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ON DYNAMIC PROPERTIES OF DIFFEOMORPHISMS WITH HOMOCLINIC TANGENCY

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Abstract. We study dynamic properties of systems in Newhouse domains near a diffeomorphism having a saddle fixed point with a homoclinic tangency in the following cases: one-dimensional, two-dimensional, where a fixed point is a saddle-focus with one real and two complex-conjugate multipliers, and four-dimensional saddle-focus with two pairs of complex-conjugate multipliers.

CONTENTS

Introduction

Poincaré homoclinic trajectories, i.e., trajectories bi-asymptotic to saddle periodic trajectories, are one of the most attractive objects of study in the theory of dynamic systems. First of all, this is due to the fact that their existence justifies the existence of a complex dynamics. So, in a neighborhood of a rough homoclinic trajectory, at whose points invariant manifolds of a saddle periodic orbit have a transversal intersection, there exists a countable set of periodic trajectory and a continuum of Poisson stable trajectories [26, 28].

If there is at least one nonrough homoclinic trajectory, or, as one says, a homoclinic tangency, then this implies that in any neighborhood of the system considered, there exists a countable set of nonroughness domains in which systems with homoclinic tangency are dense. For the first time, this phenomenon was discovered by Newhouse in the case of two-dimensional diffeomorphisms in [21]. In the higher-dimensional case, the Newhouse domains also exist in a neighborhood of any system with a homoclinic tangency in

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the parameter space of finite-parametric families [16], as well as in the space of dynamic systems in a natural way [16, 23, 24].

We note immediately that the dynamic properties demonstrated by systems from the Newhouse domains are characterized by an exceptional strangeness and complexity. Thus, in [13, 15], it was proved that even in the two-dimensional case, the description of the dynamics of a system from the Newhouse domains requires an infinite set of invariants (the so-called Ω-moduli, see [8, 9]), and, moreover, systems with homoclinic tangency of any order and with trajectories with an arbitrary degeneration are dense (in the C^r-topology with any $r \geq 3$) in the Newhouse domains (see [13, 19]).

In the present work, we continue the study of dynamic properties of systems from the Newhouse domains near a diffeomorphism having a saddle fixed point with a homoclinic tangency. Here, we restrict ourselves to the consideration of only the so-called base cases: the two-dimensional case, the three-dimensional case, where the fixed point is a saddle-focus with one real multiplier and a pair of complex-conjugate multipliers, and the four-dimensional case of a saddle-focus that has two pairs of complex-conjugate multipliers. In the general higher-dimensional case, the dynamics near the homoclinic tangency is mainly determined by the structure of the set of so-called leading multipliers of a fixed point. Generically, this is either a pair of real multipliers, or a single real multiplier and a pair of complex-conjugate multipliers, or two pairs of complex-conjugate multipliers. Therefore, it is obvious that the diffeomorphisms considered are the simplest among all the diffeomorphisms with the same tuple of leading multipliers.

The bifurcations of two-dimensional diffeomorphisms with homoclinic tangency have been intensively studied since [1]. Therefore, here, we focus our attention mainly on three- and four-dimensional diffeomorphisms, i.e., on the case of a saddle-focus. We consider three types of saddle-focuses. A fixed point with the multipliers $\lambda e^{\pm i\varphi}$ and γ, where $0 < \lambda < 1$, $0 < \varphi < \pi$, and $|\gamma| > 1$, is called a $(2,1)$ saddle-focus. A point is called a $(1,2)$-saddle-focus if it has multipliers λ and $\gamma e^{\pm i\psi}$, where $0 < |\lambda| < 1$, $\gamma > 1$, and $0 < \psi < \pi$. A point is called a $(2,2)$-saddle-focus if it has the multipliers $\lambda e^{\pm i\varphi}$ and $\gamma e^{\pm i\psi}$, where $0 < \lambda < 1$, $\gamma > 1$, $0 < \varphi < \pi$, and $0 < \psi < \pi$.

In all cases, we assume that the module J of the product of multipliers is not equal to 1. For definiteness, we assume that $J < 1$ (the case $J > 1$ reduces to this case by passing to the inverse mapping).

We show that similarly to the two-dimensional case, in the Newhouse domains, *diffeomorphisms with countably many stable periodic trajectories are dense* for $J < 1$. It was known earlier [2, 20, 23] that stable periodic trajectories can arise under the homoclinic tangency bifurcation under the condition that the unstable manifold of the corresponding saddle point is one-dimensional and the saddle quantity satisfies $\sigma \equiv |\lambda\gamma| < 1$. As our result shows, no one of these conditions is necessary. On the other hand, we note that in the case $J > 1$, the expansion of volumes near a saddle fixed point prohibits stable trajectories in a small neighborhood of the homoclinic tangency for the system itself, as well as for all systems close to it [14, 32]. Moreover, in the Newhouse domains, diffeomorphisms with countably many completely unstable periodic trajectories are dense.

As for saddle periodic trajectories, in the case of a saddle-focus with homoclinic tangency, we find the following substantially non-two-dimensional phenomenon. Precisely, we show here that under certain conditions, *diffeomorphisms having simultaneously a countable set of saddle periodic trajectories of two and even three types* (*i.e., diffeomorphisms with unstable manifolds of different dimensions*) *are dense* in the Newhouse domains. Note that the dimensions of the unstable manifolds of these periodic trajectories can be even greater than the dimension of the unstable manifold of the initial fixed point. In general, this phenomenon can be found in many cases of homoclinic bifurcations near a loop of the saddle-focus [22], near nonrough heteroclinic contours [12, 18, 32], and near homoclinic tangencies in certain cases of codimension 2 [6, 7, 30]; it was explicitly used in constructing a wild spiral attractor in [31]. In principle, the coexistence of trajectories with different numbers of Lyapunov exponents seems to be the most general of higher-dimensional systems from the Newhouse domains.

The existence of *nonrough* periodic trajectories from the Newhouse domains is another characteristic property of systems from Newhouse domains. It is known (see [1, 2]) that in the two-dimensional case, under a bifurcation of homoclinic tangency, periodic trajectories with one multiplier equal to 1 or -1 arise.

We show here that in the case of a saddle-focus, periodic trajectories with *two* or even *three* multipliers whose modules are equal to 1 can arise and the diffeomorphisms with these trajectories are dense in the corresponding Newhouse domains.

The bifurcations of a periodic motion in the case of one multiplier equal to $+1$ or -1 are well known: they are the saddle-node bifurcation and the period doubling bifurcation. In the case of a periodic trajectory with two multipliers on the unit circle, for example, $\nu_{1,2} = e^{\pm i\omega}$, its bifurcations can lead to the arising of closed invariant curves. Here, in connection with the problem of the existence of countably many nontrivial attractors, we first of all are interested in stable closed curves. Thus, we show that in the case of a homoclinic tangency to a fixed point of saddle-focus type with $J < 1$ (except for a $(2,1)$-saddle-focus for $|\lambda\gamma| < 1$, where the dynamics does not, in principle, differ from the case of a saddle), *diffeomorphisms with countably many stable invariant curves* are dense in the corresponding Newhouse domains.

In the case of a $(2,2)$-saddle-focus with $\lambda\gamma^2 > 1$, the bifurcations of homoclinic tangency lead to the arising of periodic trajectories with three multipliers on the unit circle. These cases require a separate consideration, which is not presented in this paper. We only note that, for example, in the case of multipliers $(-1,-1,+1)$, the corresponding normal form is a set of three autonomous differential equations (Morioka–Shimizu system) having a Lorenz-type attractor [25]. Correspondingly, we can expect that in the case of a $(2,2)$-saddle-focus with $\lambda\gamma^2 > 1$, diffeomorphisms with countably many strange attractors will be dense in the Newhouse domains.

The main results of the paper were announced in [17].

1. Statement of the Problem and Main Results

1.1. Main assumptions. Consider a C^r-smooth diffeomorphism f having a saddle fixed point O. Assume that the stable and unstable manifolds $W^s(O)$ and $W^u(O)$ intersect nontransversally at points of a certain homoclinic trajectory Γ_0.

Assume that the point O has no nonleading multipliers. Here, the following four main cases arise: the two-dimensional case where the multipliers of the point O are real, two three-dimensional cases where there are one real and two complex-conjugate multipliers, and one four-dimensional case where the multipliers are complex. Precisely, assume that the following condition holds.

A. The point O belongs to one of the following types:

$(1,1)$ if the multipliers λ and γ of the point are real, $|\lambda| < 1$, and $|\gamma| > 1$;

$(2,1)$ if O has a pair of complex multipliers $\lambda_{1,2} = \lambda e^{\pm i\varphi}$, where $\lambda \in (0,1)$ and $\varphi \in (0,\pi)$, and one real multiplier γ, where $|\gamma| > 1$;

$(1,2)$ if O has one real multiplier λ, where $|\lambda| < 1$, and a pair of complex multipliers $\gamma_{1,2} = \gamma e^{\pm i\psi}$, where $\gamma > 1$ and $\psi \in (0,\pi)$;

$(2,2)$ if O has two pairs of complex multipliers $\lambda_{1,2} = \lambda e^{\pm i\varphi}$ and $\gamma_{1,2} = \gamma e^{\pm i\psi}$, where $\lambda \in (0,1)$, $\gamma > 1$, and $\varphi, \psi \in (0,\pi)$.

The point O is called a saddle in the first case and a saddle-focus in all other cases. Let J be the module of the product of the multipliers of the point O. Assume that f satisfies the following condition.

B. $J < 1$, $|\lambda\gamma| \neq 1$ in the case $(2,1)$, and $\lambda\gamma^2 \neq 1$ in the case $(2,2)$.

Introduce the integer number d_e (which will be called the "effective dimension") as follows:

- $d_e = 1$ in the case $(1,1)$ and also in the case $(2,1)$ for $|\lambda\gamma| < 1$;
- $d_e = 2$ in the case (2,1) for $|\lambda\gamma| > 1$, in the case $(1,2)$, and also in the case $(2,2)$ for $\lambda\gamma^2 < 1$;
- $d_e = 3$ in the case $(2,2)$ for $\lambda\gamma^2 > 1$.

The meaning of the constants J and d_e is very simple. J is the Jacobian of the mapping f at the fixed point O. Therefore, the diffeomorphism f contracts volumes near O if $J < 1$ and expands them if $J > 1$. Also, it is obvious that if $J < 1$, then, under iterations of the mapping f, all $(d_e + 1)$-dimensional volumes near O are exponentially contracted, while d_e-dimensional volumes can be expanded.

Obviously, condition **B** is not restrictive since the case $J > 1$ reduces to the given case by passing to the inverse mapping. It should only be kept in mind that the stable manifold becomes unstable in this case, i.e., the case $(1,2)$ transforms into $(2,1)$ and vice versa. The definition of the set d_e is also changed in an obvious way.

Denote by T_0 the restriction of the diffeomorphism f to a sufficiently small neighborhood U_0 of the fixed point O. We say that T_0 is a *local mapping*. In a small neighborhood of the point $O(0,0)$, the mapping T_0 can be written as follows:

$$\bar{x} = Ax + \ldots, \quad \bar{y} = By + \ldots. \tag{1.1}$$

The stable multipliers (whose modules are less than 1) and the unstable multipliers (whose modules are greater than 1) of the point O are eigenvalues of the matrices A and B, respectively. Note that if a stable multiplier is real, then $A = \lambda$ and x is a scalar; if there is a pair of complex stable multipliers, then $x = (x_1, x_2)$ and $A = \lambda \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Also, if an unstable multiplier γ is real, then $B = \gamma$ and y is a scalar; if there is a pair of complex unstable multipliers, then $y = (y_1, y_2)$ and $B = \gamma \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$.

The intersection points of the trajectory Γ_0 with U_0 belong to the set $W^s \cap W^u$ and accumulate to O. A countable set of these points lies on W^s_{loc} and W^u_{loc}. Let $M^+ \in W^s_{\text{loc}}$ and $M^- \in W^u_{\text{loc}}$ be two certain points of a trajectory Γ_0 and let $M^+ = f^{k_0}(M^-)$ for a certain natural k_0. Let Π^+ and Π^- be certain small neighborhoods of the points M^+ and M^- lying in U_0. The mapping $T_1 \equiv f^{k_0} : \Pi^- \to \Pi^+$ is called a *global mapping*.

By condition, $T_1(W^u_{\text{loc}})$ is tangent to W^s_{loc} at the point M^+. Assume that this tangency is *simple*, i.e., the following conditions hold:

C. $T_1(W^u_{\text{loc}})$ and W^s_{loc} have a unique common tangent vector at the point M^+;

D. The tangency of surfaces $T_1 W^u_{\text{loc}}$ and W^s_{loc} at the point M^+ is quadratic.

1.2. Bifurcation parameters. Let f be a diffeomorphism with a homoclinic tangency satisfying Conditions **A**–**D**. The diffeomorphisms close to f and having a nonrough homoclinic trajectory close to Γ_0 compose a smooth bifurcation surface $\mathcal{H}$ of codimension 1 in the space of C^r-smooth diffeomorphisms equipped with the C^r-topology.

In this paper, we consider the bifurcations in parametric families f_ε transversal to $\mathcal{H}$ for $\varepsilon = 0$. In this case, the minimum number of control parameters is exactly equal to d_e. As the first parameter, we take the parameter μ estimating the splitting of $W^s(O)$ and $W^u(O)$ near the point M^+ (for a precise definition of μ in terms of coefficients of the Taylor series expansion of the global mapping T_1, see Lemma 2.3). Formally speaking, μ is a smooth functional defined for diffeomorphisms close to f such that the bifurcation surface $\mathcal{H}$ is given by the equation $\mu(f) = 0$. The family f_ε is transversal to $\mathcal{H}$ if and only if $\dfrac{\partial}{\partial\varepsilon}(\mu(f_\varepsilon)) \neq 0$ for $\varepsilon = 0$. Precisely this condition allows us to take μ as the first component of the parameter vector ε.

If $d_e \geq 2$, then, in addition to μ, we need one or two (when $d_e = 3$) more control parameters. In this case, we require that the family f_ε, $\varepsilon = 0$, be transversal not only to the bifurcation surface $\mathcal{H}$ but to the surfaces $\varphi = \text{const}$ and/or $\psi = \text{const}$, where φ and ψ are angular arguments of the complex multipliers of the saddle-focus O. This condition allows us to directly take the parameters μ, $\varphi - \varphi_0$, and $\psi - \psi_0$, where φ_0 and ψ_0 are values of φ and ψ for $\varepsilon = 0$ as control parameters.

Therefore, we set

(1) $\varepsilon = \mu$ in the case $(1,1)$ and also in the case $(2,1)$ for $|\lambda\gamma| < 1$;
(2) $\varepsilon = (\mu, \varphi - \varphi_0)$ in the case $(2,1)$ for $|\lambda\gamma| > 1$;
(3) $\varepsilon = (\mu, \psi - \psi_0)$ in the case $(1,2)$ and also in the case $(2,2)$ for $\lambda\gamma^2 < 1$;
(4) $\varepsilon = (\mu, \varphi - \varphi_0, \psi - \psi_0)$ in the case $(2,2)$ for $\lambda\gamma^2 > 1$.

Note that φ and ψ are continuous invariants on the topological conjugacy on the set of nonwandering trajectories (the so-called Ω-moduli) for systems with homoclinic tangencies in the case of the saddle-focus. As was shown in [4, 14], any change of values of these Ω-moduli (in the class of diffeomorphisms on $\mathcal{H}$, i.e., when the initial tangency is not split) can lead to the bifurcations of one-time going-around periodic trajectories.[1] In particular, this explains why one parameter μ cannot be sufficient in studying the bifurcations in the cases $(2,1)$, $(1,2)$, and $(2,2)$.

Note that all our results will hold here for arbitrary families f_ε (including the case where the number of parameters is greater than d_e) under only one assumption that the above-presented transversality conditions hold.

One of the general results on the families f_ε is the existence of the *Newhouse domains* in them. First of all, we recall the following result from [16].

Theorem (theorem on the Newhouse intervals)**.** *Let f_μ be a one-parameter family of C^r-smooth $(r \geq 3)$ diffeomorphisms transversal to the bifurcation surface $\mathcal{H}$ of a diffeomorphism satisfying conditions* ***A***–***D***.[2] *Then in any neighborhood of the point $\mu = 0$, there exists Newhouse intervals such that*

(1) *the values of the parameter μ corresponding to the case where the diffeomorphism f_μ has a simple homoclinic tangency at the point O are dense in these intervals*;

(2) *the family f_μ is transversal to the corresponding bifurcation surfaces.*

Since the Newhouse domains are open in C^2-topology in the set of dynamic systems, applying the theorem on Newhouse intervals to the family f_ε, we obtain the following corollary.

Corollary (Newhouse domains for parameter families)**.** *In the space of parameters ε, there exists a sequence of open domains δ_j accumulating to $\varepsilon = 0$ such that the values of the parameters ε for each of which the diffeomorphism f_ε has a trajectory of simple homoclinic tangency to the point O are dense in δ_j. Moreover, the family f_ε is transversal to each of the corresponding bifurcation surfaces.*

1.3. Main results. We will study the properties of the diffeomorphisms f_ε from the Newhouse domains δ_j. For analysis of bifurcations of periodic trajectories, we assume that the diffeomorphisms f_ε are sufficiently smooth, precisely, $r \geq 5$.

First of all, we consider the case $d_e = 1$ (recall that we consider one-parameter families with $\varepsilon = \mu$ here).

Theorem 1.1. *In the cases of a saddle* $(1,1)$ *and a saddle-focus* $(2,1)$ *for $|\lambda\gamma| < 1$, on the Newhouse intervals δ_j,*

(1) *the values of μ for which the diffeomorphism f_μ has a periodic trajectory with the multiplier* $+1$ *are dense*;

(2) *the values of μ for which the diffeomorphism f_μ has a periodic trajectory with the multiplier* -1 *are dense*;

(3) *the values of μ for which the diffeomorphism f_μ has countably many stable periodic trajectories are dense* (*and form a set of the second category*).

In essence, items (1) and (2) of this theorem are proved in [1] for the case of a saddle and in [2, 3] for the case of a saddle-focus. Item (3) has been known since [20], and the three-dimensional case was considered in [10, 14] (see also [23]). For completeness, we present the proof of Theorem 1.1 together with the proof of other results presented below.

Further, we consider the case $d_e \geq 2$. Here, the main focus is on those properties of the diffeomorphisms f_ε which are new as compared with the case of a saddle. These are

(1) the existence of nonrough periodic trajectories having more than one multiplier on the unit circle (Theorem 1.2);

[1]A similar situation also takes place in the case of a saddle but for two-time going-around periodic trajectories. Here, any change of the Ω-module $\theta = -\ln|\lambda|/\ln|\gamma|$ leads to bifurcations of these trajectories [8, 11]. Also, note that three-time going-around periodic trajectories in this case can be subjected to the so-called cusp bifurcations [29] when, at a critical instant, one of the multipliers is equal to $+1$ and the first Lyapunov value vanishes.

[2]Note that in [16], instead of condition **B**, we require only that $\lambda\gamma \neq 1$. Note that our condition **B** includes this requirement.

(2) the existence of countably many stable closed invariant curves (Theorem 1.3);

(3) the coexistence of countably many rough periodic trajectories of more than two different types (Theorem 1.4).

Theorem 1.2. *In the case $d_e = 2$, i.e., in the case of a saddle-focus $(2,2)$ for $\lambda\gamma^2 < 1$, a saddle-focus $(1,2)$, and also a saddle-focus $(2,1)$, for $|\lambda\gamma| > 1$, the values of the parameters ε such that the corresponding diffeomorphism f_ε has a periodic trajectory with two multipliers on the unit circle given in advance are dense in the Newhouse domains δ_i. In the case of saddle-focus $(2,2)$ for $\lambda\gamma^2 > 1$ (i.e., if $d_e = 3$) the values of the parameters ε such that the corresponding diffeomorphism f_ε has a periodic trajectory with any three multipliers on the unit circle given in advance are dense in the Newhouse domains δ_j.*

Since we consider real diffeomorphisms, note that, more precisely, Theorem 1.2 tells us about those tuples of multipliers into which each complex multiplier enters together with its complex conjugate. In particular, we obtain that in the case of homoclinic tangency to a saddle-focus, for $d_e \geq 2$, the diffeomorphisms with periodic trajectories having the multipliers $e^{\pm i\omega}$, where $0 < \omega < \pi$, are dense in the Newhouse domains. The analysis of bifurcations of these periodic trajectories and also periodic trajectories with the multipliers $(-1,-1)$ allows us to prove the following result.

Theorem 1.3. *Let a C^r-smooth $(r \geq 5)$ diffeomorphism f satisfy conditions **A**–**D**. Then in the case $d_e \geq 2$, the values of parameters for which the diffeomorphism f_ε has countably many stable closed invariant curves are dense and compose a set of the second category in the Newhouse domains δ_j.*

In this theorem, the condition $J < 1$ is essential (for $J > 1$, all trajectories are necessarily unstable). In the class of two-dimensional diffeomorphisms with $J \neq 1$, there cannot be closed invariant curves near the homoclinic tangency, since we have either a contraction of areas (for $J < 1$) or their expansion (for $J > 1$). However, in the case of codimension 2, where $J = 1$ at the instant of homoclinic tangency, even for diffeomorphisms of the plane, we can observe the arising of closed invariant curves [6, 7]. Closed invariant curves also arise under bifurcations of a nonrough heteroclinic contour with two saddles such that $J < 1$ on one saddle and $J > 1$ on the other. As was shown in [12, 18], near systems with these contours, there exist Newhouse domains in which the diffeomorphisms simultaneously having a countable set of stable and completely unstable closed invariant curves are dense.

The following theorem gives an answer to one of the main questions on the dynamics of systems from the Newhouse domains, the question on the coexistence of rough periodic trajectories of different types.

Theorem 1.4. *The values of the parameters under which the diffeomorphisms f_ε have countably many stable periodic trajectories and countably many saddle periodic trajectories with the dimensions of unstable manifolds varying from 1 up to d_e inclusively are dense and compose a set of the second category in the Newhouse domains δ_j.*

Note that here, there can be no periodic trajectories with unstable manifolds of dimension greater than d_e because of the contraction of $(d_e + 1)$-dimensional volumes [17, 32]. Thus, for example, in the case of saddle-focus $(2,2)$ with $\lambda\gamma^2 < 1$, we have saddles with one- and two-dimensional unstable manifolds, and there are no saddles with three-dimensional unstable manifolds. At the same time, for $\lambda\gamma^2 > 1$ and $\lambda\gamma < 1$, we simultaneously have saddles with one-, two-, and three-dimensional unstable manifolds.

The proof of Theorems 1.1–1.3 is based on the study of the first-return mappings near a trajectory of homoclinic tangency. We reduce the study of these mapping to the analysis of the following standard mappings:

(i) the mapping of the parabola $\bar{y} = M - y^2$ (for the case of a saddle-focus $(2,1)$ and for $|\lambda\gamma| < 1$);

(ii) the Henon mapping $\bar{x}_1 = y$, $\bar{y} = M - y^2 + Bx_1$ (for the case of a saddle-focus $(2,1)$ for $|\lambda\gamma| > 1$);

(iii) the Meer mapping $\bar{y}_1 = y_2$, $\bar{y}_2 = M + Cy_2 - y_1^2$ (for the case of a saddle-focus $(1,2)$ and also a saddle-focus $(2,2)$ for $\lambda\gamma^2 < 1$);

(iv) the three-dimensional Henon mapping $\bar{x}_1 = y_1$, $\bar{y}_1 = y_2$, $\bar{y}_2 = M + Cy_2 + Bx_1 - y_1^2$ (for the case of a saddle-focus $(2,2)$ for $\lambda\gamma^2 > 1$).

The linear analysis of fixed points of these mappings is comparatively easy (see Sec. 4), but it yields the information necessary for the proof of Theorems 1.1, 1.2, and 1.4. As for closed invariant curves (and, properly, Theorem 1.3) itself, we deduce their existence in the case of saddle-focuses $(1, 2)$ and $(2, 2)$ from the nonlinear bifurcation analysis of mappings (iii) and (iv). In the case of a saddle-focus $(2, 1)$ with $|\lambda\gamma| > 1$, the Henon mapping (ii) itself has no (asymptotically stable) invariant closed curves. Therefore, to prove Theorem 1.3 in this case, we must deal with the so-called generalized Henon mapping (see Lemma 1.2).

1.4. Rescaling lemma. In the case of diffeomorphisms close to a diffeomorphism with homoclinic tangency, the first-return mappings from a small fixed neighborhood Π^+ of the homoclinic point M^+ are written in the form of compositions $T_k = T_1 T_0^k$, where $k = \bar{k}, \bar{k}+1, \ldots$, and $\bar{k}$ can be sufficiently large. Recall that $T_0 = f_\varepsilon\big|_{U_0}$, where U_0 is a certain small neighborhood of a fixed point and $T_1 \equiv f_\varepsilon^{k_0}$ is a mapping defined in a small neighborhood Π^- of the homoclinic point M^- that maps Π^- inside Π^+. Therefore, the domain of the mapping T_k on Π^+ is a "small strip" $\sigma_k^0 = \Pi^+ \cap T_0^{-k}\Pi^-$. The domains of σ_k^0 are nonempty for all sufficiently large k (the less the sizes of the neighborhoods Π^+ or Π^- are, the greater the minimum k is), and they accumulate to $W^s_{\text{loc}} \cap \Pi^+$ as $k \to +\infty$.

The following lemma (the main technical result of our paper) shows that for a sufficiently large k, the first-return mappings T_k can be represented in a certain standard form as mappings asymptotically close as $k \to \infty$ to certain one-, two-, or three-dimensional quadratic mappings.

Lemma 1.1 (Rescaling lemma)**.** *Let f_0 be a C^r-smooth ($r \geq 5$) diffeomorphism satisfying conditions **A**–**D** and f_ε be a d_e-parameter family transversal to $\mathcal{H}$ for $\varepsilon = 0$. Then, in the parameter space, there exist a countable sequence of domains Δ_k accumulating to $\varepsilon = 0$ such that the following assertions hold.*

For $\varepsilon \in \Delta_k$, there exists a transformation of coordinates on σ_k^0 and parameters on Δ_k, C^{r-1}-smooth in coordinates and C^{r-2}-smooth in parameters resulting in the first-return mapping $T_k : (x, y) \mapsto (\bar{x}, \bar{y})$ reducing to one of the following forms:

(i)

$$\bar{y} = M - y^2 + o(1), \quad \bar{x} = o(1) \tag{1.2}$$

in the case $(1, 1)$ and also in the case $(2, 1)$ for $\lambda\gamma < 1$;

(ii)

$$\bar{x}_1 = y, \qquad \bar{y} = M - y^2 + Bx_1 + o(1), \qquad \bar{x}_2 = o(1) \tag{1.3}$$

in the case $(2, 1)$ for $\lambda\gamma > 1$;

(iii)

$$\bar{y}_1 = y_2, \qquad \bar{y}_2 = M + Cy_2 - y_1^2 + o(1), \qquad \bar{x} = o(1) \tag{1.4}$$

in the case $(1, 2)$ and also in the case $(2, 2)$ for $\lambda\gamma^2 < 1$;

(iv)

$$\bar{x}_1 = y_1, \qquad \bar{y}_1 = y_2, \qquad \bar{y}_2 = M + Cy_2 + Bx_1 - y_1^2 + o(1), \qquad \bar{x}_2 = o(1) \tag{1.5}$$

in the case $(2, 2)$ for $\lambda\gamma^2 > 1$.

Moreover, the domain of the mapping T_k is asymptotically large and, in the limit as $k \to +\infty$, covers all finite values of the coordinates (x, y).

The normalized parameters M, B, and C are related to the initial parameters μ, φ, and ψ as follows:

$$\begin{aligned} M &= M_0\gamma^{2nk}(\mu + O(|\lambda|^k + |\gamma|^{-k})), \\ B &= B_0\,(\lambda\gamma^n)^k \cos(k\varphi + \alpha_k(\varepsilon)), \\ C &= C_0\gamma^k \cos(k\psi + \beta_k(\varepsilon)), \end{aligned} \tag{1.6}$$

where $n = \dim W^u(O)$, the constants M_0, B_0, and C_0 are different from zero, and the functions α_k and β_k, together with their derivatives, are uniformly bounded in k. Moreover, when ε runs over the domain Δ_k, in turn, the values of the parameters M, B, and C run over asymptotically large domains covering all finite values in the limit as $k \to +\infty$.

Here, $o(1)$ denotes certain functions (of the coordinates and of the parameters M, B, and C) tending to zero as $k \to \infty$ together with all its derivatives up to the order $(r-2)$ inclusively and up to the order $(r-3)$ in the parameters uniformly on any bounded set of the space (x, y, M, B, C). Also, note that in the case of a saddle-focus, the domains Δ_k corresponding to finite values of B and C can consist of several connected components (by the periodicity of the coefficients B and C in φ and ψ, respectively).

In case (ii) of Lemma 1.1, we need a sharper account for asymptotically small terms of mapping (1.3), which leads to the following result.

Lemma 1.2. *In the case* $(2,1)$ *for* $\lambda\gamma > 1$, *where* $\varepsilon = (\mu, \varphi - \varphi_0) \in \Delta_k$ *and the corresponding value of the parameter* B *is bounded away from zero, the mapping* T_k *in the form* (1.3) *has a two-dimensional attracting invariant* C^{r-2}*-smooth manifold* $\mathcal{M}_k^s \subset \sigma_k^0$ *of the form* $x_2 = o(1)$ *as* $k \to \infty$ *such that the mapping* $T_k\Big|_{\mathcal{M}_k^s}$ *has the form*

$$\bar{x}_1 = y, \qquad \bar{y} = M - y^2 + Bx_1 + \frac{2J_1}{B}(\lambda^2\gamma)^k \left(x_1 y + o(1)\right), \tag{1.7}$$

where $J_1 \neq 0$ *is a certain constant* (*more precisely,* J_1 *is the Jacobian of the global mapping* T_1 *calculated at the homoclinic point* M^- *for* $\varepsilon = 0$).

Mapping (1.7) is called the generalized Henon mapping. It was introduced in [6, 7], where, in particular, it was shown that this mapping demonstrates the nondegenerate Andronov–Hopf bifurcation and has a closed stable invariant curve for the values of the parameters (M, B) in certain open domains (see Sec. 4).

The content of the paper is as follows. In Sec. 2, we study the properties of the local mapping $T_0(\varepsilon)$ and the global mapping $T_1(\varepsilon)$. In Sec. 3, we study the first-return mapping and prove Lemmas 1.1 and 1.2. In Sec. 4, we carry out the analysis of mappings (1.2)–(1.5) and (1.7) and prove Theorems 1.1–1.4.

2. Properties of the Local and Global Mappings

To study the first-return mapping $T_k = T_1 T_0^k$ for all sufficiently large k and all sufficiently small ε, we need appropriate formulas for the mappings T_0 and the form of the local mapping $T_0(\varepsilon)$. For all sufficiently small values of the parameters, this mapping has a fixed point O_ε which is assumed to be located at the origin. Extracting the linear part and choosing the coordinate axes in the appropriate way, we can represent $T_0(\varepsilon)$ in the form (1.1). Moreover, using a C^r-smooth coordinate change, we can rectify the local stable and unstable manifolds of the point O_ε. Thus, T_0 reduces to the form

$$\bar{x} = A(\varepsilon)x + p(x, y, \varepsilon), \qquad \bar{y} = B(\varepsilon)y + q(x, y, \varepsilon), \tag{2.1}$$

where the functions p and q are C^r-smooth and vanish together with their first derivatives at the origin; moreover, $p(0, y, \varepsilon) \equiv 0$ and $q(x, 0, \varepsilon) \equiv 0$. In this case,

$$W^s_{\text{loc}} = \{y = 0, v = 0\}, \qquad W^u_{\text{loc}} = \{x = 0, y = 0\}.$$

Note that only rectification of the manifolds W^s_{loc} and W^u_{loc} is not sufficient for our purposes. In essence, this is related to the fact that in the right-hand sides of Eqs. (2.1), there are too many nonresonant terms. Nevertheless, using additional coordinate changes, we can remove a certain part of these terms. Precisely, the following assertion holds.

Lemma 2.1. *Let* $r \geq 3$. *For all sufficiently small* ε, *we can introduce* C^{r-1}*-coordinates* (x, u, y, v) *on* U_0 *of class* C^{r-2} *with respect to the parameters in which the mapping* $T_0(\varepsilon)$ *is written in the form*

$$\bar{x} = A(\varepsilon)x + P(x, y, \varepsilon)x, \qquad \bar{y} = B(\varepsilon)y + Q(x, y, \varepsilon)y, \tag{2.2}$$

where

$$P(0, y, \varepsilon) = P(x, 0, \varepsilon) \equiv 0, \qquad Q(x, 0, \varepsilon) = Q(0, y, \varepsilon) \equiv 0. \tag{2.3}$$

One of the crucial merits of the form (2.2) is that, in this case, the mapping $T_0^k : U_0 \to U_0$ represented in the so-called "crossed form" is linear in the principal order for all sufficiently large k. Precisely, let $T_0(\varepsilon)$ be reduced to the form (2.2) and (x_i, y_i), $i = 0, \dots, k$, be points on U_0 such that $(x_i, y_i) = T_0(x_{i-1}, y_{i-1})$. Then the following assertion holds.

Lemma 2.2. *For any sufficiently large k and any sufficiently small ε, the mapping $T_0^k(\varepsilon) : (x_0, y_0) \to (x_k, y_k)$ can be represented in the form*

$$x_k - A_1^k(\varepsilon)x_0 = \hat{\lambda}^k \xi_k(x_0, y_k, \varepsilon), \qquad y_0 - B_1^{-k}(\varepsilon)y_k = \hat{\gamma}^{-k}\eta_k(x_0, y_k, \varepsilon), \tag{2.4}$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are certain constants such that $0 < \hat{\lambda} < |\lambda|$ and $\hat{\gamma} > |\gamma|$ and the functions ξ_k and η_k, together with their derivatives (including those in the parameters) up to the order $(r-2)$ are uniformly bounded in k.

Lemmas 2.1 and 2.2 are proved in [8, 9, 27] for various cases.

As for the global mapping $T_1(\varepsilon)$, using conditions **C** and**D** of the quadratic character of the homoclinic tangency, we also find its convenient representation. Recall that the transversality condition for the family f_ε to the bifurcation set H means that among the parameters ε, we can distinguish a splitting parameter μ of the invariant manifolds of the point O near the chosen homoclinic point M^+. In this case, the global mapping $T_1(\varepsilon)$ can be represented in the form described in the following lemma.

Lemma 2.3. *The coordinates defined in Lemma 2.1 can be introduced in U_0 in such a way that for all sufficiently small ε, the global mapping $T_1(\varepsilon)$ can be written in the form*

$$\begin{aligned} \bar{x} - x^+ &= ax + b_0(y - y^-) + \dots, \\ \bar{y} &= \mu + cx + D_0(y - y^-)^2 + \dots \end{aligned} \tag{2.5}$$

in the case $(1,1)$ (here $x \in R^1$, $y \in R^1$);

$$\begin{aligned} \bar{x} - x^+ &= ax + \begin{pmatrix} b_0 \\ 0 \end{pmatrix}(y - y^-) + \dots, \\ \bar{y} &= \mu + c_1x_1 + c_2x_2 + D_0(y - y^-)^2 + \dots \end{aligned} \tag{2.6}$$

in the case $(2,1)$ (here $x \in R^2$, $y \in R^1$);

$$\begin{aligned} \bar{x} - x^+ &= ax + b_0(y_1 - y_1^-) + b_1\bar{y}_2 + \dots, \\ \bar{y}_1 &= \mu + cx + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + ex + \dots \end{aligned} \tag{2.7}$$

in the case $(1,2)$ (here $x \in R^1$, $y \in R^2$);

$$\begin{aligned} \bar{x} - x^+ &= ax + \begin{pmatrix} b_0 \\ 0 \end{pmatrix}(y_1 - y_1^-) + b_1\bar{y}_2 + \dots, \\ \bar{y}_1 &= \mu + c_1x_1 + c_2x_2 + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + e_1x_1 + e_2x_2 + \dots, \end{aligned} \tag{2.8}$$

where $b_0 \neq 0$, $c \neq 0$, $D_0 \neq 0$, $d_2 \neq 0$, $x^+ \neq 0$, and $y^- \neq 0$ in the case $(2,2)$ (here $x \in R^2$, $y \in R^2$).

Note that in formulas (2.7) and (2.8), the global mapping T_1 is represented in the crossed form with respect to the coordinate y_2, i.e., the right-hand sides are functions of (x, y_1) and $\bar{y}_2$.

In essence, formulas (2.5)–(2.8) are Taylor-series expansions for $y^-(\varepsilon)$ chosen in an appropriate way; by dots we denote nonlinear terms (except for one quadratic term explicitly written). Also, note that the coefficients $a, \dots, e_2$, as well as x^+ and y^-, and also the terms denoted by dots depend on the parameters ε in general (where, as usual, $\varepsilon_1 = \mu$). Here, the corresponding class of smoothness in ε is C^{r-3}: in the coordinates of Lemma 2.1, the mapping T_1, together with its derivative in (x, y), is C^{r-2}-smooth in ε (see [27]); therefore, the coefficient $D_0(\varepsilon)$ of the quadratic term is C^{r-3}-smooth.

Proof of Lemma 2.3. We use the coordinates of Lemma 2.1 on U_0.

Let the homoclinic points $M^+ \in W^s_{\rm loc}$ and $M^- \in W^u_{\rm loc}$ have the coordinates $M^+ = M^+(x^+, 0)$ and $M^- = M^-(0, y^-)$, where $\|x^+\| \neq 0$ and $\|y^-\| \neq 0$. Since $T_1 M^- = M^+$ for $\varepsilon = 0$, we can write the mapping $T_1(\varepsilon)$ in the following form for sufficiently small ε:

$$\begin{aligned} \bar{x} - x^+(\varepsilon) &= \hat{a}x + \hat{b}(y - y^-(\varepsilon)) + \cdots, \\ \bar{y} &= y^+(\varepsilon) + \hat{c}x + \hat{d}(y - y^-(\varepsilon)) + \cdots \end{aligned} \tag{2.9}$$

where the dots denotes the terms of order of smallness not less than two, all coefficients depend on ε in general, and $y^+(0) = 0$. Moreover,

$$\det \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \neq 0. \tag{2.10}$$

Using conditions **C** and **D**, we find relations which the coefficients in (2.9) satisfy. Consider condition **C**. It means that for $\varepsilon = 0$, the surface $T_1 W^u_{\rm loc}$ has *exactly one* tangent vector to the plane $W^s_{\rm loc}$ at the point M^+. Since $W^u_{\rm loc}$ has the equation $x = 0$ and $W^s_{\rm loc}$ the equation $\bar{y} = 0$, it follows from (2.9) that the intersection of the tangent planes to $T_1 W^u_{\rm loc}$ and $W^s_{\rm loc}$ at the point M^+ is one-dimensional if and only if the system $\hat{d}(y - y^-) = 0$ has a one-parameter family of solutions for $\varepsilon = 0$. Thus, in the case where $y \in R^1$ and $\hat{d}$ is a scalar (i.e., in the cases $(1, 1)$ and $(2, 1)$), we have

$$\hat{d} = 0 \quad \text{for} \quad \varepsilon = 0. \tag{2.11}$$

If $y \in R^2$ (the cases $(1, 2)$ and $(2, 2)$), then $\hat{d}$ is a (2×2)-matrix, and

$$\det \hat{d} = 0 \quad \text{and} \quad \operatorname{rank} \hat{d} = 1 \quad \text{for} \quad \varepsilon = 0. \tag{2.12}$$

In the case where $y \in R^1$, the second equation in (2.9) can be written in the form

$$\bar{y} = \hat{y}^+(\varepsilon) + \hat{c}x + D_0(y - y^-)^2 + \ldots, \tag{2.13}$$

where several first terms of the Taylor-series expansion, including linear terms and second-order terms, are explicitly written. Condition **D** of the quadratic character of the homoclinic tangency means that $D_0 \neq 0$. Note that the right-hand side of Eq. (2.13) does not contain a term linear in $(y - y^-)$; this can always be attained for sufficiently small ε assuming that y^- depends on ε in an appropriate way.

Finally, in the case $(1, 1)$, we obtain formula (2.5) for the mapping T_1, where $b_0 = \hat{b}$, $c = \hat{c}$, and $b_0 c \neq 0$ by (2.10) and (2.12). Also note that in (2.5), we have set $\mu = \hat{y}^+(\varepsilon)$, thus stressing the fact that $\hat{y}^+(\varepsilon)$ is a splitting parameter of the manifolds $W^s(O)$ and $W^u(O)$ near the homoclinic point M^+.

In the case $(2, 1)$, to obtain formula (2.6), we make one more coordinate change, precisely the linear rotation in the x-plane in order to transform the vector $b = (\hat{b}_1, \hat{b}_2)$ into $(b_0, 0)$, where $b_0 = \sqrt{\hat{b}_1^2 + \hat{b}_2^2} \neq 0$. It is easily seen that this can be attained by using the rotation $x \mapsto R_\omega x$, where $\omega = \arctan(-\hat{b}_2/\hat{b}_1)$. In this case, we note that

$$c_1 = \frac{\hat{b}_1\hat{c}_1 - \hat{b}_2\hat{c}_2}{b_0}, \quad c_2 = \frac{\hat{b}_2\hat{c}_1 + \hat{b}_1\hat{c}_2}{b_0},$$

and $c_1^2 + c_2^2 \neq 0$ by (2.10).

Now we consider the cases where $y \in R^2$ (i.e., the cases $(1, 2)$ and $(2, 2)$). The equations for $\bar{y}$ from (2.9) become

$$\begin{aligned} \bar{y}_1 &= \hat{y}_1^+(\varepsilon) + \hat{c}_1 x + \hat{d}_{11}(y_1 - y_1^-) + \hat{d}_{12}(y_2 - y_2^-) + \ldots, \\ \bar{y}_2 &= \hat{y}_2^+(\varepsilon) + \hat{c}_2 x + \hat{d}_{21}(y_1 - y_1^-) + \hat{d}_{22}(y_2 - y_2^-) + \ldots. \end{aligned} \tag{2.14}$$

Note that the rotation of the coordinates in the y-plane leaves the form of Eq. (2.14) the same in principle, but the coefficients can change. For $\varepsilon = 0$, since $\det \hat{d} = 0$, we can rotate the y-coordinates in such a way that the following relations hold:

$$\hat{d}_{11} = 0, \quad \hat{d}_{12} = 0. \tag{2.15}$$

Without loss of generality, we assume that these relations hold for $\varepsilon = 0$. Since $\operatorname{rank} \hat{d} = 1$ for $\varepsilon = 0$, at least one of the coefficients $\hat{d}_{21}$ or $\hat{d}_{22}$ is nonzero. Assume that

$$\hat{d}_{22} \neq 0. \tag{2.16}$$

If this is not true (i.e., $\hat{d}_{22} = 0$, and hence $\hat{d}_{21} \neq 0$), then we take another homoclinic point, precisely, the point $T_0^{-1}(M^-)$, and consider it as a new point M^-. For the new global mapping ($T_{1\,\text{new}} = T_1 T_0$), the new matrix $\hat{d}$ is written as

$$\hat{d}_{\text{new}} = \hat{d} \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

By (2.15), we obtain

$$\hat{d}_{\text{new}} = \begin{pmatrix} 0 & 0 \\ \hat{d}_{21}\cos\varphi + \hat{d}_{22}\sin\varphi & -\hat{d}_{21}\sin\varphi + \hat{d}_{22}\cos\varphi \end{pmatrix}.$$

If $\hat{d}_{22} = 0$, then, after passing to the new homoclinic point, we indeed obtain that (2.16) holds (since $\hat{d}_{21} \neq 0$ and $\sin\varphi > 0$ since $\varphi \in (0, \pi)$).

Now we also take into account the quadratic terms; then Eq. (2.14) for $\bar{y}_1$ is written in the following form for $\varepsilon = 0$:

$$\bar{y}_1 = c_1 x + D_1(y_1 - y_1^-)^2 + D_2(y_1 - y_1^-)(y_2 - y_2^-) + D_3(y_2 - y_2^-)^2 + \dots. \tag{2.17}$$

Since $\hat{d}_{22} \neq 0$, the second equation in (2.14) can be resolved with respect to $(y_2 - y_2^-)$. Respectively, for $\varepsilon = 0$, we have

$$y_2 - y_2^- = d_1(y_1 - y_1^-) + d_2\bar{y}_2 + ex + \dots, \tag{2.18}$$

where $d_1 = -\hat{d}_{21}/\hat{d}_{22}$ and $d_2 = \hat{d}_{22}^{-1}$. Substituting (2.18) in (2.17), we obtain

$$\bar{y}_1 = c_1 x + D_0(y_1 - y_1^-)^2 + \tilde{D}_1(y_1 - y_1^-)\bar{y}_2 + \tilde{D}_2\bar{y}_2^2 + \dots, \tag{2.19}$$

where

$$D_0 \equiv D_1 + d_1 D_2 + d_1^2 D_3 \tag{2.20}$$

and $\tilde{D}_{1,2}$ are certain coefficients. Therefore, for $\varepsilon = 0$, the mapping T_1 in the crossed form is written as

$$\begin{aligned} \bar{x} - x^+ &= ax + b_0(y_1 - y_1^-) + b_1\bar{y}_2 + \dots, \\ \bar{y}_1 &= cx + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + ex + \dots \end{aligned} \tag{2.21}$$

in the case $(1, 2)$ and as

$$\begin{aligned} \bar{x}_1 - x_1^+ &= a_{11}x_1 + a_{12}x_2 + b_0(y_1 - y_1^-) + b_{11}\bar{y}_2 + \dots, \\ \bar{x}_2 - x_2^+ &= a_{21}x_1 + a_{22}x_2 + b_{12}\bar{y}_2 + \dots, \\ \bar{y}_1 &= c_1x_1 + c_2x_2 + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + e_1x_1 + e_2x_2 + \dots \end{aligned} \tag{2.22}$$

in the case $(2, 2)$ with certain new coefficients a, b, c, d, and e (in the case $(2, 2)$, we make the coefficient of $(y_1 - y_1^-)$ in the equation for $\bar{x}_2$ zero by using an appropriate rotation of x-coordinates as in the case $(2, 1)$). In this case, condition (2.10) is rewritten as

$$\det \frac{\partial(\bar{x}, \bar{y}_1)}{\partial(x, y_1)} \neq 0, \tag{2.23}$$

which yields $b_0 \neq 0$ and $c \neq 0$ in both cases.

Since $x = 0$ on $W^u_{\rm loc}$, as follows from (2.21), (2.22), the image $T_1 W^u_{\rm loc}$ of this surface is given by the equation

$$y_1 = \frac{D_0}{b_0^2}(x - x^+)^2 + \dots \tag{2.24}$$

near the point M^+ in the case $(1, 2)$ and by the equations

$$x_2 - x_2^+ = b_{12} y_2 + \dots, \qquad y_1 = \frac{D_0}{b_0^2}(x_1 - x_1^+)^2 + \dots \tag{2.25}$$

in the case $(2, 2)$. In any case, it is obvious that condition **D** for the quadratic tangency of this surface to $W^s_{\rm loc}$: $y = 0$ is equivalent to the condition $D_0 \neq 0$.

As above, for $\varepsilon \neq 0$, the mapping T_1 is given by Eqs. (2.21) and (2.22): since $D_0(\varepsilon) \neq 0$ for all sufficiently small ε, we can always choose $y_1^-(\varepsilon)$ and $y_2^-(\varepsilon)$ and additionally rotate the y-coordinates so that the coefficients $d_{11}(\varepsilon)$ and $d_{12}(\varepsilon)$ will be identically equal to zero for all sufficiently small ε. The only distinction from the case $\varepsilon = 0$ is thus the appearance of the nonzero free term $y_1^+(\varepsilon)$ in the equation for $\bar y_1$. As above, the transversality condition of the family f_ε to the bifurcation surface $\mathcal{H}$ allows us to set $y_1^+(\varepsilon) = \mu$. This completes the proof of Lemma 2.3. □

3. Proof of the Rescaling Lemmas

In this section, we study the first-return mapping

$$T_k(\varepsilon) \equiv T_1 T_0^k : \sigma_k^0 \to \sigma_k^0$$

for all sufficiently large k, $k = \bar k, \bar k + 1, \dots,$ and small ε, $\|\varepsilon\| \leq \varepsilon_0$. In this case, for the mapping $T_0^k : \sigma_k^0 \to \sigma_k^1$, we use formula (2.4) of Lemma 2.2, where $(x_0, y_0) \in \Pi^+, (x_k, y_k) \in \Pi^-$, and for the global mapping $T_1(\varepsilon)$, we use the corresponding formulas of Lemma 2.3. According to Lemma 2.2, for any small x_0 and y_k and for any sufficiently large k, the corresponding coordinates x_k and y_0 are uniquely defined. Therefore, we can use (x_0, y_k) as coordinates on σ_k^0; the coordinate y_0 is calculated by the formula

$$y_0 = B_1^{-k}(\varepsilon) y_k + \hat\gamma^{-k} \eta_k(x_0, y_k, \varepsilon)$$

(see Lemma 2.2). Note that in the coordinates (x_0, y_k), the size of the small strip σ_k^0 is bounded away from zero in all directions for any k. So, if we define the neighborhoods Π^+ and Π^-as $\{\|x - x^+\| \leq \rho_0, \|y\| \leq \rho_0\}$ and $\{\|x\| \leq \rho_0, \|y - y^-\| \leq \rho_0\}$, respectively, where ρ_0 is a small positive constant, then each small strip σ_k^0 is defined by the relation

$$\{\|x_0 - x^+\| \leq \rho_0,\ \|y_k - y^-\| \leq \rho_0\}.$$

3.1. First-return mappings in the case $(1, 1)$**.** Here, the coordinates x and y are one-dimensional and $A = \lambda$ and $B = \gamma$. By (2.4) and (2.5), the first-return mapping $T_k \equiv T_1 T_0^k$ can be represented in the following form for all sufficiently large k and small ε:

$$\begin{aligned} \bar x_0 - x^+(\varepsilon) &= a\lambda^k x_0 + b_0(y_k - y^-) + O\left((y_k - y^-)^2 + |\lambda|^k |x_0||y_k - y^-| + \hat\lambda^k |x_0|\right), \\ \gamma^{-k} \bar y_k + \hat\gamma^{-k} O(|\bar x_0| + |\bar y_k|) &= \mu + c x_0 \lambda^k + D_0(y_k - y^-)^2 \\ &\quad + O\left((y_k - y^-)^3 + |\lambda|^k |x_0||y_k - y^-| + \hat\lambda^k |x_0|\right), \end{aligned} \tag{3.1}$$

where $b_0 \neq 0$, $c \neq 0$, and $D_0 \neq 0$. Note that here and in what follows, we choose $\hat\lambda$ sufficiently close to $|\lambda|$ (but always less than $|\lambda|$), so that, in particular, $\hat\lambda > \lambda^2$.

Shift the origin:

$$x = x_0 - x^+(\varepsilon) + \tilde\nu_k^1, \qquad y = y_k - y^- + \tilde\nu_k^2$$

in such a way that the first equation in (3.1) contains no free terms, i.e., the terms depending on ε only, and the second equation contains no linear term in y. This can always be done for $\tilde{\nu}_k^j = O(\lambda^k + \hat{\gamma}^{-k})$ chosen in an appropriate way. As a result, system (3.1) is rewritten in the form

$$\begin{gathered} \bar{x} = O(|\lambda|^k|x| + |y|), \\ \bar{y} + (\hat{\gamma}/\gamma)^{-k}O(|\bar{x}| + |\bar{y}|) = \gamma^k M_1 + \tilde{D}_0\gamma^k y^2 + \gamma^k O(|y|^3 + |\lambda|^k|x|), \end{gathered} \tag{3.2}$$

where

$$M_1 \equiv \mu - \gamma_1^{-k}y^-(1 + \dots) + c\lambda_1^k x^+(1 + \dots)$$

and $\tilde{D}_0 = D_0(1 + \beta_k)$; here $\beta_k = O(\lambda^k + \hat{\gamma}^{-k})$ is a certain small quantity.

We now normalize the coordinates as follows:

$$x = -\frac{\gamma^{-k}}{\rho^k}x_{\mathrm{new}}, \qquad y = -\frac{1}{\tilde{D}_0}\gamma^{-k}y_{\mathrm{new}}, \tag{3.3}$$

where ρ is a certain number from the interval

$$\max\left\{|\lambda\gamma|, |\gamma|^{-1}\right\} < \rho < 1. \tag{3.4}$$

Since $|\lambda\gamma| < 1$ and $|\gamma| > 1$, such ρ really exists and, moreover, the normalization factors in (3.3) are asymptotically small as $k \to \infty$. Hence, since the size of the small strip σ_k^0 is bounded away from zero in the coordinates (x_0, y_k), the range of the orthonormal coordinates (x, y) becomes infinitely large under the increase of k.

In the new coordinates, system (3.2) is rewritten as

$$\begin{gathered} \bar{x} = O(\rho^k|y| + |\lambda|^k|x|), \\ \bar{y} + (\hat{\gamma}/\gamma)^{-k}O(\rho^{-k}|\bar{x}| + |\bar{y}|) = -\tilde{D}_0\gamma^{2k}M_1 - y^2 + O\left(|\gamma|^{-k}|y|^3 + \frac{|\lambda\gamma|^k}{\rho^k}|x|\right). \end{gathered} \tag{3.5}$$

Now, by (3.4) and taking into account the inequalities $|\lambda\gamma| < 1$ and $\hat{\lambda} < |\lambda|$, we reduce system (3.5) to the desired form (1.2), where we set

$$M = -\tilde{D}_0\gamma_1^{2k}[\mu - \gamma_1^{-k}y^-(1 + \dots) + c\lambda_1^k x^+(1 + \dots)]. \tag{3.6}$$

Note that the parameter M, as well as the coordinates (x, y), can assume arbitrary finite values for large k.

3.2. First-return mapping in the case $(2,1)$**.** Here, $x = (x_1, x_2)$ is two-dimensional, y is one-dimensional, and

$$A \equiv \lambda\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \qquad B \equiv \gamma.$$

By (2.4) and (2.6), the first-return mapping $T_k \equiv T_1 T_0^k$ can be represented in the following form for each sufficiently large k and all small ε:

$$\begin{aligned} \bar{x}_{01} - x_1^+(\varepsilon) &= \lambda^k A_{11}(k\varphi)x_{01} + \lambda^k A_{12}(k\varphi)x_{02} + b_0(y_k - y^-) \\ &\qquad + O\left((y_k - y^-)^2 + \lambda^k\|x_0\||y_k - y^-| + \hat{\lambda}^k\|x_0\|\right), \\ \bar{x}_{02} - x_2^+(\varepsilon) &= \lambda^k A_{21}(k\varphi)x_{01} + \lambda^k A_{22}(k\varphi)x_{02} \\ &\qquad + O\left((y_k - y^-)^2 + \lambda^k\|x_0\||y_k - y^-| + \hat{\lambda}^k\|x_0\|\right), \\ \gamma^{-k}\bar{y}_k + \hat{\gamma}^{-k}\eta_k(\bar{x}_0, \bar{y}_k, \varepsilon) &= \mu + D_0(y_k - y^-)^2 \\ &\qquad + \lambda^k\left[x_{01}(c_1\cos k\varphi + c_2\sin k\varphi)x_{01} + (c_2\cos k\varphi - c_1\sin k\varphi)x_{02}\right] \\ &\qquad + O\left((y_k - y^-)^3 + \lambda^k\|x_0\||y_k - y^-| + \hat{\lambda}^k\|x_0\|\right), \end{aligned} \tag{3.7}$$

where

$$
\begin{aligned}
A_{11}(k\varphi) &= a_{11}\cos k\varphi - a_{12}\sin k\varphi, \qquad & A_{12}(k\varphi) &= a_{12}\cos k\varphi + a_{11}\sin k\varphi,\\
A_{21}(k\varphi) &= a_{21}\cos k\varphi + a_{22}\sin k\varphi, \qquad & A_{22}(k\varphi) &= a_{22}\cos k\varphi - a_{21}\sin k\varphi.
\end{aligned}
\tag{3.8}
$$

Shift the origin:

$$
x_1 = x_{01} - x_1^+(\varepsilon) + \tilde{\nu}_k^1, \qquad x_2 = x_{02} - x_2^+(\varepsilon) + \tilde{\nu}_k^2, \qquad y = y_k - y^-(\varepsilon) + \tilde{\nu}_k^3,
$$

in such a way that the first and second equations in (3.7) contain no free term and the third equation contains no term linear in y (here $\tilde{\nu}_k^i = O(\lambda^k + \hat{\gamma}^{-k})$). In this case, we also replace $\bar{x}_{01}$ and $\bar{x}_{02}$ in the terms independent of $\bar{y}$ in the left-hand side of the third equation in (3.7) by their expressions from the first and second equations. Then (3.7) is rewritten as

$$
\begin{aligned}
\bar{x}_1 &= \lambda^k A_{11}(k\varphi)x_1 + \lambda^k A_{12}(k\varphi)x_2 + b_0 y + O(y^2 + \lambda^k|y| + \hat{\lambda}^k\|x\|),\\
\bar{x}_2 &= \lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + O(y^2 + \lambda^k|y| + \hat{\lambda}^k\|x\|),\\
\bar{y} + (\hat{\gamma}/\gamma)^{-k}O(|\bar{y}|) &= \gamma^k M_1 + \tilde{D}_0\gamma^k y^2 + \lambda^k\gamma^k\Big[(c_1\cos k\varphi + c_2\sin k\varphi)x_1\\
&+(c_2\cos k\varphi - c_1\sin k\varphi)x_2 + \Big((\hat{\lambda}/\lambda)^k + \hat{\gamma}^{-k}\Big)O(\|x\|)\Big] + \gamma^k O\Big(|y|^3 + \lambda^k\|x\||y|\Big),
\end{aligned}
\tag{3.9}
$$

where

$$
M_1 \equiv \mu - \gamma^{-k}y^-(1+\ldots) + C_0\lambda^k\left(\cos(k\varphi + \vartheta_1) + \ldots\right)
$$

and

$$
C_0 = \sqrt{(c_1^2 + c_2^2)(x_1^{+2} + x_2^{+2})}, \qquad \sin\vartheta_1 = \frac{c_1x_2^+ - c_2x_1^+}{C_0}, \qquad \cos\vartheta_1 = \frac{c_1x_1^+ + c_2x_2^+}{C_0}, \tag{3.10}
$$

$\tilde{D}_0 = D_0(1+\beta_k)$, and $\beta_k = O(\lambda^k + \hat{\gamma}^{-k})$ is a certain small coefficient.

Consider the case $|\lambda\gamma| < 1$. As in the case $(1,1)$, we normalize the coordinates as follows:

$$
x = \frac{\gamma^{-k}}{\rho^k}x_{\text{new}}, \qquad y = -\tilde{D}_0^{-1}\gamma^{-k}y_{\text{new}},
$$

where ρ is a certain number from interval (3.4). In the new coordinates, system (3.9) becomes

$$
\begin{aligned}
&\bar{x}_1 = \rho^k O(y) + \lambda^k O(x), \qquad \bar{x}_2 = \rho^k\gamma^{-k}O(y^2) + \lambda^k O(x),\\
&\bar{y} + (\hat{\gamma}/\gamma)^{-k}O(\bar{y}) = M - y^2 + \frac{\lambda^k\gamma^k}{\rho^k}O(x) + |\gamma|^{-k}O(y^3),
\end{aligned}
\tag{3.11}
$$

where

$$
M \equiv -\tilde{D}_0\gamma^{2k}\Big[\mu - \gamma^{-k}y^-(1+\ldots) + \lambda^k(C_0\cos(k\varphi+\vartheta_1) + \ldots)\Big]. \tag{3.12}
$$

Therefore, by (3.4), after resolving the latter equation with respect to $\bar{y}$, mapping (3.11) reduces to the desired form (1.2).

Now we consider the case where $|\lambda\gamma| > 1$ (but, as above, $|\lambda^2\gamma| < 1$ here). We normalize the coordinates in (3.9) as follows:

$$
x_1 = -(b_0\tilde{D}_0^{-1})\gamma^{-k}x_{1\,\text{new}}, \qquad x_2 = -\rho^k(b_0\tilde{D}_0^{-1})\gamma^{-k}x_{2\,\text{new}}, \qquad y = -\tilde{D}_0^{-1}\gamma^{-k}y_{\text{new}},
$$

where ρ is a certain constant from the interval

$$
|\gamma|^{-1} < \lambda < \rho < |\lambda\gamma|^{-1}, \tag{3.13}
$$

which is nonempty since $1 > \dfrac{1}{|\lambda\gamma|} = \dfrac{\lambda}{|\lambda^2\gamma|} > \lambda$ by $|\lambda^2\gamma| < 1$.

In the new coordinates, system (3.9) becomes

$$\begin{aligned}
\bar{x}_1 &= y + \lambda^k O(\|x\| + |y|),\\
\bar{x}_2 &= \rho^{-k}\lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + \rho^{-k}\lambda^k O(y) + \rho^{-k}\hat{\lambda}^k O(x),\\
\bar{y} + (\hat{\gamma}/\gamma)^{-k} O(\bar{y}) &= M - y^2 + \lambda^k\gamma^k b_0\Big\{(c_1\cos k\varphi + c_2 \sin k\varphi + \nu_k^1)x_1\\
+\rho^k(c_2\cos k\varphi - c_1\cos k\varphi &+ \nu_k^2)x_2\Big\} + O\left(|\gamma|^{-k}|y|^3 + \lambda^k\|x\||y| + (\hat{\lambda}^k + \lambda^k\hat{\gamma}^{-k})\|x\|^2\right),
\end{aligned} \tag{3.14}$$

where, as above, formula (3.12) holds for the parameter M, ν_k^1 and ν_k^2 are certain small coefficients, and $\nu_k^{1,2} = O\left((\hat{\lambda}/\lambda)^k + \hat{\gamma}^{-k}\right)$. Resolve the third equation in (3.14) with respect to $\bar{y}$ and normalize the coordinate y: $y_{\text{new}} = y(1 + \hat{\beta}_k)$, where $\hat{\beta}_k = O\left((\hat{\gamma}/\gamma)^{-k}\right)$ is a certain small quantity, once again. Then (3.14) is rewritten as

$$\begin{aligned}
\bar{x}_1 &= y + \lambda^k O(\|x\| + |y|),\\
\bar{x}_2 &= \rho^{-k}\lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + \rho^{-k}\lambda^k O(|y|) + \rho^{-k}\hat{\lambda}^k O(\|x\|),\\
\bar{y} &= M - y^2 + \lambda^k\gamma^k b_0\Big\{(c_1\cos k\varphi + c_2 \sin k\varphi + \nu_k^1)x_1\\
+\rho^k(c_2\cos k\varphi - c_1\cos k\varphi &+ \nu_k^2)x_2\Big\} + O\left(|\gamma|^{-k}|y|^3 + \lambda^k\|x\||y| + \hat{\lambda}^k\|x\|^2\right),
\end{aligned} \tag{3.15}$$

where the new coefficients M and $\nu_k^{1,2}$ differ from the old coefficients by small quantities of order $O\left((\hat{\gamma}/\gamma)^{-k}\right)$; the coefficient b_0 remains the same.

Note that since $|\lambda\gamma| > 1$, the coefficient

$$B_k(\varphi) \equiv b_0\lambda^k\gamma^k(c_1\cos k\varphi + c_2\sin k\varphi + \nu_k^1) \tag{3.16}$$

in the third equation in (3.15) is no longer small. Nevertheless, $B_k(\varphi)$ can assume arbitrary finite values (for all large k) if the parameter φ varies near those values at which $c_1\cos k\varphi + c_2\sin k\varphi = 0$, i.e., near the values

$$\varphi = \frac{\vartheta_2}{k} \pm \frac{\pi}{2k} + 2\pi\frac{j}{k}, \qquad j \in Z, \tag{3.17}$$

where $\vartheta_2 \in [0, 2\pi)$ is such that

$$\cos\vartheta_2 = c_1/\sqrt{c_1^2 + c_2^2}, \qquad \sin\vartheta_2 = c_2/\sqrt{c_1^2 + c_2^2}.$$

Since $c \neq 0$ (by Lemma 2.3), values (3.17) of the angle φ densely fill in the interval $(0, \pi)$ for all possible k and j.

Denote $B = B_k(\varphi)$, thus stressing that along with M, B is one more control parameter. Note that the parameter M (see formula (3.12)) can also assume arbitrary finite values when μ varies near the value $\mu_k^0 = \gamma_1^{-k}y^- - C_0\lambda^k\cos(k\varphi_0 + \vartheta_1)$.

Introduce the new coordinate $y_{\text{new}} = y + \lambda^k O(\|x\| + |y|)$ such that $\bar{x}_1 = y$ in the new coordinates. Then by (3.13), mapping (3.15) assumes the desired form (1.3).

3.3. Proof of Lemma 1.2.

We continue the study of the case $(2, 1)$ for $|\lambda\gamma| > 1$ and $|\lambda^2\gamma| < 1$. Assume that $B \neq 0$ in (3.15). Since $\lambda^k\gamma^k\rho^k \to 0$ as $k \to \infty$, we can introduce the new coordinate

$$x_{1\,\text{new}} = x_1 + \frac{1}{B}b_0\lambda^k\gamma^k\rho^k(c_2\cos k\varphi - c_1\cos k\varphi + \nu_k^2)x_2.$$

Then (3.15) is rewritten as

$$\begin{aligned}\bar{x}_1 &= y + \frac{b_0 A_{21}(k\varphi)}{B}(c_2 \cos k\varphi - c_1 \sin k\varphi + \nu_k^2)\lambda^{2k}\gamma^k x_1 + O(\lambda^k),\\ \bar{x}_2 &= O\left(\frac{\lambda^k}{\rho^k}\right), \qquad \bar{y} = M - y^2 + Bx_1 + O(\lambda^k).\end{aligned} \tag{3.18}$$

Note that this mapping is an exponential contraction with respect to the coefficient x_2 (with the contraction coefficient $O(\lambda^k\rho^{-k})$); then in domains of the phase space where there is a contraction with respect to the coordinates x_1 and y, the contraction coefficient with respect to these variables is bounded away from zero for $B \neq 0$. Then [27, Theorem 4.4] implies that for any $Q, R > 0$ and for each sufficiently large k, mapping (3.18) has a C^{r-2}-smooth nonlocal asymptotically stable invariant central manifold $\mathcal{M}_k^c$ of the form $x_2(x_1, y, M, B) = O(\lambda^k\rho^{-k})$ in the domain $\|(x, y)\| \leq Q$ for $\|(M, B)\| \leq R$. The restriction of mapping (3.18) to $\mathcal{M}_k^c$ has the form

$$\begin{aligned}\bar{x}_1 &= y + \frac{b_0 A_{21}(k\varphi)}{B}(c_2 \cos k\varphi - c_1 \sin k\varphi + \nu_k^2)\lambda^{2k}\gamma^k x_1 + O(\lambda^k),\\ \bar{y} &= M - y^2 + Bx_1 + O(\lambda^k).\end{aligned} \tag{3.19}$$

In the domain where B is uniformly bounded, $|B| < Q$, we find from (3.16) that $c_1 \cos k\varphi + c_2 \sin k\varphi = O(\lambda^{-k}\gamma^{-k})$. Since $|\lambda\gamma| > 1$, we obtain

$$c_2 \cos k\varphi - c_1 \sin k\varphi = \pm\sqrt{c_1^2 + c_2^2} + \dots,$$

where the dots stand for the terms tending to zero as $k \to \infty$. Also (see (3.8)),

$$A_{21}(k\varphi) = a_{21} \cos k\varphi + a_{22} \sin k\varphi = \pm\frac{a_{21}c_2 - a_{22}c_1}{\sqrt{c_1^2 + c_2^2}} + \dots.$$

Therefore,

$$b_0 A_{21}(k\varphi)(c_2 \cos k\varphi - c_1 \sin k\varphi + \nu_k^2) = b_0(a_{21}c_2 - a_{22}c_1) + \dots.$$

It is easy to see from (2.6) that the constant $J_1 = b_0(a_{21}c_2 - a_{22}c_1)$ is exactly the Jacobian of the global mapping T_1 calculated at the point $(x = 0, y_1 = y^-)$ for $\varepsilon = 0$. Also, note that in this case, $\lambda^{2k}\gamma^k$ is the principal part of the Jacobian of the local mapping T_0^k. Denote $J_k = J_1\lambda^{2k}\gamma^k$. Then mapping (3.19) can be written in the form

$$\bar{x}_1 = y + \frac{J_k}{B}x_1 + o(J_k), \qquad \bar{y} = M - y^2 + Bx_1 + O(\lambda^k). \tag{3.20}$$

Make one more change of coordinates:

$$x_{1\,\text{new}} = x_1, \qquad y_{\text{new}} = y + \frac{J_k}{B}x_1 + o(J_k) \equiv \bar{x}_1.$$

Then (3.20) becomes

$$\bar{x}_1 = y, \qquad \bar{y} = M - y^2 + Bx_1 + \frac{J_k}{B}y + \frac{2J_k}{B}x_1 y + o(J_k). \tag{3.21}$$

By additional shifts of the coordinate y and the parameter M,

$$y_{\text{new}} = y - \frac{J_k}{2B}, \qquad M_{\text{new}} = M - \frac{J_k^2}{4B^2},$$

we reduce mapping (3.21) to the form (1.7). Lemma 1.2 is proved.

3.4. First-return mapping in the case $(1,2)$**.** Here x is one-dimensional, $y = (y_1, y_2)$ is two-dimensional, and

$$A \equiv \lambda, \qquad B \equiv \gamma \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

By (2.4) and (2.7), for any sufficiently large k and all small ε, the first-return mapping $T_k \equiv T_1 T_0^k$ can be written as follows:

$$\begin{aligned}
&\bar{x}_0 - x^+ = a\lambda^k x_0 + b_0(y_{k1} - y_1^-) + b_1\gamma^{-k}(\cos k\psi \cdot \bar{y}_{k2} + \sin k\psi \cdot \bar{y}_{k1}) \\
&\qquad + O\Big((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-|(|\lambda|^k|x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k|x_0| + \hat{\gamma}^{-k}(|\bar{x}_0| + \|\bar{y}_k\|)\Big), \\
&\gamma^{-k}(\cos k\psi \cdot \bar{y}_{k1} - \sin k\psi \cdot \bar{y}_{k2}) = \mu + c\lambda^k x_0 + D_0(y_{k1} - y_1^-)^2 \\
&\qquad + O\Big((y_{k1} - y_1^-)^3 + |y_{k1} - y_1^-|(|\lambda|^k|x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k|x_0| + \hat{\gamma}^{-k}(|\bar{x}_0| + \|\bar{y}_k\|)\Big), \\
&y_{k2} - y_2^- = e\lambda^k x_0 + d_1(y_{k1} - y_1^-) + d_2\gamma^{-k}(\cos k\psi \cdot \bar{y}_{k2} + \sin k\psi \cdot \bar{y}_{k1}) \\
&\qquad + O\Big((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-|(|\lambda|^k|x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k|x_0| + \hat{\gamma}^{-k}(|\bar{x}_0| + \|\bar{y}_k\|)\Big);
\end{aligned} \tag{3.22}$$

recall that $0 < \hat{\lambda} < |\lambda|$ and $\hat{\gamma} > \gamma$, and, moreover, it is assumed that $\hat{\lambda}$ and $\hat{\gamma}$ are sufficiently close to $|\lambda|$ and γ, respectively.

Shift the origin:

$$x_{\text{new}} = x_0 - x^+(\varepsilon) + \tilde{\nu}_k^1, \qquad y_{1\,\text{new}} = y_{k1} - y_1^-(\varepsilon) + \tilde{\nu}_k^2, \qquad y_{2\,\text{new}} = y_{k2} - y_2^-(\varepsilon) + \tilde{\nu}_k^3,$$

in such a way that the first and third equations in (3.22) contain no terms depending only on ε and the second equation contains no terms linear in $(y_1 - y_1^-)$. Here, $\tilde{\nu}_k^i(\varepsilon) = O(\gamma^{-k})$. If we additionally resolve the first equation with respect to $\bar{x}$ and substitute the corresponding expression in the right-hand sides of the remaining equations, then (3.22) is rewritten as follows:

$$\begin{aligned}
&\bar{x} = O\left(|y_1| + |\lambda|^k|x| + \gamma^{-k}\|\bar{y}\|\right), \\
&\gamma^{-k}\left\{\cos k\psi \cdot \bar{y}_1 - \sin k\psi \cdot \bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{y}\|)\right\} = M_1 + c\lambda^k\, x + \tilde{D}_0\, y_1^2 \\
&\qquad + O\Big(|y_1|^3 + |\lambda|^k|x||y_1| + \hat{\lambda}^k|x| + \gamma^{-k}\|\bar{y}\||y_1|\Big), \\
&y_2 - e\lambda^k\, x - \tilde{d}_1 y_1 = d_2\gamma^{-k}\left\{(\cos k\psi + \nu_k^1)\bar{y}_2 + (\sin k\psi + \nu_k^2)\bar{y}_1\right\} \\
&\qquad + O\Big(y_1^2 + |\lambda|^k|x||y_1| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\lambda}^k|x| + \hat{\gamma}^{-k}\|\bar{y}\|^2\Big),
\end{aligned} \tag{3.23}$$

where $\nu_k^{1,2} = O(\hat{\gamma}^{-k}\gamma^k)$ and the coefficients $\tilde{D}_0$ and $\tilde{d}_1$ differ from D_0 and d_1, respectively, by small quantities of order $O(\gamma^{-k})$. Also, we denote

$$M_1 \equiv \mu - \gamma^{-k}E_0\, \cos(k\psi - \vartheta_2 + \dots) + c\lambda^k(x^+ + \dots), \tag{3.24}$$

where

$$E_0 = \sqrt{(y_1^-)^2 + (y_2^-)^2}, \qquad \cos\vartheta_2 = y_1^-/E_0, \qquad \sin\vartheta_2 = y_2^-/E_0. \tag{3.25}$$

Make one more change of coordinates:

$$x_{\text{new}} = x, \qquad y_{1\,\text{new}} = y_1, \qquad y_{2\,\text{new}} = y_2 - \tilde{d}_1 y_1.$$

Then mapping (3.23) is rewritten as

$$\begin{aligned}
&\bar{x} = O\left(|y_1| + |\lambda|^k|x| + \gamma^{-k}\|\bar{y}\|\right),\\
&(\cos k\psi - d_1 \sin k\psi)\bar{y}_1 - \sin k\psi \cdot \bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{y}\|)\\
&\qquad = M_1\gamma^k + c\lambda^k\gamma^k x + \tilde{D}_0\gamma^k y_1^2 + O\left(\gamma^k|y_1|^3 + |\lambda|^k\gamma^k|x||y_1| + \hat{\lambda}^k\gamma^k|x| + \|\bar{y}\||y_1|\right),\\
&y_2 - e\lambda^k x = d_2\gamma^{-k}\left\{(\cos k\psi + \nu_k^1)\bar{y}_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^2)\bar{y}_1\right\}\\
&\qquad + O\left(y_1^2 + |\lambda|^k|x||y_1| + \hat{\lambda}^k|x| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\gamma}^{-k}\|\bar{y}\|^2\right)
\end{aligned} \tag{3.26}$$

with certain coefficients $\nu_k^{1,2} = O(\hat{\gamma}^{-k}\gamma^k)$.

Introduce the new coordinates y_1 and y_2 by the formulas

$$\begin{aligned}
y_{1\,\text{new}} &= (\cos k\psi + \nu_k^1)y_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^2)y_1,\\
y_{2\,\text{new}} &= \frac{1}{d_2}\gamma^k\left(y_2 - e\lambda^k x\right).
\end{aligned} \tag{3.27}$$

The old coordinates are expressed through the new coordinates by the formulas

$$\begin{aligned}
y_2 &= d_2\gamma^{-k}y_{2\,\text{new}} + e\lambda^k x,\\
y_1 &= \frac{1}{s_0}y_{1\,\text{new}} - \frac{1}{s_0}(\cos k\psi + \nu_k^1)(d_2\gamma^{-k}y_{2\,\text{new}} + e\lambda^k x),
\end{aligned} \tag{3.28}$$

where

$$s_0 \equiv s_0(k\psi) = \sin k\psi + d_1\cos k\psi + \nu_k^2. \tag{3.29}$$

Consider only those ψ for which $s_0 \neq 0$. Then the coordinate change (3.27) is nondegenerate and (3.26) is rewritten in the form

$$\begin{aligned}
&\bar{x} = O(|y_1| + |\lambda|^k|x| + \gamma^{-k}(|y_2| + |\bar{y}|)),\\
&\gamma^k\bar{y}_1(\cos k\psi - d_1\sin k\psi + \nu_k^3) - d_2\bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\bar{y}_2) + |\lambda\gamma|^k O(\bar{x})\\
&\qquad = \gamma^{2k}s_0 M_1 + cs_0\lambda^k\gamma^{2k}x + \tilde{D}_0(s_0)^{-1}\gamma^{2k}y_1^2\\
&\qquad\quad + \gamma^{2k}O\left(y_1^3 + |\lambda|^k|x||y_1| + \gamma^{-k}\|\bar{y}\|\|y\| + \hat{\lambda}^k|x| + \gamma^{-k}\|y\|^2 + \hat{\gamma}^{-k}\|\bar{y}\|^2)\right),\\
&y_2 = \bar{y}_1 + \gamma^k O\left(y_1^2 + |\lambda|^k|x||y_1| + \gamma^{-k}\|\bar{y}\|\|y\| + \gamma^{-k}\|\bar{y}\|^2 + \hat{\lambda}^k|x|\right),
\end{aligned} \tag{3.30}$$

where $\nu_k^3 = O(\hat{\gamma}^{-k}\gamma^k)$ is a certain small coefficient.

Now, normalize the coordinates as follows:

$$x = \rho^k\gamma^{-2k}x_{\text{new}}, \qquad y_1 = \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k}y_{1\,\text{new}}, \qquad y_2 = \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k}y_{2\,\text{new}}, \tag{3.31}$$

where

$$1 < \rho < \frac{1}{|\lambda|\gamma^2}$$

(recall that $|\lambda\gamma^2| < 1$ by condition, and we also assume that s_0 is bounded away from zero).

Then, after normalizations (3.31), mapping (3.30) in the new coordinates is written as follows:

$$\begin{aligned}
&\bar{x}_1 = \phi_k^1(x, y, \bar{y}),\\
&\frac{1}{d_2}\gamma^k\bar{y}_1(\cos k\psi - d_1\sin k\psi + \nu_k^3) - \bar{y}_2 = \tilde{M} + y_1^2 + \phi_k^2(x, y, \bar{y}),\\
&y_2 = \bar{y}_1 + \phi_k^3(x, y, \bar{y}),
\end{aligned} \tag{3.32}$$

where

$$\tilde{M} = \gamma^{4k}\frac{D_0}{d_2^2}\left[\mu - \gamma^{-k}(y_1^- \cos k\psi - y_2^- \sin k\psi + \dots) + c\lambda^k(x^+ + \dots)\right] \tag{3.33}$$

and $\phi_k^l = o(1)$ as $k \to \infty$.

We note that the trigonometrical term

$$C(k\psi) \equiv \frac{1}{d_2}\gamma^k(\cos k\psi - d_1 \sin k\psi + \nu_k^3)$$

in (3.32) is bounded for large k only if $\cos k\psi - d_1 \sin k\psi$ is close to zero, i.e., for the values of ψ close to

$$\psi = \frac{1}{k}\arctan\left(\frac{1}{d_1}\right) + \pi\frac{j}{k}, \quad j \in Z. \tag{3.34}$$

For these values of ψ, the coefficient s_0 in (3.29) is bounded away from zero: $s_0^2 = 1 + d_1^2 + \dots$.

Note that values (3.34) of the angle ψ are dense in $(0,\pi)$ for all possible j and k. This means that for any $Q > 0$, in any neighborhood of each point $\psi_0 \in (0,\pi)$, there exist intervals ϑ_k (of size of order γ^{-k}) of values of ψ such that the coefficient $C(k\psi)$ assumes all values from the interval $[-Q, Q]$ when ψ runs over ϑ_k.

In the range of ψ where C is finite, we can resolve system (3.32) with respect to $\bar{y}$. The mapping T_k becomes

$$\begin{aligned}
\bar{x}_1 &= \tilde{\phi}_k^1(x, y, M, C),\\
\bar{y}_2 &= M - y_1^2 + Cy_2 + \tilde{\phi}_k^2(x, y, M, C),\\
\bar{y}_1 &= y_2 + \tilde{\phi}_k^3(x, y, M, C),
\end{aligned} \tag{3.35}$$

where $M = -\tilde{M}, C = C(k\psi)$, and $\tilde{\phi}_k = o(1)$. Now setting $y_{2\,\mathrm{new}} = y_2 + \tilde{\phi}_k^3$, we obtain exactly mapping (1.4) of Lemma 1.1.

3.5. First-return mapping in the case (2, 2).

Here $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two-dimensional and

$$A \equiv \lambda\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}, \qquad B \equiv \gamma\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}.$$

By (2.4) and (2.8), for each sufficiently large k and all sufficiently small ε, the first-return mapping $T_k \equiv T_1T_0^k$ can be written as

$$\begin{aligned}
\bar{x}_{01} - x_1^+ &= b_0(y_{k1} - y_1^-) + b_{11}\gamma^{-k}(\cos k\psi\ \bar{y}_{k2} + \sin k\psi\ \bar{y}_{k1})\\
&\quad + O\left((y_{k1} - y_1^-)^2 + \gamma^{-k}|y_{k1} - y_1^-|\|\bar{y}_k\| + \lambda^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\right),\\[1ex]
\bar{x}_{02} - x_2^+ &= b_{12}\gamma^{-k}(\cos k\psi\ \bar{y}_{k2} + \sin k\psi\ \bar{y}_{k1})\\
&\quad + O\left((y_{k1} - y_1^-)^2 + \gamma^{-k}|y_{k1} - y_1^-|\|\bar{y}_k\| + \lambda^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\right),\\[1ex]
&\gamma^{-k}(\cos k\psi\cdot\bar{y}_{k1} - \sin k\psi\cdot\bar{y}_{k2})\\
&\quad = \mu + \lambda^k C_1(k\varphi)x_{01} + \lambda^k C_2(k\varphi)x_{02} + D_0(y_{k1} - y_1^-)^2\\
&\quad + O\Big((y_{k1} - y_1^-)^3 + |y_{k1} - y_1^-|(\lambda^k\|x_0\| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\Big),\\[1ex]
y_{k2} - y_2^- &= \lambda^k E_1(k\varphi)x_{01} + \lambda^k E_2(k\varphi)x_{02} + d_1(y_{k1} - y_1^-)\\
&\quad + d_2\gamma^{-k}(\cos k\psi\cdot\bar{y}_{k2} + \sin k\psi\cdot\bar{y}_{k1})\\
&\quad + O\Big((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-|(\lambda^k\|x_0\| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\Big),
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
C_1 &= c_1\cos k\varphi + c_2\sin k\varphi, \qquad & C_2 &= c_2\cos k\varphi - c_1\sin k\varphi,\\
E_1 &= e_1\cos k\varphi + e_2\sin k\varphi, \qquad & E_2 &= e_2\cos k\varphi - e_1\sin k\varphi,
\end{aligned} \tag{3.37}$$

and, as above, $0 < \hat{\lambda} < \lambda$ and $\hat{\gamma} > \gamma$.

Introduce the following new coordinates (shifts of coordinates):

$$x_{1\,\text{new}} = x_1 - x_1^+(\varepsilon) + \tilde{\nu}_k^1, \qquad x_{2\,\text{new}} = x_2 - x_2^+(\varepsilon) + \tilde{\nu}_k^2,$$
$$y_{1\,\text{new}} = y_{k1} - y_1^- + \tilde{\nu}_k^3, \qquad y_{2\,\text{new}} = y_{k2} - y_2^- + \tilde{\nu}_k^4.$$

Here the small shifts $\tilde{\nu}_k^i(\varepsilon)$ (of order $O(|\gamma|^{-k})$) are chosen so that the first, second, and fourth equations in (3.36) contain no free term and the third equation contains no term linear in y_1. In this case, we resolve the first and second equations with respect to $\bar{x}$ and substitute the obtained expression into the third and fourth equations. As a result, system (3.36) is rewritten as

$$\begin{aligned}
&\bar{x}_1 = b_0 y_1 + O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + |\bar{y}|)\right),\\
&\bar{x}_2 = O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + |\bar{y}|)\right),\\
&\gamma^{-k}\left\{\cos k\psi\cdot\bar{y}_1 - \sin k\psi\cdot\bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{y}\|)\right\}\\
&\qquad = M_1 + C_1\lambda^k x_1 + C_2\lambda^k x_2 + \tilde{D}_0\, y_1^2 + O\Big(|y_1|^3 + \lambda^k\|x\||y_1| + \hat{\lambda}^k\|x\| + \gamma^{-k}\|\bar{y}\||y_1|\Big),\\
&y_2 - \tilde{d}_1 y_1 - E_1\lambda^k x_1 - E_2\lambda^k x_2 = d_2\gamma^{-k}\left\{(\cos k\psi + \nu_k^1)\bar{y}_2 + (\sin k\psi + \nu_k^2)\bar{y}_1\right\}\\
&\qquad + O\Big(y_1^2 + \lambda^k\|x\||y_1| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\lambda}^k\|x\| + \hat{\gamma}^{-k}\|\bar{y}\|^2\Big),
\end{aligned}\tag{3.38}$$

where $\nu_k^{1,2} = O(\hat{\gamma}^{-k}\gamma^k)$ and the coefficients $\tilde{D}_0$ and $\tilde{d}_1$ differ from D_0 and d_1, respectively, by certain small quantities of order $O(\gamma^{-k})$. Also, we have denoted

$$M_1 \equiv \mu - \gamma^{-k}E_0\cos(k\psi + \vartheta_2 + \dots) + \lambda^k C_0\,\cos(k\varphi - \vartheta_1 + \dots)\tag{3.39}$$

(see formulas (3.25) and (3.10)).

Introduce the new coordinate y_2 by the formula $y_{2\,\text{new}} = y_2 - \tilde{d}_1 y_1$. Then mapping (3.38) is rewritten in the form

$$\begin{aligned}
&\bar{x}_1 = b_0 y_1 + O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right),\\
&\bar{x}_2 = O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right),\\
&(\cos k\psi - \tilde{d}_1\sin k\psi)\bar{y}_1 - \sin k\psi\cdot\bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{y}\|)\\
&\qquad = M_1\gamma^k + C_1\lambda^k\gamma^k x_1 + C_2\lambda^k\gamma^k x_2 + \tilde{D}_0\gamma^k y_1^2 + O\Big(\gamma^k|y_1|^3 + \lambda^k\gamma^k\|x\||y_1| + \hat{\lambda}^k\gamma^k\|x\| + \|\bar{y}\||y_1|\Big),\\
&y_2 - \lambda^k E_1 x_1 - \lambda^k E_2 x_2 = d_2\gamma^{-k}\left\{(\cos k\psi + \nu_k^3)\bar{y}_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^4)\bar{y}_1\right\}\\
&\qquad + O\Big(y_1^2 + \lambda^k\|x\||y_1| + \hat{\lambda}^k|x| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\gamma}^{-k}\|\bar{y}\|^2\Big),
\end{aligned}\tag{3.40}$$

where $\nu_k^{3,4} = O(\hat{\gamma}^{-k}\gamma^k)$. Introduce the new coordinates y by the formulas

$$\begin{aligned}
y_{1\,\text{new}} &= (\cos k\psi + \nu_k^3)y_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^4)y_1,\\
y_{2\,\text{new}} &= \gamma^k\frac{1}{d_2}\Big(y_2 - E_1\lambda^k x_1 - E_2\lambda^k x_2\Big).
\end{aligned}\tag{3.41}$$

For the old coordinates (y_1, y_2), we have

$$\begin{aligned}
y_2 &= \gamma^{-k}d_2 y_{2\,\text{new}} + E_1\lambda^k x_1 + E_2\lambda^k x_2,\\
y_1 &= \frac{1}{s_0}y_{1\,\text{new}} - \frac{d_1}{s_0}(d_2\cos k\psi + \nu_k^3)(\gamma^{-k}y_{2\,\text{new}} + E_1\lambda^k x_1 + E_2\lambda^k x_2),
\end{aligned}\tag{3.42}$$

where

$$s_0 \equiv s_0(k\psi) = \sin k\psi + d_1 \cos k\psi + \nu_k^4. \tag{3.43}$$

Also consider those ψ for which s_0 is uniformly bounded away from zero. Then (3.40) is rewritten in the form

$$\begin{aligned}
&\bar{x}_1 = \tfrac{b_0}{s_0} y_1 + O\left(y_1^2 + \lambda^k \|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right), \\
&\bar{x}_2 = O\left(y_1^2 + \lambda^k \|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right), \\
&\frac{1}{d_2}\gamma^k \bar{y}_1(\cos k\psi - d_1 \sin k\psi + \nu_k^5 + O(\bar{y}_1)) - \bar{y}_2(1 + \nu_k^6 + O(\bar{y}_2)) + (\lambda\gamma)^k O(\bar{x}) \\
&\qquad = \gamma^{2k} s_0 M_1 + \tilde{D}_0(s_0)^{-1}\gamma^{2k} y_1^2 + \tilde{C}_1 s_0 \lambda^k \gamma^{2k} x_1 + \tilde{C}_2 s_0 \lambda^k \gamma^{2k} x_2 \\
&\qquad + \gamma^{2k} O\left(|y_1|^3 + \lambda^k \|x\| |y_1| + \gamma^{-k}(\|\bar{y}\|\|y\| + \|y\|^2) + \hat{\lambda}^k \|x\|^2 + \hat{\gamma}^{-k}\|\bar{y}\|^2\right), \\
&y_2 = \bar{y}_1 + O\left(\gamma^k \|y\|^2 + \lambda^k \gamma^k \|x\| \|y\| + \|\bar{y}\|\|y\| + \|\bar{y}\|^2 + \hat{\lambda}^k \gamma^k \|x\|\right),
\end{aligned} \tag{3.44}$$

where $\nu_k^{5,6} = O(\hat{\gamma}^{-k}\gamma^k)$ and the coefficients $\tilde{C}_1$ and $\tilde{C}_2$ differ from C_1 and C_2, respectively, by quantities of order $O(\hat{\lambda}^k \lambda^{-k})$.

Consider the case $\lambda\gamma^2 < 1$. Normalize the coordinates in (3.44) as follows:

$$\begin{aligned}
x_1 &= \rho^{-k}\frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} x_{1\,\text{new}}, & x_2 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} x_{2\,\text{new}}, \\
y_1 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} y_{1\,\text{new}}, & y_2 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} y_{2\,\text{new}},
\end{aligned} \tag{3.45}$$

where ρ is a certain number such that $\lambda\gamma^2 < \rho < 1$.

Note that since the normalization coefficients in (3.45) are asymptotically small, the domains of the new coordinates (x, y) grow when k increases, and in the limit as $k \to \infty$, they cover all finite values. This allows us to assume that our mapping is defined on the domain $\|(x_{\text{new}}, y_{\text{new}})\| \le Q$ for a certain $Q > 0$, and, moreover, the constant Q can be arbitrarily large. In this case, mapping (3.44) is rewritten in the following form in coordinates (3.45) for sufficiently large k:

$$\begin{aligned}
&\bar{x}_1 = \rho^k O(y_1) + \gamma^{-k} O(\|(x, y, \bar{y})\|), \\
&\bar{x}_2 = \gamma^{-k} O(\|(x, y, \bar{y})\|), \\
&\frac{1}{d_2}\gamma^k C(k\psi)\bar{y}_1 - \bar{y}_2 \;=\; \tilde{M} + y_1^2 + \left(\frac{\lambda^k \gamma^{2k}}{\rho^k} + \gamma^{-k}\right) O(\|(x, y, \bar{y})\|), \\
&y_2 = \bar{y}_1 + \gamma^{-k} O(\|(x, y, \bar{y})\|).
\end{aligned} \tag{3.46}$$

where

$$\tilde{M} = \gamma^{4k}\frac{\tilde{D}_0}{d_2^2} M_1, \tag{3.47}$$

formula (3.39) holds for M_1, and

$$C(k\psi) = \cos k\psi - d_1 \sin k\psi + \nu_k^5. \tag{3.48}$$

Note that the coefficients $\tilde{M}$ and $C = d_2^{-1} C(k\psi)\gamma^k$ can obviously assume arbitrary finite values for large k under the variation of the initial parameters μ and ψ.

In this case, C is uniformly bounded only in the case where $\cos k\psi - d_1 \sin k\psi$ is asymptotically close to zero, i.e., for values of ψ close to those given by formulas (3.34). As we have already noted, for these ψ, the quantity s_0 from (3.43) is uniformly bounded away from zero: $|s_0| = \sqrt{1 + d_1^2}(1 + \dots)$. In what follows, we will consider only these ψ.

As a result, in any bounded domain of values of (x, y, M, C), mapping (3.44) can be rewritten in the form

$$\begin{aligned} \bar{x}_1 &= o(1), & \bar{x}_2 &= o(1), \\ \bar{y}_1 &= y_2 + o(1), & \bar{y}_2 &= -\tilde{M} + Cy_2 - y_1^2 + o(1), \end{aligned} \tag{3.49}$$

where by $o(1)$ we have denoted the functions of all coordinates and parameters which tend to zero uniformly in any bounded domains of values of (x, y, M, C) as $k \to \infty$ together with all derivatives up to the order $(r-2)$ in coordinates and up to the order $(r-3)$ in parameters. If we set $M = -\tilde{M}$ in (3.49), then we obtain exactly the desired mapping (1.4).

Now, let us consider the case $\lambda\gamma^2 > 1$ (as above, $\lambda\gamma < 1$). Now normalize the coordinates in (3.44) as follows:

$$\begin{aligned} x_1 &= \frac{d_2 b_0}{\tilde{D}_0}\gamma^{-2k} x_{1\,\mathrm{new}}, & x_2 &= q^k \gamma^{-2k} x_{2\,\mathrm{new}}, \\ y_1 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} y_{1\,\mathrm{new}}, & y_2 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} y_{2\,\mathrm{new}}, \end{aligned} \tag{3.50}$$

where q is a certain number from the interval $q \in \left(\gamma^{-1}, (\lambda\gamma^2)^{-1}\right)$. This interval is nonempty and lies in $(0, 1)$ since

$$1 > \frac{1}{\lambda\gamma^2} = \frac{\gamma^{-1}}{\lambda\gamma} > \gamma^{-1}.$$

Now mapping (3.44) is written in coordinates (3.50) in the following form:

$$\begin{aligned} &\bar{x}_1 = y_1 + \gamma^{-k} O(\|(x, y, \bar{y})\|), \\ &\bar{x}_2 = \frac{\gamma^{-k}}{q^k} O(\|(x, y, \bar{y})\|), \\ &\frac{1}{d_2}\gamma^k C(k\psi)\bar{y}_1 - \bar{y}_2 = M + y_1^2 + \frac{b_0}{d_2}\lambda^k\gamma^{2k}\left(c_{11}\cos k\varphi + c_{12}\sin\varphi + l_k\right) x_1 \\ &\qquad + \lambda^k\gamma^{2k} q^k O(x_2) + \gamma^{-k} O(\|(x, y, \bar{y})\|), \\ &y_2 = \bar{y}_1 + \gamma^{-k} O(\|(x, y, \bar{y})\|), \end{aligned} \tag{3.51}$$

where $l_k = O((\hat{\lambda}/\lambda)^k)$ is a certain small parameter and M and $C(k\psi)$ are defined by the formulas (3.47) and (3.48).

As compared with (3.46), in mapping (3.51), along with M and $C = \gamma^k C(k\psi)$, there arises an additional independent parameter

$$B = B(k\varphi) \equiv \frac{b_0}{d_2}\lambda^k\gamma^{2k}(c_{11}\cos k\varphi + c_{12}\sin\varphi + l_k).$$

Since $\lambda\gamma^2 > 1$, the coefficient $B(k\varphi)$ is no longer small (as in the case $\lambda\gamma^2 < 1$), and under a variation of φ, it can assume any finite values for large k. The values of φ near

$$\varphi = -\frac{1}{k}\arctan\left(\frac{c_1}{c_2}\right) + \pi\frac{j}{k}, \qquad j \in Z, \tag{3.52}$$

correspond to bounded values of B. In the domain of bounded values of (x, y, M, B, C), mapping (3.51) can be written in the form

$$\begin{aligned} \bar{x}_1 &= y_1 + o(1), & \bar{x}_2 &= o(1), \\ \bar{y}_2 &= -M - Bx_1 + Cy_1 - y_1^2 + o(1), & \bar{y}_1 &= y_2 + o(1). \end{aligned}$$

If we change the signs of M and B, then we obtain mapping (1.5).

Therefore, the rescaling lemma is proved.

4. Proof of the Main Theorems

The proof of Theorems 1.1–1.4 is based on the rescaling lemmas. These lemmas allow us to carry out a comparatively simple analysis of the first-return mappings $T_k(\varepsilon)$, which assume the form of standard quadratic mappings for $\varepsilon \in \Delta_k$. It is convenient to prove Theorems 1.1–1.4 in the following order: we first prove Theorem 1.2, then Theorem 1.4 (for which, in fact, only the linear analysis of fixed points of the first-return mapping is needed), and then prove Theorem 1.3. Theorem 1.1 is deduced in proving Theorems 1.2 and 1.4.

4.1. Proof of Theorems 1.2 and items 1 and 2 of Theorem 1.1. We first carry out the analysis of fixed points of the first-return mappings (1.2)–(1.5) and (1.7) in order to find the values of the parameters M, B, and C under which the above mappings have fixed points with multipliers on the unit circle.

4.1.1. Mapping (1.2). Consider the one-dimensional mapping of the parabola

$$\bar{y} = M - y^2.$$

Let $\nu_1 \neq 0$ be a multiplier of a certain fixed point of it. Then the coordinate y of this fixed point satisfies the equations $M = y + y^2$ and $2y = -\nu_1$, and we obtain then that the mapping of the parabola has a fixed point with this multiplier ν_1 for

$$M = \frac{\nu_1^2}{4} - \frac{\nu_1}{2}. \tag{4.1}$$

Since mapping (1.2) is closed to the mapping of the parabola together with sufficiently many derivatives, it also has a fixed point with the multiplier ν_1 for $M = M_k(\nu_1)$ that is asymptotically close to the value of (4.1) as $k \to \infty$. Other multipliers of the fixed point (one in the case (1,1) and two in the case (2,1)) have modules always less than 1: they tend to zero as $k \to \infty$.

4.1.2. Mapping (1.3). Consider the Henon mapping (a limit mapping for (1.3)):

$$\bar{x} = y, \quad \bar{y} = M + Bx - y^2.$$

Let ν_1 and ν_2 be multipliers of a certain fixed point of it (then they are both real or compose a complex-conjugate pair of numbers; also, we set $\nu_1\nu_2 \neq 0$). The coordinates $x = y$ of this fixed point satisfy the equation $M = y(1 - B) + y^2$. In this case, the characteristic equation has the form $\nu^2 + 2y\nu - B = 0$. Then it is easily found that

$$B(\nu_1, \nu_2) = -\nu_1\nu_2, \quad M(\nu_1, \nu_2) = \frac{\nu_1 + \nu_2}{4}(\nu_1 + \nu_2 - 2\nu_1\nu_2 - 2). \tag{4.2}$$

Clearly, mapping (1.3) also has a fixed point with these multipliers ν_1 and ν_2 for values of M and B asymptotically close to those which are given by formula (4.2). The module of the third multiplier is always less than 1 (it tends to zero as $k \to \infty$).

4.1.3. Mapping (1.4). Consider the following mapping (limit for (1.4)):

$$\bar{y}_1 = y_2, \quad \bar{y}_2 = M + Cy_2 - y_1^2.$$

Let ν_1 and ν_2 be the multipliers of a certain fixed point of it (then they either are both real or compose a complex-conjugate pair again; again, we assume that $\nu_1\nu_2 \neq 0$). The coordinates $y_1 = y_2 = y$ of this fixed point satisfy the equation $M = y(1-C)+y^2$ and the characteristic equation has the form $\nu^2 - C\nu + 2y = 0$. Then it is easily found that

$$C = \nu_1 + \nu_2, \quad M = \frac{\nu_1\nu_2}{2}(1 - C) + \frac{(\nu_1\nu_2)^2}{4}. \tag{4.3}$$

Mapping (1.4) also has a fixed point with these multipliers ν_1 and ν_2 for M and C asymptotically close to those given by formula (4.3). The modules of other multipliers (of the third in the case (1,2) and the fourth in the case (2,2)) are always less than 1.

4.1.4. Mapping (1.5). Consider the following three-dimensional mapping (limit for (1.5)):

$$\bar{x} = y_1, \quad \bar{y}_1 = y_2, \quad \bar{y}_2 = M + Bx + Cy_2 - y_1^2. \tag{4.4}$$

Let ν_1, ν_2, and ν_3 be nonzero multipliers of a certain fixed point of it (then either they all are real or one of them is real and the other two form a complex-conjugate pair of numbers). The coordinates $x = y_1 = y_2$ of this fixed point satisfy the equation $M = x(1 - B - C) + x^2$ and the characteristic equation has the form $-\nu^3 + C\nu^2 - 2x\nu + B = 0$. Then it is easily found that

$$\begin{gathered} B = \nu_1\nu_2\nu_3, \qquad C = \nu_1 + \nu_2 + \nu_3, \\ M = (\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)(1 - B - C) + \frac{(\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)^2}{4}. \end{gathered} \tag{4.5}$$

Obviously, the initial mapping (1.5) also has a fixed point with these multipliers ν_1, ν_2, and ν_3 for values of the parameters M, B, and C asymptotically close to those which are given by formula (4.5). The module of one more multiplier of this point is always less than 1 for large k.

Therefore, for any tuple $\{\nu_1, \dots, \nu_{d_e}\}$ of d_e multipliers given in advance (where $d_e = 1$ in the case of mapping (1.2), $d_e = 2$ for mappings (1.3) and (1.4), and $d_e = 3$ for mapping (1.5)), for each of the mappings (1.2)–(1.5) there are values of the parameters $M = M_k$, $B = B_k$, and $C = C_k$ for which there exists a fixed point whose d_e multipliers are exactly equal to $\nu_1, \dots, \nu_{d_e}$. Note that in this case, the values M_k, B_k, and C_k of the parameters mentioned above are uniformly bounded in k. By (1.6), this implies that for the corresponding values of the initial parameters, $(\mu, \varphi, \psi) = (\mu_k, \varphi_k, \psi_k)$ as $k \to +\infty$, we have that first, $\mu_k \to 0$, and, second, in the case $d_e \geq 2$, we can always find a subsequence of (φ_k, ψ_k) converging to the point (φ_0, ψ_0), where φ_0 and ψ_0 are values of the angular arguments of the complex multipliers for the point O of the diffeomorphism f_0. Therefore, we obtain the following assertion.

Corollary 4.1. *For any tuple $\{\nu_1, \dots, \nu_{d_e}\}$ of multipliers given in advance, there exists a sequence $\varepsilon_k \to 0$ of values of the parameters ε such that for $\varepsilon = \varepsilon_k$, the diffeomorphism f_ε has a one-time going-around periodic trajectory, d_e multipliers of which are exactly equal to $\nu_1, \dots, \nu_{d_e}$, and other multipliers of which strictly lie inside the unit disk.*

Theorem 1.2 immediately follows from this assertion. Indeed, in the Newhouse domain δ_j, near any $\varepsilon \in \delta_j$, there exist values of the parameters corresponding to homoclinic tangencies to the point O for which Conditions A–D hold; since we have just proved that arbitrarily small perturbations in the framework of the same family f_ε yield periodic trajectories (one-time going around with respect to the repeated homoclinic tangencies) with any tuple of d_e multipliers given in advance on the unit circle in a complete correspondence with Theorem 1.2.

Note that in the case of Theorem 1.1, we have $d_e = 1$, because we speak about the periodic trajectories having either the multiplier $\nu_1 = +1$ or $\nu_1 = -1$. Corollary 4.1 immediately yields items 1 and 2 of Theorem 1.1 for the Newhouse intervals δ_j.

4.2. Proof of Theorem 1.4 and item 3 of Theorem 1.1. We again use Corollary 4.1 but for rough periodic trajectories such that the module of each of their multipliers $\nu_1, \dots, \nu_{d_e}$ is not equal to 1. Then we obtain $d_e + 1$ different types of rough trajectories in totality in accordance with how many of these d_e multipliers lie inside the unit disk: $0, 1, \dots,$ or all d_e trajectories. The first case corresponds to a stable periodic trajectory.

Recall that arbitrarily close to any value of the parameters from the Newhouse domain δ_j, there exists a value of ε for which the point O has a trajectory of a simple homoclinic tangency. According to Corollary 4.1, arbitrarily close to this value of ε, there exists a value of the parameter for which f_ε has a rough periodic trajectory exactly with d multipliers outside the unit disk for any $d = 0, \dots, d_e$ given in advance. Since this trajectory is rough, it exists in a certain domain of the range of parameters. Repeating the arguments, inside this domain, we find a smaller one that controls the existence of one more rough periodic trajectory with d multipliers outside the unit disk with the same d or any other d from 0 up to d_e, and so on. Repeating this procedure infinitely many times for each $d = 0, \dots, d_e$, we obtain a sequence of embedded domains such that the values of ε in the intersection of these domains correspond to the existence of infinitely many rough periodic trajectories with all possible tuples of $0, 1, \dots, d_e$ multipliers lying outside the unit disk. By construction, the obtained set of values of ε is the intersection of countably many open and dense sets in δ_j, i.e., sets of the second category. The theorem is proved.

4.3. Proof of Theorem 1.3. As in the proof of Theorem 1.4, it suffices to show that the first-return mappings $T^{(k)}$ have a stable closed invariant curve in certain domains of the parameters (M, B), (M, C), or (M, C, B). To obtain countably many closed invariant curves, we apply the construction with embedded domains, which is the same as in the proof of Theorem 1.4.

We first consider the case of saddle-focus (1,2) and saddle-focus (2,2) with $\lambda\gamma^2 < 1$. In this case, according to Lemma 1.1, the mapping $T^{(k)}$ reduces to the form

$$\bar{x} = o(1), \qquad \bar{y}_1 = y_2, \qquad \bar{y}_2 = M + Cy_2 - y_1^2 + o(1). \tag{4.6}$$

The limit mapping

$$\bar{y}_1 = y_2, \qquad \bar{y}_2 = M + Cy_2 - y_1^2 \tag{4.7}$$

has a fixed point with the multiplier $\nu_{1,2} = e^{\pm i\omega}$ for values of the parameters (M, C) on the curve $L : \{M = \frac{3}{4} - \frac{1}{2}C,\ C = 2\cos\omega\}$ (i.e., $|C| < 2$; see formula (4.3)). For $\omega \neq \pi/2$ and $2\pi/3$, the stability of a closed invariant curve that appears under bifurcations of such a fixed point is determined by the sign of the first Lyapunov quantity. Recall that the Lyapunov quantity G_1 is the coefficient of the cubic term of the normal form $\bar{\rho} = \rho + G_1\rho^3 + o(\rho^3)$, $\bar{\theta} = \theta + \omega + O(r^2)$ of the mapping written in polar coordinates (ρ, θ) near the fixed point. For mapping (4.7), it is easy to calculate that $G_1 = -1 - \dfrac{1}{2(1 - \cos\omega)}$, so that the Lyapunov quantity is always negative here. Since G_1 is the coefficient of the cubic term, the Lyapunov quantity remains negative for all mappings C^3-close to (4.7).

Now we consider mapping (4.6). For each sufficiently large k, it also has a curve in the parameter space near the curve L controlling the existence of a fixed point with two multipliers $e^{\pm i\omega}$ (the absolute values of other multipliers are less than 1). Denote this curve by L_k. The restriction of mapping (1.4) to the central manifold near the fixed point is C^{r-2}-close to (4.7). Since $r \geq 5$, we obtain that the corresponding Lyapunov quantity for mapping (1.4) is negative, and hence, in passing the values of the parameters through L_k, a closed stable invariant curve arises; it exists in a certain domain of parameters; this is what was required to be proved.

In the case of saddle-focus (2,2) with $\lambda\gamma^2 > 1$, the first return mapping $T^{(k)}$ reduces to the form

$$\bar{x}_2 = o(1), \qquad \bar{x}_1 = y_1, \qquad \bar{y}_1 = y_2, \qquad \bar{y}_2 = M + Cy_2 + Bx_1 - y_1^2 + o(1). \tag{4.8}$$

Here, for small B, the Lyapunov quantity is also negative for the fixed point having the multipliers $\nu_{1,2} = e^{\pm i\omega}$, $\nu_3 = B + o(1)$, and $\nu_4 = o(1)$. This is directly implied by the fact that for $B = 0$, mapping (4.8) degenerates as $k \to +\infty$ into mapping (4.7) with respect to the coordinates y_1 and y_2, while the negativity of the Lyapunov quantity for the latter mapping has just been proved. Therefore, in this case, we also obtain that for each sufficiently large k and for values of the parameters from a certain domain, the first-return mapping has a stable closed invariant curve.

In the case of saddle-focus (2,1) with $d_e = 2$, i.e., for $\lambda\gamma > 1$, to find stable invariant curves, we will use the refined form of the first-return mapping deduced in Lemma 1.2. This is the so-called generalized Henon mapping of the form

$$\bar{x}_1 = y, \qquad \bar{y} = M - y^2 + Bx_1 + Q_k x_1 y + o(Q_k), \tag{4.9}$$

where $Q_k \neq 0$ and $Q_k \to 0$ as $k \to +\infty$. This mapping was studied in [6, 7], where, in particular, it was shown that for values of the parameters (M, B) in a certain domain, mapping (4.9) has a stable closed invariant curve for sufficiently large k. Precisely such a domain arises adjoining the point $(M = M_k^*, B = B_k^*)$, where

$$M_k^* = 3 - Q_k + o(Q_k), \qquad B_k^* = -1 + Q_k/2 + o(Q_k),$$

at which mapping (4.9) has a fixed point with multipliers $(-1, -1)$.

Therefore, in all cases with $d_e \geq 2$, the family f_ε has a countable sequence of domains $\tilde{\Delta}_k \subset \Delta_k$ accumulated to $\varepsilon = 0$ as $k \to \infty$ such that for $\varepsilon \in \tilde{\Delta}_k$, the diffeomorphism f_ε has a stable closed invariant curve. To obtain finitely many closed invariant curves for a dense set of values of the parameters from the Newhouse domains δ_j, it suffices to apply the construction with embedded domains from the proof of Theorem 1.4. This completes the proof of Theorem 1.3.

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