavior of the Derivatives If $\tau \to \infty$*

In Section 3.2 we obtained the estimates for the derivatives of solutions of the boundary value problem. Below we study the convergences of these derivatives if $\tau \to \infty$.

By virtue of the inequalities (3.31)–(3.33) and (3.35)–(3.37), some of the derivatives tend to zero if $\tau \to \infty$ because their norms go to zero. It is clear also that some of the derivatives do not have limits since their norms are unbounded. In this subsection we consider derivatives that are uniformly bounded by some positive constants. Namely, we study here the convergences of the following functions:

$$
\begin{aligned}
&\left.\frac{\partial^{|k_1|+|k_3|}(u,v)}{\partial\left(u^0,\mu\right)^{k_1}\partial t^{k_3}}\right|_{t=t_0} && \text{if } 0<|k_1|\alpha<\beta,\\
&\left.\frac{\partial^{|k_1|+|k_3|}(u,v)}{\partial\left(u^0\right)^{k_1}\partial(t,\mu)^{k_3}}\right|_{t=t_0} && \text{if } \alpha<0<\beta,\\
&\left.\frac{\partial^{|k_1|+|k_3|}(u,v)}{\partial\left(u^0\right)^{k_1}\partial t^{k_3}}\right|_{t=t_0} && \text{if } \alpha<\beta<0,\\
&\left.\frac{\partial^{|k_2|+|k_3|}(u,v)}{\partial\left(v^1\right)^{k_2}\partial(\tau-t)^{k_3}}\right|_{\tau-t=t_0} && \text{if } 0<\alpha<\beta,\\
&\left.\frac{\partial^{|k_2|+|k_3|}(u,v)}{\partial\left(v^1\right)^{k_2}\partial(\tau-t,\mu)^{k_3}}\right|_{\tau-t=t_0} && \text{if } \alpha<0<\beta,\\
&\left.\frac{\partial^{|k_2|+|k_3|}(u,v)}{\partial\left(v^1,\mu\right)^{k_2}\partial(\tau-t)^{k_3}}\right|_{\tau-t=t_0} && \text{if } \alpha<|k_2|\beta<0.
\end{aligned}
\tag{3.51}
$$

Unfortunately, in the general case, the derivatives (3.51) do not have limits if $\tau \to \infty$, but, under some additional conditions, convergence takes place. The next two theorems, 3.4 and 3.5, give us these conditions.

Below, for any functions $\phi(t)$, $\psi(t)$ and constants ϱ, τ, we use the following denotations:

$$
\begin{aligned}
\operatorname{Dist}(\phi;\ \psi)_{\varrho,\tau} &\equiv \sup_{t\in(0;\tau)}\left(\|\phi(t)-\psi(t)\|\, e^{-\varrho t}\right),\\
\operatorname{Dist}(\phi;\ \psi)^-_{\varrho,\tau} &\equiv \sup_{t\in(0;\tau)}\left(\|\phi(t)-\psi(t)\|\, e^{-\varrho(\tau-t)}\right).
\end{aligned}
\tag{3.52}
$$

Theorem 3.4. *Let $\tau \to +\infty$ and $\mu \to \mu_*$. Also let the boundary conditions u^0, v^1 vary in such a way that the solution (3.7) of the boundary value problem (3.1), (3.6) lies in a bounded domain and the point $(u^0, v^0) \equiv (u(t=0; u^0, v^1, \tau, \mu), v(t=0; u^0, v^1, \tau, \mu))$ converges to a point (u^0_*, v^0_*). Then there exist functions $A_{k_1k_3}(t)$, $B_{k_1k_3}(t)$, and $C_{k_1k_3}(t)$*

such that

$$\begin{aligned}
&\mathrm{Dist}\left(\frac{\partial^{|k_1|+|k_3|}(u,v)}{\partial\left(u^0,\mu\right)^{k_1}\partial t^{k_3}};\ A_{k_1k_3}\right)_{\sigma,\tau}\to 0 && \textit{if } 0<|k_1|\alpha<\sigma<\beta,\\
&\mathrm{Dist}\left(\frac{\partial^{|k_1|+|k_3|}(u,v)}{\partial\left(u^0\right)^{k_1}\partial(t,\mu)^{k_3}};\ B_{k_1k_3}\right)_{\sigma,\tau}\to 0 && \textit{if } \alpha<0<\beta,\quad \alpha<\sigma<\beta,\\
&\mathrm{Dist}\left(\frac{\partial^{|k_1|+|k_3|}(u,v)}{\partial\left(u^0\right)^{k_1}\partial t^{k_3}};\ C_{k_1k_3}\right)_{\sigma,\tau}\to 0 && \textit{if } \alpha<\sigma<\beta<0,
\end{aligned}\tag{3.53}$$

where $|k_1|+|k_3|\le k$*.*

Notice, by virtue of the changes (3.34), Theorem 3.4 directly implies the following result.

Theorem 3.5. *Let* $\tau\to+\infty$ *and* $\mu\to\mu_*$*. Let also the boundary conditions* u^0*,* v^1 *vary in such a way that the solution (3.7) of the boundary value problem (3.1), (3.6) lies in a bounded domain and the point* $(u^1,v^1)\equiv(u(t=\tau;u^0,v^1,\tau,\mu),v(t=\tau;u^0,v^1,\tau,\mu))$ *converges to a point* (u^1_*,v^1_*)*. Then there exist functions* $A^-_{k_2k_3}(t)$*,* $B^-_{k_2k_3}(t)$*, and* $C^-_{k_2k_3}(t)$ *such that*

$$\begin{aligned}
&\mathrm{Dist}\left(\frac{\partial^{|k_2|+|k_3|}(u,v)}{\partial\left(v^1\right)^{k_2}\partial(\tau-t)^{k_3}};\ A^-_{k_2k_3}\right)^-_{\sigma,\tau}\to 0 && \textit{if } 0<\alpha<\sigma<\beta,\\
&\mathrm{Dist}\left(\frac{\partial^{|k_2|+|k_3|}(u,v)}{\partial\left(v^1\right)^{k_2}\partial(\tau-t,\mu)^{k_3}};\ B^-_{k_2k_3}\right)^-_{\sigma,\tau}\to 0 && \textit{if } \alpha<0<\beta,\quad \alpha<\sigma<\beta,\\
&\mathrm{Dist}\left(\frac{\partial^{|k_2|+|k_3|}(u,v)}{\partial\left(v^1,\mu\right)^{k_2}\partial(\tau-t)^{k_3}};\ C^-_{k_2k_3}\right)^-_{\sigma,\tau}\to 0 && \textit{if } \alpha<\sigma<|k_2|\beta<0,
\end{aligned}\tag{3.54}$$

where $|k_2|+|k_3|\le k$*.*

Now we shall prove Theorem 3.4 for the first case, i.e., under the condition $0<|k_1|\alpha<\sigma<\beta$. Note that, by the considerations of the previous subsection, the derivatives with respect to μ and t can be calculated via derivatives with respect to u^0. So, we must prove the statement for $\frac{\partial^{|k_1|}(u,v)}{\partial(u^0)^{k_1}}$ only.

First, let us prove the theorem for k_1 such that $|k_1|=1$, i.e., for $\frac{\partial(u,v)}{\partial u^0}$. Let the sequences u^0_i, v^1_i, τ_i, and μ_i, $(i=1,2,3,\ldots)$ satisfy the theorem conditions. In this case, for any fixed τ_0, the appropriate solutions $(u_i(t;u^0_i,v^1_i,\tau_i,\mu_i),v_i(t;u^0_i,v^1_i,\tau_i,\mu_i))$ converge uniformly on $t\in[0,\tau_0]$ to a solution of the initial value problem $(u_*(t;u^0_*,v^0_*,\mu_*),v_*(t;u^0_*,v^0_*,\mu_*))$ that starts from the point (u^0_*,v^0_*), i.e.,

$$\sup_{t\in(0,\tau_0)}\|(u_i(t),v_i(t))-(u_*(t),v_*(t))\|\to 0.\tag{3.55}$$

It means, in particular, that for any $\delta > 0$,

$$\operatorname{Dist}\left(\frac{\partial(f,g)}{\partial\left(u^0,v^1\right)}(u_i,v_i,\mu_i,s)\,;\ \frac{\partial(f,g)}{\partial\left(u^0,v^1\right)}(u_*,v_*,\mu_*,s)\right)_{\delta,\tau_i} \to 0 \qquad \text{if } \tau_i \to \infty. \tag{3.56}$$

We shall show that the sequence of the derivatives $\frac{\partial(u_i,v_i)}{\partial u^0}$ is a Cauchy sequence in the following sense: For any $\varepsilon > 0$, there exists a constant $N(\varepsilon)$ such that

$$\operatorname{Dist}\left(\frac{\partial(u_l,v_l)}{\partial u^0};\ \frac{\partial(u_p,v_p)}{\partial u^0}\right)_{\sigma,\tau_l} \le \varepsilon,$$

if $l > N(\varepsilon)$, $p > N(\varepsilon)$, and $\alpha < \sigma < \beta$.[10] Since the solution (u_i, v_i), $(i = l, p)$ is a fixed point of the operator T (see (3.17)), the derivative $\frac{\partial(u_i,v_i)}{\partial u^0}$ satisfies the following equality:

$$\begin{cases} \dfrac{\partial u_i}{\partial u^0} = e^{At} + \displaystyle\int_0^t e^{A(t-s)} \frac{\partial f}{\partial\left(u^0,v^1\right)}(u_i,v_i,\mu_i,s)\,\frac{\partial(u_i,v_i)}{\partial u^0}ds, \\ \dfrac{\partial v_i}{\partial u^0} = \displaystyle\int_{\tau_i}^t e^{B(t-s)} \frac{\partial g}{\partial\left(u^0,v^1\right)}(u_i,v_i,\mu_i,s)\,\frac{\partial(u_i,v_i)}{\partial u^0}ds. \end{cases} \tag{3.57}$$

Therefore, for any fixed $t \in [0, \tau_l]$, by (3.2)–(3.5), (3.31), (3.52), (3.56), and (3.57), we have

$$\begin{aligned}
&\left\|\frac{\partial u_l}{\partial u^0} - \frac{\partial u_p}{\partial u^0}\right\| \\
&\le \int_0^t e^{\alpha(t-s)} \left\| \left.\frac{\partial f}{\partial\left(u^0,v^1\right)}\right|_{(u_l,v_l,\mu_l)} \frac{\partial(u_l,v_l)}{\partial u^0} - \left.\frac{\partial f}{\partial\left(u^0,v^1\right)}\right|_{(u_p,v_p,\mu_p)} \frac{\partial(u_p,v_p)}{\partial u^0} \right\| ds \\
&\le \int_0^t e^{\alpha(t-s)} \left\| \left.\frac{\partial f}{\partial\left(u^0,v^1\right)}\right|_{(u_l,v_l,\mu_l)} \right\| \operatorname{Dist}\left(\frac{\partial(u_l,v_l)}{\partial u^0};\frac{\partial(u_p,v_p)}{\partial u^0}\right)_{\sigma,\tau_l} e^{\sigma s} ds \\
&\quad + \int_0^t e^{\alpha(t-s)} \operatorname{Dist}\left(\left.\frac{\partial f}{\partial\left(u^0,v^1\right)}\right|_{(u_l,v_l,\mu_l)};\ \left.\frac{\partial f}{\partial\left(u^0,v^1\right)}\right|_{(u_p,v_p,\mu_p)} \right)_{\delta,\tau_l} e^{\delta s} \left\|\frac{\partial(u_p,v_p)}{\partial u^0}\right\| ds \\
&\le \varepsilon_1 \operatorname{Dist}\left(\frac{\partial(u_l,v_l)}{\partial u^0};\ \frac{\partial(u_p,v_p)}{\partial u^0}\right)_{\sigma,\tau_l} e^{\sigma t} + \varepsilon_2(N)e^{(\alpha+\delta)t},
\end{aligned} \tag{3.58}$$

[10] Without loss of generality, we assume that $\tau_l \le \tau_p$.

$$\left\| \frac{\partial v_l}{\partial u^0} - \frac{\partial v_p}{\partial u^0} \right\|$$

$$\leq \int_t^{\tau_l} e^{\beta(t-s)} \left\| \frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_l, v_l, \mu_l)} \frac{\partial(u_l, v_l)}{\partial u^0} - \frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_p, v_p, \mu_p)} \frac{\partial(u_p, v_p)}{\partial u^0} \right\| ds$$

$$+ \int_{\tau_l}^{\tau_p} e^{\tilde{\beta}(t-s)} \left\| \frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_p, v_p, \mu_p)} \frac{\partial(u_p, v_p)}{\partial u^0} \right\| ds$$

$$\leq \int_t^{\tau_l} e^{\tilde{\beta}(t-s)} \left\| \frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_l, v_l, \mu_l)} \right\| \operatorname{Dist}\left(\frac{\partial(u_l, v_l)}{\partial u^0};\ \frac{\partial(u_p, v_p)}{\partial u^0}\right)_{\sigma, \tau_l} e^{\sigma s} ds$$

$$+ \int_t^{\tau_l} e^{\tilde{\beta}(t-s)} \operatorname{Dist}\left(\frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_l, v_l, \mu_l)};\ \frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_p, v_p, \mu_p)} \right)_{\delta, \tau_l} e^{\delta s} \left\| \frac{\partial(u_p, v_p)}{\partial u^0} \right\| ds$$

$$+ \int_{\tau_l}^{\tau_p} e^{\tilde{\beta}(t-s)} \left\| \frac{\partial g}{\partial\left(u^0, v^1\right)}\bigg|_{(u_p, v_p, \mu_p)} \right\| \left\| \frac{\partial(u_p, v_p)}{\partial u^0} \right\| ds$$

$$\leq \varepsilon_3 \operatorname{Dist}\left(\frac{\partial(u_l, v_l)}{\partial u^0};\ \frac{\partial(u_p, v_p)}{\partial u^0}\right)_{\sigma, \tau_l} e^{\sigma t} + \varepsilon_4(N) e^{(\alpha+\delta)t} + \varepsilon_5 e^{\beta(t-\tau_l)+\alpha\tau_l}, \tag{3.59}$$

where the constants $\varepsilon_1, \varepsilon_3, \delta$ can be made arbitrarily small and $\varepsilon_2(N) \to 0$, $\varepsilon_4(N) \to 0$ if $N \to \infty$. By the inequalities (3.58) and (3.59), the following relation holds:

$$\operatorname{Dist}\left(\frac{\partial(u_l, v_l)}{\partial u^0};\ \frac{\partial(u_p, v_p)}{\partial u^0}\right)_{\sigma, \tau_l} \leq \varepsilon_6 \operatorname{Dist}\left(\frac{\partial(u_l, v_l)}{\partial u^0};\ \frac{\partial(u_p, v_p)}{\partial u^0}\right)_{\sigma, \tau_l} + \varepsilon_7(N), \tag{3.60}$$

where $\varepsilon_6 < 1$ and $\varepsilon_7(N) \to 0$ if $N \to \infty$. Formula (3.60) implies that $\operatorname{Dist}(\frac{\partial(u_l, v_l)}{\partial u^0};\ \frac{\partial(u_p, v_p)}{\partial u^0})_{\sigma, \tau_l} \to 0$ at $N \to \infty$, i.e., the sequence of the derivatives $\frac{\partial(u_i, v_i)}{\partial u^0}$ is a Cauchy sequence and, therefore, it has a limit.

So, we have proved the statement of Theorem 3.4 for the derivatives $\frac{\partial^{|k_1|}(u,v)}{\partial(u^0)^{k_1}}$ such that $|k_1| = 1$. The statement of the theorem for the higher order derivatives $(2 \leq |k_1| < k)$ may be proved by induction on $|k_1|$.

3.4. A Boundary Value Problem for the System X_μ

In this subsection we evaluate behavior of orbits of the system X_μ near equilibria O using the results obtained above.

It is well known that in a neighborhood of the saddle O one can introduce local coordinates (x, y, z), $x \in R^n$, $y \in R^1$, $z \in R^{m-1}$, such that the system X_μ takes the form

$$\begin{cases} \dot{x} &= Ax + f^x(x, y, z, \mu), \\ \dot{y} &= \gamma y + f^y(x, y, z, \mu), \\ \dot{z} &= Bz + f^z(x, y, z, \mu), \end{cases} \tag{3.61}$$

where A is a matrix $(n \times n)$ and Spectr $A = \{\lambda_1 \ldots \lambda_n\}$, B is a matrix $(m-1 \times m-1)$

and Spectr $B = \{\gamma_2 \dots \gamma_m\}$, $\gamma \equiv \gamma_1$. Remember that, by the condition **(A)**, we have the relations

$$\max_{i=1,\dots,n} \operatorname{Re} \lambda_i < 0 < \gamma < \min_{i=2,\dots,m} \operatorname{Re} \gamma_i . \tag{3.62}$$

The functions f^x, f^y, f^z are C^r-smooth ($k \geq 1$) with respect to the phase variables (x, y, z) and the parameter μ. Moreover, these functions satisfy the following equalities:

$$(f^x, f^y, f^z)|_{(x,y,z)=0} = 0, \quad \left. \frac{\partial\,(f^x, f^y, f^z)}{\partial (x, y, z)} \right|_{(x,y,z,\mu)=0} = 0. \tag{3.63}$$

Below we are interested in the solutions of the system (3.61) satisfying the following boundary conditions:

$$x(0) = x^0, \qquad y(0) = y^0, \qquad z(\tau) = z^1, \qquad \tau > 0, \tag{3.64}$$

with small x^0, y^0, and z^1. So, the problem (3.61), (3.64) is a boundary value problem of the kind (3.1), (3.6).[11]

By virtue of Theorem 3.2 (see (3.31)), the solution

$$(x(t), y(t), z(t)) \equiv \big(x(t; x^0, y^0, z^1, \tau, \mu), y(t; x^0, y^0, z^1, \tau, \mu), z(t; x^0, y^0, z^1, \tau, \mu)\big) \tag{3.65}$$

of the boundary value problem (3.61), (3.64) satisfies the following inequalities:

$$\left\| \frac{\partial (x, y, z)}{\partial t} \right\| \leq C, \qquad \left\| \frac{\partial (x, y, z)}{\partial (x^0, y^0, \mu)} \right\| \leq C\, e^{\alpha t}, \qquad \left\| \frac{\partial (x, y, z)}{\partial (z^1, \tau)} \right\| \leq C\, e^{\beta (t-\tau)}, \tag{3.66}$$

where

$$\max_{i=1,\dots,n} \operatorname{Re} \lambda_i < 0 < \gamma < \alpha < \beta < \min_{i=2,\dots,m} \operatorname{Re} \gamma_i . \tag{3.67}$$

Moreover, by Theorems 3.4 and 3.5, if $\mu \to 0$, $\tau \to \infty$, and

$$(x^0, y^0, z^0) \equiv (x(0), y(0), z(0)) \to (x^+, y^+, z^+) \in W^s,$$

$$(x^1, y^1, z^1) \equiv (x(\tau), y(\tau), z(\tau)) \to (x^-, y^-, z^-) \in W^u,$$

then

$$\left. \frac{\partial (x, y, z)}{\partial \left(x^0, y^0, \mu, t\right)} \right|_{t=0} \to \mathcal{A}, \qquad \left. \frac{\partial (x, y, z)}{\partial \left(z^1, \tau, t\right)} \right|_{t=\tau} \to \mathcal{B}, \tag{3.68}$$

where $\mathcal{A}$ and $\mathcal{B}$ are some matrices.

---

[11] We use (x, y) for the variable u and the variable z for v.

4. Local and Global Maps

The proof of the main Theorem 2.1 is based on the investigation of the Poincaré map along the orbits of the system X_μ in a neighborhood of the homoclinic loop. This map may be represented as a superposition of two maps: T_{loc} and T_{gl}, where T_{loc} is defined by the flow of the system near the equilibrium point and T_{gl} is defined by the flow near the global piece of the homoclinic trajectory Γ. Using the form (3.61) for the system X_μ, we shall construct cross sections S^0, S^1 near the equilibria O and investigate properties of the first return maps $T_{loc} : S^0 \to S^1$, $T_{gl} : S^1 \to S^0$.

By virtue of the system (3.61), the stable manifold W^s of the point O is an n-dimensional surface that, when $\mu = 0$, is tangent to the plane $(y, z) = 0$ at the point $O = (0, 0, 0)$. This means that W^s is locally the graph of a smooth function,

$$(y, z) = \left(y^s(x, \mu), z^s(x, \mu)\right),$$

where

$$\left(y^s(0, \mu), z^s(0, \mu)\right) = 0, \qquad \left.\frac{\partial\,(y^s, z^s)}{\partial x}\right|_{(x,\mu)=0} = 0.$$

The unstable manifold W^u of O is locally the graph of a smooth function

$$x = x^u(y, z, \mu),$$

where

$$x^u(0, 0, \mu) = 0, \qquad \left.\frac{\partial x^u}{\partial\,(y, z)}\right|_{(y,z,\mu)=0} = 0.$$

If $\mu = 0$ and $t \to +\infty$, the orbit $\Gamma \subset W^s \cap W^u$ tends to O. Therefore, there exist small enough $\varepsilon > 0$, $\delta > 0$, and μ such that the surface

$$S^0 = \left\{(x, y, z) \mid \|x\| = \varepsilon,\ \left\|\left(x - x^+, y - y^+, z - z^+\right)\right\| \leq \delta\right\} \tag{4.1}$$

is a cross section for the orbits close to Γ; here (x^+, y^+, z^+) are the coordinates of a point of the first intersection of Γ with the surface $\|x\| = \varepsilon$ (see Figure 9).

Since the orbit $\Gamma \subset W^s \cap W^u$ does not belong to the strong-unstable submanifold W^{uu} (see condition (**C**)), Γ leaves the equilibrium point O tending toward the y-axis, which corresponds to the leading direction. Without loss of generality, we assume that Γ leaves O in the positive direction of the y-axis. In this case, if $\delta > 0$, $y^- > 0$, and μ are small enough, the surface,

$$S^1 = \left\{(x, y, z) \mid y = y^-,\ \left\|\left(x - x^-, z - z^-\right)\right\| \leq \delta\right\}, \tag{4.2}$$

is a cross section for the orbits close to Γ; here (x^-, y^-, z^-) are the coordinates of a point of the first intersection of Γ with the surface $y = y^-$.

Thus, we have constructed two cross sections in a small neighborhood of the equilibrium point O: S^1 and S^0. It is clear that the dimension of the cross sections equals $(n + m - 1)$ and, without loss of generality, we may consider the coordinates $(x_1, \ldots, x_n, z_1, \ldots, z_{m-1})$ as coordinates (x^1, z^1) on the cross section S^1 and the coordinates $(x_1, \ldots, x_{n-1}, y, z_1, \ldots, z_{m-1})$ as coordinates (x^0, y^0, z^0) on the cross section S^0.

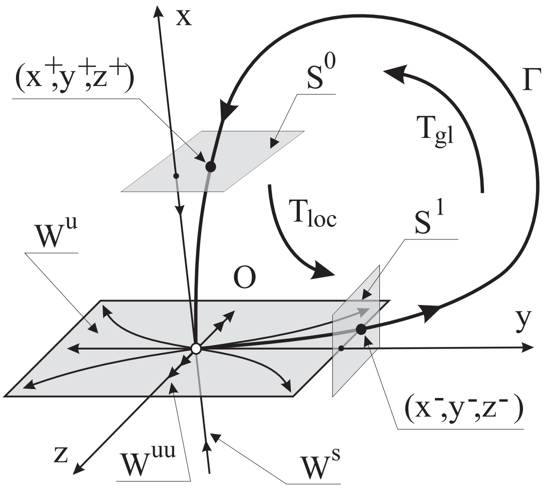

Fig. 9. Two cross sections S^0 and S^1 can be constructed for orbits of X_μ near the loop $\mathcal{L} = O \cup \Gamma$.

The Poincaré map $T : S^0 \to S^0$ is the superposition of $T_{loc} : S^0 \to S^1$ and $T_{gl} : S^1 \to S^0$. The map $T_{gl} : S^1 \to S^0$ is a diffeomorphism because the flight time from S^1 to S^0 is bounded. Therefore T_{gl}^{-1}: $S^0 \to S^1$ may be represented in the following form:

$$\begin{pmatrix} x^1 - x^-(\mu) \\ z^1 - z^-(\mu) \end{pmatrix} = \begin{pmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{pmatrix} \begin{pmatrix} x^0 - x^+(\mu) \\ y^0 - y^+(\mu) \\ z^0 - z^+(\mu) \end{pmatrix} + \begin{pmatrix} \phi(x^0, y^0, z^0, \mu) \\ \psi(x^0, y^0, z^0, \mu) \end{pmatrix}, \quad (4.3)$$

where

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0.$$

Here $a_{11}, a_{22}, a_{12}, a_{21}$ are matrices of the dimensions $(n \times n)$, $(m-1 \times m-1)$, $(n \times m-1)$, $(m-1 \times n)$, respectively. The smooth functions $\phi(x^0, y^0, z^0, \mu)$, $\psi(x^0, y^0, z^0, \mu)$ contain only the nonlinear terms on $(x^0 - x^+, y^0 - y^+, z^0 - z^+)$ and are defined in the domain

$$\big\{(x^0, y^0, z^0, \mu) \mid \big\|\big(x^0 - x^+, y^0 - y^+, z^0 - z^+, \mu\big)\big\| \leq \delta\big\}. \quad (4.4)$$

Note that we assume that the manifold W^{s+} transversely intersects the leaves of the foliation F^u (condition (**D**)). This assumption means exactly the same as the condition

$$\det(a_{11}) \neq 0. \quad (4.5)$$

Consider the local map $T_{loc} : S^0 \to S^1$. The study of this map is not so trivial because the flight time of orbits that go from S^0 to S^1 is unbounded. Moreover the infimum τ^* of the flight times can be made arbitrarily big by choosing small δ. In order to obtain the local map, we use the boundary value problem (3.61), (3.64). According to this problem, for given τ and small x^0, y^0, z^1, there exists a unique orbit

$$(x(t), y(t), z(t)) \equiv \big(x(t; x^0, y^0, z^1, \tau, \mu), y(t; x^0, y^0, z^1, \tau, \mu), z(t; x^0, y^0, z^1, \tau, \mu)\big),$$

which, at $t = 0$, starts with a point (x^0, y^0, z^0) and reaches a point (x^1, y^1, z^1) at $t = \tau$. This means that the following equalities are fulfilled:

$$\begin{aligned} x^1 &= x(\tau; x^0, y^0, z^1, \tau, \mu), \\ y^1 &= y(\tau; x^0, y^0, z^1, \tau, \mu), \\ z^0 &= z(0; x^0, y^0, z^1, \tau, \mu). \end{aligned} \tag{4.6}$$

If we fix $y^1 = y^- > 0$ and $\|x^0\| = \varepsilon$, the first and third equations of the system (4.6) give implicitly the map $T_{loc} : (x^0, y^0, z^0) \mapsto (x^1, z^1)$ from S^0 to S^1 where τ should be expressed from the second equation of the system (4.6) as a function of (x^0, y^0, z^1, μ):

$$\begin{aligned} x^1 &= x(\tau(x^0, y^0, z^1, \mu); x^0, y^0, z^1, \tau(x^0, y^0, z^1, \mu), \mu), \\ z^0 &= z(0; x^0, y^0, z^1, \tau(x^0, y^0, z^1, \mu), \mu), \end{aligned} \tag{4.7}$$

where

$$y^- \equiv y(\tau(x^0, y^0, z^1, \mu); x^0, y^0, z^1, \tau(x^0, y^0, z^1, \mu), \mu). \tag{4.8}$$

We shall show (see (4.17)) that

$$\left.\frac{\partial y}{\partial t}\right|_{t=\tau} + \left.\frac{\partial y}{\partial \tau}\right|_{t=\tau} \neq 0.$$

Therefore, the flight time $\tau > \tau^*(\delta)$ can indeed be found from the equation (4.8) and the local map T_{loc} may be represented in the following form:

$$\begin{aligned} x^1 &= f(x^0, y^0, z^1, \mu), \\ z^0 &= g(x^0, y^0, z^1, \mu). \end{aligned} \tag{4.9}$$

Since $y^- > 0$, the functions f, g are defined in the domain

$$\left\{(x^0, y^0, z^1, \mu) \mid \left\|\left(x^0 - x^+, y^0 - y^+, z^1 - z^-, \mu\right)\right\| \le \delta,\ y^0 > y^s(x^0, \mu)\right\}, \tag{4.10}$$

where the function $y = y^s(x^0, \mu)$ gives the y-coordinate of the intersection of the stable manifold W^s with the cross section S^0.

Notice the relation that follows from the identity (4.8):

$$\frac{\partial \tau}{\partial(x^0, y^0, z^1, \mu)} = -\left(\left.\frac{\partial y}{\partial t}\right|_{t=\tau} + \left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}\right)^{-1} \left.\frac{\partial y}{\partial(x^0, y^0, z^1, \mu)}\right|_{t=\tau}. \tag{4.11}$$

By formulas (4.7), (4.9), and (4.11) we have

$$\begin{aligned} &\frac{\partial f}{\partial(x^0, y^0, z^1, \mu)} \equiv \frac{\partial x^1}{\partial(x^0, y^0, z^1, \mu)} \\ &= \left.\frac{\partial x}{\partial(x^0,y^0,z^1,\mu)}\right|_{t=\tau} - \left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau}\right)\left(\left.\frac{\partial y}{\partial t}\right|_{t=\tau} + \left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}\right)^{-1} \left.\frac{\partial y}{\partial(x^0,y^0,z^1,\mu)}\right|_{t=\tau}, \end{aligned}$$

$$\begin{aligned} &\frac{\partial g}{\partial(x^0, y^0, z^1, \mu)} \equiv \frac{\partial z^0}{\partial(x^0, y^0, z^1, \mu)} \\ &= \left.\frac{\partial z}{\partial(x^0, y^0, z^1, \mu)}\right|_{t=0} - \left.\frac{\partial z}{\partial \tau}\right|_{t=0} \left(\left.\frac{\partial y}{\partial t}\right|_{t=\tau} + \left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}\right)^{-1} \left.\frac{\partial y}{\partial(x^0, y^0, z^1, \mu)}\right|_{t=\tau}. \end{aligned} \tag{4.12}$$

By the identity (3.40), we also have the relation

$$\left.\frac{\partial y}{\partial z^1}\right|_{t=\tau}\left.\frac{\partial z}{\partial t}\right|_{t=\tau}+\left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}\equiv 0. \tag{4.13}$$

The orbit Γ leaves the equilibria O tending in the leading direction (positive y-axis); therefore,

$$\left.\frac{\partial z}{\partial t}\right|_{t=\tau}=o\left(\left.\frac{\partial y}{\partial t}\right|_{t=\tau}\right)\qquad \text{at } y^-\to 0. \tag{4.14}$$

The estimates (3.66) imply that

$$\left\|\left.\frac{\partial y}{\partial z^1}\right|_{t=\tau}\right\|\leq C. \tag{4.15}$$

So, by (4.13), (4.14), and (4.15),

$$\left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}=o\left(\left.\frac{\partial y}{\partial t}\right|_{t=\tau}\right)\qquad \text{at } y^-\to 0. \tag{4.16}$$

Since $\left.\frac{\partial y}{\partial t}\right|_{t=\tau}\sim\gamma y^-$, formula (4.16) implies that

$$\left.\frac{\partial y}{\partial t}\right|_{t=\tau}+\left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}\sim\gamma y^->0. \tag{4.17}$$

Since y^- is fixed, formula (4.17) implies that

$$\left(\left.\frac{\partial y}{\partial t}\right|_{t=\tau}+\left.\frac{\partial y}{\partial \tau}\right|_{t=\tau}\right)^{-1}<\frac{2}{\gamma y^-}, \tag{4.18}$$

i.e., $(\frac{\partial y}{\partial t}|_{t=\tau}+\frac{\partial y}{\partial \tau}|_{t=\tau})^{-1}$ is bounded. Now, according to the estimates (3.66) for the boundary value problem (3.61), (3.64), and by (4.12), (4.18), we have the following relations:

$$\begin{aligned}&\left\|\frac{\partial x^1}{\partial(x^0,y^0,\mu)}\right\|\equiv\left\|\frac{\partial f}{\partial(x^0,y^0,\mu)}\right\|\leq D\,e^{\alpha\tau}, &&\left\|\frac{\partial x^1}{\partial z^1}\right\|\equiv\left\|\frac{\partial f}{\partial z^1}\right\|\leq d,\\ &\left\|\frac{\partial z^0}{\partial(x^0,y^0,\mu)}\right\|\equiv\left\|\frac{\partial g}{\partial(x^0,y^0,\mu)}\right\|\leq d, &&\left\|\frac{\partial z^0}{\partial z^1}\right\|\equiv\left\|\frac{\partial g}{\partial z^1}\right\|\leq D\,e^{-\beta\tau},\end{aligned} \tag{4.19}$$

where D and d are some constants and the constants α and β satisfy inequalities (3.67). Moreover, if $\mu\to 0$, $\tau\to\infty$, $(x^0,y^0,z^0)\to(x^+,y^+,z^+)\in W^s$, and $(x^1,y^1,z^1)\to(x^-,y^-,z^-)\in W^u$, by (3.68), we have

$$\frac{\partial x^1}{\partial z^1}\equiv\frac{\partial f}{\partial z^1}\to\bar{\mathcal{A}},\qquad \frac{\partial z^0}{\partial(x^0,y^0,\mu)}\equiv\frac{\partial g}{\partial(x^0,y^0,\mu)}\to\bar{\mathcal{B}}, \tag{4.20}$$

where $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are fixed matrices. Without loss of generality, we may assume that $\bar{\mathcal{A}}\equiv 0$ and $\bar{\mathcal{B}}\equiv 0$. Otherwise, we shall change the variables $x^1\mapsto x^1_{new}$ on S^1 and $z^0\mapsto z^0_{new}$ on S^0 in the following way:

$$x^1_{new}=x^1-\bar{\mathcal{A}}z^1,\qquad z^0_{new}=z^0-\bar{\mathcal{B}}(x^0,y^0,\mu). \tag{4.21}$$

So, we have

$$d=o(1)\qquad \text{if } \tau\to+\infty. \tag{4.22}$$

5. Redefinition of the Local and Global Maps

In order to establish the existence of the center manifold $\mathcal{M}^{cs}$, we must show that the cross section S^0 contains an invariant set with respect to the operator $T = T_{gl} \circ T_{loc}$ and that this set is the graph of some vector-function $z^0 = h^0_*(x^0, y^0, \mu)$. Below we use the regular method, which is based on proving the contractibility for the graph transformations (see Section 6). Namely, we show that the operator T induces a contraction operator P in a complete metric space H^0 of functions $z^0 = h^0(x^0, y^0, \mu)$. However, to construct the operator P, we must redefine the maps T_{gl} and T_{loc} in an extended domain and, by that, redefine the Poincaré map $T = T_{gl} \circ T_{loc}$.

Consider first the global map $T_{gl} : S^1 \to S^0$. As we have seen in Section 3.4, the inverse map $T_{gl}^{-1} : S^0 \to S^1$ is given by the formula (4.3). We redefine T_{gl}^{-1} by the following formulas:

$$
\begin{aligned}
x^1 &= \Phi(x^0, y^0, z^0, \mu) \\
&\equiv x^-(\mu) + a_{11}(\mu) \begin{pmatrix} x^0 - x^+(\mu) \\ y^0 - y^+(\mu) \end{pmatrix} + a_{12}(\mu)\left(z^0 - z^+(\mu)\right) \\
&\quad + \vartheta(\|(x^0 - x^+, y^0 - y^+)\|/\delta)\phi(x^0, y^0, z^0, \mu), \\
z^1 &= \Psi(x^0, y^0, z^0, \mu) \equiv z^-(\mu) \\
&\quad + \left(a_{21}(\mu) \begin{pmatrix} x^0 - x^+(\mu) \\ y^0 - y^+(\mu) \end{pmatrix} + a_{22}(\mu)\left(z^0 - z^+(\mu)\right) + \psi(x^0, y^0, z^0, \mu)\right) \\
&\quad \times \vartheta(\|(x^0 - x^+, y^0 - y^+)\|/\delta),
\end{aligned}
\tag{5.1}
$$

where $\vartheta(t)$ is a C^∞-function such that

$$
\vartheta(t) = \begin{cases} 1, & \text{if } t \le 1, \\ 0, & \text{if } t \ge 2, \end{cases} \qquad \left|\frac{\partial \vartheta(t)}{\partial t}\right| < 2. \tag{5.2}
$$

Observe that the redefined map coincides with the initial map if $\|(x^0 - x^+, y^0 - y^+)\| < \delta$, but functions Φ and Ψ are defined in the larger domain

$$
\Omega_{gl} = \left\{(x^0, y^0, z^0, \mu) \mid \left\|\left(z^0 - z^+, \mu\right)\right\| \le \delta \ \left(x^0, y^0\right) \in R^n\right\}. \tag{5.3}
$$

Let us consider now the local map $T_{loc} : S^0 \to S^1$ (see (4.9)). We overwrite T_{loc} as

$$
\begin{aligned}
x^1 &= F(x^0, y^0, z^1, \mu) \\
&\equiv \vartheta(\|(x^0 - x^+, y^0 - y^+)\|/\delta)\, f(x^0, |y^0 - y^s(x^0, \mu)| + y^s(x^0, \mu), z^1, \mu), \\
z^0 &= G(x^0, y^0, z^1, \mu) \\
&\equiv \vartheta(\|(x^0 - x^+, y^0 - y^+)\|/\delta) g(x^0, |y^0 - y^s(x^0, \mu)| + y^s(x^0, \mu), z^1, \mu).
\end{aligned}
\tag{5.4}
$$

The functions f and g, defining the local map (4.9), coincide with the functions F and G, respectively, if $\|(x^0 - x^+, y^0 - y^+)\| < \delta$ and $y^0 > y^s(x^0, \mu)$, but the functions F and G are defined for any $(x^0, y^0) \in R^n$, i.e., in the domain

$$\Omega_{loc} = \left\{(x^0, y^0, z^1, \mu) \mid \left\|\left(z^1 - z^-, \mu\right)\right\| \leq \delta,\ (x^0, y^0) \in R^n\right\}. \tag{5.5}$$

Moreover, by (4.19), (4.22), (5.2), and (5.4), F and G satisfy the following estimates:

$$\begin{aligned} &\left\|\frac{\partial F}{\partial(x^0, y^0, \mu)}\right\| \leq M\, e^{\alpha\tau}, \qquad \left\|\frac{\partial F}{\partial z^1}\right\| = o(1) \quad \text{if } \tau \to \infty, \\ &\left\|\frac{\partial G}{\partial(x^0, y^0, \mu)}\right\| = o(1) \qquad \text{if } \tau \to \infty, \qquad \left\|\frac{\partial G}{\partial z^1}\right\| \leq M\, e^{-\beta\tau}. \end{aligned} \tag{5.6}$$

Here M is a constant and the constants α and β satisfy inequalities (3.67).

6. Existence of the Lipschitz Center Manifold

Introduce the space H^0 of bounded $(m-1)$-dimensional vector-functions $z^0 = h^0(x^0, y^0, \mu)$ defined in the domain

$$D^0 = \left\{(x^0, y^0, \mu) \mid (x^0, y^0) \in R^n,\ \|\mu\| < \delta\right\}. \tag{6.1}$$

Let the functions $h^0 \in H^0$ satisfy the following Lipschitz condition:

$$\left\|h^0(x_1^0, y_1^0, \mu_1) - h^0(x_2^0, y_2^0, \mu_2)\right\| \leq \ell \left\|(x_1^0, y_1^0, \mu_1) - (x_2^0, y_2^0, \mu_2)\right\|. \tag{6.2}$$

Here $\ell > 0$ is a small constant, which we define below. The space H^0 with the uniform norm

$$\left\|h^0\right\|_{D^0} \equiv \sup_{(x^0, y^0, \mu) \in D^0} \left\|h^0(x^0, y^0, \mu)\right\| \tag{6.3}$$

is a complete metric space.

Introduce also the space H^1 of bounded $(m-1)$-dimensional vector-functions $z^1 = h^1(x^1, \mu)$ defined in the domain

$$D^1 = \left\{(x^1, \mu) \mid x^1 \in R^n,\ \|\mu\| < \delta\right\}. \tag{6.4}$$

Let the functions $h^1 \in H^1$ satisfy the Lipschitz condition

$$\left\|h^1(x_1^1, \mu_1) - h^1(x_2^1, \mu_2)\right\| \leq L\ \left\|(x_1^1, \mu_1) - (x_2^1, \mu_2)\right\|, \tag{6.5}$$

where L is some, maybe big, constant, which we define later. The space H^1 with the norm

$$\left\|h^1\right\|_{D^1} \equiv \sup_{(x^1, \mu) \in D^1} \left\|h^1(x^1, \mu)\right\| \tag{6.6}$$

is a complete metric space.

Lemma 6.1. *The map T_{gl}^{-1} induces an operator $P_{gl} : H^0 \to H^1$, i.e., T_{gl}^{-1} transforms points of the graph of any function $h^0 \in H^0$ to points of the graph of some function $h^1 \in H^1$. Moreover the map P_{gl} satisfies the property of the limited expansion, i.e., for any $h_1^0 \in H^0$, $h_2^0 \in H^0$, the following estimate holds:*

$$\left\| P_{gl}\left(h_1^0\right) - P_{gl}\left(h_2^0\right)\right\|_{D^1} \leq \; Q \left\| h_1^0 - h_2^0 \right\|_{D^0}, \tag{6.7}$$

where Q is some constant.

To prove the lemma we must show that T_{gl}^{-1} induces an operator $P_{gl} : H^0 \to H^1$. Let a point with the coordinates $(x^0, y^0, h^0(x^0, y^0, \mu))$ be mapped by T_{gl}^{-1} to a point $(x^1, h^1(x^1, \mu))$. By virtue of formulas (5.1), we have the equalities

$$\begin{aligned} x^1 &= \Phi(x^0, y^0, h^0(x^0, y^0, \mu), \mu), \\ h^1(x^1, \mu) &= \Psi(x^0, y^0, h^0(x^0, y^0, \mu), \mu). \end{aligned} \tag{6.8}$$

Let us fix $h^0 \in H^0$. We assume that the constant ℓ in (6.2) is small enough. In this case the function $h^0 \in H^0$ satisfies the Lipschitz condition with the small constant ℓ and, since $\det(a_{11}) \neq 0$ (see (4.5)), the first equation in (6.8) may be resolved with respect to x^0 and y^0, i.e.,

$$x^0 = x^0(x^1, \mu), \qquad y^0 = y^0(x^1, \mu). \tag{6.9}$$

Substituting $x^0(x^1, \mu)$ and $y^0(x^1, \mu)$ into the right-hand side of the second equation in (6.8), we obtain the function $h^1(x^1, \mu)$. Therefore, the operator P_{gl} is correctly defined. It is easy to check that there exists a constant L such that the function $h^1(x^1, \mu)$ satisfies the Lipschitz condition (6.5) and that the function $h^1(x^1, \mu)$ is bounded. This implies that the operator P_{gl} maps any function from H^0 into H^1.

Let us examine the property of the limited expansion. Let the functions $h_1^0 \in H^0$ and $h_2^0 \in H^0$ be mapped by P_{gl} to $h_1^1 \in H^1$ and $h_2^1 \in H^1$, respectively. Let us estimate the norm $\|h_1^1 - h_2^1\|_{D^1}$. Assume that a point $(x^0, y^0, h_1^0(x^0, y^0, \mu))$ is mapped to a point $(x_1^1, h_1^1(x_1^1, \mu))$ and a point $(x^0, y^0, h_2^0(x^0, y^0, \mu))$ is mapped to a point $(x_2^1, h_2^1(x_2^1, \mu))$ (see Figure 10). By (6.8), (6.5), and (5.1) we obtain

$$\begin{aligned} &\|h_1^1(x_1^1, \mu) - h_2^1(x_1^1, \mu)\| \\ &\quad \leq \|h_1^1(x_1^1, \mu) - h_2^1(x_2^1, \mu)\| + \|h_2^1(x_2^1, \mu) - h_2^1(x_1^1, \mu)\| \\ &\quad \leq \|h_1^1(x_1^1, \mu) - h_2^1(x_2^1, \mu)\| + L\ \|x_2^1 - x_1^1\| \\ &\|\Psi(x^0, y^0, h_1^0(x^0, y^0, \mu), \mu) - \Psi(x^0, y^0, h_2^0(x^0, y^0, \mu), \mu)\| \\ &\qquad + \ L\ \|\Phi(x^0, y^0, h_1^0(x^0, y^0, \mu), \mu) - \Phi(x^0, y^0, h_2^0(x^0, y^0, \mu), \mu)\| \\ &\quad \leq \left(\left\| \frac{\partial \Psi}{\partial z^0} \right\|_{\Omega_{gl}} + L\ \left\| \frac{\partial \Phi}{\partial z^0} \right\|_{\Omega_{gl}} \right) \left\| h_1^0 - h_2^0 \right\|_{D^0} \leq Q\ \left\| h_1^0 - h_2^0 \right\|_{D^0}, \end{aligned} \tag{6.10}$$

where Q is some constant and $\|\cdot\|_{\Omega_{gl}}$ denotes $\sup_{(x^0, y^0, z^0, \mu) \in \Omega_{gl}} \|\cdot\|$. The property of the limited expansion immediately follows from the estimate (6.10). Lemma 6.1 is proved.

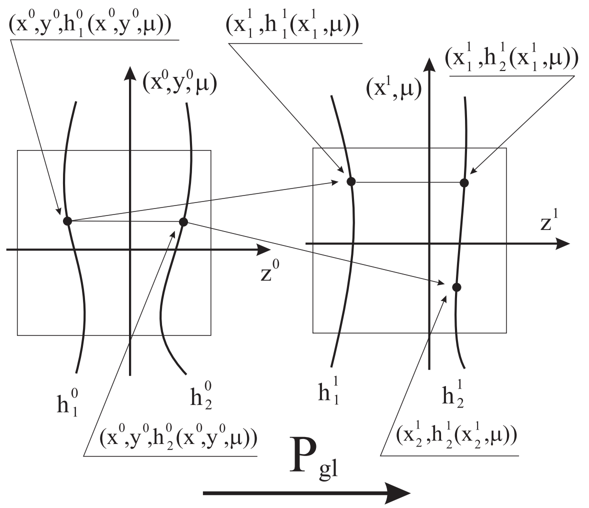

Fig. 10. The map T_{gl}^{-1} induces a map $P_{gl} : H^0 \to H^1$, satisfying the property of the limited expansion.

Lemma 6.2. *The map T_{loc}^{-1} induces an operator $P_{loc} : H^1 \to H^0$, i.e., T_{loc}^{-1} transforms points of the graph of any function $h^1 \in H^1$ to points of the graph of some function $h^0 \in H^0$. Moreover the operator P_{loc} satisfies the property of the strong contraction, i.e., for any $h_1^1 \in H^1$, $h_2^1 \in H^1$,*

$$\left\| P_{loc}\left(h_1^1\right) - P_{loc}\left(h_2^1\right)\right\|_{D^0} \leq q \left\|h_1^1 - h_2^1\right\|_{D^1}, \tag{6.11}$$

where the constant q can be made arbitrarily small.

To prove the lemma, first we check that T_{loc}^{-1} induces an operator $P_{loc} : H^1 \to H^0$. Assume that the points of the graph $z^0 = h^0(x^0, y^0, \mu)$ are mapped by T_{loc} to the points of the graph $z^1 = h^1(x^1, \mu) \in H^1$. By (5.4) we have the equalities

$$\begin{aligned} x^1 &= F(x^0, y^0, h^1(x^1, \mu), \mu), \\ h^0(x^0, y^0, \mu) &= G(x^0, y^0, h^1(x^1, \mu), \mu), \end{aligned} \tag{6.12}$$

which give us the operator $P_{loc} : H^1 \to H^0$. First, let us check that the formulas (6.12) determine the operator P_{loc} correctly, i.e., it is possible to calculate $z^0 = h^0(x^0, y^0, \mu)$ if we know $h^1 \in H^1$. Then we check that h^0 belongs to H^1. Let us fix $h^1 \in H^1$ and $(x^0, y^0, \mu) \in D^0$. Since h^1 is a Lipschitz function with respect to x^1 (see (6.5)), the map $x^1 \mapsto \overline{x}^1$ defined by the formula

$$\overline{x}^1 = F(x^0, y^0, h^1(x^1, \mu), \mu) \tag{6.13}$$

is a contraction map from R^n to R^n. Indeed, it follows from formulas (6.13), (6.5), and (5.6), that

$$\left\|\overline{x}_1^1 - \overline{x}_1^1\right\| \equiv \left\|F(x^0, y^0, h^1(x_1^1, \mu), \mu) - F(x^0, y^0, h^1(x_2^1, \mu), \mu)\right\|$$

$$\leq L \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} \left\|x_1^1 - x_1^1\right\|, \tag{6.14}$$

where

$$L \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} \leq p < 1. \tag{6.15}$$

This implies that the first equation of the system (6.12) may be resolved with respect to x^1, i.e.,

$$x^1 = x^1(x^0, y^0, \mu). \tag{6.16}$$

Substituting (6.16) into the second equation of the system (6.12), we obtain $h^0(x^0, y^0, \mu)$. Hence, the operator P_{loc} is correctly defined. By (5.4), (5.2) the function G is bounded and, therefore, h^0 is also bounded. In order to show that the function h^0 belongs to H^0, we must prove that h^0 satisfies the Lipschitz condition (6.2) in the domain D^0 (see (6.1)). First, we check this property for $\tau \in [\tau^0; \theta\tau^0]$, i.e., in the domain

$$D^0_{\tau^0} = \left\{(x^0, y^0, \mu) \mid (x^0, y^0) \in R^n,\ \|\mu\| < \delta,\ \tau \in [\tau^0; \theta\tau^0]\right\}, \tag{6.17}$$

where the constant θ satisfies the following equalities:

$$1 < \theta < \beta/\alpha. \tag{6.18}$$

We denote the corresponding range of the variables (x^0, y^0, z^1, μ) by

$$\Omega_{loc}^{\tau^0} = \left\{(x^0, y^0, z^1, \mu) \mid \left\|\left(z^1 - z^-, \mu\right)\right\| \leq \delta,\ (x^0, y^0) \in R^n,\ \tau \in [\tau^0; \theta\tau^0]\right\}. \tag{6.19}$$

Let $(x_1^0, y_1^0, \mu_1) \in D^0_{\tau^0}$ and $(x_2^0, y_2^0, \mu_2) \in D^0_{\tau^0}$. By (6.12), (6.5), and (6.16), we obtain

$$\begin{aligned}
&\left\|\left(x^1(x_1^0, y_1^0, \mu_1), \mu_1\right) - \left(x^1(x_2^0, y_2^0, \mu_2), \mu_2\right)\right\| \\
&\equiv \left\|\left(F\left(x_1^0, y_1^0, h^1(x^1(x_1^0, y_1^0, \mu_1), \mu_1), \mu_1\right), \mu_1\right)\right. \\
&\quad \left. - \left(F\left(x_2^0, y_2^0, h^1(x^1(x_2^0, y_2^0, \mu_2), \mu_2), \mu_2\right), \mu_1\right)\right\| \\
&\leq \left\| \frac{\partial(F, \mu)}{\partial(x^0, y^0, \mu)} \right\|_{\Omega_{loc}^{\tau^0}} \left\|\left(x_1^0, y_1^0, \mu_1\right) - \left(x_2^0, y_2^0, \mu_2\right)\right\| \\
&\quad + L \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}^{\tau^0}} \left\|\left(x^1(x_1^0, y_1^0, \mu_1), \mu_1\right) - \left(x^1(x_2^0, y_2^0, \mu_2), \mu_2\right)\right\|.
\end{aligned} \tag{6.20}$$

Here we use the following notation: $\|\cdot\|_{\Omega_{loc}^{\tau^0}} \equiv \sup_{(x^0, y^0, z^1, \mu) \in \Omega_{loc}^{\tau^0}} \|\cdot\|$. It follows from (6.15) and (6.20) that the function $(x^1(x^0, y^0, \mu), \mu)$ satisfies the Lipschitz property

$$\begin{aligned}
&\left\|\left(x^1(x_1^0, y_1^0, \mu_1), \mu_1\right) - \left(x^1(x_2^0, y_2^0, \mu_2), \mu_2\right)\right\| \\
&\leq \left\| \frac{\partial(F, \mu)}{\partial(x^0, y^0, \mu)} \right\|_{\Omega_{loc}^{\tau^0}} \left(1 - L \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}^{\tau^0}}\right)^{-1} \left\|\left(x_1^0, y_1^0, \mu_1\right) - \left(x_2^0, y_2^0, \mu_2\right)\right\|.
\end{aligned} \tag{6.21}$$

By virtue of the relations (5.6) (6.5), (6.12), (6.15), (6.16), and (6.21), the function $h^0(x^0, y^0, \mu)$ satisfies the estimate

$$
\begin{aligned}
&\left\| h^0(x_1^0, y_1^0, \mu_1) - h^0(x_2^0, y_2^0, \mu_2) \right\| \\
&\qquad \equiv \left\| G\left(x_1^0, y_1^0, h^1(x^1(x_1^0, y_1^0, \mu_1), \mu_1), \mu_1\right) \right. \\
&\qquad\quad \left. - \, G\left(x_2^0, y_2^0, h^1(x^1(x_2^0, y_2^0, \mu_2), \mu_2), \mu_2\right) \right\| \\
&\qquad \le \left\| \frac{\partial G}{\partial(x^0, y^0, \mu)} \right\|_{\Omega_{loc}^{\tau^0}} \left\| \left(x_1^0, y_1^0, \mu_1\right) - \left(x_2^0, y_2^0, \mu_2\right) \right\| \\
&\qquad\quad + \, L \, \left\| \frac{\partial G}{\partial z^1} \right\|_{\Omega_{loc}^{\tau^0}} \left\| \frac{\partial(F, \mu)}{\partial(x^0, y^0, \mu)} \right\|_{\Omega_{loc}^{\tau^0}} \left(1 - L \, \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}^{\tau^0}} \right)^{-1} \\
&\qquad\quad \times \, \left\| \left(x_1^0, y_1^0, \mu_1\right) - \left(x_2^0, y_2^0, \mu_2\right) \right\| \\
&\qquad \le M_0 \, e^{\varrho\tau^0} \left\| \left(x_1^0, y_1^0, \mu_1\right) - \left(x_2^0, y_2^0, \mu_2\right) \right\|,
\end{aligned}
\tag{6.22}
$$

where $M_0 > 0$ and $\varrho < 0$ are some constants. Note that $\tau^0 \ge \tau^*(\delta)$ and $\tau^*(\delta) \to \infty$ if $\delta \to 0$. Therefore for any small ℓ, there exists δ such that the constant $M_0 \, e^{\varrho\tau^0}$ will be less than ℓ, i.e., the function $h^0(x^0, y^0, \mu)$ satisfies the Lipschitz property (6.2) in the domain $D^0_{\tau^0}$. Note that $h^0(x^0, y^0, \mu)$ is a continuous function, D^0 is a convex domain, and

$$
\overline{\bigcup_{\tau^0 \ge \tau^*(\delta)} D^0_{\tau^0}} = D^0.
$$

Therefore the function $h^0(x^0, y^0, \mu)$ satisfies the Lipschitz property (6.2) in the domain D^0, i.e., h^0 belongs to H^0.

Now let us prove the strong contractibility of the map $P_{loc} : H^1 \to H^0$. Assume the functions $h_1^1 \in H^1$ and $h_2^1 \in H^1$ are mapped by P_{loc} to $h_1^0 \in H^0$ and $h_2^0 \in H^0$, respectively. Let us estimate the norm $\|h_1^0 - h_2^0\|_{D^0}$. Let a point $(x_1^1, h_1^1(x_1^1, \mu))$ be mapped to a point $(x^0, y^0, h_1^0(x^0, y^0, \mu))$ and a point $(x_2^1, h_2^1(x_2^1, \mu))$ be mapped to a point $(x^0, y^0, h_2^0(x^0, y^0, \mu))$ (see Figure 11). Then, by (6.12), we have

$$
\begin{aligned}
\left\| x_1^1 - x_2^1 \right\| &\equiv \left\| F\left(x^0, y^0, h_1^1(x_1^1, \mu), \mu\right) - F\left(x^0, y^0, h_2^1(x_2^1, \mu), \mu\right) \right\| \\
&\le \left\| F\left(x^0, y^0, h_1^1(x_1^1, \mu), \mu\right) - F\left(x^0, y^0, h_2^1(x_1^1, \mu), \mu\right) \right\| \\
&\quad + \left\| F\left(x^0, y^0, h_2^1(x_1^1, \mu), \mu\right) - F\left(x^0, y^0, h_2^1(x_2^1, \mu), \mu\right) \right\| \\
&\le \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} \left\| h_1^1 - h_2^1 \right\|_{D^1} + \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} L \, \left\| x_1^1 - x_2^1 \right\|.
\end{aligned}
\tag{6.23}
$$

Therefore, by (6.23) and (6.15), we obtain

$$
\left\| x_1^1 - x_2^1 \right\| \le \left(1 - \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} L \right)^{-1} \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} \left\| h_1^1 - h_2^1 \right\|_{D^1}. \tag{6.24}
$$

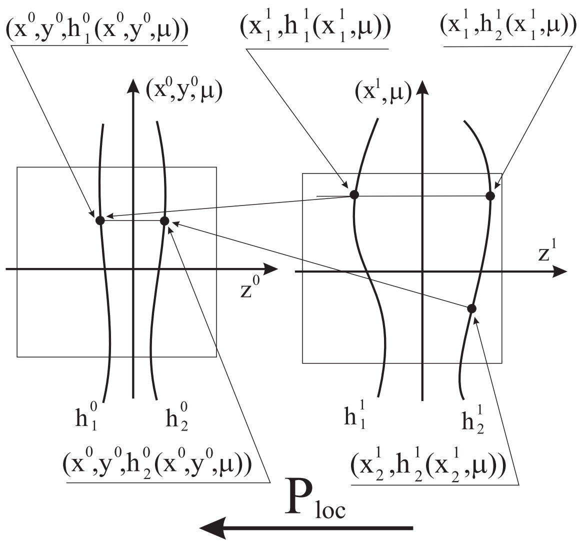

Fig. 11. The map T_{loc}^{-1} induces a map $P_{loc} : H^1 \to H^0$, satisfying the property of the strong contraction.

By virtue of the relations (5.6), (6.5), (6.12), (6.24), and (6.15), the following estimate takes place:

$$
\begin{aligned}
&\left\| h_1^0(x^0, y^0, \mu) - h_2^0(x^0, y^0, \mu) \right\| \\
&\qquad \equiv \left\| G\left(x^0, y^0, h_1^1(x_1^1, \mu), \mu\right) - G\left(x^0, y^0, h_2^1(x_2^1, \mu), \mu\right) \right\| \\
&\qquad \le \left\| G\left(x^0, y^0, h_1^1(x_1^1, \mu), \mu\right) - G\left(x^0, y^0, h_2^1(x_1^1, \mu), \mu\right) \right\| \\
&\qquad\quad + \left\| G\left(x^0, y^0, h_2^1(x_1^1, \mu), \mu\right) - G\left(x^0, y^0, h_2^1(x_2^1, \mu), \mu\right) \right\| \\
&\qquad \le \left\| \frac{\partial G}{\partial z^1} \right\|_{\Omega_{loc}} \left\| h_1^1 - h_2^1 \right\|_{D^1} + \left\| \frac{\partial G}{\partial z^1} \right\|_{\Omega_{loc}} L \left(1 - \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} L \right)^{-1} \\
&\qquad \times \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} \left\| h_1^1 - h_2^1 \right\|_{D^1} \le M_1 \, e^{\varrho \tau^*} \left\| h_1^1 - h_2^1 \right\|_{D^1} ,
\end{aligned}
\tag{6.25}
$$

where $M_1 > 0$ and $\varrho < 0$ are some constants. Note that $\tau^*(\delta) \to \infty$ if $\delta \to 0$. Therefore for any small $q > 0$, there exists $\delta > 0$ such that the constant $M_1 \, e^{\varrho \tau^*}$ is less than q, i.e., P_{loc} is the strong contraction operator. Lemma 6.2 is proved.

By virtue of Lemmas 6.1 and 6.2, we have the following statement.

Lemma 6.3. *The superposition $P = P_{loc} \circ P_{gl} : H^0 \to H^0$ is the contraction map, i.e., for any $h_1^0 \in H^0$, $h_2^0 \in H^0$, the following estimate holds:*

$$
\left\| P\left(h_1^0\right) - P\left(h_2^0\right) \right\|_{D^0} \le \; p \left\| h_1^0 - h_2^0 \right\|_{D^0} ,
\tag{6.26}
$$

where the constant $p = q \; Q$ is strictly less 1.

By virtue of Lemma 6.3 and Banach's principle, there exists a unique fixed point $h^0_* = P(h^0_*) \in H^0$ that is a limit of the functions $h^0_0, h^0_1, h^0_2, \ldots$, obtained by the iterations $h^0_{i+1} = P(h^0_i)$, $(i = 0, 1, 2, \ldots)$ with any initial function $h^0_0 \in H^0$. The graph of the function h^0_* is an invariant set with respect to the Poincaré map T and, therefore, orbits of the system X_μ passing through the points of this graph form the invariant manifold $\mathcal{M}^{cs}$. Since the contraction map P is induced by the inverse Poincaré map $T^{-1} = T^{-1}_{loc} \circ T^{-1}_{gl}$, the manifold $\mathcal{M}^{cs}$ is a repelling manifold, i.e., any orbit not lying in $\mathcal{M}^{cs}$ leaves the neighborhood U of the homoclinic orbit as t tends to $+\infty$.

7. Smoothness of the Invariant Functions h^0_* and h^1_*

In order to establish that $\mathcal{M}^{cs}$ is a smooth manifold, we must prove that $h^0_*(x^0, y^0, \mu) \in H^0$ is a smooth function with respect to (x^0, y^0, μ).

In our case, whereas the map T^{-1}_{gl} is a diffeomorphism, the following statement takes place.

Lemma 7.1. *The operator P_{gl} maps a smooth function $h^0 \in H^0$ to a smooth function $h^1 \in H^1$.*

To calculate the derivatives of the function $h^1(x^1, \mu)$, we use the formulas (6.8), (6.9). By these formulas we have the following identity:

$$\left(x^1, \mu\right) \equiv \left(\Phi\left(x^0(x^1, \mu), y^0(x^1, \mu), h^0(x^0(x^1, \mu), y^0(x^1, \mu), \mu), \mu\right), \mu\right), \tag{7.1}$$

which directly implies the relation

$$\frac{\partial(x^0(x^1, \mu), y^0(x^1, \mu), \mu)}{\partial(x^1, \mu)} \equiv \left(\frac{\partial(\Phi, \mu)}{\partial(x^0, y^0, \mu)} + \frac{\partial(\Phi, \mu)}{\partial z^0}\frac{\partial h^0}{\partial(x^0, y^0, \mu)}\right)^{-1}. \tag{7.2}$$

Therefore, by the relations (6.8), (6.9), and (7.2),

$$\begin{aligned}\frac{\partial h^1(x^1, \mu)}{\partial(x^1, \mu)} \equiv & \left[\left(\frac{\partial \Psi}{\partial(x^0, y^0, \mu)} + \frac{\partial \Psi}{\partial z^0}\frac{\partial h^0}{\partial(x^0, y^0, \mu)}\right)\right. \\ & \times \left.\left(\frac{\partial(\Phi, \mu)}{\partial(x^0, y^0, \mu)} + \frac{\partial(\Phi, \mu)}{\partial z^0}\frac{\partial h^0}{\partial(x^0, y^0, \mu)}\right)^{-1}\right]_{\substack{x^0=x^0(x^1,\mu)\\ y^0=y^0(x^1,\mu)\\ z^0=h^0(x^0(x^1,\mu),y^0(x^1,\mu),\mu)}}.\end{aligned} \tag{7.3}$$

Notice, by virtue of the relations (5.1), (5.2), (6.2), and (4.5), the operator

$$\left(\frac{\partial(\Phi, \mu)}{\partial(x^0, y^0, \mu)} + \frac{\partial(\Phi, \mu)}{\partial z^0}\frac{\partial h^0}{\partial(x^0, y^0, \mu)}\right)$$

is an invertible operator. Thus, the derivative may indeed be calculated via formula (7.3).

Consider the space N^0 of continuous functions $\eta^0(x^0, y^0, \mu)$ defined in the domain D^0 (see (6.1)). We assume that the functions $\eta^0 \in N^0$ are uniformly bounded by the constant ℓ (see (6.2)), i.e.,

$$\left\|\eta^0(x^0, y^0, \mu)\right\| \le \ell. \tag{7.4}$$

The space N^0 with the uniform norm

$$\left\|\eta^0\right\|_{D^0} \equiv \sup_{(x^0, y^0, \mu)\in D^0} \left\|\eta^0(x^0, y^0, \mu)\right\| \tag{7.5}$$

is a complete metric space.

Let us introduce the space N^1 of continuous functions $\eta^1(x^1, \mu)$ defined in the domain D^1 (see (6.4)). Let the functions $\eta^1 \in N^1$ be uniformly bounded by the constant L (see (6.5)), i.e.,

$$\left\|\eta^1(x^1, \mu)\right\| \le L. \tag{7.6}$$

The space N^1 with the norm

$$\left\|\eta^1\right\|_{D^1} \equiv \sup_{(x^1, \mu)\in D^1} \left\|\eta^1(x^1, \mu)\right\| \tag{7.7}$$

is also a complete metric space.

The relation (7.3) induces a family of operators $S_{gl}^{h^0}$. For any $h^0 \in H^0$, we define the operator $S_{gl}^{h^0}\left(\eta^0\right) \mapsto \eta^1$ by the following rule:

$$\begin{aligned} \eta^1(x^1, \mu) \equiv & \left[\left(\frac{\partial \Psi}{\partial(x^0, y^0, \mu)} + \frac{\partial \Psi}{\partial z^0}\eta^0(x^0, y^0, \mu)\right)\right. \\ & \left.\times \left(\frac{\partial(\Phi, \mu)}{\partial(x^0, y^0, \mu)} + \frac{\partial(\Phi, \mu)}{\partial z^0}\eta^0(x^0, y^0, \mu)\right)^{-1}\right]_{\substack{x^0 = x^0(x^1, \mu) \\ y^0 = y^0(x^1, \mu) \\ z^0 = h^0(x^0(x^1, \mu), y^0(x^1, \mu), \mu)}} . \end{aligned} \tag{7.8}$$

Here the functions $x^0(x^1, \mu)$ and $y^0(x^1, \mu)$ (see (6.9)) are obtained from the first equation (6.8). These functions depend continuously on the function $h^0 \in H^0$. Indeed, let the sequence of functions $(x_i^0(x^1, \mu), y_i^0(x^1, \mu))$, $(i = 1, 2, 3, \ldots)$ correspond to the sequence of functions $h_i^0 \in H^0$. Let also $h_i^0 \to h_*^0 \in H^0$, i.e., $\|h_i^0 - h_*^0\|_{D^0} \to 0$. In this case the sequence $(x_i^0(x^1, \mu), y_i^0(x^1, \mu))$ converges to the function $(x_*^0(x^1, \mu), y_*^0(x^1, \mu))$, which corresponds to the function $h_*^0 \in H^0$, i.e., $\|(x_i^0(x^1, \mu), y_i^0(x^1, \mu)) - (x_*^0(x^1, \mu), y_*^0(x^1, \mu))\|_{D^1} \to 0$.

This fact, along with the formulas (7.8), (5.1), and (4.5), implies the following statement.

Lemma 7.2. *For any $h^0 \in H^0$, the operator $S_{gl}^{h^0}$ maps any function η^0 from N^0 to $\eta^1 \in N^1$. This operator depends continuously on $h^0 \in H^0$. Namely, if the sequence of functions $h_i^0 \in H^0$ converges to $h_*^0 \in H^0$ as $i \to \infty$, i.e., $\|h_i^0 - h_*^0\|_{D^0} \to 0$ then, for any*

$\eta^0 \in N^0$, the sequence $S_{gl}^{h_i^0}(\eta^0)$ converges to $S_{gl}^{h_*^0}(\eta^0)$, i.e., $\|S_{gl}^{h_i^0}(\eta^0) - S_{gl}^{h_*^0}(\eta^0)\|_{D^1} \to 0$. Moreover, the operator $S_{gl}^{h^0}$ satisfies the property of the limited expansion, i.e., for any $\eta_1^0 \in N^0$, $\eta_2^0 \in N^0$, and $h^0 \in H^0$, the following estimate holds:

$$\left\| S_{gl}^{h^0}\left(\eta_1^0\right) - S_{gl}^{h^0}\left(\eta_2^0\right)\right\|_{D^1} \leq \; Q \left\|\eta_1^0 - \eta_2^0\right\|_{D^0}, \tag{7.9}$$

where Q is some constant.

Let us now explore the properties of the operator P_{loc}.

Lemma 7.3. *The operator P_{loc} maps a smooth function $h^1 \in H^1$ to a smooth function $h^0 \in H^0$.*

To prove Lemma 7.3, let us assume first that (x^0, y^0) does not belong to the stable manifold, i.e., $\tau \neq \infty$. In this case, if $h^1 \in H^1$ is a smooth function, by (6.12), (6.16), the following relation takes place:

$$\frac{\partial\left(x^1(x^0,y^0,\mu),\mu\right)}{\partial\left(x^0,y^0,\mu\right)} = \left(E - \frac{\partial(F,\mu)}{\partial z^1}\,\frac{\partial h^1}{\partial(x^1,\mu)}\right)^{-1} \frac{\partial(F,\mu)}{\partial(x^0,y^0,\mu)}, \tag{7.10}$$

where E is the identity matrix. By (5.6) and (6.5), we have

$$\left\|\frac{\partial(F,\mu)}{\partial z^1}\right\|_{\Omega_{loc}} \left\|\frac{\partial h^1}{\partial(x^1,\mu)}\right\|_{D^1} < 1, \tag{7.11}$$

and, therefore,

$$\left(E - \frac{\partial(F,\mu)}{\partial z^1}\,\frac{\partial h^1}{\partial(x^1,\mu)}\right) \tag{7.12}$$

is an invertible operator. Hence, formula (7.10) makes sense. By the relations (6.12), (6.16), and (7.10),

$$\begin{aligned}\frac{\partial h^0}{\partial\left(x^0,y^0,\mu\right)} &= \frac{\partial G}{\partial(x^0,y^0,\mu)}\\ &+ \frac{\partial G}{\partial z^1}\frac{\partial h^1}{\partial(x^1,\mu)}\left(E - \frac{\partial(F,\mu)}{\partial z^1}\,\frac{\partial h^1}{\partial(x^1,\mu)}\right)^{-1} \frac{\partial(F,\mu)}{\partial(x^0,y^0,\mu)}.\end{aligned} \tag{7.13}$$

Formula (7.13) allows us to extend the domain of definition up to the points that belong to the stable manifold, i.e., for $\tau = \infty$. In order to show this, we consider the following function:

$$\mathcal{F}(x^0,y^0,z^1,\eta^1,\mu) \equiv \begin{cases} \dfrac{\partial G}{\partial(x^0,y^0,\mu)} + \dfrac{\partial G}{\partial z^1}\,\eta^1 & \\ \times\left(E - \dfrac{\partial(F,\mu)}{\partial z^1}\,\eta^1\right)^{-1} \dfrac{\partial(F,\mu)}{\partial(x^0,y^0,\mu)}, & \text{if } \tau \neq \infty; \\ 0, & \text{if } \tau = \infty. \end{cases} \tag{7.14}$$

The continuity of the function $\mathcal{F}$, at the points provided that $\tau \neq \infty$, follows from the fact that $\mathcal{F}$ is an algebraic combination of the continuous functions. The continuity, at the points provided that $\tau = \infty$, follows from the fact that $\|\mathcal{F}\| \to 0$ if $\tau \to \infty$ (see (5.6), (7.6), and (7.14)). So, the superposition

$$\mathcal{F}\left(x^0, y^0, h^1(x^1,\mu), \frac{\partial h^1}{\partial(x^1,\mu)}, \mu\right), \tag{7.15}$$

where $x^1 = x^1(x^0, y^0, \mu)$ (see (6.12), (6.16)), depends continuously on (x^0, y^0, μ) because this function is a superposition of continuous functions. Therefore, the function (7.15) is an extension of the function $\frac{\partial h^0}{\partial(x^0,y^0,\mu)}$ up to the points that belong to the stable manifold, i.e., provided that $\tau = \infty$. This fact, along with the continuity of the function $h^0(x^0, y^0, \mu)$, implies the smoothness of $h^0(x^0, y^0, \mu)$. Thus, we have the equality

$$\frac{\partial h^0}{\partial(x^0,y^0,\mu)} = \mathcal{F}\left(x^0, y^0, h^1(x^1,\mu), \frac{\partial h^1}{\partial(x^1,\mu)}, \mu\right), \qquad \text{where } x^1 = x^1(x^0, y^0, \mu), \tag{7.16}$$

for all points $(x^0, y^0, \mu) \in D^0$. Lemma 7.3 is proved.

Observe that $\frac{\partial h^0}{\partial(x^0,y^0,\mu)}$ is a uniformly continuous function because it is continuous and equals zero if $\|(x^- x^+, y^0 - y^+)\| \geq \rho$ (see (5.2), (5.4), and (6.12)). By Lemmas 7.1 and 7.3, the following statement holds.

Lemma 7.4. *The operator $P = P_{loc} \circ P_{gl}$ maps any smooth function $h^0 \in H^0$ to a smooth function $\bar{h}^0 \in H^0$.*

The relation (7.16) induces a family of operators $S^{h^1}_{loc}$. For any $h^1 \in H^1$, we define the operator $S^{h^1}_{loc}(\eta^1) \mapsto \eta^0$ by the formulas

$$\eta^0(x^0, y^0, \mu) = \mathcal{F}\left(x^0, y^0, h^1(x^1,\mu), \eta^1(x^1,\mu), \mu\right), \qquad \text{where } x^1 = x^1(x^0, y^0, \mu). \tag{7.17}$$

Here we come to the following lemma.

Lemma 7.5. *For any $h^1 \in H^1$, the operator $S^{h^1}_{loc}$ maps any function $\eta^1 \in N^1$ to $\eta^0 \in N^0$. This operator depends continuously on $h^1 \in H^1$. Namely, if the sequence of functions $h^1_i \in H^1$ $(i \to \infty)$ converges to $h^1_* \in H^1$, i.e., $\|h^1_i - h^1_*\|_{D^1} \to 0$, then for any $\eta^1 \in N^1$, the sequence $S^{h^1_i}_{loc}(\eta^1)$ converges to $S^{h^1_*}_{loc}(\eta^1)$, i.e., $\|S^{h^1_i}_{loc}(\eta^1) - S^{h^1_*}_{loc}(\eta^1)\|_{D^0} \to 0$. Moreover, the operator $S^{h^1}_{loc}$ satisfies the property of the strong contraction, i.e., for any $\eta^1_1 \in N^1$, $\eta^1_2 \in N^1$, and $h^1 \in H^1$, the following estimate is valid:*

$$\left\| S^{h^1}_{loc}\left(\eta^1_1\right) - S^{h^1}_{loc}\left(\eta^1_2\right) \right\|_{D^0} \leq q \left\|\eta^1_1 - \eta^1_2\right\|_{D^1}, \tag{7.18}$$

where the constant q can be made arbitrarily small.

The function $\eta^0(x^0, y^0, \mu)$ (see (7.17)) is a continuous function since it is a superposition of continuous functions. It follows from the relations (7.14), (5.2), (5.4), (5.6) that the norm $\|\mathcal{F}\|_{D^0}$ is bounded (it tends to zero as $\tau^*(\delta) \to \infty$). Hence, the function $\eta^0(x^0, y^0, \mu)$ is bounded by the constant ℓ (see (7.4)) and, therefore, $\eta^0(x^0, y^0, \mu) \in N^0$. The continuity of the operator $S_{loc}^{h^1}$ with respect to h^1 follows from the following considerations. Assume that the sequence of functions $h_i^1 \in H^1$ converges to $h_*^1 \in H^1$ if $i \to \infty$, i.e., $\|h_i^1 - h_*^1\|_{D^1} \to 0$. In this case there exists the sequence $x_i^1 = x_i^1(x^0, y^0, \mu)$, which corresponds to the sequence $h_i^1 \in H^1$ (see (6.16)). By virtue of the relation (6.24), the sequence $x_i^1 = x_i^1(x^0, y^0, \mu)$ uniformly converges to the function $x_*^1 = x_*^1(x^0, y^0, \mu)$, which corresponds to h_*^1. Indeed,

$$\begin{aligned} &\left\| x_i^1(x^0, y^0, \mu) - x_*^1(x^0, y^0, \mu) \right\|_{D^0} \\ &\quad \le \left(1 - \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} L \right)^{-1} \left\| \frac{\partial F}{\partial z^1} \right\|_{\Omega_{loc}} \|h_i^1 - h_*^1\|_{D^1} \to 0 \,. \end{aligned} \tag{7.19}$$

The functions h_*^1 and η^1 are uniformly continuous. Therefore, the sequence $h_i^1(x_i^1(x^0, y^0, \mu))$ converges to $h_*^1(x_*^1(x^0, y^0, \mu))$ and $\eta^1(x_i^1(x^0, y^0, \mu))$ converges to $\eta^1(x_*^1(x^0, y^0, \mu))$, i.e.,

$$\left\| h_i^1(x_i^1(x^0, y^0, \mu)) - h_*^1(x_*^1(x^0, y^0, \mu)) \right\|_{D^0} \to 0,$$

$$\left\| \eta^1(x_i^1(x^0, y^0, \mu)) - \eta^1(x_*^1(x^0, y^0, \mu)) \right\|_{D^0} \to 0.$$

Since $\mathcal{F}$ is a uniformly continuous function, we obtain that

$$\begin{aligned} &\left\| \eta_i^0(x^0, y^0, \mu) - \eta_*^0(x^0, y^0, \mu) \right\|_{D^0} \\ &\quad = \Big\| \mathcal{F}\left(x^0, y^0, h_i^1(x_i^1(x^0, y^0, \mu), \mu), \eta^1(x_i^1(x^0, y^0, \mu), \mu), \mu\right) \\ &\qquad - \mathcal{F}\left(x^0, y^0, h_*^1(x_*^1(x^0, y^0, \mu), \mu), \eta^1(x_*^1(x^0, y^0, \mu), \mu), \mu\right) \Big\|_{D^0} \to 0. \end{aligned} \tag{7.20}$$

The property of the strong contraction for the operator $S_{loc}^{h^1}$ follows from the relations (7.17), (7.14), (7.6), and (5.6). Indeed,

$$\begin{aligned} &\left\| \eta_1^0(x^0, y^0, \mu) - \eta_2^0(x^0, y^0, \mu) \right\|_{D^0} \\ &\quad = \Bigg\| \frac{\partial G}{\partial z^1} \Bigg(\eta_1^1(x^1, \mu) \left(E - \frac{\partial (F, \mu)}{\partial z^1}\, \eta_1^1(x^1, \mu) \right)^{-1} \\ &\qquad - \eta_2^1(x^1, \mu) \left(E - \frac{\partial (F, \mu)}{\partial z^1}\, \eta_2^1(x^1, \mu) \right)^{-1} \Bigg) \frac{\partial (F, \mu)}{\partial (x^0, y^0, \mu)} \Bigg\|_{D^0} \\ &\quad \le q \left\| \eta_1^1(x^1, \mu) - \eta_2^1(x^1, \mu) \right\|_{D^1} , \end{aligned} \tag{7.21}$$

where the constant q may be made arbitrarily small. Lemma 7.5 is proved.

By virtue of Lemmas 7.2 and 7.5, the following statement takes place.

Lemma 7.6. *For any $h^0 \in H^0$, the superposition operator $S^{h^0} \equiv S_{loc}^{h^1} \circ S_{gl}^{h^0}$, where $h^1 = P_{gl}\left(h^0\right)$, maps any function η^0 from N^0 to $\eta^0 \in N^0$. This operator depends continuously* *on the function $h^0 \in H^0$. Moreover, the operator S^{h^0} satisfies the contraction property, i.e., for any $\eta_1^0 \in N^0$, $\eta_2^0 \in N^0$, and $h^0 \in H^0$, the following estimate holds:*

$$\left\| S^{h^0}\left(\eta_1^0\right) - S^{h^0}\left(\eta_2^0\right) \right\|_{D^0} \leq \ p \left\| \eta_1^0 - \eta_2^0 \right\|_{D^0}, \tag{7.22}$$

where the constant $p = q\ Q$ is strictly less than 1.

Below we use the following statement, which is a parametric variant of Banach's contraction principle.

Lemma 7.7. *("Fiber contraction theorem" [Hirsch & Pugh, 1970]) Let H^0 and N^0 be metric spaces. We assume also that N^0 is a complete space. Let P be an operator in H^0, i.e., $P : H^0 \to H^0$, and let any sequence $h_0^0, h_1^0, h_2^0, \ldots$ obtained by the iterations $h_{i+1}^0 = P(h_i^0)$, $(i = 0, 1, 2, \ldots)$, converge to a unique fixed point $h_*^0 = P(h_*^0) \in H$ for any initial element $h_0^0 \in H^0$. For any element $h^0 \in H^0$, let there be an operator $S^{h^0} : N^0 \to N^0$. We assume that the family of the operators S^{h^0} satisfies the following conditions.*

1. *For any $h^0 \in H^0$, the operator S^{h^0} satisfies the contraction property, i.e.,*

$$\operatorname{dist}\left(S^{h^0}(\eta_1^0), S^{h^0}(\eta_2^0)\right) \leq p \ \operatorname{dist}\left(\eta_1^0, \eta_2^0\right),$$

   where $\eta_1^0 \in N^0$, $\eta_2^0 \in N^0$, and the constant p is less than 1.
2. *The family of the operators S^{h^0} depends continuously on $h^0 \in H^0$, i.e., if a sequence $h_i^0 \in H^0$ $(i \to \infty)$ tends to $h_*^0 \in H^0$, then the sequence $S^{h_i^0}(\eta^0)$ tends to $S^{h_*^0}(\eta^0)$ for any $\eta^0 \in N^0$.*

Then, the operator $\Omega : H^0 \times N^0 \to H^0 \times N^0$ defined by the formula $\Omega(h^0, \eta^0) \equiv (P(h^0), S^{h^0}(\eta^0))$ has a unique fixed point, $(h_*^0, \eta_*^0) = \Omega(h_*^0, \eta_*^0)$. Moreover, any sequence $(h_0^0, \eta_0^0), (h_1^0, \eta_1^0), (h_2^0, \eta_2^0), \ldots$, obtained by the iterations $(h_{i+1}^0, \eta_{i+1}^0) = \Omega(h_i^0, \eta_i^0)$, $(i = 0, 1, 2, \ldots)$ with any initial element (h_0^0, η_0^0), converges to the point (h_*^0, η_*^0).

By the conditions of the lemma, the operator P has a unique fixed point, i.e., $h_*^0 = P(h_*^0) \in H^0$. Since N^0 is a complete space and $S^{h_*^0}$ is a contractive operator, there exists a unique fixed point $\eta_*^0 = S^{h_*^0}(\eta_*^0) \in N^0$. The point $(h_*^0, \eta_*^0) \in H^0 \times N^0$ is a fixed point of the operator Ω because $(h_*^0, \eta_*^0) = \Omega(h_*^0, \eta_*^0) \equiv (P(h_*^0), S^{h_*^0}(\eta_*^0))$. The uniqueness of the fixed point for the operator Ω follows from the uniqueness of the fixed point for the operator P. Let us show that any sequence $(h_0^0, \eta_0^0), (h_1^0, \eta_1^0), (h_2^0, \eta_2^0), \ldots$, obtained by the iterations $(h_{i+1}^0, \eta_{i+1}^0) = \Omega(h_i^0, \eta_i^0)$, $(i = 0, 1, 2, \ldots)$ with any initial element (h_0^0, η_0^0), converges to the point (h_*^0, η_*^0). It is clear that

$$\begin{aligned} \operatorname{dist}\left(\eta_*^0, \eta_{i+1}^0\right) &= \operatorname{dist}\left(\eta_*^0, S^{h_i^0}(\eta_i^0)\right) \\ &\leq \operatorname{dist}\left(\eta_*^0, S^{h_i^0}(\eta_*^0)\right) + \operatorname{dist}\left(S^{h_i^0}(\eta_*^0), S^{h_i^0}(\eta_i^0)\right) \\ &\leq \operatorname{dist}\left(\eta_*^0, S^{h_i^0}(\eta_*^0)\right) + p \ \operatorname{dist}\left(\eta_*^0, \eta_i^0\right). \end{aligned} \tag{7.23}$$

By the conditions of the lemma, the family of the operators $S^{h_i^0}$ depends continuously on h_i^0 and $h_i^0 \to h_*^0$ if $i \to \infty$. Therefore,

$$\operatorname{dist}\left(\eta_*^0, S^{h_i^0}(\eta_*^0)\right) \to \operatorname{dist}\left(\eta_*^0, S^{h_*^0}(\eta_*^0)\right) \equiv 0 \qquad \text{at } i \to \infty,$$

i.e., for any $\varepsilon > 0$, there exists i_0 such that, for any $i > i_0$, $\operatorname{dist}(\eta_*^0, S^{h_i^0}(\eta_*^0)) < \varepsilon$. Hence, for $i = i_0 + j$, we have the inequality

$$\begin{aligned} \operatorname{dist}\left(\eta_*^0, \eta_i^0\right) &\equiv \operatorname{dist}\left(\eta_*^0, \eta_{i_0+j+1}^0\right) \le \varepsilon + p \operatorname{dist}\left(\eta_*^0, \eta_{i_0+j}^0\right) \\ &\le \varepsilon + p\,\varepsilon + \cdots + p^{j-1}\,\varepsilon + p^j \operatorname{dist}\left(\eta_*^0, \eta_{i_0}^0\right) \\ &\le \frac{\varepsilon}{1-p} + p^j \operatorname{dist}\left(\eta_*^0, \eta_{i_0}^0\right). \end{aligned} \tag{7.24}$$

Observe that there exists j_0 such that $p^j \operatorname{dist}(\eta_*^0, \eta_{i_0}^0) \le \varepsilon$ for any $j > j_0$. Therefore, the inequality $\operatorname{dist}(\eta_*^0, \eta_i^0) \le \varepsilon(1/(1-p)+1)$ holds true for any $i > i_0 + j_0$, i.e., $\eta_i^0 \to \eta_*^0$. Lemma 7.7 is proved.

By Lemmas 6.3, 7.6, and 7.7, any sequence (h_0^0, η_0^0), (h_1^0, η_1^0), (h_2^0, η_2^0), ..., obtained by the iterations $(h_{i+1}^0, \eta_{i+1}^0) = (P(h_i^0), S^{h_i^0}(\eta_i^0))$, $(i = 0, 1, 2, \ldots)$, converges to the fixed point (h_*^0, η_*^0). Let us choose the point $(0, 0)$ as the initial point (h_0^0, η_0^0). By Lemma 7.4 and the formulas (7.16), (7.17), the relation

$$\eta_i^0 = \frac{\partial h_i^0}{\partial(x^0, y^0, \mu)} \tag{7.25}$$

holds true for any $i = 0, 1, 2 \ldots$. Since h_i^0 uniformly converges to h_*^0 and η_i^0 uniformly converges to η_*^0, the equality

$$\eta_*^0 = \frac{\partial h_*^0}{\partial(x^0, y^0, \mu)} \tag{7.26}$$

also holds true, i.e., h_*^0 is a C^1-smooth function with respect to all of its variables. Notice that, using the results of Section 3, by induction it is possible to prove that $h_*^0 \in C^{k+\varepsilon}$ if $X_\mu \in C^{k+\varepsilon}$ and $\operatorname{Re} \gamma_2/\gamma_1 > k + \varepsilon$.

To finish the proof of the main Theorem 2.1 we must show that orbits passing through the invariant curve h_*^0 (h_*^1) form a smooth manifold $\mathcal{M}^{cs}$. A proof of this fact is in the following section.

8. Smoothness of the Invariant Manifold $\mathcal{M}^{cs}$

Here we show that orbits passing through the points of the graphs of functions h_*^0 and h_*^1 form a smooth invariant manifold $\mathcal{M}^{cs}$.

Note that the flight time of the orbits is bounded outside any neighborhood of the equilibrium point O. Therefore, outside the neighbourhood, the smoothness of the manifold follows from the smoothness of the function h_*^1. The following theorem guarantees the smoothness of the manifold near the equilibrium point $O = (x = 0, y = 0, z = 0)$.

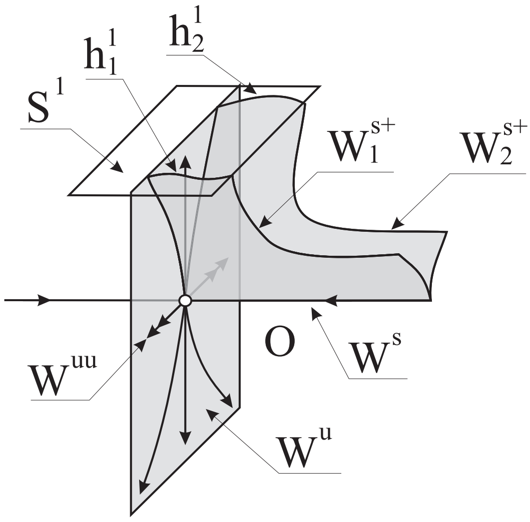

Fig. 12. The functions h^1_1 and h^1_2 uniquely determine the invariant manifolds W^{s+}_1 and W^{s+}_2, correspondingly. The manifolds have a common tangent everywhere on W^s.

Theorem 8.1. *Let $z^1 = h^1(x^1, \mu)$ ($h^1 \in H^1$) be a smooth function. Then orbits passing through the points of the graph h^1 form an invariant manifold. This manifold is locally a graph of some smooth function $z^0 = W^{s+}(x^0, y^0, \mu)$. Two different functions h^1_1 and h^1_2 uniquely determine two different manifolds W^{s+}_1 and W^{s+}_2, such that the manifolds have a common tangent space everywhere on the stable manifold (see Figure 12).*

In order to find the function $z^0 = W^{s+}(x^0, y^0, \mu)$, we shall use the boundary value problem (3.61), (3.64). Namely, let us consider an orbit $(x(t), y(t), z(t))$ that starts, at $t = 0$, with a point (x^0, y^0, z^0) on the manifold W^{s+} and reaches, at $t = \tau$, a point (x^1, y^1, z^1) on the cross section S^1. Since the coordinate x^1 on the cross section S^1 corresponds to the coordinate $x^1 - \bar{A}z^1$ of the initial system (see (4.21)) and (x^1, y^1, z^1) belongs to the graph h^1, we have the following equalities:

$$y^1 = y^- , \qquad z^1 = h^1\left(x^1 - \bar{A}z^1, \mu\right). \tag{8.1}$$

According to the boundary value problem (3.61), (3.64), we have the following relations:

$$\begin{aligned} x^1 &= x(\tau; x^0, y^0, z^1, \tau, \mu), \\ y^1 &= y(\tau; x^0, y^0, z^1, \tau, \mu), \\ z^0 &= z(0; x^0, y^0, z^1, \tau, \mu). \end{aligned} \tag{8.2}$$

Since $y^1 = y^-$ (see (8.1)), by (4.17), the flight time τ may be expressed from the second equation of the system (8.2) as a function $\tau = \tau(x^0, y^0, z^1, \mu)$. Thus we have

$$\begin{aligned} x^1 &= x^1(x^0, y^0, z^1, \mu) \equiv x(\tau(x^0, y^0, z^1, \mu); x^0, y^0, z^1, \tau(x^0, y^0, z^1, \mu), \mu), \\ z^0 &= z^0(x^0, y^0, z^1, \mu) \equiv z(0; x^0, y^0, z^1, \tau(x^0, y^0, z^1, \mu), \mu). \end{aligned} \tag{8.3}$$

The derivatives of these functions may be expressed by (4.12) and they satisfy the estimates (4.19), (4.20). Now, by (8.1) and (8.3), we have

$$z^1 = h^1\left(x^1(x^0, y^0, z^1, \mu) - \bar{\mathcal{A}}z^1, \mu\right). \tag{8.4}$$

Using formulas (4.19), (4.20), and (6.5), we obtain the following estimate:

$$\left\| \frac{\partial h^1}{\partial\left(x^1 - \bar{\mathcal{A}}z^1\right)} \left(\frac{\partial x^1}{\partial z^1} - \bar{\mathcal{A}}\right)\right\| \le q < 1. \tag{8.5}$$

By *the implicit function theorem*, the last inequality means that z^1 can be expressed from the equation (8.4) as a function

$$z^1 = z^1(x^0, y^0, \mu). \tag{8.6}$$

Note that, by (8.4), (4.19), (4.20), and (6.5), we have the following relation:

$$\begin{aligned}\frac{\partial z^1}{\partial(x^0, y^0)} &= \left(E - \frac{\partial h^1}{\partial\left(x^1 - \bar{\mathcal{A}}z^1\right)} \left(\frac{\partial x^1}{\partial z^1} - \bar{\mathcal{A}}\right)\right)^{-1} \frac{\partial h^1}{\partial\left(x^1 - \bar{\mathcal{A}}z^1\right)} \frac{\partial x^1}{\partial(x^0, y^0)} \\ &= O(e^{\alpha\tau}).\end{aligned} \tag{8.7}$$

Substituting the function (8.6) in the second equation of (8.3), we obtain that the invariant manifold W^{s+} is a graph of some function

$$z^0 = z^0(x^0, y^0, z^1(x^0, y^0, \mu), \mu) \equiv W^{s+}(x^0, y^0, \mu). \tag{8.8}$$

The derivative of this function

$$\frac{\partial W^{s+}(x^0, y^0, \mu)}{\partial(x^0, y^0)} = \frac{\partial z^0}{\partial(x^0, y^0)} + \frac{\partial z^0}{\partial z^1} \frac{\partial z^1}{\partial(x^0, y^0)} \tag{8.9}$$

may be calculated via formulas (4.12) and (8.7). By (4.19), (4.20), and (8.7), we have

$$\frac{\partial W^{s+}(x^0, y^0, \mu)}{\partial(x^0, y^0)} \to \bar{\mathcal{B}}, \qquad \text{if} \quad \tau \to \infty, \tag{8.10}$$

i.e., the derivative has the fined limit if the initial point $\left(x^0, y^0, z^0 = W^{s+}(x^0, y^0, \mu)\right)$ goes to the stable manifold W^s. Moreover, the limit does not depend on the function h^1. Thus, all of the manifolds W^{s+} have the same tangent space everywhere on W^s. Notice that the manifold has zero tangent space $\{z = 0\}$ at the point O, since the stable manifold W^s tangents to the x-axis and orbits, passing through the points $S^1 \cap W^u$ tangent to the y-axis. Theorem 8.1 is proved.

That also concludes the proof of the main Theorem 2.1.

9. An Example

The following three-dimensional system of differential equations gives an example of a C^1-smooth vector field having a global center invariant manifold:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x + \mu_1 y + \mu_2 xy + \mu_3 y^2 + \mu_4 x^2 + \mu_5 z/ln(z^2), \\ \dot{z} &= \mu_6 z + \mu_7 y/ln(y^2).\end{aligned} \tag{9.1}$$

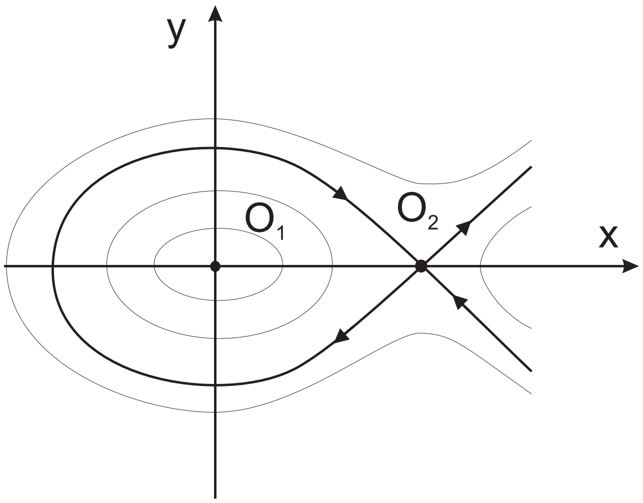

Fig. 13. The phase portrait of system (9.3) if $\mu_1 = \mu_2 = 0$.

Here we assume that

$$\mu_6 < \frac{1}{2}\left(\mu_1 + \frac{\mu_2}{\mu_4} - \sqrt{\left(\mu_1 + \frac{\mu_2}{\mu_4}\right)^2 + 8}\right), \quad \mu_3 < 0, \quad \mu_4 > 0, \quad \mu_3 + \mu_4 > 0. \quad (9.2)$$

Note that the right-hand side of (9.1) contains C^1-smooth functions $y/ln(y^2)$, and $z/ln(z^2)$, which are not $C^{1+\varepsilon}$-smooth.

The system has two equilibria O_1 and O_2 with coordinates $(x_1, y_1, z_1) = (0, 0, 0)$ and $(x_2, y_2, z_2) = (1/\mu_4, 0, 0)$, correspondingly. If $\mu_7 = 0$, the system (9.1) has a stable global center manifold $\{z = 0\}$. Therefore, the vector field takes the following form on this two-dimensional manifold:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \mu_1 y + \mu_2 xy + \mu_3 y^2 + \mu_4 x^2. \end{aligned} \tag{9.3}$$

Equations (9.3) were studied by Bautin (see Bautin and Leontovich [1976]). If $\mu_1 = \mu_2 = 0$, the system is conservative (see Figure 13). Moreover, by (9.2), separatrices of the saddle $O_2 = (1/\mu_4, 0)$ form a homoclinic orbit. In this case the integral of the system (9.3) takes the form

$$H(x, y) = \left(\frac{\mu_4}{\mu_3} x^2 + y^2 + \frac{\mu_4 - \mu_3}{\mu_3^2} x + \frac{\mu_4 - \mu_3}{2\mu_3^3}\right) e^{-2\mu_3 x} = h.$$

Therefore, homoclinic orbits to the saddle O_2 satisfy the following relation:

$$H(x, y) = H(1/\mu_4, 0).$$

Thus, if $(\mu_1, \mu_2, \mu_7) = (0, 0, 0)$, the initial system (9.1) has a homoclinic orbit to O_2. Moreover, by (9.2), this system satisfies Theorem 2.2. Therefore, by the theorem, we have that for any μ_3, μ_4, μ_5, μ_6, and small μ_1, μ_2, μ_7, the system has a smooth center manifold $\mathcal{M}^{cu}$ that depends smoothly on the parameters.

Acknowledgments

The authors are grateful to L. P. Shilnikov, V. N. Belykh, L. M. Lerman, S. V. Gonchenko, O. V. Stenkin, B. Fiedler, and A. J. Homburg for useful discussions. This research was supported by the EC-Russia Collaborative Project ESPRIT P 9282–ACTCS, by the ISF grant R98300, and the grant of INTAS-93-0570-ext. M. Shashkov was also supported by DAAD (1995) to work at Institut für Mathematik I, Freie Universität, Berlin.

References

[1] Bautin, N. N, & Leontovich, E. A. [1976] *Methods and Tricks of Qualitative Investigation of Dynamical Systems on Planes* (Nauka, Russia) Chap. 16, pp. 299–307.

[2] Bykov, V. V. [1978] "On the structure of a neighborhood of a separatrix contour with a saddle-focus," *Methods of the Qualitative Theory of Differential Equations*, Gorky, 3–32 (in Russian).

[3] Bykov, V. V. [1980] "On bifurcations of dynamical systems with a separatrix contour containing a saddle-focus," *Methods of the Qualitative Theory of Differential Equations*, Gorky, 44–72 (in Russian).

[4] Bykov, V. V. [1993] "The bifurcations of separatrix contours and chaos," *Physica* **D62**, 290–299.

[5] Chow, S. N., Deng, B., & Fiedler, B. [1990a] "Homoclinic bifurcation at resonant eigenvalues," *J. Dyn. Diff. Eq*. **2**, 177–244.

[6] Chow, S. N., Deng, B., & Terman, D. [1990b] "The bifurcation of homoclinic and periodic orbits from two heteroclinic orbits," *SIAM J. Math. Anal*. **21**, 179–204.

[7] Deng, B. [1989] "The Shilnikov problem, exponential expansion, strong λ-Lemma, C^1-linearization, and homoclinic bifurcation," *J. Diff. Eq*. **79**, 189–231.

[8] Deng, B. [1991] "The bifurcations of countable connections from a twisted heteroclinic loop," *SIAM J. Math. Anal*. **22**, 653–679.

[9] Fenichel, N. [1971] "Persistence and smoothness of invariant manifolds for flows," *Indiana Univ. Math. J*. **21**, 193–266.

[10] Fiedler, B., & Turaev, D. V. [1996] "Coalescence of reversible homoclinic orbits causes elliptic resonance," *Int. J. Bifurcation Chaos* **6**, 1007–1027.

[11] Gambaudo, J.-M., Glendinning, P., & Tresser, C. [1988] "The gluing bifurcation: 1. Symbolic dynamics on the closed curves," *Nonlinearity* **1**, 203–214.

[12] Hirsch, M., & Pugh, C. [1970] "Stable manifolds and hyperbolic sets," *Proc. Symp. Pure Math*. **14**, 133–163.

[13] Hirsch, M., Pugh, C., & Shub, M. [1977] "Invariant manifolds," *Lecture Notes in Math*. **583**, (Springer-Verlag).

[14] Homburg, A. J. [1993] "Some global aspects of homoclinic bifurcations of vector fields," Ph.D. Dissertation, University of Groningen.

[15] Homburg, A. J. [1996] "Global aspects of homoclinic bifurcations of vector fields," *Mem. A.M.S*. **578**.

[16] Kelley, A. [1967] "The stable, center-stable, center, center-unstable, and unstable manifolds," Appendix C of *Transversal Mappings and Flows*, ed. Abraham, R., & Robbin, J. (Benjamin, New York).

[17] Ovsyannikov, I. M., & Shilnikov, L. P. [1986] "On systems with a saddle-focus homoclinic curve," *Mat. Sb*. **130(172)**, 552–570; English transl. in [1987] *Math. USSR Sb*. **58**, 557–574.

[18] Ovsyannikov, I. M., & Shilnikov, L. P. [1991] "Systems with a homoclinic curve of multidimensional saddle-focus type, and spiral chaos," *Mat. Sb*. **182**, 1043–1073; English transl. in [1992] *Math. USSR Sb*. **73**, 415–443.

[19] Pliss, V. A. [1964] "A reduction principle in the theory of stability of motion," *Izv. Akad. Nauk SSSR*, *Ser Mat*. **28**, 1297–1324.

[20] Sandstede, B. [1995] "Center manifolds for homoclinic solutions," preprint.

[21] Shashkov, M. V. [1991a] "On existence of a smooth invariant two-dimensional attractive manifold for systems with separatrix contours," *Methods of Qualitative Theory and Theory of Bifurcations*, Nizhny Novgorod State University, 61–73 (in Russian).

[22] Shashkov, M. V. [1991b] "On bifurcations of heteroclinic contours with two saddle equilibrium points," *Methods of Qualitative Theory and Theory of Bifurcations*, Nizhny Novgorod State University, 74–83 (in Russian).

[23] Shashkov, M. V. [1992] "On bifurcations of separatrix contours with two saddles," *Int. J. Bifurcation Chaos* **2**, 911–915.

[24] Shashkov, M. V. [1994] "Bifurcations of Separatrix Contours," Ph.D. Dissertation, University of Nizhny Novgorod (in Russian).

[25] Shashkov M. V., & Turaev, D. V. [1996] "On the Complex Bifurcation Set for a System with Simple Dynamics," *Int. J. Bifurcation Chaos* **6**, 949–968.

[26] Shilnikov, L. P. [1963] "Some cases of generation of periodic motion from singular trajectories," *Mat. Sb*. **61**, 443–466 (in Russian).

[27] Shilnikov, L. P. [1965] "A case of the existence of a denumerable set of periodic motions," *Dokl. Acad. Nauk USSR* **160**, 558–561; English transl. in [1965] *Sov. Math. Dokl*. **6**, 163–166.

[28] Shilnikov, L. P. [1967] "On a Poincaré-Birkhoff problem," *Mat. Sb*. **74**, 378–397; English transl. in [1967] *Math. USSR Sb*. **3**, 415–443.

[29] Shilnikov, L. P. [1968] "On the generation of a periodic motion from a trajectory doubly asymptotic to an equilibrium state of saddle type," *Mat. Sb*. **77**, 461–472; English transl. in [1968] *Math. USSR Sb*. **6**, 415–443.

[30] Shilnikov, L. P. [1969] "On a new type of bifurcation of multidimensional dynamical systems," *Sov. Math. Dokl*. **10**, 1369-1371.

[31] Shilnikov, L. P. [1970] "A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type," *Mat. Sb*. **81**, 92–103; English transl. in [1970] *Math. USSR Sb*. **10**, 91–102.

[32] Shoshitaishvili, A. N. [1975] "Bifurcations of the topological type of a vector field near a singular point," *Trudy Seminarov I.G.Petrovskogo* **1**, 279–309 (in Russian).

[33] Turaev, D. V. [1984] "On a case of bifurcations of a contour composed by two homoclinic curves of a saddle," *Methods of Qualitative Theory of Differential Equations*, Gorky State University, 162–175 (in Russian).

[34] Turaev, D. V. [1991] "On bifurcations of dynamical systems with two homoclinic curves of the saddle," Ph.D. Dissertation, University of Nizhny Novgorod (in Russian).

[35] Turaev, D. V. [1996] "On the dimension of nonlocal bifurcational problems," *Int. J. Bifurcation Chaos* **6**, 911–948.

[36] Turaev, D. V., & Shilnikov [1986] "On bifurcations of a homoclinic 'figure eight' for a saddle with a negative saddle value," *Dokl. Akad. Nauk SSSR* **290**, 1301-1304; English transl. in [1987] *Sov. Math. Dokl*. **34**, 397–401.