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HOMOCLINIC TANGENCIES OF AN ARBITRARY ORDER IN NEWHOUSE DOMAINS

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1. Introduction

This paper is devoted to the study of complex and unexpected phenomena that are observed in two-dimensional mappings (or in three-dimensional flows) with homoclinic tangencies. In particular, we show that, in the C^r-topology, for an arbitrary finite r, in any neighborhood of a system with a quadratic homoclinic tangency there are nonrough systems with homoclinic tangencies of arbitrarily high orders, i.e., systems of an arbitrarily high codimension. Such phenomena were not observed in bifurcation theory previously.

The study of systems with homoclinic tangencies was initiated in [4]. First and foremost, three classes of such systems were distinguished in that paper. Namely, let L be a saddle periodic motion, and let Γ be a homoclinic trajectory along which the stable and unstable invariant manifolds of L are quadratically tangent to each other (see Fig. 1). Let λ and γ be multipliers of L, $|\lambda| < 1$, and let $|\gamma| > 1$. Assume that $|\lambda\gamma| \neq 1$; moreover, without loss of generality, we can assume that $|\lambda\gamma| < 1$. Let U be a small neighborhood of the closure $\Gamma \cup L$ of the homoclinic trajectory, and let N be the set of all trajectories that lie entirely in U. Depending on the signs of multipliers and on the signs of certain coefficients that characterize the way in which the stable and unstable manifolds adjoin to Γ, the systems with homoclinic tangencies fall into one of the following three classes:

(1) for systems of the first class, the set N is trivial: $N = \{L, \Gamma\}$;

(2) for systems of the second class, N is a nontrivial, nonuniformly hyperbolic set that admits a complete description in the language of symbolic dynamics (via some quotient system of the topological Bernoulli scheme consisting of three symbols);

(3) for systems of the third class, N still contains nontrivial, hyperbolic subsets, but, generally speaking, the set N is not exhausted by them; moreover, the everywhere dense nonroughness takes place on bifurcation films of systems of the third class. (In [10, 11], a similar classification was carried out for the multidimensional case, including the case of systems with homoclinic tangencies of an arbitrary finite order.)

To be more specific, according to [4], systems that have nonrough periodic motions are dense in any one-parameter family of systems with homoclinic tangencies of the third class in which the quantity

$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|} \tag{1.1}$$

varies monotonocally. Subsequently, it was shown in [10, 12] that systems with countable sets of stable periodic motions (for $\theta > 1$; if $\theta < 1$, then systems with countable sets of unstable periodic motions) and systems with secondary homoclinic tangencies were everywhere dense in one-parameter families of this kind. This is due to the fact that the structure of the set N essentially depends on the value of θ in the case of systems of the third class. Indeed, even from the results of [4], it follows that θ is an *invariant of the Ω-equivalence* (i.e., of topological equivalence on the set of nonwandering trajectories) for systems of the third class (see [11, 13]). In other words, systems with different values of θ cannot be Ω-equivalent; therefore, *arbitrarily small changes in θ necessarily lead to bifurcations in the nonwandering set.*

In particular, one can obtain one more trajectory with homoclinic tangency by an arbitrarily small change of θ (moreover, the original homoclinic tangency does not vanish under this procedure). This fact has far-reaching implications. Namely, by using localized small smooth additions, it is proved that *in the*

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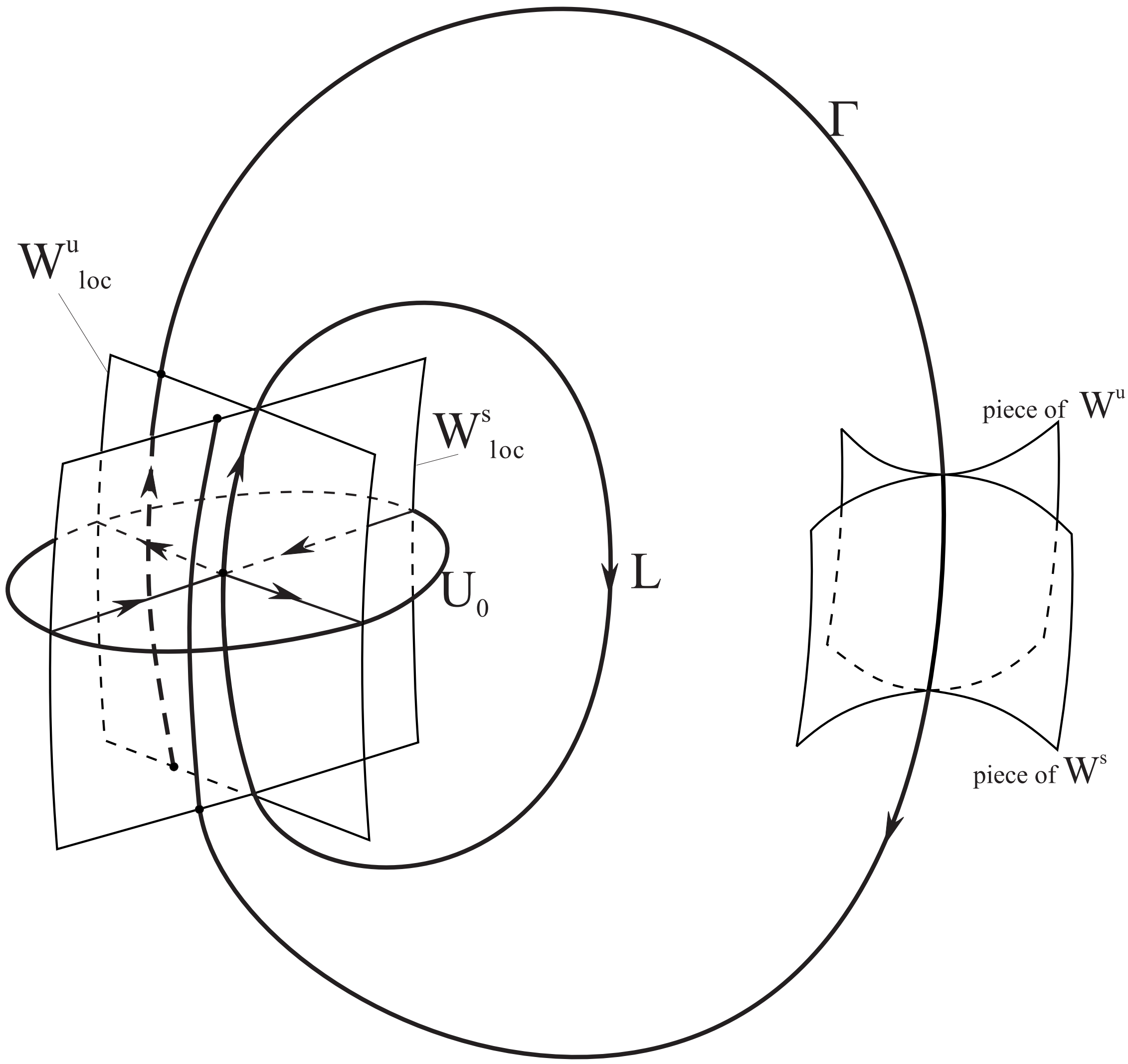

Fig. 1

set of systems with homoclinic tangencies of the third class, systems having infinitely many saddle periodic trajectories with homoclinic tangencies of the third class are dense.[1]

We note that the latter statement means that such systems have *infinitely many independent continuous invariants* (*moduli*) *of the* Ω*-equivalence* (because for each individual tangency of the third class, the corresponding value of θ is such an invariant; we do not insist that the totality of all those values serve as a complete invariant; other invariants are also possible, for instance, τ from [11, 12, 13]).

The construction that involves an infinite set of trajectories with homoclinic tangencies is the basic element in the proof of the following statement, which is called the Main Theorem in view of its importance.

Main Theorem. *Systems that have homoclinic tangencies of an arbitrarily high order are dense in the set of systems with quadratic homoclinic tangencies of the third class.*

We note that systems of the third class exist near any system with a quadratic homoclinic tangency. Therefore, the result of the Main Theorem contradicts the scheme of resolution of degeneracies, which is, so to say, traditional in the singularity theory: here by small and arbitrarily smooth perturbations of a system, we pass without obstruction from the quadratic tangency to the degeneracies (tangencies) of higher order.

[1]Here and throughout this paper, we mean density in the C^r-topology for an arbitrary finite r. If we consider C^∞-smooth systems, then density in the C^r-topology for an arbitrary finite r means density in the C^∞-topology by definition.

The study of the first return mapping near the tangency of the nth order allows one to deduce from the Main Theorem the following result: *systems that have infinitely many nonrough periodic trajectories of an arbitrary order of degeneracy*[2] *are dense in the set of systems of the third class.*

Thus, a complete description of dynamics of systems with homoclinic tangencies of the third class (in particular, a complete description of bifurcations of periodic motions of such systems) within the framework of some finite-parameter family is not possible in principle.

Systems with quadratic homoclinic tangencies that are close to the original tangency form bifurcation surfaces of codimension one. The above-presented results are obtained for the most part by examining perturbations under which a system does not leave such a bifurcation surface. However, it is natural to consider initially the perturbations that "split" the original tangency of invariant manifolds. Let μ be a bifurcation parameter that controls the splitting of the separatrices, and let X_μ be a family of systems in which μ varies monotonically. Thus, X_μ transversally intersects the surfaces of systems with homoclinic tangencies when $\mu = 0$. The following fact is of fundamental importance here: *in any transversal one-parameter family X_μ, there exists a sequence of intervals, accumulating to $\mu = 0$, in which values of the parameter that correspond to quadratic homoclinic tangencies are dense* (*moreover, X_μ is transversal to each of the corresponding bifurcation surfaces*).

This result was proved by Newhouse for two-dimensional diffeomorphisms[3] in [25]. Roughly speaking, this means that although each individual homoclinic tangency can be eliminated by small perturbations of a system, such perturbations, generally speaking, do not allow one to remove homoclinic tangencies completely.

Domains of everywhere dense nonroughness in the space of C^r-smooth ($r \geq 2$) dynamical systems in which systems with homoclinic tangencies are dense are called the *Newhouse domains* (the above intervals of values of the parameter at which the transversal family X_μ intersects the Newhouse domains are called the *Newhouse intervals*).

The most popular result (proved in [24]) on the dynamics of two-dimensional mappings in Newhouse domains consists of the fact that if the saddle quantity $\sigma = |\lambda\gamma|$, i.e., the absolute value of the product of multipliers of a periodic trajectory L, is less than unity, then *systems having infinitely many stable periodic trajectories are dense in Newhouse domains.*[4]

This assertion is an almost immediate consequence of the density of values of the parameter that correspond to homoclinic tangencies and the earlier result in [4] stating that if $\sigma < 1$, then in a transversal family, we have a sequence (accumulating to $\mu = 0$) of intervals of values of μ corresponding to the existence of a stable periodic motion.[5]

As was already noted, systems with heteroclinic tangencies of the third class are arbitrarily close to any system with a homoclinic tangency of any kind. Therefore, they are dense in Newhouse domains, and our Main Theorem immediately implies the following fundamental fact.

Systems with infinitely many homoclinic tangencies as well as systems with infinitely many periodic trajectories of an arbitrarily high order of degeneracy are dense in Newhouse domains.

Thus, the infinite degeneracies do not vanish when the bifurcation surface corresponding to the homoclinic tangency is left. Quite the contrary, arbitrarily near any such surface, we have domains in which the infinite degeneracies are dense.

From a purely mathematical standpoint, this result is indicative of the fact that the structure of the partition of the space of dynamical systems into classes of the Ω-equivalence is far from trivial. What is

[2]We mean the trajectories with one multiplier equal to 1 or to -1 and with arbitrarily many zero Lyapunov values, i.e., successive coefficients of nonlinear terms in the normal form of the Poincaré mapping on the central manifold.

[3]In [8], it was carried over to the general higher-dimensional case; the higher-dimensional case was also considered in [27] under the condition that the unstable manifold of the saddle periodic trajectory was one-dimensional.

[4]If $\sigma > 1$, then systems having infinitely many completely unstable periodic trajectories are dense in Newhouse domains. In the higher-dimensional case, the general property of systems in Newhouse domains consists of the coexistence (of infinitely many) periodic trajectories with stable manifolds of distinct dimensions, i.e., with distinct numbers of positive or negative Lyapunov exponents (see [21]); the criteria for existence of an infinite set of stable periodic trajectories in the higher-dimensional case are also given in this paper; see also [9]; a special case was considered in [27].

[5]If we have no any special interest in one-parameter families, then we can use the fact that systems with homoclinic tangencies of the third class are dense in Newhouse domains, and that, in turn, systems with countably many stable periodic motions are dense (see [10]).

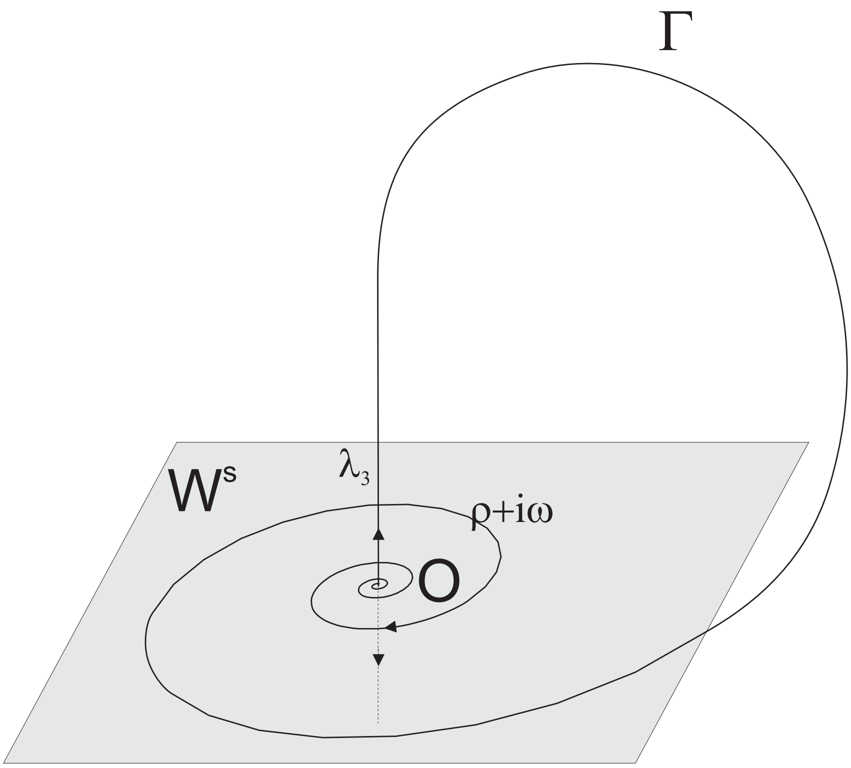

Fig. 2

more important, this result is directly related to the study of specific dynamical models. The point is that homoclinic tangencies (and, therefore, Newhouse domains in the parameter space) can be found in a great variety of specific families of systems with complex dynamics. Thus, they exist in the Henon mapping (and, in general, in any family of two-dimensional mappings that are sufficiently close to one-dimensional ones after doubling of a period); they appear when invariant tori are blown up (see [2, 26]), i.e., under the transition from the quasiperiodic regime to chaos, they can be found in Lorentz-type models in domains beyond a boundary of the domain of existence of a Lorentz attractor (see [3, 28]), in systems with wild pseudohyperbolic attractors (see [16]), and in systems with spiral chaos.

A system with spiral chaos is a system whose attractor can contain a homoclinic loop of an equilibrium state of saddle-focus type. In the three-dimensional case, it is a saddle equilibrium state at which the roots λ_1, λ_2, and λ_3 of the characteristic equation satisfy the following conditions:

$$\lambda_{1,2} = \rho \pm i\omega,\ \rho < 0,\ \omega \neq 0,\ \lambda_3 > 0,$$
$$\rho + \lambda_3 > 0, \qquad \lambda_1 + \lambda_2 + \lambda_3 < 0.$$

The unstable manifold of such an equilibrium state is one-dimensional and consists (less the equilibrium state) of two trajectories, the unstable separatrices. If one of the separatrices returns to the saddle-focus as $t \to +\infty$ (i.e., if this separatrix lies in the two-dimensional stable manifold; see Fig. 2), then one says that there is a homoclinic loop. Systems with homoclinic loops of saddle-focus type form bifurcation surfaces of codimension one. It was proved in [17] (in [19] for the higher-dimensional case) that nontrivial hyperbolic sets are present in any neighborhood of a loop. Moreover, according to [14], the following systems are dense on the bifurcation film[6]: (1) systems that have nonrough periodic motions; (2) systems with countably many stable periodic motions; (3) *systems with homoclinic tangencies*. The latter implies that near any system with a loop of saddle-focus type, there exist Newhouse domains and infinite degeneracies that are described by our Main Theorem.

The presence of degeneracies of an arbitrarily high order in the Newhouse domains renders the problem of a complete qualitative description of a great variety of models with complex dynamics (such as those enumerated above) unrealistic. Thus, the periodic and homoclinic trajectories of the nth order of degeneracy

[6]More precisely, in any one-parameter family on the film in which the quantity $\nu = -\lambda_3/\rho < 1$ varies monotonically.

occur in a generic n-parameter family that intersects surfaces of a homoclinic tangency.[7] But if we take into account additional parameters, we would necessarily obtain new degeneracies of higher order, and so on.

This results were announced in [6]. A detailed scheme of the proof was presented in [7, 22]; some technical passages that were omitted in these papers are contained in [21, 22] in one form or another. These problems and the method for proving the above results have aroused certain interest recently; therefore, we believe that it is appropriate to give a complete presentation of proofs. It should be noted that similar results are valid also in the higher-dimensional case (see [9, 21]), but here we restrict ourselves to the consideration of two-dimensional diffeomorphisms and three-dimensional flows for better clarity.

The paper consists of six sections. Section 2 is of a preparatory nature. In this section, we give the necessary formulas for the local and global mappings, carry out the subdivision of systems with homoclinic tangencies into classes, and prove Theorem 1, which states that in a neighborhood of any system with a homoclinic tangency, there exist systems with homoclinic tangencies of the third class.

In Sec. 3, we study the structure of nontrivial hyperbolic subsets for the case of systems belonging to the third class; we show that this structure essentially depends on the quantity θ (Theorem 2).

In Sec. 4, we consider the main kinds of bifurcations in one-parameter families of systems of the third class (the role of parameter is played by the Ω-module of θ). We prove Theorem 3, which states that the values of the parameter that correspond to secondary homoclinic tangencies are dense in such families. Theorem 4 is also proved in this section; it states that systems with infinitely many homoclinic tangencies are dense in the set of systems with homoclinic tangencies of the third class.

In Sec. 5, we prove the Main Theorem (Theorem 5) on the density of systems with homoclinic tangencies of an arbitrarily high order. We note that here we use the inductive method of proof, passing by small perturbations from a tangency of order n to a tangency of order $n+1$.[8] In this chain of arguments, in essence, the final step is of fundamental importance, i.e., the transition from the tangency of order $r-1$ to the tangency of order r . The latter (like a tangency of curves $y=0$ and $y=x^{r+1}$ at zero) is a tangency of an "uncertain order" in the case of the C^r-topology (with an arbitrary finite r), since such a tangency can be transformed "into anything" by arbitrarily small C^r-smooth perturbations; for instance, one can attain the local coincidence of curves. Using this method, we prove the following result, which, in some way, makes the assertions of the Main Theorem more specific.

Systems with homoclinic tangencies corresponding to the local coincidence of stable and unstable manifolds are dense in the bifurcation surface of systems with homoclinic tangencies of the third class (*and in Newhouse domains as well*).

In Sec. 6, we prove that there exist nonrough periodic trajectories of an arbitrarily high order of degeneracy (Theorem 6). Here the proof is based on the rescaling of the first-return mapping near a trajectory with a degenerate homoclinic tangency. Thus, we find that in the case of tangency of order n, where $n<r$, there exist periodic trajectories with multipliers $+1$ and -1 and with zero first n and first $[n/2]-1$ Lyapunov values, respectively (the restriction of the corresponding mapping to the central manifold is written in the form $\bar{x}=\pm x+lx^{n+1}+...$, where $l\neq 0$). We prove that if $n\geq r$, that is, in the case of homoclinic tangency corresponding to the local coincidence of stable and unstable manifolds, there exist periodic trajectories for which either the restriction of the Poincaré mapping to the central manifold is the identity mapping or the square of this restriction is the identity mapping (i.e., it is a mapping of the form $\bar{x}=x$ in the first case, and it is a mapping of the form $\bar{x}=-x$ in the second case).

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[7] For $n=2$, bifurcations leading to the appearance of periodic trajectories with degeneracy of order two (the so-called cusps) were studied in [15] in the case of systems with nonrough Poincaré homoclinic trajectories; in [1], such bifurcations were studied in the case of systems with homoclinic loops of saddle-focus type.

[8] Here we use the lemma (Lemma 2) stating that it is possible to make up, by small perturbations, one tangency of order $n+1$ from two tangencies (of orders n and 1, respectively).

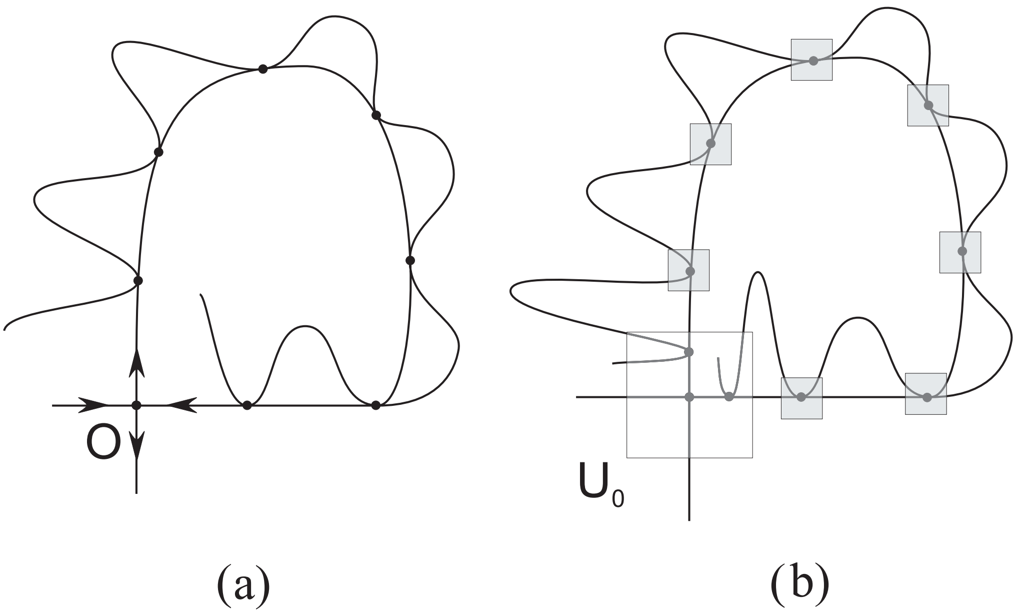

Fig. 3

2. Three Classes of Homoclinic Tangencies

We consider a C^r-smooth $(3 \le r \le \infty)$ two-dimensional diffeomorphism f_0. Let the following conditions be satisfied: **(A)** f_0 has a rough saddle periodic trajectory L_0 with multipliers λ and γ, where $|\lambda| < 1,\ |\gamma| > 1$.

(B) The saddle quantity $\sigma \equiv |\lambda\gamma|$ is less than unity.

(C) The stable manifold (W_0^s) and the unstable manifold (W_0^u) of the periodic trajectory are quadratically tangent to each other at some points of a homoclinic trajectory Γ_0 (Fig. 3a).

Let O be some point of the trajectory L_0. It is a fixed point for some iteration of the diffeomorphism under consideration. We denote by T_0 the restriction of this iteration to a small neighborhood U_0 of the point O (Fig. 3b) and say that T_0 is a *local mapping*. By definition, the multipliers λ and γ are the eigenvalues of the matrix of the linearization of T_0 at the point O.

In some C^{r-1}-coordinates (x, y) on U_0, the mapping T_0 can be written in the following form (see [11, 13]):

$$\bar{x} = \lambda x + h(x,y)x^2 y, \quad \bar{y} = \gamma y + g(x,y)xy^2, \tag{2.1}$$

where $h(x,y)\cdot xy$ and $g(x,y)\cdot xy$ are functions of class C^{r-1}. In particular, in these coordinate systems, the origin corresponds to the fixed point O, the equation for the local stable manifold W^s_{loc} of the point O is $y = 0$, and the equation for the local unstable manifold W^u_{loc} of the point O is $x = 0$.

The representation of T_0 in the form (2.1) is rather convenient because the mapping T_0^k in these coordinates is linear in the principal order, uniformly for all sufficiently large k. Namely, in the case where $\sigma < 1$, we have the following representation (see [11, 13, 29]) for the mapping $T_0^k : (x_0, y_0) \mapsto (x_k, y_k)$:

$$\begin{aligned} x_k &= \lambda^k x_0 + |\lambda|^k \xi_k(x_0, y_k), \\ y_0 &= \gamma^{-k} y_k + |\gamma|^{-k} \eta_k(x_0, y_k), \end{aligned} \tag{2.2}$$

where the functions ξ_k and η_k tend to zero as $k \to \infty$, together with all their derivatives up to the order $(r-1)$. Furthermore,

$$\|\xi_k, \eta_k\|_{C^{r-2}} = O(|\gamma|^{-k}). \tag{2.3}$$

U_0 contains a countable set of points of the trajectory Γ_0 (the fact that Γ_0 is a homoclinic trajectory means that positive iterations of any point of Γ_0 accumulate to O along W^s_{loc}, while the negative ones accumulate to this point along W^u_{loc}). We choose a pair of such homoclinic points and denote them by M^+ and M^-, where $M^+(x^+, 0) \in W^s_{\mathrm{loc}}$ and $M^-(0, y^-) \in W^u_{\mathrm{loc}}$. Without loss of generality, we can assume that $x^+ > 0$ and $y^- > 0$. Let Π^+ and Π^- be some sufficiently small rectangular neighborhoods of the homoclinic points M^+ and M^-, respectively. They are defined as follows:

$$\begin{array}{l} \Pi^+ = \{(x,y) \mid |x-x^+| \le \varepsilon_0, |y| \le \varepsilon_0\}, \\ \Pi^- = \{(x,y) \mid |x| \le \varepsilon_1, |y-y^-| \le \varepsilon_1\}. \end{array} \tag{2.4}$$

Obviously, for small ε_0 and ε_1, we have that $T_0(\Pi^+) \cap \Pi^+ = \emptyset$ and $T_0(\Pi^-) \cap \Pi^- = \emptyset$. Let q be a positive integer such that $f_0^q(M^-) = M^+$ (such q always exists, since M^- and M^+ are points of the same trajectory). The mapping $T_1 \equiv f^q : \Pi^- \to U_0$ is called the *global mapping*. It can be written in the following form:

$$\begin{array}{rcl} \bar{x} - x^+ & = & ax + b(y-y^-) + ..., \\ \bar{y} & = & cx + d(y-y^-)^2 + ..., \end{array} \tag{2.5}$$

where the dots denote the higher-order terms ($o(|x| + |y-y^-|)$ in the first equation and $o(|x| + |y-y^-|^2) + O(|x||y-y^-|)$ in the second one). We note that we have $bc \neq 0$ in (2.5) because T_1 is a diffeomorphism, and we have $d \neq 0$ there because the tangency of $T_1(W^u_{\text{loc}})$ with W^s_{loc} at the point M^+ is a quadratic one by assumption.

The signs of quantities c and d will be of importance for us. Of course, these signs are also determined by the choice of the orientation of the axes x and y (we recall that the orientation was fixed by setting $x^+ > 0$ and $y^- > 0$). A positive d corresponds to the tangency of $T_1(W^u_{\text{loc}} \cap \Pi^-)$ with W^s_{loc} from above, while a negative d corresponds to a tangency from below. If $\gamma < 0$, then each iteration of the mapping T_0 reverses the orientation along the y axis, and we can choose the homoclinic point M^+ in such a way that d would be positive. For $\lambda < 0$, the iterations of T_0 reverse the orientation along the x axis, and we can choose homoclinic points in such a way that c would be positive. In accordance with what was said above (see also [4]), one can distinguish 10 different types of homoclinic tangencies; a certain combination of signs of the parameters λ, γ, c, and d corresponds to each of these types (see Table 1). We note that in this table, the symbol $+(-)$ means that the sign of the corresponding parameter for this type of homoclinic tangency can be changed if some other pair of homoclinic points is chosen.

Table 1.

	H_1^1	H_1^2	H_1^3	H_2	H_3^1	H_3^2	H_3^3	H_3^4	H_3^5	H_3^6
λ	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$
γ	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
d	$-$	$-$	$-$	$+$	$+$	$+$	$+(-)$	$+(-)$	$+(-)$	$+(-)$
c	$+$	$-$	$+(-)$	$-$	$+$	$+(-)$	$+$	$-$	$+$	$-$

A similar construction arises when one considers a three-dimensional flow with a saddle periodic trajectory and with a nonrough homoclinic trajectory along which the stable and unstable manifolds are quadratically tangent to each other (Fig. 1). As U_0, we take a small secant line of the periodic trajectory and denote by T_0 the Poincaré mapping on U_0. The point O at which the periodic trajectory intersects the secant is a saddle fixed point for T_0; therefore, here we also have formulas (2.1)–(2.3). Now, as M^+ and M^- we take a pair of points at which the homoclinic trajectory intersects U_0, namely, M^+ on the local stable manifold and M^- on the local unstable manifold. Since these points lie on the same trajectory, the mapping T_1 is defined along the trajectories of the flow from the small neighborhood Π^- of the point M^- on U_0 to the small neighborhood Π^+ of the point M^+ on U_0. Since the time of transition of the trajectory from the neighborhood of the point M^- on U_0 to the neighborhood of the point M^+ on U_0 is finite and depends on the initial point smoothly, T_1 is a diffeomorphism. Therefore, we can write the Taylor series expansion of T_1 at the point M^-, which is given by the same formula (2.5).

Throughout the rest of this paper, we deal only with the pair of mappings (T_0, T_1); therefore, we will not distinguish between the cases of two-dimensional diffeomorphisms and three-dimensional flows (we use the term "system" in both cases). We note that the set of systems f each of which is C^r-close to f_0 and has a nonrough homoclinic trajectory Γ that is close to Γ_0 forms a smooth Banach submanifold H of codimension one in the space of C^r-diffeomorphisms or C^r-flows.

We denote by U a sufficiently small neighborhood of the set $O \cup \Gamma_0$ (Fig. 3b). We study the structure of the set N of trajectories that lie entirely in U.

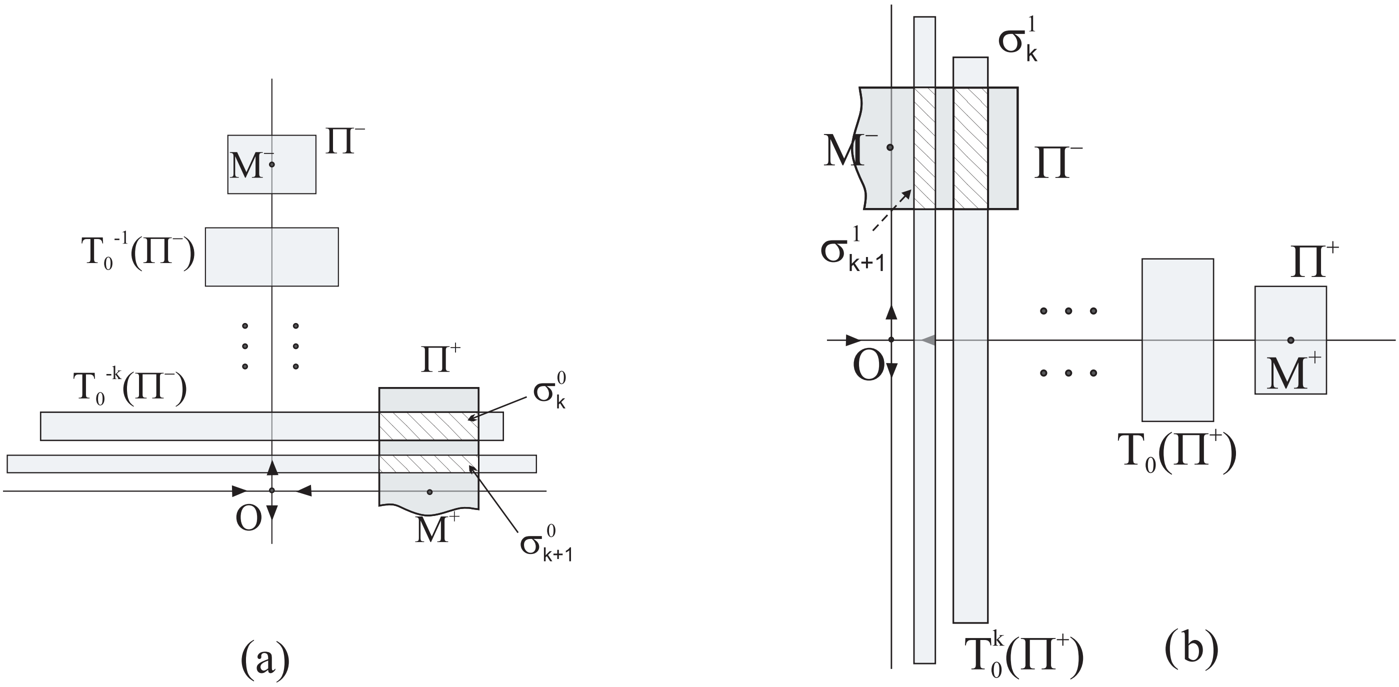

Fig. 4

Any trajectory from N (except for O) should intersect the neighborhoods Π^+ and Π^- (otherwise, such trajectory will not be close to Γ_0). However, not for every initial point of Π^+ does its trajectory get into Π^-. The set of those points on Π^+ whose iterations get into Π^- under the action of T_0 is the union of countably many strips $\sigma_k^0 = \Pi^+ \cap T_0^{-k}\Pi^-$, $k = \bar{k}, \bar{k}+1, ...$, which accumulate to the segment $\Pi^+ \cap W^s_{\text{loc}}$ (Fig. 4). In turn, the images of strips σ_k^0 under the action of T_0^k are vertical strips $\sigma_k^1 = \Pi^- \cap T_0^k\Pi^+$ on Π^-, which accumulate to the segment $\Pi^- \cap W^u_{\text{loc}}$ (Fig. 4b). It follows from (2.2)–(2.4) that

$$\begin{aligned}&\sigma_k^0 = \{(x,y)\big|\ |x-x^+| \le \varepsilon_0,\\ &\gamma^{-k}(y^- - \varepsilon_1 + O(\gamma^{-k})) \le y \le \gamma^{-k}(y^- + \varepsilon_1 + O(\gamma^{-k}))\},\\ &\sigma_k^1 = \{(x,y)\big|\ \lambda^k(x^+ - \varepsilon_0 + O(\gamma^{-k})) \le x\\ &\le \lambda^k(x^+ + \varepsilon_0 + O(\gamma^{-k})),\ |y-y^-| \le \varepsilon_1\}.\end{aligned} \tag{2.6}$$

Following [4], we assign those homoclinic tangencies to the *first class* for which the following holds:

$$\gamma > 0 \ \text{ and } \ d < 0 \tag{2.7}$$

(the first three columns of Table 1). The tangencies for which

$$\gamma > 0,\ \lambda > 0,\ c < 0,\ d > 0 \tag{2.8}$$

belong to the *second class* (the fourth row of Table 1). All other combinations of signs of γ, λ, c, and d correspond to homoclinic tangencies of the *third class* (the last six columns of Table 1). We note that, according to (2.7) and (2.8), any quadratic tangency in the case where $\gamma < 0$ is a tangency of the third class, and in the case where $\gamma > 0$ and $\lambda < 0$, any quadratic tangency "from above" is a tangency of the third class. In the case where $\gamma > 0$ and $\lambda > 0$, the combination $c > 0$ and $d > 0$ corresponds to a tangency of the third class. According to (2.5), such a tangency is a tangency "from above" ($d > 0$), and a right half-neighborhood of a nonrough homoclinic point in Π^- is mapped under the corresponding global mapping T_1 into a half-neighborhood that is adjacent to the parabola $T_1(W^u_{\text{loc}}) \cap \Pi^+$ "from above" ($c > 0$).

For the case where $\lambda > 0$ and $\gamma > 0$, the classification of homoclinic tangencies is illustrated by Fig. 5. Here, Fig. 5a and Fig. 5b correspond to homoclinic tangencies of the first class (the tangencies are "from below" in both cases), and Fig. 5c and Fig. 5d correspond to homoclinic tangencies of the second and third classes, respectively (the tangencies are "from above" in both cases).

The following statement from [4] gives a complete description of the set N for the homoclinic tangencies of the first and second classes.

In the case of a tangency of the first class, the set N has a trivial structure: $N = \{L_0, \Gamma_0\}$. *In the case of a tangency of the second class, the set N is (nonuniformly) hyperbolic; moreover, the trajectories from N are in a one-to-one correspondence with the trajectories of the quotient system obtained from the Bernoulli topological scheme consisting of three symbols $\{0, 1, 2\}$ by means of identification of two homoclinic trajectories* $(..., 0, ..., 0, 1, 0, ..., 0, ...)$ *and* $(..., 0, ..., 0, 2, 0, ..., 0, ...)$.

The structure of the set N in the case of a tangency of the third class is studied in the subsequent sections. This is important to study systems of precisely this type because tangencies of the third class exist near any system with a homoclinic tangency. That is, the following theorem holds.

Theorem 1. *Let f_μ be a one-parameter family that is transversal to a bifurcation surface of systems with quadratic homoclinic tangencies when $\mu = 0$. Then, in any neighborhood of the point $\mu = 0$, there are values of the parameter that correspond to quadratic homoclinic tangencies of the third class.*

Proof. Let f_0 have a nonrough homoclinic trajectory of the first or second class. We embed f_0 into a smooth one-parameter family f_μ that for $\mu = 0$ is transversal to the surface H of codimension one consisting of C^r-systems that are C^r-close to f_0 and have a trajectory with a homoclinic tangency that is close to Γ_0.

Now the mappings T_0 and T_1 depend on μ. The change of variables reducing the mapping T_0 to the form (2.1) depends on μ C^{r-2}-smoothly (see [29]). Moreover, the functions ξ and η in formula (2.2) for T_0^k also depend on μ C^{r-2}-smoothly and estimates (2.3) hold for the derivatives with respect to μ as well (see [11, 13, 29]). Then both λ and γ should be considered as functions of μ (of class C^{r-2}).

The global mapping $T_1 \equiv T_1(\mu)$ is written in the form

$$\begin{aligned} \bar{x} - x^*(\mu) &= ax + b(y - y^-) + \varphi_1(x, y - y^-, \mu), \\ \bar{y} &= y^*(\mu) + cx + d(y - y^-)^2 + \varphi_2(x, y - y^-, \mu), \end{aligned} \tag{2.9}$$

where $x^*(0) = x^+$ and $y^*(0) = 0$; the coefficients a, b, c, and also y^- are now functions of μ (of class C^{r-2}; we assume that y^- depends on μ in order to exclude from the equation for $\bar{y}$ the term that is linear in $(y - y^-)$); the functions $\varphi_{1,2}$ do not contain linear terms, and, moreover, the coefficient of $(y - y^-)^2$ in φ_2 turns to zero for $\mu = 0$. Therefore,

$$\begin{aligned} \varphi_1 &= O[(|x| + |y - y^-|)^2], \\ |\varphi_2| &= O(x^2 + |x||y - y^-| + |\mu|(y - y^-)^2) + o((y - y^-)^2). \end{aligned} \tag{2.10}$$

We recall that the local stable and unstable manifolds of the point O are rectified, i.e., $W^u_{\text{loc}} = \{x = 0\}$ and $W^s_{\text{loc}} = \{y = 0\}$. Respectively, the piece $T_1(W^u_{\text{loc}} \cap \Pi^-)$ of the unstable manifold near the point M^+ is defined by the following equation:

$$\bar{y} = y^*(\mu) + \frac{d}{b^2}(x - x^*)^2 + o((x - x^*)^2). \tag{2.11}$$

The fact that the family is transversal to the bifurcation surface means that the homoclinic tangency is split with a nonzero "rate" when μ changes, i.e.,

$$\frac{d}{d\mu} y^*(\mu) \neq 0.$$

Therefore, without loss of generality, we can assume that

$$y^*(\mu) \equiv \mu$$

in (2.9), and thus, $T_1(\mu)$ can be rewritten as

$$\begin{aligned} \bar{x} - x^*(\mu) &= ax + b(y - y^-) + \varphi_1(x, y - y^-, \mu), \\ \bar{y} &= \mu + cx + d(y - y^-)^2 + \varphi_2(x, y - y^-, \mu). \end{aligned} \tag{2.12}$$

We consider first the case where Γ_0 is a nonrough homoclinic trajectory of the second class (Fig. 6); here $\gamma > 0$, $\lambda > 0$, $c < 0$, and $d > 0$. We take a strip σ_i^0 with a sufficiently large number i and consider those $\mu > 0$, for which the vertex of the curve $T_1(\mu)(W^u_{\text{loc}})$ lies below the strip σ_i^0 on Π^+, and the curve itself intersects this strip along two segments of the curves u_i^1 and u_i^2. The vertex of curve (2.11) is the point $(x^*(\mu), \mu)$; it lies below the strip σ_i^0 if $\mu \ll \gamma^{-i}$ (see (2.6)). We note that this is certainly so when $\mu \sim \lambda^i$ (since $\lambda\gamma < 1$ by assumption, and i is taken sufficiently large), and we will consider precisely such μ. The equations of the

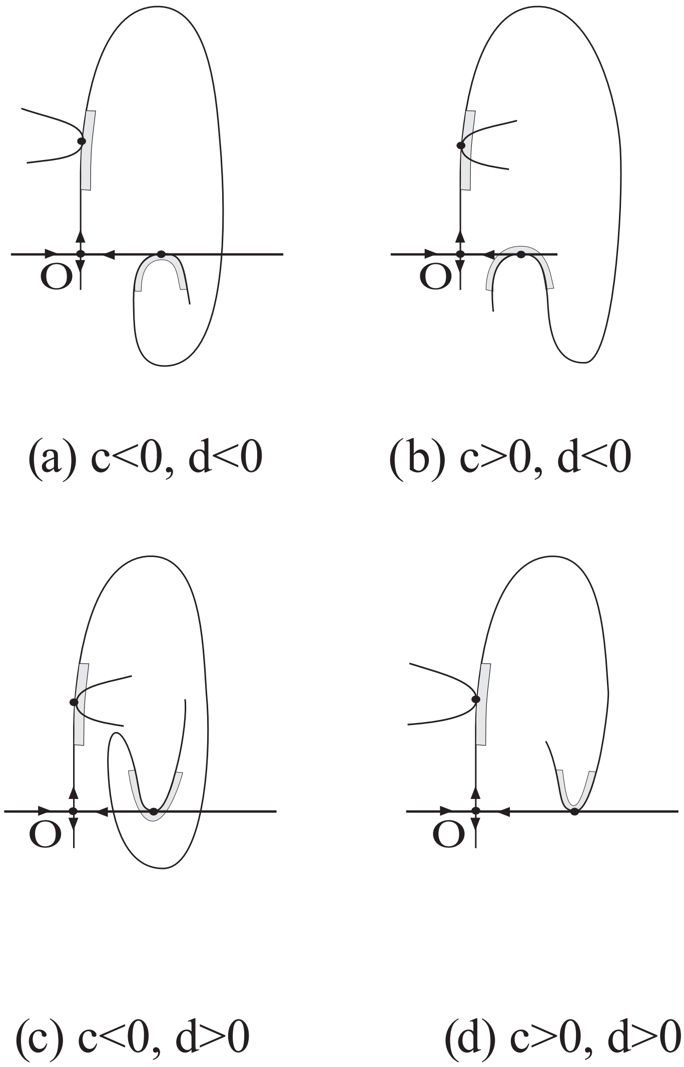

Fig. 5

curves u_i^1 and u_i^2 on the strip σ_i^0 are obtained from system (2.12), in which it is necessary to set $x = 0$ and to take into account that the coordinates $\bar{y}$ of the points on the strip σ_i^0 should satisfy the following inequalities:

$$\gamma^{-i}(y^- - \varepsilon_1) \le \bar{y} \le \gamma^{-i}(y^- + \varepsilon_1) \tag{2.13}$$

(see (2.6)). Thus, we obtain that the curves u_i^1 and u_i^2 in parametric form are defined by the equation

$$x_0 - x^*(\mu) = bt + \dots \ , \quad y_0 = \mu + dt^2 + \dots \ , \tag{2.14}$$

where t runs over the values from the following interval (we take into account the fact that $\mu \ll \gamma^{-i}$):

$$\gamma^{-i/2}\sqrt{\frac{y^- - \varepsilon_1}{d}} \le t \le \gamma^{-i/2}\sqrt{\frac{y^- + \varepsilon_1}{d}} \tag{2.15}$$

for the curve u_i^1; for u_i^2, t runs over the interval

$$-\gamma^{-i/2}\sqrt{\frac{y^- + \varepsilon_1}{d}} \le t \le -\gamma^{-i/2}\sqrt{\frac{y^- - \varepsilon_1}{d}}. \tag{2.16}$$

Inequalities (2.15) and (2.16) define (if we set $t = y - y^-$ and $x = 0$) the segments l_1 and l_2 that are two connected components of the set $T_1^{-1}(\sigma_i^0) \cap W^u_{\mathrm{loc}}(O)$. Since we are considering the tangency of the second class, small neighborhoods that are adjacent to l_1 and l_2 on the side of positive x are mapped under the action of T_1 below (in the direction of decreasing values of y) the curves $u_i^{1,2}$, i.e., to the right (in the direction of increasing values of x) of the curve u_i^1 and to the left of the curve u_i^2 for $b > 0$ (Fig. 6b) and, conversely, they are mapped to the left of u_i^1 and to the right of u_i^2 for $b < 0$.

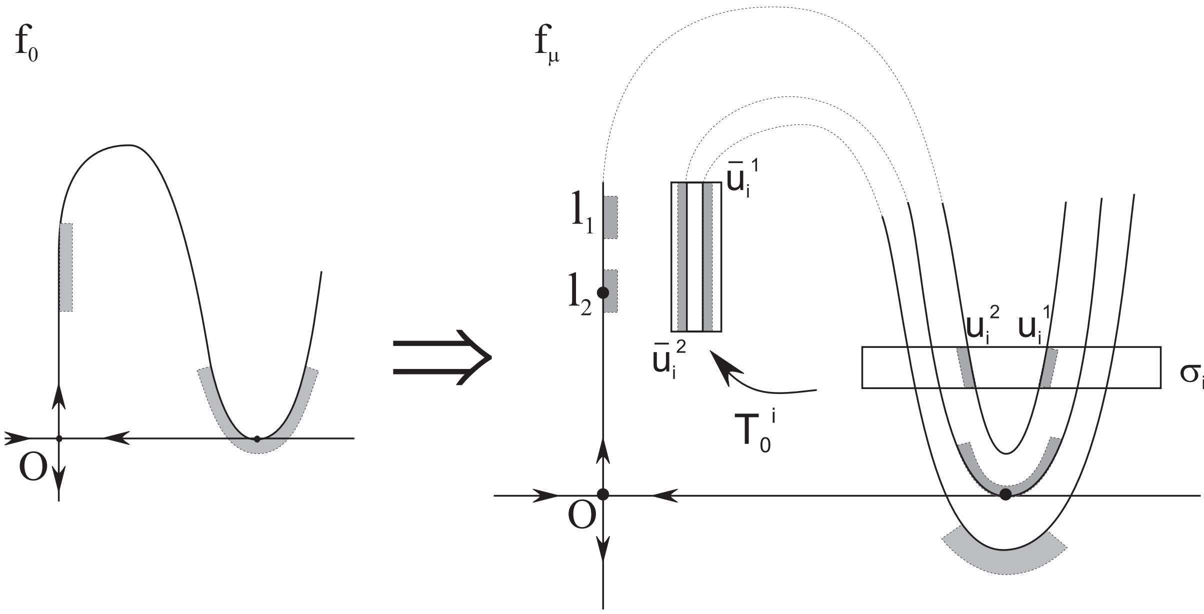

Fig. 6

Under the mapping T_0^i, the strip σ_i^0 transforms into the strip $\sigma_i^1 \subset \Pi^-$, and the curves u_i^1 and u_i^2 transform into the curves $\bar{u}_i^1$ and $\bar{u}_i^2$, respectively; by virtue of (2.2) and (2.14), the equations of these curves are the following ones:

$$x = \lambda^i(x^* + bt + \dots), \quad \gamma^{-i}(y + \dots) = \mu + dt^2 + \dots\ , \tag{2.17}$$

where t runs over the values from the interval (2.15) for the curve $\bar{u}_i^1$, or it runs over the values from interval (2.16) for the curve $\bar{u}_i^2$. Since $\lambda > 0$, left half-neighborhoods of the curves $u_i^{1,2}$ are mapped to the left (in the direction of decreasing values of x) of $\bar{u}_i^{1,2}$, respectively.

It is seen from (2.17), (2.10), (2.2), (2.15), and (2.16) that the curves $\bar{u}_i^{1,2}$ are close to $y = \text{const}$. Namely, on these curves

$$\left\| \frac{\partial x_1}{\partial(y_1, \mu)} \right\| + \left\| \frac{\partial^2 x_1}{\partial y_1^2} \right\| = O(\lambda^i \gamma^{-i/2}). \tag{2.18}$$

Moreover,

$$\begin{aligned} \bar{u}_i^1 \ &: \ x = \lambda^i x^+ + \lambda^i b \sqrt{\frac{\gamma^{-i} y^-}{d}} + \dots\ , \\ \bar{u}_i^2 \ &: \ x = \lambda^i x^+ - \lambda^i b \sqrt{\frac{\gamma^{-i} y^-}{d}} + \dots \end{aligned} \tag{2.19}$$

(we have taken into account the fact that $\mu \ll \gamma^{-i}$). Now it follows from (2.12) that the curves $T_1(\bar{u}_i^1)$ and $T_1(\bar{u}_i^2)$ are quadratically tangent to $W^s_{\text{loc}} = \{y = 0\}$ for

$$\mu = \mu_i^{1,2} = |c|\lambda^i(x^+ \pm b\sqrt{\frac{\gamma^{-i} y^-}{d}} + \dots). \tag{2.20}$$

These curves are the images of the segments $l_{1,2}$ on the local unstable manifold (with respect to the new global mapping $T_1 T_0^i T_1$), i.e., these values of μ correspond to the homoclinic tangencies. It follows from (2.12) and (2.18) that these tangencies are split with a nonzero rate when μ changes; therefore, the family f_μ is transversal to the corresponding bifurcation films.

As was already noted, for $b > 0$, the right half-neighborhood of the segment l_2 and, for $b < 0$, the right half-neighborhood of the segment l_1 are mapped under the action of T_1 to the left of the curves u_i^2 and u_i^1. Futhermore, under the action of T_0^i, they are mapped into the left half-neighborhoods of the curves $\bar{u}_i^2$ and $\bar{u}_i^1$, respectively, which, in turn (since $c < 0$), are mapped above the curves $T_1(\bar{u}_i^1)$ and $T_1(\bar{u}_i^2)$, respectively.

Thus, $\mu = \mu_i^1$ for $b < 0$ and $\mu = \mu_i^2$ for $b > 0$ correspond to the homoclinic tangencies of the third class (see Fig. 6b for the case $b > 0$).

To complete the proof, it remains to consider the case of the tangency of the first class (the first three columns in Table 1). This means that $\gamma > 0$ and $d < 0$, i.e., for $\mu = 0$, we have the tangency from below (as in Fig. 5a and Fig. 5b). For $\mu < 0$, the stable and unstable manifolds do not have intersections, and for $\mu > 0$, there appear two transversal intersections and the piece $T_1(W^u_{\text{loc}} \cap \Pi^-)$ of the unstable manifold enters the domain $y > 0$. We show that in this case, there exist values of $\mu > 0$ for which the homoclinic tangency from above appears; moreover, the family f_μ is transversal to the corresponding bifurcation surface. Such a tangency is either of the third class (in the case where $\lambda < 0$ it is always so) and then the theorem is proved, or it belongs to the second class. The appearance of tangencies of the third class in this case for the values of parameters that are close to each other was just proved above.

We take a strip σ_i^0 with a sufficiently large number i. We consider those $\mu > 0$ for which the vertex of the curve $T_1(W^u_{\text{loc}} \cap \Pi^-)$ lies inside the strip σ_i^0 (Fig. 7 for the case where $\lambda > 0, c < 0$ and Fig. 10a for the case where $\lambda > 0, c > 0$), i.e., (see (2.11), (2.6))

$$\gamma^{-i}(y^- - \varepsilon_1) \le \mu \le \gamma^{-i}(y^- + \varepsilon_1).$$

The equation of the piece $T_1^{-1}(W^s_{\text{loc}} \cap \Pi^+)$ of the stable manifold of the point O near the point M^- has the following form (we set $\bar{y}$ as in (2.12) and also use (2.10)):

$$x = -\frac{\mu}{c} + \frac{|d|}{c}(y - y^-)^2 + \dots. \tag{2.21}$$

Since $\mu \sim \gamma^{-i} \gg |\lambda|^i$, this curve always has two connected components $\nu_i^{1,2}$ in the intersection with the vertical strip σ_i^1, which is at a distance of about $|\lambda|^i$ from $y = 0$ (see (2.6)). According to (2.2) and (2.3), the images $\tilde{\nu}_i^{1,2}$ of these components with respect to the mapping T_0^{-i} are defined in a neighborhood of the point M^+ by the following parametric equation:

$$\lambda^i(x + \dots) = -\frac{\mu}{c} + \frac{|d|}{c}t^2 + \dots, \quad y = \gamma^{-i}(y^- + t + \dots), \tag{2.22}$$

where t is a parameter that runs over the values from $-\varepsilon_1$ to ε_1 and x takes values near x^+. Since $\mu \gg |\lambda|^i$, it is easy to see that the curves $\tilde{\nu}_i^{1,2}$ are close (at least in C^2 and also in C^1 with respect to μ) to the straight lines

$$y = \gamma^{-i}\left(y^- \pm \sqrt{\frac{\mu}{|d|}} + \dots\right).$$

Now it is obvious (see (2.11)) that $T_1(W^u_{\text{loc}} \cap \Pi^-)$ has the quadratic tangency with the curves $\tilde{\nu}_i^{1,2}$ for

$$\mu = \mu_i^{\pm} = \gamma^{-i}\left(y^- \pm \sqrt{\frac{\gamma^{-i}}{|d|}} + \dots\right).$$

For both values of μ, the curve $T_1(W^u_{\text{loc}} \cap \Pi^-)$ is tangent to the corresponding curve $\tilde{\nu}_i^{1,2}$ from below, i.e., on the side corresponding to the decrease of y. Since $\gamma > 0$, the image $T_0^i T_1(W^u_{\text{loc}} \cap \Pi^-)$ is also adjacent to $T_1^{-1}(W^s_{\text{loc}} \cap \Pi^+)$ from below in both cases (Figs. 8, 9, and 10). This means that the piece $T_1 T_0^i T_1(W^u_{\text{loc}} \cap \Pi^-)$ of the unstable manifold W^u is tangent to W^s_{loc} from different sides for $\mu = \mu_i^+$ and $\mu = \mu_i^-$. Thus, one of these tangencies is a tangency of the first class (as the original one is) (Fig. 9 and Fig. 10b), while the other is either a tangency of the second class (Fig. 10c) or a tangency of the third class (Fig. 8), which completes the proof of the theorem.

3. Nontrivial Hyperbolic Subsets of Systems with Homoclinic Tangencies of the Third Class

Before turning to the study of the set of trajectories that lie entirely in a small neighborhood of a nonrough homoclinic trajectory, we will make more precise the method by which such neighborhoods are chosen. This method is completely determined by the choice of neighborhoods Π^+ and Π^- of the homoclinic points M^+ and M^-, respectively; this is convenient first of all from the instrumental point of view. Namely, we choose the neighborhoods in such a way that, for some sufficiently large $\bar{k}$, these neighborhoods would

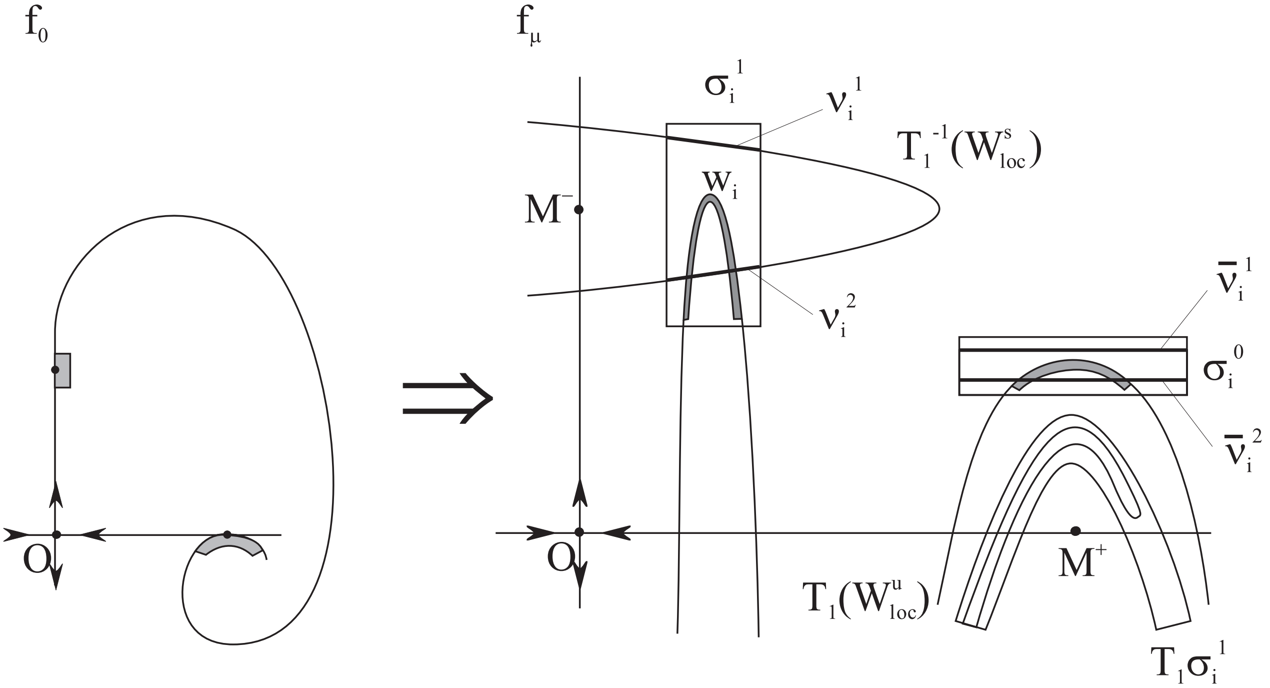

Fig. 7

contain entirely all strips σ_k^0 and σ_k^1 with numbers $k \geq \bar{k}$ and would not intersect strips whose numbers are less than $\bar{k}$. Such neighborhoods are called *special*; the way in which they are constructed is described, e.g., in [5, 11].

Schematically, a special neighborhood is constructed as follows (Fig. 11). The initial arbitrarily small neighborhoods Π^+ and Π^- are further diminished by removing "everything unnecessary" from them. Namely, we remove the points that get from Π^+ to Π^- for less than $\bar{k}$ iterations of the mapping T_0 and remove the points that get from Π^- to Π^+ for less than $\bar{k}$ iterations of the mapping T_0^{-1}. Thus, by virtue of (2.2) we retain in Π^+ only those points for which $|y| \leq |\gamma|^{-\bar{k}}(y^- + \varepsilon_1)$, and in Π^-, we retain only those points for which $|x| \leq |\lambda|^{\bar{k}}(x^+ + \varepsilon_0)$.

Furthermore, we retain in Π^+ only such a "minimal" rectangular neighborhood of the point M^+ that contains $T_1(\Pi^-) \cap \Pi^+$, while, in Π^-, we retain only such a "minimal" rectangular neighborhood of the point M^- that contains $T_1^{-1}(\Pi^+) \cap \Pi^-$ (see Fig. 11). As a result, we obtain (see [5, 11]) that without loss of generality, we can choose Π^+ and Π^- as follows:

$$\begin{aligned} \Pi^+ &= \{(x,y) \mid |x - x^+| \leq \rho_{\bar{k}}, |y| \leq |\gamma|^{-\bar{k}}(y^- + \rho_{\bar{k}})\}, \\ \Pi^- &= \{(x,y) \mid |x| \leq |\lambda|^{\bar{k}}(x^+ + \rho_{\bar{k}}), |y - y^-| \leq \rho_{\bar{k}}\}, \end{aligned} \tag{3.1}$$

where $\rho_{\bar{k}} = C|\gamma|^{-\bar{k}/2}$ and C is some positive constant that does not depend on $\bar{k}$.

The images $T_1\sigma_k^1$ of the strips σ_k^1 have the form of horseshoes (see Fig. 12) that accumulate to the piece $l_u = T_1(W^u_{\text{loc}}) \cap \Pi^+$ of the unstable manifold of the point O as $k \to \infty$. It is obvious that the trajectories from the set N (from the set of all trajectories lying entirely in the special neighborhood) should cross Π^+ at points of intersections of horseshoes $T_1\sigma_i^1$ with strips σ_j^0 for various $i, j \geq \bar{k}$. Thus, it is clear that the structure of the set N essentially depends on the character of these intersections.

A nonempty intersection of a horseshoe with a strip can be tame or it can be wild (different types of intersections are shown in Fig. 13). Namely, the intersection of the horsehoe $T_1\sigma_i^1$ with the strip σ_j^0 is *tame* if the set $T_1\sigma_i^1 \cap \sigma_j^0$ consists of two connected components σ_{ji}^{01} and σ_{ji}^{02} (Fig. 14), and the restriction of the mapping $\tilde{T}_i \equiv T_1 T_0^i$ to the inverse images $\tilde{T}_i^{-1}\sigma_{ji}^{01}$ and $\tilde{T}_i^{-1}\sigma_{ji}^{02}$ (to the substrips σ_i^{01} and σ_i^{02} on σ_i^0) is a saddle mapping in the sense of [18] (i.e., it is a contracting mapping along the x axis and it is an expanding mapping along the y axis; the exact definition will be given below).

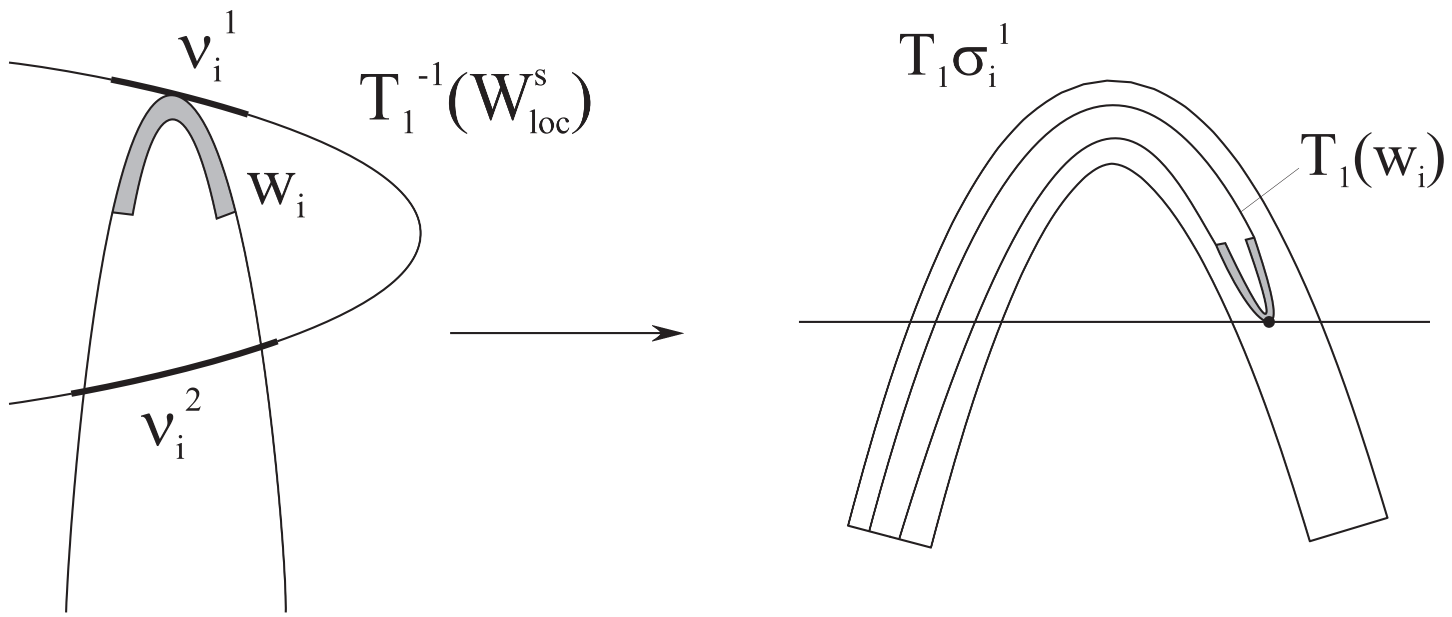

Fig. 8

The following result was proved in [5] (see also [11, 12]).

Lemma 1. *There exist a sufficiently large $\bar{k}$ and a positive constant S_1, depending only on f_0 and independent of $\bar{k}$ such that*

(1) *if the inequality*

$$d[\gamma^{-j}y^- - c\lambda^i x^+] > S_{\bar{k}}(i,j), \tag{3.2}$$

where $S_{\bar{k}}(i,j) = S_1(|\lambda|^i + |\gamma|^{-j}) \cdot |\gamma|^{-\bar{k}/2}$, holds for some integers $i, j \geq \bar{k}$, then the intersection of the horseshoe $T_1\sigma_i^1$ with the strip σ_j^0 is tame;

(2) *if the inequality*

$$d[\gamma^{-j}y^- - c\lambda^i x^+] < -S_{\bar{k}}(i,j) \tag{3.3}$$

holds for some integers $i, j \geq \bar{k}$, then $T_1\sigma_i^1 \cap \sigma_j^0 = \emptyset$.

It is convenient to restate this lemma in the following way: if the horseshoe $T_1\sigma_i^1$ has a wild intersection with the strip σ_j^0 (i.e., if, for instance, the intersection $T_1\sigma_i^1 \cap \sigma_j^0$ consists of one connected component, or if the corresponding mappings are not saddle ones), then we necessarily have

$$|d| \cdot |\gamma^{-j}y^- - c\lambda^i x^+| \leq S_{\bar{k}}(i,j). \tag{3.4}$$

Also, if $T_1\sigma_i^1 \cap \sigma_j^0 \neq \emptyset$, then the inequality

$$d[\gamma^{-j}y^- - c\lambda^i x^+] \geq -S_{\bar{k}}(i,j) \tag{3.5}$$

holds.

Inequalities (3.2)–(3.5) have a relatively simple geometric sense. The strip σ_j^0 is a thin horizontal rectangle on Π^+ with the central line $y = \gamma^{-j}y^-$, and the strip σ_i^1 is a thin vertical rectangle on Π^- with the central line $x = \lambda^i x^+$. By (2.2), the strip σ_i^1 is mapped under the action of T_1 into the horseshoe whose central line is the parabola $y = c\lambda^i x^+ + d((x - x^+)/b)^2$. The condition $d[\gamma^{-j}y^- - c\lambda^i x^+] > 0$ implies that the straight line $y = \gamma^{-j}y^-$ and the above parabola intersect at two points, while the condition $d[\gamma^{-j}y^- - c\lambda^i x^+] < 0$ implies that they do not intersect. Allowance is made for the nonzero thickness of both the strip and the horseshoe by means of the coefficient $S_{\bar{k}}(i,j)$ in (3.2)–(3.5).

We recall that in order for the intersection to be tame, it is also necessary to verify that the mapping $\tilde{T}_i = T_1 T_0^i$ is a saddle mapping on $\tilde{T}_i^{-1}\sigma_{ji}^{01}$ and on $\tilde{T}_i^{-1}\sigma_{ji}^{02}$. We will adhere to the following definition: the mapping $(x_1, y_1) \mapsto (x_2, y_2)$ (where $(x_1, y_1) \in X_1 \times Y_1$, $(x_2, y_2) \in X_2 \times Y_2$, and X_1, X_2, Y_1, and Y_2 are some

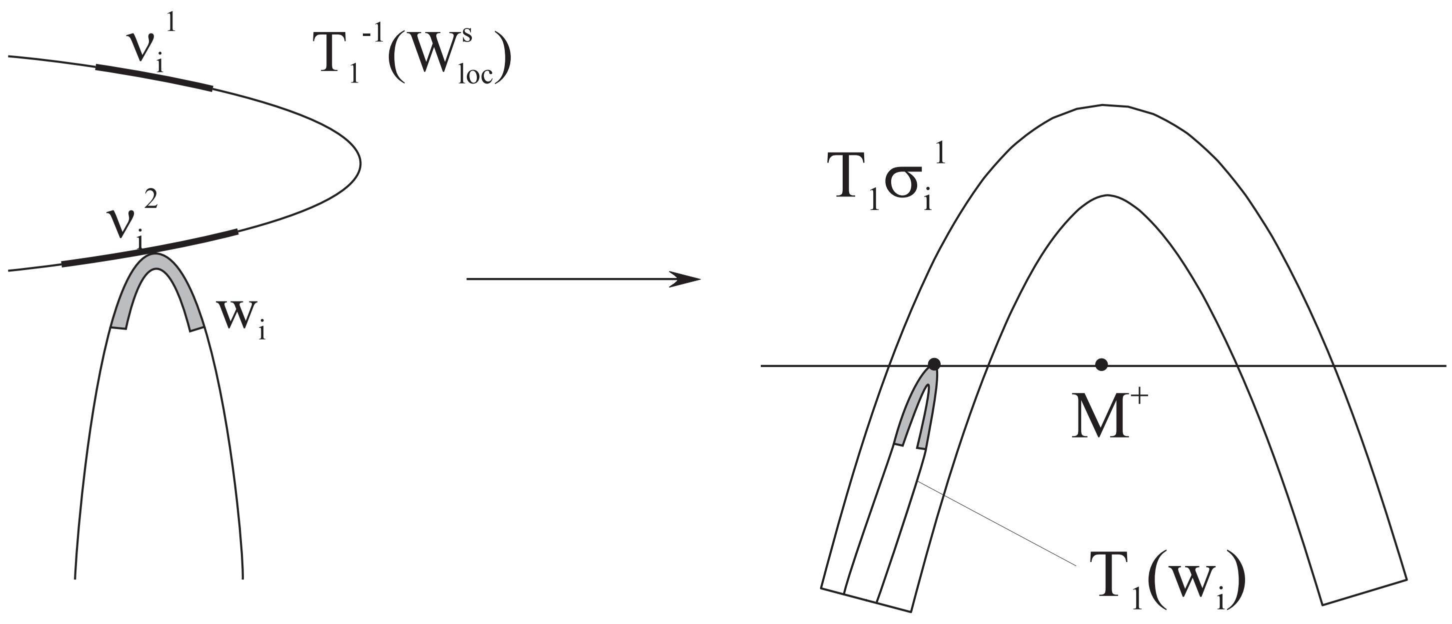

Fig. 9

closed subsets of Banach spaces) is said to be saddle if x_2 and y_1 are uniquely defined by any $x_1 \in X_1$ and $y_2 \in Y_2$ and, moreover, the mapping of the correspondence $(x_1, y_2) \mapsto (x_2, y_1)$ (the so-called cross mapping) is contractive in the metric $\max\{\|x\|, \|y\|\}$.

We note that from the definition it follows directly that the composition of saddle mappings is a saddle mapping itself. It is also obvious that, for any function $y_1 \mapsto x_1$ with Lipschitz constant ≤ 1, the saddle mapping transforms the graph of this function into the graph of the function $y_2 \mapsto x_2$ that is defined for all $y_2 \in Y_2$ and is a Lipschitzian function with Lipschitz constant that is uniformly less than 1; moreover, the mapping is expansive along the y axis. Similarly, the inverse mapping transforms a graph of any function $x_2 \mapsto y_2$ with a Lipshitz constant ≤ 1 into the graph of the function $x_1 \mapsto y_1$, which is defined for all $x_1 \in X_1$ and is a Lipshitz function with Lipschitz constant that is uniformly less than 1; moreover, it is expanding along the x axis. Therefore, a fixed point[9] of the saddle mapping acting from the product $X \times Y$ into this same product is a rough saddle (hyperbolic) point; moreover, the stable and unstable manifolds of this point are the graphs of the functions (respectively $x \mapsto y$ and $y \mapsto x$) that are defined for all $x \in X$ (respectively, for all $y \in Y$) and have Lipschitz constants <1.

It is obvious that, to a certain extent, the definition of the saddle mapping depends on the choice of coordinates in $X_1 \times Y_1$ and $X_2 \times Y_2$. On the strips σ_i^0, we use coordinates that are introduced in the following way. First of all, let $(x_i, y_i) = T_0^i(x_0, y_0)$ for any point $(x_0, y_0) \in \sigma_i^0$. Then, by (2.2), x_i and y_0 are uniquely defined by (x_0, y_i); therefore, (x_0, y_i) can be chosen as coordinates on σ_i^0. We denote such coordinates by (x', y') (here we also have $|x' - x^+| \leq \rho_{\bar{k}}$ and $|y' - y^-| \leq \rho_{\bar{k}}$; see (3.1)).

From (2.2) and (2.5), we obtain the following relation for the restriction of the mapping $\tilde{T}_i$ to $\tilde{T}_i^{-1}(T_1\sigma_i^1 \cap \sigma_j^0)$:

$$\begin{aligned} \bar{x}' - x^+ &= a(\lambda^i x' + \lambda^i \xi_i(x', y')) + b(y' - y^-) + ..., \\ \gamma^{-j}\bar{y}' + \gamma^{-j}\eta_j(\bar{x}', \bar{y}') &= c(\lambda^i x' + \lambda^i \xi_i(x', y')) + d(y' - y^-)^2 + ... \end{aligned} \tag{3.6}$$

[9]Such a fixed point exists and is unique by virtue of the contraction mapping principle: obviously, the fixed points of the original and cross mappings coincide.

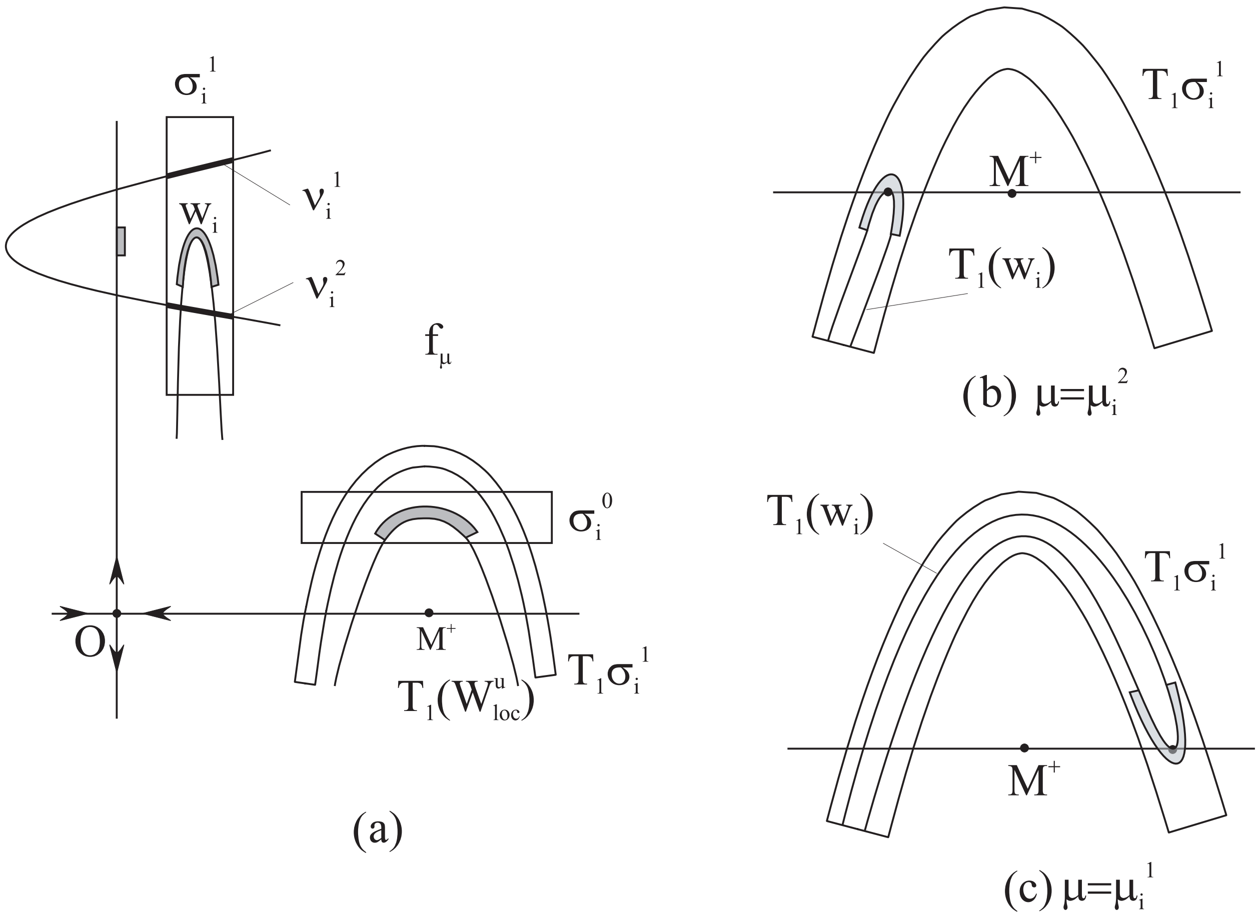

Fig. 10

If inequality (3.2) holds with appropriate S_1, then relation (3.6) can be rewritten in a cross form as

$$\begin{aligned} \bar{x}' &= x^+ + \lambda^i(x' + \dots) \pm b\sqrt{\gamma^{-j}(\frac{1}{d}\bar{y}' + \dots) - \lambda^i(\frac{c}{d}x' + \dots)}, \\ y' &= y^- \pm \sqrt{\gamma^{-j}(\frac{1}{d}\bar{y}' + \dots) - \lambda^i(\frac{c}{d}x' + \dots)}, \end{aligned} \tag{3.7}$$

where the signs "+" and "−" correspond to the mapping $\tilde{T}_i$ on the components $\tilde{T}_i^{-1}\sigma_{ji}^{01}$ and $\tilde{T}_i^{-1}\sigma_{ji}^{02}$, respectively. Dots denotes terms that tend to zero as $i, j \to +\infty$. Obviously, if (3.2) is satisfied, then the cross mapping (3.7) is contractive and, therefore, the mapping $\tilde{T}_i|_{\tilde{T}_i^{-1}(T_1\sigma_i^1 \cap \sigma_j^0)}$ is, in fact, a saddle mapping.

Lemma 1 allows us to give a rather detailed description of the structure of hyperbolic subsets in a neighborhood of a homoclinic tangency. We note that for each trajectory from the set $N \backslash O$, we can define in a natural way its *coding*, i.e., the following sequence of positive integers:

$$(..., k_{-s}, ..., k_s, ...), \tag{3.8}$$

where k_s ($k_s \geq \bar{k}$) is the number of the strip to which the sth point M_s of the intersection of the trajectory with Π^+ belongs. Codings that are infinite on both sides correspond to those trajectories from N that do not lie either in the stable manifold of the point O or in the unstable manifold of this point. For each trajectory from W^s, its last point of intersection with Π^+ belongs to $W^s_{\rm loc}$, and its further iterations under the action of T_0 do not leave a small neighborhood of the point O (and tend to O). Correspondingly, the codings of such trajectories are finite on the right, and we end them by the symbol $+\infty$. Codings for trajectories from W^u (i.e., for trajectories that are α-limit with respect to O) are finite on the left and begin with the symbol $+\infty$. Codings for trajectories that are homoclinic with respect to O (i.e., for trajectories belonging to the intersection of the stable manifold with the unstable one) are finite on both sides.

A pair of integers (k_s, k_{s+1}) with $k_s \geq \bar{k}$ and $k_{s+1} \geq \bar{k}$ is said to be *inadmissible* if inequality (3.3) holds for $j = k_s$ and $i = k_{s+1}$; otherwise (if the inequality (3.5) is fulfilled), this pair is said to be *admissible*. An admissible pair (k_s, k_{s+1}) for which inequality (3.2) is fulfilled (for $j = k_s$ and $i = k_{s+1}$) is said to be *tame*. A sequence $\{k_s\}$ of integers with $k_s \geq \bar{k}$ is said to be *admissible* if any pair (k_s, k_{s+1}) in this sequence is admissible. Finally, an admissible sequence is said to be *tame* if any pair (k_s, k_{s+1}) is tame. It is obvious that in the case of a tangency of the second class ($\gamma > 0, \lambda > 0, d > 0$, and $c < 0$), all sequences with $k_s \geq \bar{k}$ are tame (except for the sequence $(+\infty, +\infty)$, which is the coding of the original trajectory Γ_0 with the homoclinic tangency), provided that $\bar{k}$ is sufficiently large. For tangencies of the first class ($\gamma > 0$ and $d < 0$), any pair (i, j) with $j \leq i$ is not admissible (by virtue of the fact that $|\lambda\gamma| < 1$); therefore, in this case, there are no admissible sequences that are infinite on the left[10]; the only admissible sequence bounded by the symbol $+\infty$ on the left that is possible in this case is the sequence $(+\infty, +\infty)$.

For the systems with homoclinic tangencies of the third class, the set of admissible and/or tame sequences has a nontrivial structure.

In the case where $\gamma > 0, \lambda > 0, c > 0$, and $d > 0$ (the fifth column of Table 1), taking the logarithm of inequalities (3.2) and (3.5), we see that the pair (i, j) is admissible if

$$j \leq i\theta - \tau + S|\gamma|^{-\bar{k}/2}; \tag{3.9}$$

this pair is tame if

$$j < i\theta - \tau - S|\gamma|^{-\bar{k}/2}, \tag{3.10}$$

where

$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|}, \quad \tau = \frac{1}{\ln|\gamma|} \ln\left|\frac{cx^+}{y^-}\right|,$$

and S is some positive constant.

In the case where $\lambda > 0, \gamma < 0, c > 0$, and $d > 0$ (the sixth column of Table 1), it follows from (3.2)–(3.5) that any pairs (i, j) with an odd i are not admissible, while the pairs with an even i are admissible (or tame) if inequality (3.9) (respectively, (3.10)) holds.

In the case where $\lambda > 0, \gamma < 0, c < 0$, and $d > 0$ (the seventh column of Table 1), any pair (i, j) with an even i is tame, and that with an odd i is admissible if the inequality

$$j \geq i\theta - \tau - S|\gamma|^{-\bar{k}/2} \tag{3.11}$$

holds; this pair is tame if the inequality

$$j > i\theta - \tau + S|\gamma|^{-\bar{k}/2} \tag{3.12}$$

holds.

In the case where $\gamma > 0, \lambda < 0, d > 0$, and $c > 0$ (the eighth column of Table 1), any pair (i, j) with an odd j is tame, while a pair with an even j is admissible (or tame) if inequality (3.9) (respectively, (3.10)) holds.

In the case where $\gamma < 0, \lambda < 0, c > 0$, and $d > 0$ (the ninth column of Table 1), pairs (i, j) with an even i and an odd j are always tame, and those with an odd i and an even j are always not admissible; pairs (i, j) with both i and j even are admissible (or tame) if inequality (3.9) (respectively, (3.10)) holds; pairs (i, j) with both i and j odd are admissible (or tame) if inequality (3.11) (respectively, (3.12)) holds.

Finally, in the case where $\gamma < 0, \lambda < 0, c < 0$, and $d > 0$ (the tenth row of Table 1), pairs (i, j) with both i and j even are always tame, while those with both i and j odd are always inadmissible; pairs (i, j) with i odd and j even are admissible (or tame) if inequality (3.9) (respectively, (3.10)) holds; pairs (i, j) with i even and j odd are admissible (or tame) if inequality (3.11) (respectively, (3.12)) holds.

The following theorem is a strengthened version of the theorem proved in [4].

Theorem 2. (1) *A coding of any trajectory from $N \backslash O$ is an admissible one.*

(2) *For any infinite tame sequence $\mathcal{K} = \{k_s\}$, there is a continuum of trajectories with the coding $\mathcal{K}$ in N, and if $\mathcal{K}$ is a finite tame sequence of length n, then there are exactly 2^{n-1} rough and homoclinic with respect*

[10] Here the values of k_s in the admissible sequence should decrease with decrease in s, but $k_s \geq \bar{k}$ by condition.

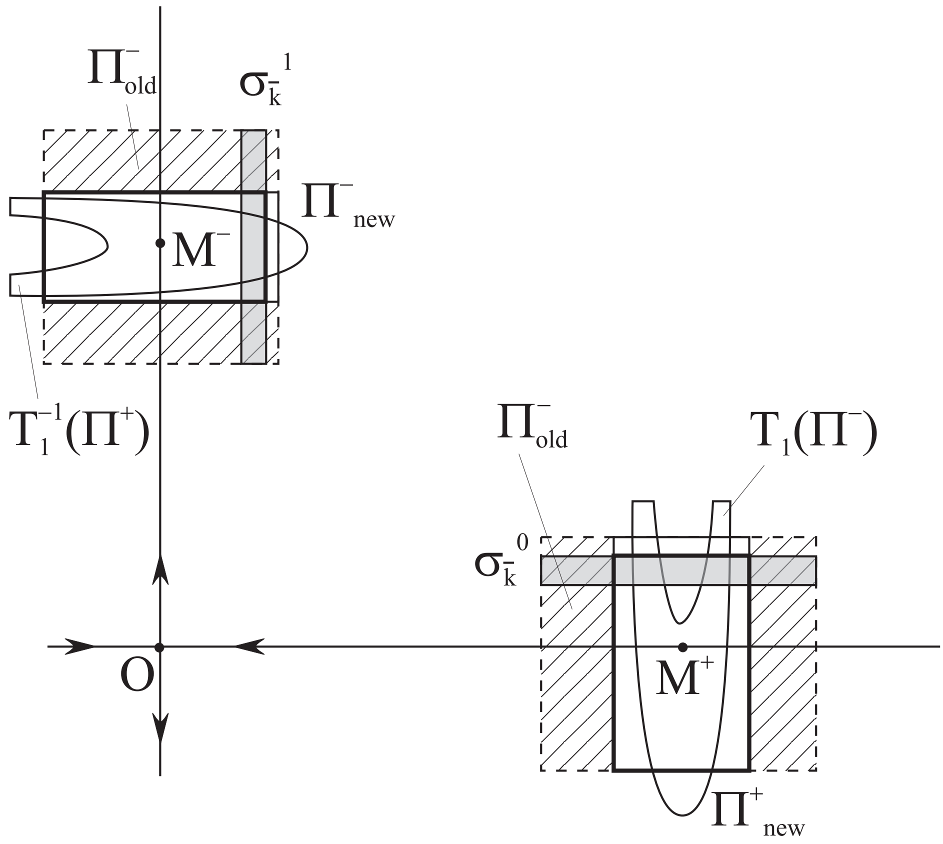

Fig. 11

to O trajectories with the coding $\mathcal{K}$ in N. Namely, for any sequence $\{\alpha_s\}$ composed of the symbols "1" and "2," there exists a unique trajectory such that the successive points M_s of intersection of this trajectory with Π^+ lie in the components $\sigma^{0\alpha_s}_{k_s,k_{s-1}}$ of the intersection $T_1\sigma^1_{k_{s-1}} \cap \sigma^0_{k_s}$.

Proof. By definition, we have

$$M_s \in T_1\sigma^1_{k_{s-1}} \cap \sigma^0_{k_s}, \tag{3.13}$$

where $\{M_s\}$ is the sequence of points of intersection of the trajectory with Π^+ and $\{k_s\}$ is the coding of the trajectory. It should be noted that this relation makes sense also in the case of finite (on the left or on the right) codings for $\sigma^1_\infty \equiv W^u_{\mathrm{loc}} \cap \Pi^-$ and $\sigma^0_\infty \equiv W^s_{\mathrm{loc}} \cap \Pi^+$.

Now the assertion of item 1 of the theorem follows immediately from (3.13) and item 2 of Lemma 1. To prove item 2 of the theorem, we note that if the sequence $\mathcal{K}$ is tame, then it follows from item 1 of Lemma 1 that for any pair (k_{s-1}, k_s), the intersection $T_1\sigma^1_{k_{s-1}} \cap \sigma^0_{k_s}$ consists of two components, $\sigma^{01}_{k_sk_{s-1}}$ and $\sigma^{02}_{k_sk_{s-1}}$. Therefore, the trajectory has the coding $\mathcal{K}$ if and only if, for some sequence $\{\alpha_s\}$ consisting of the symbols 1 and 2, we have

$$M_s \in \sigma^{0\alpha_s}_{k_sk_{s-1}} \tag{3.14}$$

for all points M_s. By definition,

$$M_s = \tilde{T}_{k_{s-1}} M_{s-1}; \tag{3.15}$$

moreover, since the pair (k_{s-1}, k_s) is tame, the mapping $\tilde{T}_{k_{s-1}}$ is a saddle one on $\sigma^{0\alpha_s}_{k_sk_{s-1}}$.

Thus, for each tame sequence $\mathcal{K}$ and for each sequence $\{\alpha_s\}$, we have the sequence of spaces $\sigma^{0\alpha_s}_{k_sk_{s-1}}$ and the sequence of saddle mappings $\tilde{T}_{k_s}$ acting from $\sigma^{0\alpha_s}_{k_sk_{s-1}}$ into $\sigma^{0\alpha_{s+1}}_{k_{s+1}k_s}$. Now if $\mathcal{K}$ is infinite on both sides, then

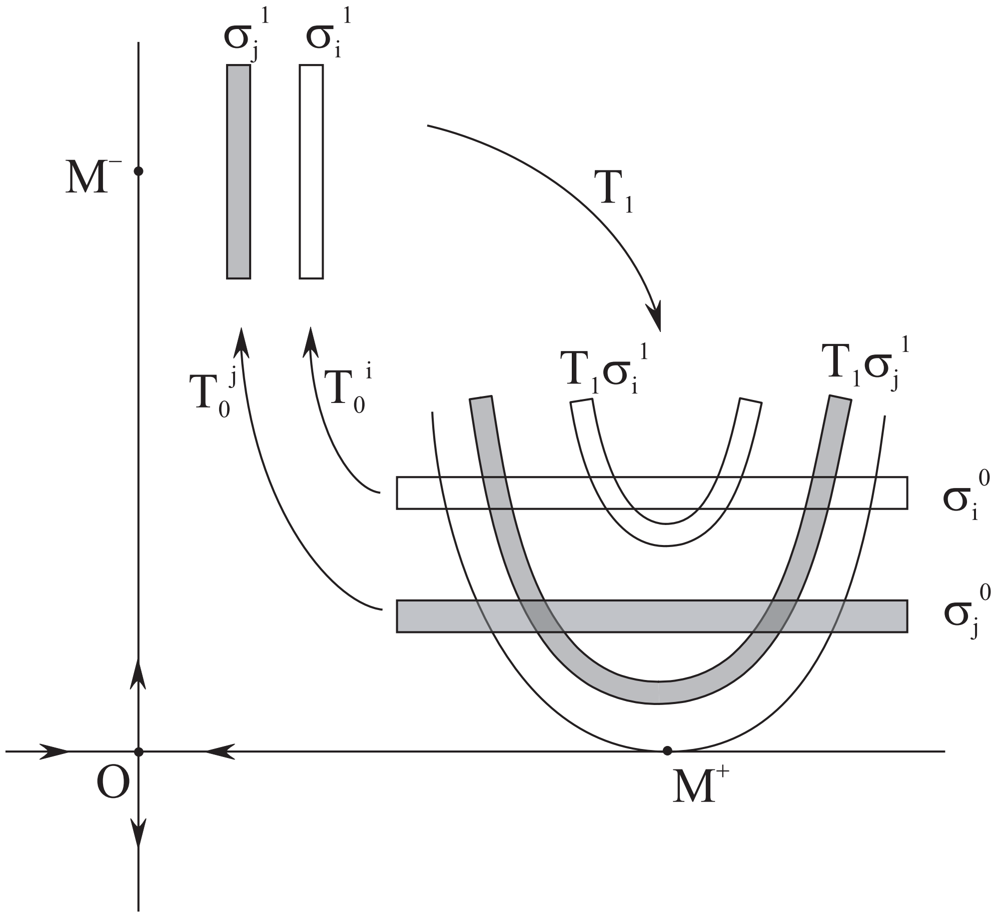

Fig. 12

the existence and uniqueness of the sequence M_s satisfying (3.14) and (3.15) follows from the lemma (see [18]) on a fixed point of a sequence of saddle mappings in the countable product of spaces.[11]

In order to prove the theorem for the case of a finite on the left tame sequence $\mathcal{K}$, we note that the curve $T_1(W^u_{\rm loc}\cap\Pi^-)$, in its intersection with any strip σ_j^0 such that the pair $(+\infty, j)$ is tame, is defined by the first formula in (3.7), in which it is necessary to set $i=+\infty$. Therefore, it is a *vertical curve*, i.e., a curve of the form $x'=\varphi(y')$, where the function φ is defined for all y' and has Lipschitz constant that is less than 1. By the property of saddle mappings, all further images of $W^u_{\rm loc}\cap\Pi^-$, which are defined according to the rule

$$l_s=(\tilde T_{k_{s-1}}l_{s-1})\cap\sigma^{0\alpha_s}_{k_sk_{s-1}},\quad l_1=\sigma^{0\alpha_1}_{k_1+\infty},$$

likewise are vertical curves, and the mappings $\tilde T_{k_s}|_{l_s}$ are expanding along the y' axis. The lemma on a fixed point of a sequence of contractive mappings in the countable product of spaces (see [18]), when applied to the sequence

$$l_1\xleftarrow{\tilde T^{-1}_{k_1}} l_2\xleftarrow{\tilde T^{-1}_{k_2}}\cdots\xleftarrow{\tilde T^{-1}_{k_{s-1}}} l_s\xleftarrow{\tilde T^{-1}_{k_s}}\ldots,$$

implies that the sequence of points M_s such that

$$M_s\in l_s,\qquad M_{s+1}=\tilde T_{k_s}M_s$$

exists and is unique. By construction, it is exactly the sequence of points of intersection of Π^+ with the trajectory with the coding $\mathcal{K}$; the existence and uniqueness of it for every given sequence $\{\alpha_s\}$ is stated in item 2 of Theorem 2.

[11] We look for a sequence of points $(x'_s,y'_s)\in\sigma^{0\alpha_s}_{k_sk_{s-1}}$ such that $\tilde T_{k_s}(x'_s,y'_s)\equiv(\bar x'_{s+1},\bar y'_{s+1})=(x'_{s+1},y'_{s+1})$, but since the mappings $\tilde T_{k_s}$ are all saddle ones, the cross mapping $(\{x_s\},\{\bar y_s\})\mapsto(\{\bar x_s\},\{y_s\})$ is a contractive mapping on the space of sequences with the norm $\max_s\max\{|x_s|,|y_s|\}$.

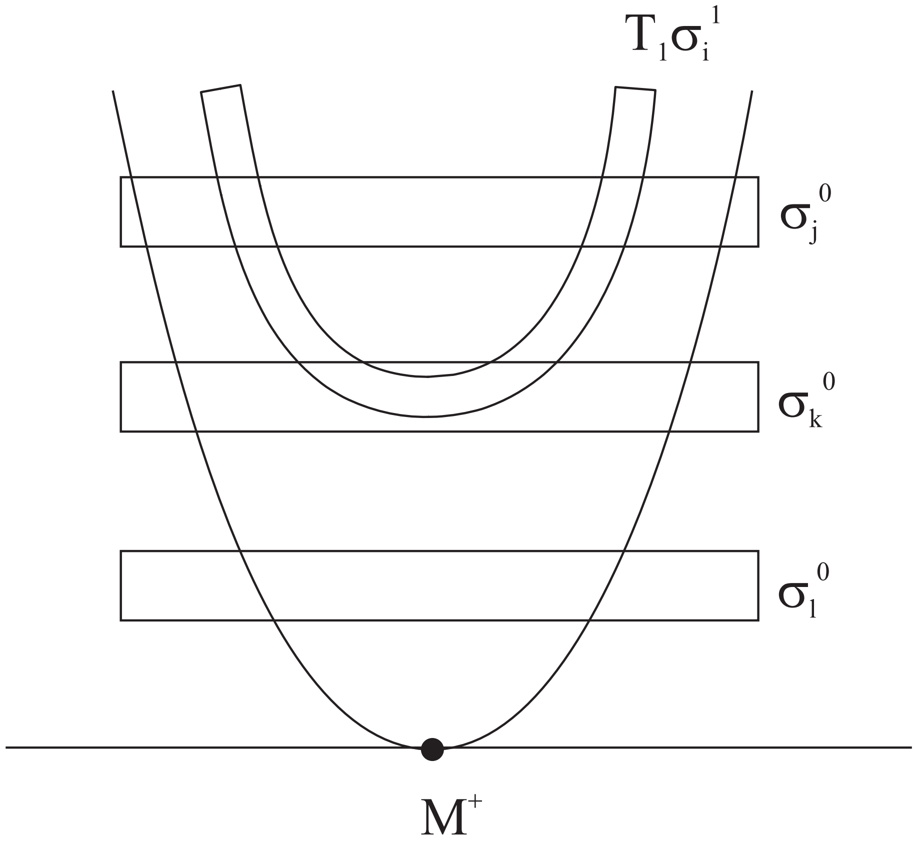

Fig. 13

In the case of a finite on the right tame sequence $\mathcal{K}$, the proof is similar; here the corresponding inverse images of $W^s_{\text{loc}} \cap \Pi^+$ are *horizontal curves*, i.e., curves of the form $y' = \psi(x')$, where the function ψ is defined for all x' and has Lipschitz constant that is less than 1.

Finally, in the case of a finite tame sequence $\mathcal{K}$, each homoclinic point is found as the unique point of the transversal intersection of the corresponding vertical curve (the image of $W^u_{\text{loc}} \cap \Pi^-$) with the corresponding horizontal curve (the inverse image of $W^s_{\text{loc}} \cap \Pi^+$). The theorem is proved.

We denote by $\tilde{N}$ the set of trajectories with tame codings from N that was constructed in Theorem 2. For any trajectory with a coding $\{k_s\}$ from $\tilde{N}$ one can construct a *refined coding* of this trajectory, i.e., the sequence of symbols "0," "1," and "2" that is obtained from the initial coding by replacing each symbol k_s with the sequence $\underbrace{0\ \ldots\ 0}_{k_s}1$ or with the sequence $\underbrace{0\ \ldots\ 0}_{k_s}2$, depending on to which component, $\sigma^{01}_{k_{s+1}k_s}$ or $\sigma^{02}_{k_{s+1}k_s}$, belongs the corresponding point M_{s+1}, at which the trajectory intersects Π^+ (if the first on the left (or the last on the right) $k_s = +\infty$, then we simply replace it by an infinite sequence of zeros). By Theorem 2, the trajectories from $\tilde{N}$ are uniquely reconstructed by the refined coding. Therefore, if we define a dynamical system on the set of points at which trajectories from $\tilde{N}$ intersect a small neighborhood U_0 of a point O in the following way: the mapping T_0 acts on points from $U_0 \backslash \Pi^-$, and the mapping T_1 acts on Π^-, then the *system thus obtained would be topologically conjugate to the mapping of the shift on the set of refined codings.*

In particular, a periodic trajectory corresponds to a periodic coding. Each of the points at which this trajectory intersects Π^+ is a fixed point for the product of the sequence of saddle mappings for the period of the coding. Since such a product is itself a saddle mapping, this periodic trajectory is a rough saddle one, and, for every point at which this trajectory intersects Π^+, the stable and unstable manifolds of these points are the horizontal and vertical curves, respectively. Heteroclinic (or homoclinic) trajectories of transversal intersection of stable and unstable manifolds of the corresponding periodic trajectories correspond to the codings that are asymptotically periodic on both sides.

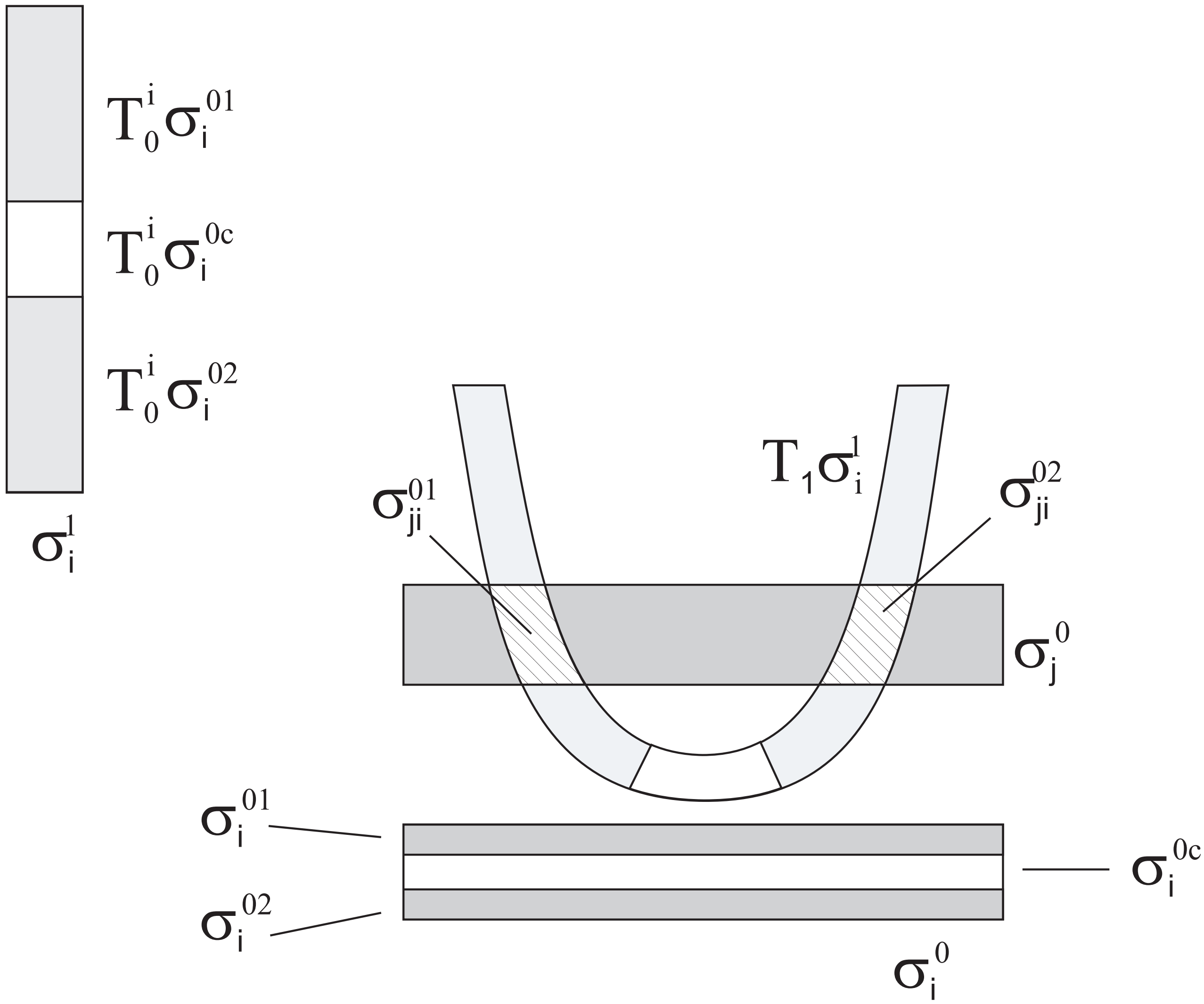

Fig. 14

We note that any point at which every trajectory from $\tilde{N}$ intersects Π^+ has, in general, stable and unstable manifolds that are horizontal and vertical curves, respectively, i.e., *all trajectories from $\tilde{N}$ are saddle ones, and the set $\tilde{N}$ is (nonuniformly) hyperbolic* (see [4, 5]).

Although the set $\tilde{N}$ may not coincide with the entire set $N\backslash\{O,\Gamma_0\}$, in any case, it is a good approximation of the latter set. Indeed, trajectories of the nonsaddle type from N should have at least two successive points of intersection with Π^+ that belong to the strips σ_i^0 and σ_j^0 whose numbers should satisfy the inequality

$$|j - i\theta + \tau| \leq \hat{S}|\gamma|^{-\overline{k}/2}, \tag{3.16}$$

which is equivalent to (3.4). The solutions of the latter inequality are given by the set of points (i, j) (with integral coordinates) that lie in a narrow band on the plane (the greater the $\bar{k}$, the narrower this band). It is clear that the structure of this set essentially depends on θ and τ. For instance, in the case where θ is rational, $\theta = p/q$, $\tau q \notin \mathbb{Z}$, this set is empty for a sufficiently large $\bar{k}$ (which depends on θ and τ). It is easy to see that in this case, the set of admissible coding (without $(+\infty, +\infty)$) coincides with the set of tame codings, and we have the following result, which was obtained earlier in [5].

Let $\theta = p/q$, and let $\tau q \notin \mathbb{Z}$. Then there exists $\overline{k} = \overline{k}(\theta, \tau)$ such that $N\backslash\{L_0, \Gamma_0\} = \tilde{N}$, that is, all trajectories from $N\backslash\Gamma_0$ are saddle trajectories.

From the geometric standpoint, the fact that the set of integer solutions to inequality (3.16) is empty for rational θ and suitable τ means that the vertices of all horseshoes get into the spaces between strips for such θ and τ. Since the number of strips is infinite, this situation is not rough. Thus, if θ is irrational, then inequality (3.16) has countably many integer solutions for any $\overline{k}$. Therefore, countably many strips and horseshoes can have wild intersections, which results in a rather nontrivial dynamics that will be considered below. It is important for us that the sets of integer solutions to inequalities (3.9)–(3.12) change under an arbitrarily small change in θ, and, by Theorem 2, the sets N and $\tilde{N}$ necessarily change.

4. Coexistence of Homoclinic Tangencies of the Third Class

Let f_0 be a system with a saddle periodic trajectory L_0 and with a nonrough trajectory Γ_0 of the third class homoclinic to L_0. The set of systems f that are C^r-close to f_0 and have a homoclinic trajectory Γ that is close to Γ_0 form a smooth Banach submanifold H of codimension one.

We consider a one-parameter family f_ν of systems containing f_0 on H. We assume that the *quantity* $\theta = -\ln|\lambda|/\ln|\gamma|$ *varies monotonically as* ν *changes*, i.e.,

$$\theta'(\nu) \neq 0. \tag{4.1}$$

As was noted in the preceding section, the sets of integer solutions to inequalities (3.9)–(3.12) change under an arbitrarily small change in θ. Therefore, for arbitrarily small changes in the values of the parameter, bifurcations should take place in the set N of all trajectories lying in a small neighborhood U of the set $L_0 \cup \Gamma_0$. Thus, the following result holds.

On the interval of variability of ν, the subsets B_1^+, B_1^-, and B_2 are dense. These subsets are such that

(1) *the system f_ν has a nonrough twicely going around*[12] *periodic trajectory of saddle-node type for $\nu \in B_1^+$, and it has a nonrough twicely going around periodic trajectory with multiplier equal to -1 for $\nu \in B_1^-$;*

(2) *for $\nu \in B_2$, the system f_ν has infinitely many stable twicely going around periodic trajectories in U.*

Item (1) was proved in [4, 12], and item (2) was proved in [12]. Our further considerations are based on the following result of the same kind.

Theorem 3. ([11]). *The values of the parameter ν for which f_ν has a trajectory of a quadratic homoclinic tangency in U that is different from Γ are dense on the interval of variability of ν.*

Proof. We take arbitrary $\nu_1 \neq \nu_2$ that are sufficiently close to each other, so that $\theta_1 \equiv \theta(\nu_1) \neq \theta(\nu_2) \equiv \theta_2$, and, for definiteness, we assume that $\theta_1 > \theta_2$. We consider the case where $\lambda > 0, \gamma > 0, c > 0$, and $d > 0$ (Fig. 5d) first.

We note that for any arbitrarily large $\bar{k}$, there exist integers $i \geq \bar{k}$ and $j \geq \bar{k}$ such that

(1) the pairs (i, i) and (j, i) are tame for all ν, i.e., the horseshoes $T_1\sigma_i^1$ and $T_1\sigma_j^1$ intersect the strip σ_i^0 tamely (Fig. 15);

(2) for $\nu = \nu_1$, the pair (i, j) is tame, and for $\nu = \nu_2$ it is inadmissible, i.e., for $\nu = \nu_1$, the horseshoe $T_1\sigma_i^1$ intersects the strip σ_j^0 tamely (Fig. 15a), and for $\nu = \nu_2$ it does not intersect this strip at all (Fig. 15b).

To attain this (see the preceding section), it is sufficient to choose $i \geq \bar{k}$ and $j \geq \bar{k}$ in such a way that the following inequalities are fulfilled simultaneously:

$$j - i\theta_1 + \tau(\nu_1) < -S|\gamma(\nu_1)|^{-\bar{k}/2} \tag{4.2}$$

and

$$j - i\theta_2 + \tau(\nu_2) > S|\gamma(\nu_2)|^{-\bar{k}/2}, \tag{4.3}$$

where (4.2) is equivalent to (3.2), and (4.3) is equivalent to (3.3). It is clear that the set of such pairs (i, j) is countable for any $\theta_1 > \theta_2$. We fix one such pair and denote it by (i^*, j^*).

Let p_i be a point on the strip σ_i^0 that is a fixed point for the first return mapping $T_i \equiv T_1 T_0^i : \sigma_i^0 \to \sigma_i^0$. In general, for each sufficiently large i, there are exactly two such points in the case under consideration. In the class of diffeomorphisms on H, both these points are rough saddle ones. Let p_i be one of them.

We denote by $\hat{W}_i^s$ a connected component of the set $W^s(p_i) \cap \sigma_i^0$ that contains the point p_i. We show that for any sufficiently large i, the curve $\hat{W}_i^s$ has the form of a segment that crosses the strip σ_i^0 horizontally

[12] That is, intersecting Π^+ exactly at two points.

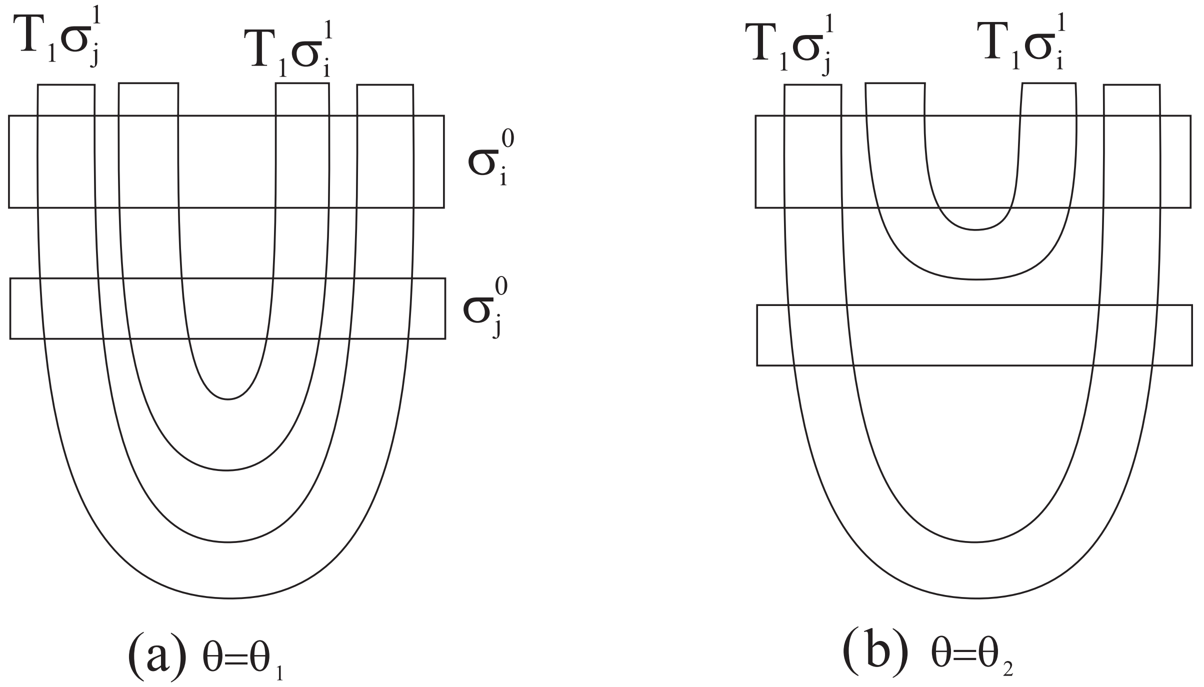

Fig. 15

(and which is asymptotically close to the straight line $y = \gamma^{-i}y^-$) (Fig. 16). We denote by $\hat{W}_i^u$ the connected component of the set $W^u(p_i) \cap (T_1(\sigma_i^1)$ that contains p_i. We show that $\hat{W}_i^u$ has the form of a segment of a parabola (defined by an equation of the type $y = cx^+\lambda^i + db^{-2}(x - x^+)^2 + ...$) which crosses the horseshoe $T_1(\sigma_i^1)$ along its whole length (Fig. 16).

When $\nu = \nu_2$, since the inequality (4.3) holds for $i = i^*$ and $j = j^*$, we have $\hat{W}_{i^*}^u \cap \hat{W}_{j^*}^s = \emptyset$ because the horseshoe $T_1(\sigma_{i^*}^1)$ and the strip $\sigma_{j^*}^0$ do not intersect (Fig. 17a). On the other hand, for $\nu = \nu_1$, the curves $\hat{W}_{i^*}^u$ and $\hat{W}_{j^*}^s$ intersect exactly at two points, since the horseshoe $T_1(\sigma_{i^*}^1)$ and the strip $\sigma_{j^*}^0$ intersect tamely (for $i = i^*$, $j = j^*$, and $\theta = \theta_1$, inequality (4.2) holds). Thus, we obtain that under a continuous change in the value of the parameter ν, there exist some $\nu_* \in (\nu_1, \nu_2)$ such that $f(\nu_*)$ has a nonrough heteroclinic trajectory in which points the unstable manifold of the point p_{i^*} and the stable manifold of the point p_{j^*} are tangent to each other. Since the intersection of the horseshoe $T_1(\sigma_{j^*}^1)$ with the strip $\sigma_{i^*}^0$ is always tame (because $j^* > i^*$), it follows that there exists a rough heteroclinic trajectory lying in the intersection of the stable manifold of the point p_{i^*} with the unstable manifold of the point p_{j^*} (in particular, the curves $\hat{W}_{j^*}^u$ and $\hat{W}_{i^*}^s$ always intersect transversally). Thus, the diffeomorphism $f(\nu_*)$ has a nonrough heteroclinic contour containing twicely going around periodic trajectories that intersect Π^+ at the points p_{i^*} and p_{j^*}, respectively, and two heteroclinic trajectories, one of which is nonrough (Fig. 17b). This contour is the simplest one in the sense that already the curves $\hat{W}_{i^*}^u \subset W^u(p_{i^*})$ and $\hat{W}_{j^*}^s \subset W^s(p_{j^*})$ have tangency. Heteroclinic contours of this type are denoted by C_{ij}.

It is clear that we can obtain homoclinic tangencies of the stable and unstable manifolds of either the point p_{i^*} or the point p_{j^*} by an arbitrarily small (smooth) perturbation of the contour $C_{i^*j^*}$. The simplest tangency of this class, which, in addition, can be directly calculated (see below), is the homoclinic tangency of the curve $\hat{W}_{i^*}^u \subset W^u(p_{i^*})$ with one of the connected components of the set $W^s(p_{i^*}) \cap \sigma_{j^*}^0$, which are defined in the following way. Since the horseshoe $T_1(\sigma_{j^*}^1)$ intersects the strip $\sigma_{i^*}^0$ tamely, the intersection $T_1(\sigma_{j^*}^1) \cap \hat{W}_{i^*}^s$ consists of two segments $l_{i^*}^1$ and $l_{i^*}^2$ (Fig. 18). The inverse images of these segments with respect to the mapping $T_0^{-j^*}T_1^{-1} : \sigma_{i^*}^0 \to \sigma_{j^*}^0$ are the horizontal curves $\hat{W}_{i^*j^*}^{s1}$ and $\hat{W}_{i^*j^*}^{s2}$ on the strip $\sigma_{j^*}^0$ (Fig. 17), which are the pieces of the manifold $W^s(p_{i^*})$. The instances of the corresponding homoclinic tangencies are shown in Fig. 18a and Fig. 18b.

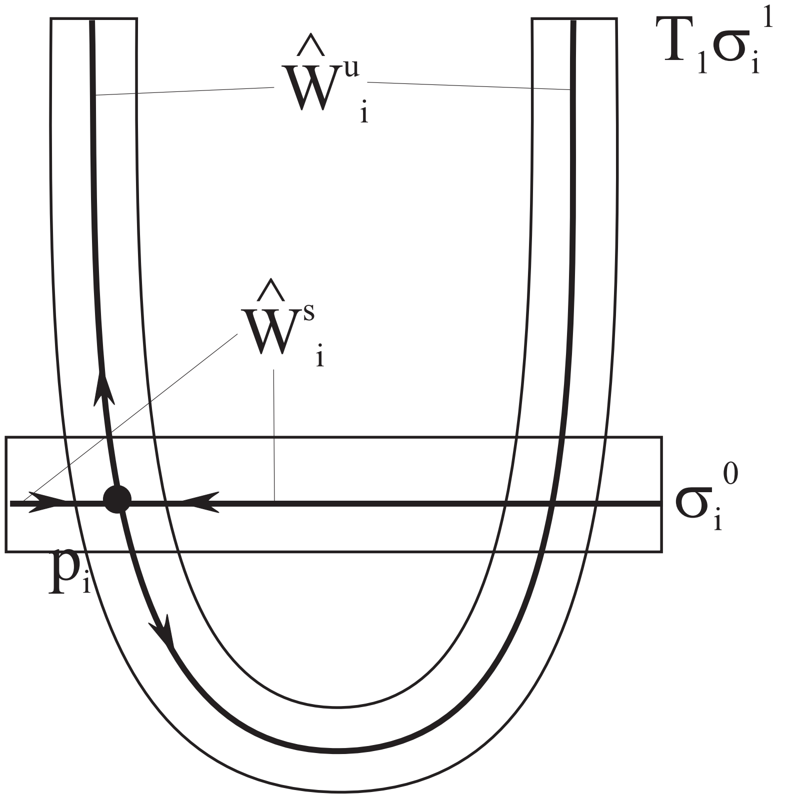

Fig. 16

To complete the proof of the theorem, it remains to make the corresponding calculations (for the curves $\hat{W}_i^s$, $\hat{W}_i^u$, $\hat{W}_{i^*j^*}^{s1}$, and $\hat{W}_{i^*j^*}^{s2}$) in the case where $\gamma > 0, \lambda > 0, c > 0$, and $d > 0$. We note that $\theta > 1$ because $|\lambda\gamma| < 1$. Since j^* and i^* are sufficiently large, it follows from (4.3) that $j^* > i^*$. This implies that the pairs (i^*, i^*) and (j^*, i^*) are always tame because they satisfy inequality (3.2), in which it is necessary to set $j = i = i^*$ in the first case and $j = i^*$ and $i = j^*$ in the second case.

In order to find the equations of the curves $\hat{W}_{i^*}^s$ and $\hat{W}_{i^*}^u$, we use the representation of the mapping $\tilde{T}_i \equiv T_1 T_0^i$ on the strip $\sigma_{i^*}^0$ in the form (3.7) (in which it is necessary to set $j = i = i^*$). Using this formula, we immediately determine the coordinates of the point p_{i^*}:

$$x = x^+ + O(|\gamma|^{-i^*/2}),\ y = \gamma^{-i^*} y^- + O(|\gamma|^{-3i^*/2}),$$

since the "cross" coordinates (x', y') and the proper coordinates (x, y) on the strip σ_i^0 are related (see (2.2) and (2.3)) by the following formula:

$$x = x', \qquad y = \gamma^{-i} y' + \gamma^{-i} \eta_i(x', y'), \tag{4.4}$$

where $\eta_j \to 0$ as $j \to +\infty$, together with all its derivatives up to the order $(r-1)$ (possibly, except for the $(r-1)$th derivative with respect to the parameter ν), and where $\|\eta_i\|_{C^{r-2}} = O(|\gamma|^{-i})$.

As was noted in the preceding section, $\hat{W}_i^s$ is a horizontal curve in the coordinates (x', y') (i.e., it is a curve of the form $y' = y^- + \varphi(x')$, where the function φ is defined for all $|x - x^+| \le \rho_{\bar{k}}$ and $|\varphi| \le \rho_{\bar{k}}$, $\left|\dfrac{d}{dx'}\varphi\right| < 1$; here $\rho_{\bar{k}} = C|\gamma|^{-\bar{k}/2}$ is the size of the neighborhoods $\Pi^{\pm}$). By the definition of a stable manifold, we have that

$$W_i^s \subset \tilde{T}_i^{-1} W_i^s,$$

and formula (3.7) immediately implies that $\tilde{T}_i^{-1} W_i^s \cap \sigma_i^0$ is given by the following equation:

$$y = \gamma^{-i} y^- + O(|\gamma|^{-3i/2}), \tag{4.5}$$

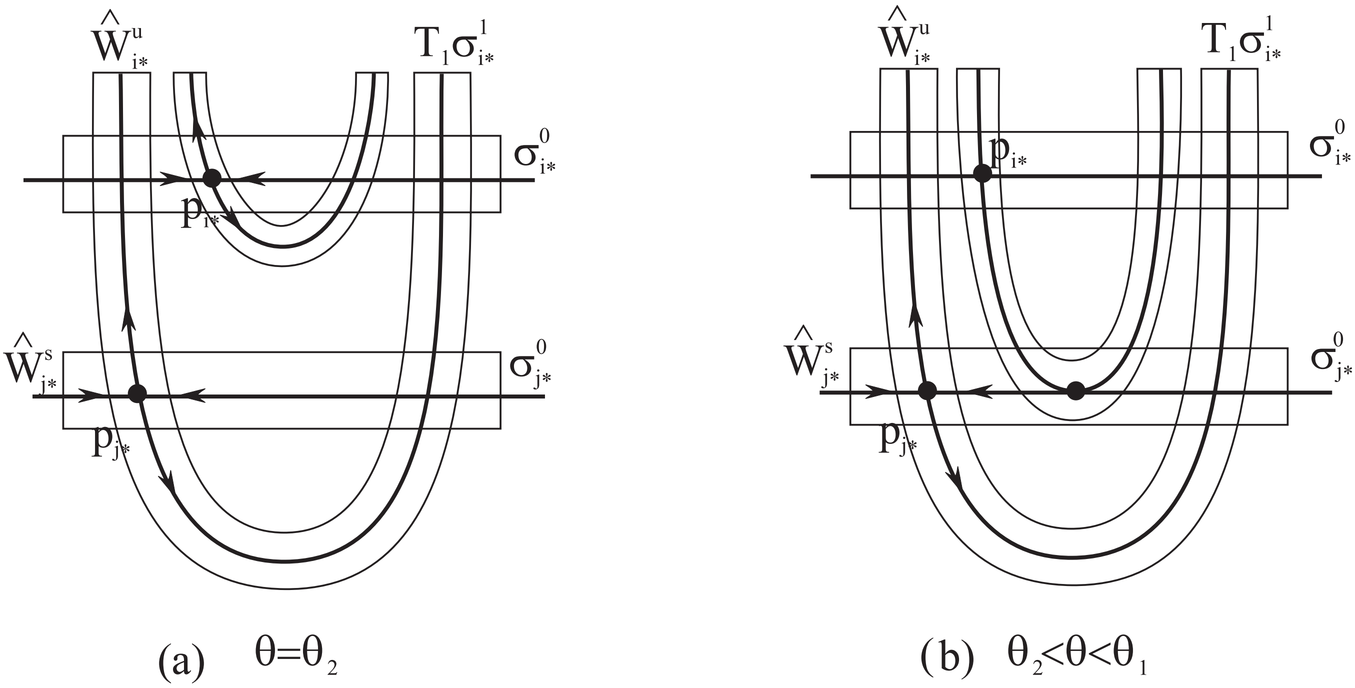

Fig. 17

i.e., in the given coordinates, it is the curve that deviates from the straight line $y' = \gamma^{-i}y^-$ not more than by a distance of about $O(|\gamma|^{-3i/2})$, doing so together with all its derivatives.[13]

By the property of saddle mappings, each of the two components of the inverse image of

$$\tilde{T}_{j^*}^{-1}\hat{W}^s_{i^*} \equiv T_0^{-j^*}(T_1^{-1}\hat{W}^s_{i^*})$$

is also a horizontal curve (in the cross coordinates (x', y')). We recall that the pair (j^*, i^*) is tame, and, therefore, the mapping $\tilde{T}_{j^*}$ is a saddle mapping on the set $\tilde{T}_{j^*}^{-1}(T_1\sigma^1_{j^*} \cap \sigma^0_{i^*})$. We denote these components by $\hat{W}^{s1}_{i^*j^*}$ and $\hat{W}^{s2}_{i^*j^*}$. The mapping $\tilde{T}_{j^*}$ is given by formula (3.7), in which it is necessary to set $i = j^*$ and $j = i^*$. Moreover (we recall that $j^* > i^*$, and, therefore, $|\gamma|^{-i^*} \gg |\lambda|^{j^*}$), the inverse image of curve (4.5) under the action of $\tilde{T}_{j^*}$ is also given by an equation of the form (4.5), but already in the coordinates (x', y'), which are related to the strip $\sigma^0_{j^*}$ by the formula (4.4) with $i = j^*$. Correspondingly (see (4.5)), the curves $\hat{W}^{s1}_{i^*j^*}$ and $\hat{W}^{s2}_{i^*j^*}$ have the form

$$y = \gamma^{-j^*}y^- + |\gamma|^{-j^*}p_\alpha(x, \nu), \tag{4.6}$$

where $\alpha = 1, 2$ and (see (2.3))

$$\|p_\alpha\|_{C^{r-2}} = O(|\gamma|^{-i^*/2}) \ \text{ and } \ \frac{d^{r-1}p_\alpha}{dx^{r-1}} \to 0 \ \text{ as } i^*, j^* \to +\infty. \tag{4.7}$$

We denote by W^u_i a connected component of the unstable manifold $W^u(p_i) \cap \sigma^0_i$ that contains p_i. Since p_i is a fixed point of the saddle mapping, W^u_i is a vertical curve of the form $x' = x^+ + \psi(y')$, where $|\psi| \le \rho_{\bar{k}}$ and $\left|\dfrac{d}{dx'}\psi\right| < 1$. Since

$$W^u_i \subset \tilde{T}_i W^u_i,$$

it immediately follows from formula (3.7) (in which it is necessary to set $i = j$) that W^u_i is given by the equation

$$x' = x^+ + O(|\gamma|^{-i/2}).$$

[13]Up to the order $(r-1)$ (possibly, except for the $(r-1)$th derivative with respect to the parameter ν; this derivative may not exist because we are bringing the system to the form (2.1); see [29] for details).

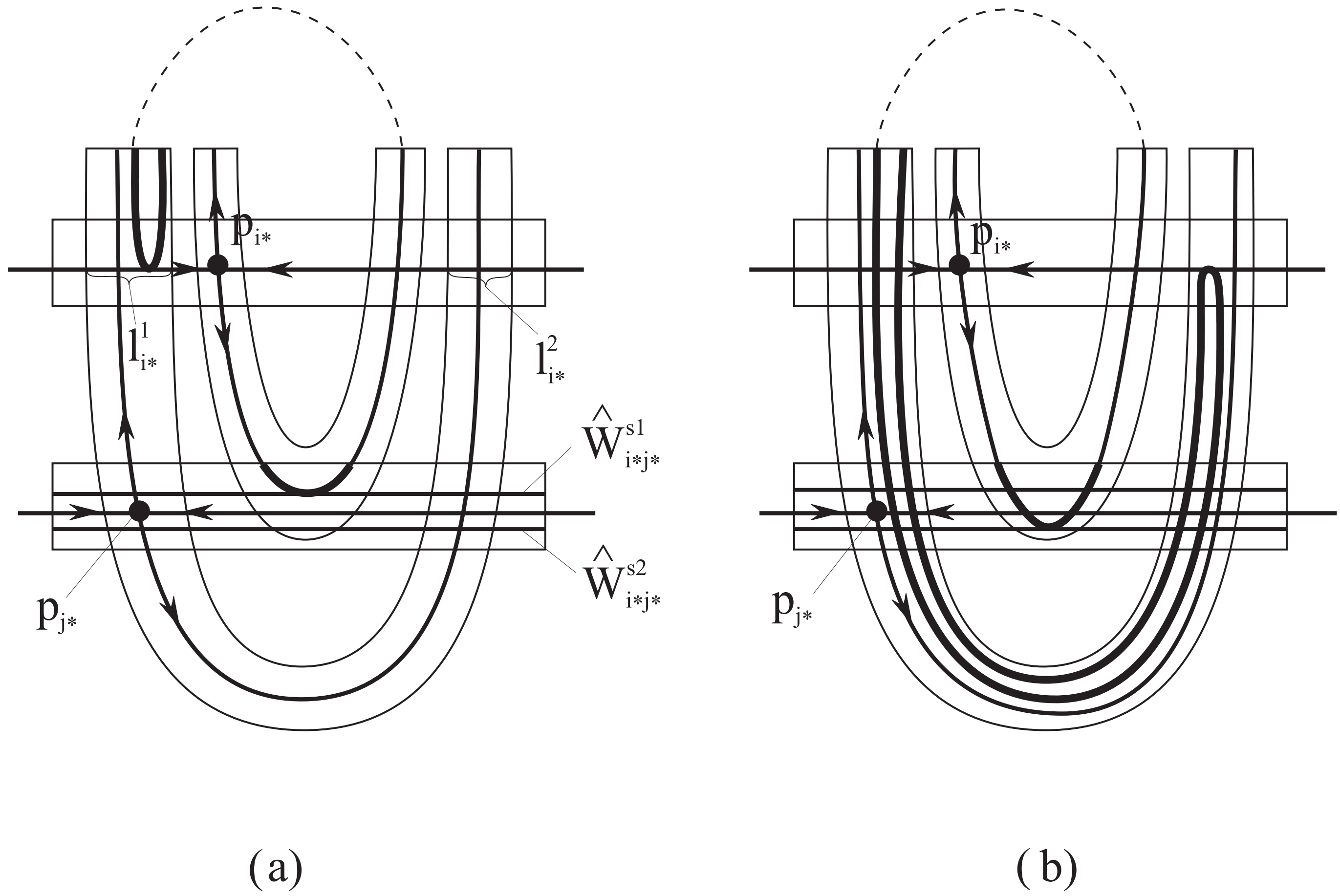

Fig. 18

Now, according to formula (2.2) (in which it is necessary to set $k = i, y_k = y' = y, x_0 = x'$, and $x_k = x$), the image of $T_0^i W_i^u$ in σ_i^1 has the form

$$x = \lambda^i x^+ + |\lambda|^i q(y, \nu), \tag{4.8}$$

where (see (2.3))

$$\|q\|_{C^{r-2}} = O(|\gamma|^{-i/2}) \text{ and } \frac{d^{r-1}q}{dy^{r-1}} \to 0 \text{ as } i \to +\infty. \tag{4.9}$$

According to formula (2.5), the image of $T_1 T_0^{i^*} W_{i^*}^u$ in Π^+ has the form

$$y = cx^+\lambda^{i^*} + \frac{d}{b^2}(x - x^+)^2 + o((x - x^+)^3) + |\lambda|^{i^*} O(|x - x^+| + |\gamma|^{-i^*/2}), \tag{4.10}$$

i.e., this curve (we denote it by $\hat{W}_{i^*}^u$) is close to the central line of the horseshoe $T_1\sigma_{i^*}^1$ (Fig. 16).

We have chosen i^* and j^* in such a way that the horseshoe $T_1\sigma_{i^*}^1$ does not intersect the strip $\sigma_{j^*}^0$ for $\nu = \nu_2$ (Fig. 15b), but it intersects this strip tamely for $\nu = \nu_1$ (Fig. 15). Thus, the piece $\hat{W}_{i^*}^u$ of the unstable manifold of the point p_{i^*} does not intersect the piece $\hat{W}_{i^*j^*}^{s1,2}$ of the stable manifold of the same point for $\nu = \nu_2$, and it intersects this piece transversally at two points for $\nu = \nu_1$. Therefore, for some ν that lies between ν_1 and ν_2, the curves $\hat{W}_{i^*}^u$ and $\hat{W}_{i^*j^*}^{s1,2}$ should be tangent to each other.

Comparing the derivatives of the right-hand sides of (4.10) and (4.6) with respect to x, we see that the curves $\hat{W}_{i^*}^u$ and $l_s^{1,2}(i^*)$ should have a tangency at the point with the coordinate

$$x = x^+ + O(|\lambda|^{i^*} + |\gamma|^{-(j^*+i^*/2)}).$$

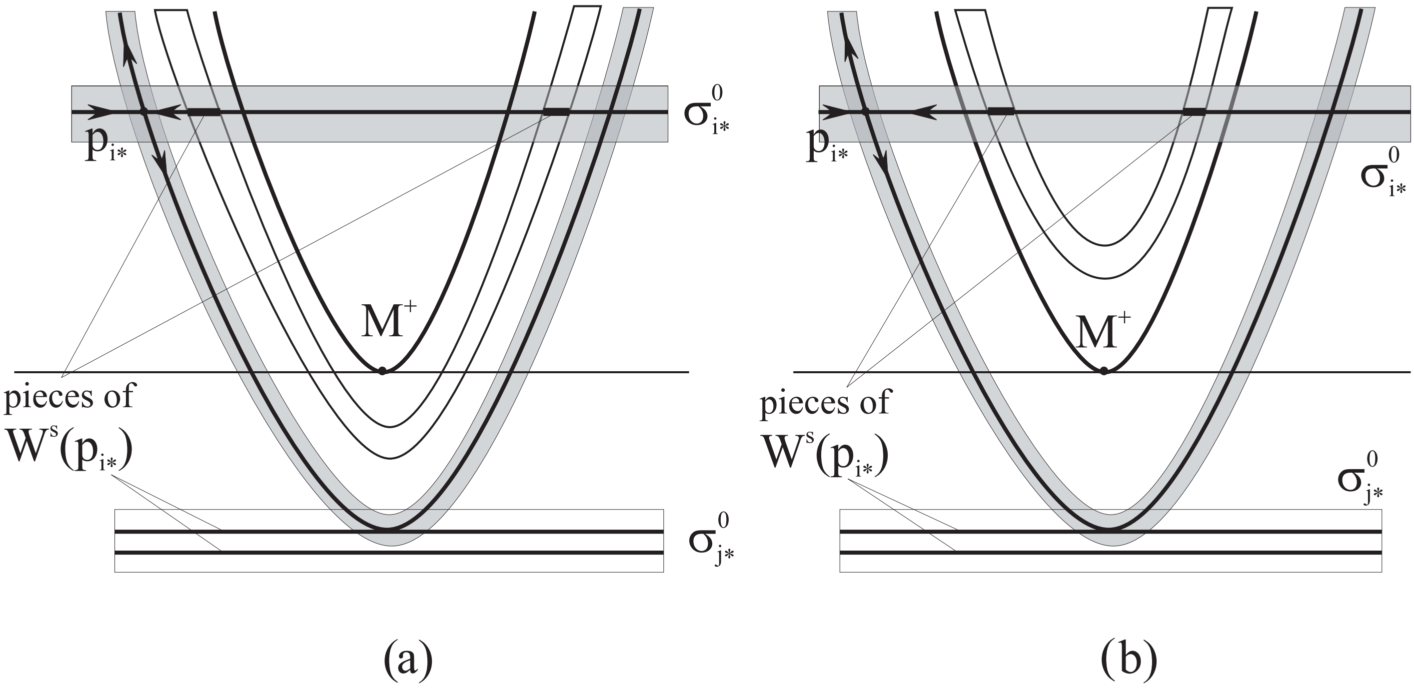

Fig. 19

The difference between the y coordinates of the corresponding points on the curves $\hat{W}^u_{i^*}$ and $\hat{W}^{s1,2}_{i^*j^*}$ is equal to

$$D_{i^*j^*} = \gamma^{-j^*}y^- - cx^+\lambda^{i^*} + |\gamma|^{-i^*/2}O(|\lambda|^{i^*} + |\gamma|^{-j^*}). \tag{4.11}$$

It is seen from (4.2) and (4.3) that this quantity changes sign when ν changes from ν_1 to ν_2. By construction, the value of ν for which $D_{i^*j^*}$ vanishes corresponds to the tangency of the curves $\hat{W}^u_{i^*}$ and $\hat{W}^{s1,2}_{i^*j^*}$, and since they are pieces of the stable and unstable manifolds of the saddle periodic point p_{i^*}, we have the desired homoclinic tangency. Since the second derivative of the right-hand side of (4.10) is bounded away from zero and the second derivative of the right-hand side of (4.6) is small, the obtained tangency is quadratic. It follows from (4.1) that for a sufficiently large i^*, the derivative $\dfrac{d}{d\nu}D_{i^*j^*}$ is different from zero for $D_{i^*j^*} = 0$. This means that the family f_ν is transversal on H to the bifurcation surface of systems (of codimension one on H) that have two nonrough homoclinic trajectories, one of which is Γ, while the other corresponds to homoclinic tangencies close to that constructed above (i.e., to the tangency of the pieces $\hat{W}^u_{i^*}$ and $\hat{W}^{s1,2}_{i^*j^*}$ of the manifolds of the point that is close to p_{i^*}).

It remains to consider the cases with negative λ or γ. In the cases that correspond to the sixth, seventh, and ninth columns of Table 1, the proof is completely similar if we take the corresponding i^* and j^* to be even. A nonrough contour $C_{i^*j^*}$ also exists in each of these cases.

The cases that correspond to the eighth and tenth columns of Table 1 (where $\lambda > 0, \gamma < 0, d > 0, c < 0$ and $\lambda < 0, \gamma < 0, d > 0, c < 0$, respectively) are somewhat different from the one considered above: here i^* is even, j^* is odd, and there is no fixed point p_{j^*} because $T_1(\sigma^1_{j^*})\cap\sigma^0_{j^*} = \emptyset$ (Fig. 19). Thus, there are no nonrough contours $C_{i^*j^*}$ in these cases, but homoclinic tangencies of the curves $\hat{W}^u_{i^*}$ and $\hat{W}^{s1,2}_{i^*j^*}$ still exist (Fig. 19), and the proof of the existence of such tangencies is similar to that presented above.[14] This completes the proof of the theorem.

Remark. According to Theorem 1, it follows from the transversality of the family of f_ν on H to the bifurcation surfaces corresponding to the constructed homoclinic tangencies that the values of ν that correspond to the homoclinic tangencies of the third class are dense in the family of f_ν.

[14]We also note that in both of these cases, the proof of the corresponding assertions for f cannot be obtained automatically by passing to f^2.

We emphasize that the secondary homoclinic tangencies such as those that were constructed above appear necessarily in any family of systems on H in which θ varies monotonically. The value of θ depends only on the values of the multipliers at the point O; therefore, new homoclinic tangencies can be obtained by small perturbations of a system that are localized in an arbitrarily small neighborhood of the point O.

Theorem 4. *Systems with infinitely many saddle periodic trajectories each of which has a nonrough homoclinic trajectory of the third class are dense on the film H formed by systems with homoclinic tangencies of the third class.*

Proof. In order to obtain new homoclinic tangencies, we use small perturbations under which a system does not leave H and which are localized in an arbitrarily small neighborhood of the point O. These perturbations are found by construction.

Namely, we take an arbitrary $\delta > 0$ and assume that δ_m is a sequence of positive numbers such that

$$\sum_{m=1}^{\infty} \delta_m < \delta. \tag{4.12}$$

For an arbitrary system f_0 on H, we construct the system f^*, which is δ-close to f_0 in the C^r-topology and has infinitely many homoclinic tangencies. We are perturbing f_0 locally in a small neighborhood of the point O; therefore, the global mapping T_1 does not change and we only keep track of the mapping T_0.

In order not to lose smoothness, we do not bring the mapping T_0 to the form (2.1). Instead, we rectify the local stable and unstable manifolds of the point O (this can be done by a C^r-smooth change of coordinates), so that the mapping T_0 becomes

$$\bar{x} = \lambda x + h(x,y), \quad \bar{y} = \gamma y + g(x,y), \tag{4.13}$$

where $h(0,y) = 0$ and $g(x,0) = 0$. We choose f^* in such a way that the mapping T_0 takes the form

$$\bar{x} = \lambda x + h(x,y), \quad \bar{y} = (\gamma + \sum_{m=1}^{\infty} \nu_m \delta_m \chi(\kappa_m^{-1} \cdot (x,y)))y + g(x,y), \tag{4.14}$$

where the positive quantities κ_m and ν_m are to be determined. In any case, we require that the series in (4.14) converge, while the function $\chi(\cdot)$, which is a C^∞-smooth function on the plane, positive in the unit circle, and identically equal to zero outside this circle, should satisfy the condition $\|\chi\|_{C^r} \le 1$ under this procedure (the existence of such functions is a standard fact). We choose a sufficiently small κ_m such that $\nu_m \in [0, \kappa_m^r]$, so this perturbation is localized in a small neighborhood of the point O, and its C^r-norm is less than some given δ. Specifically, the perturbation is localized in a neighborhood of size $\sup_{m\ge 1} \kappa_m$. In addition, by virtue of (4.12) and the choice of a small ν_m, we also attain the uniform C^r-convergence of the series in (4.14). Since we do not intend to change the mapping T_1 that acts from a small neighborhood of the homoclinic point M^- into a small neighborhood of the homoclinic point M^+, we should require that

$$\sup_{m\ge 1} \kappa_m < d_0 \equiv \min(x^+, y^-). \tag{4.15}$$

The positions of the local stable and unstable manifolds of the point O do not change under these perturbations (these manifolds remain to be the straight lines); therefore, the initial homoclinic tangency of $T_1(W^u_{\text{loc}} \cap \Pi^+)$ with W^u_{loc} does not vanish (we recall that the mapping T_1 does not change).

The system f^* serves as the limit for the sequence of systems f_M with one and the same mapping T_1 and with the mapping T_0 represented as

$$\bar{x} = \lambda x + h(x,y), \quad \bar{y} = (\gamma + \sum_{m=1}^{M} \nu_m \delta_m \chi(\kappa_m^{-1} \cdot (x,y)))y + g(x,y). \tag{4.16}$$

We choose the values of ν_m and κ_m successively, so that for each $M \ge 1$, the system f_M has M saddle periodic trajectories with homoclinic tangencies of the third class (in addition to L_0 and Γ_0). These trajectories are separated from the point O; therefore, any perturbation of the system that is localized in some small

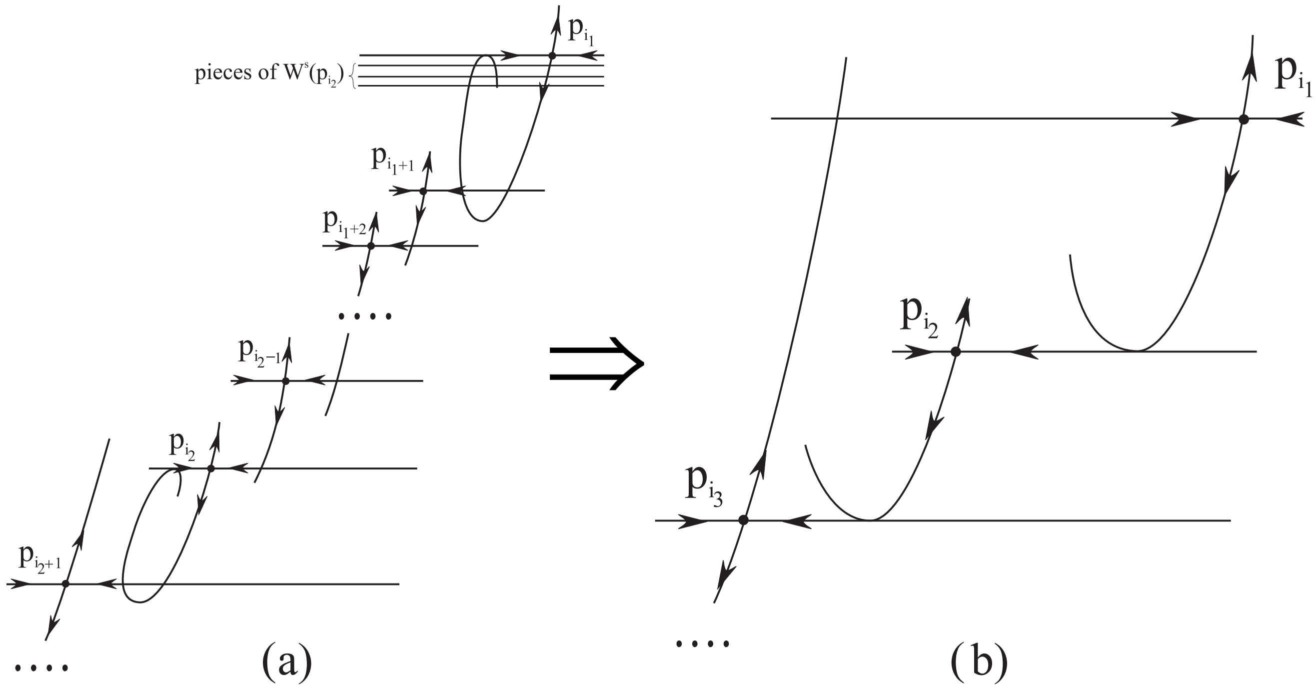

Fig. 20

neighborhood of the point O does not split these tangencies. We denote the size of this neighborhood by d_M and require

$$\sup_{m>M} \kappa_m < d_M. \tag{4.17}$$

This means that the difference between f_M and all subsequent systems $f_{M+1}, f_{M+2}, \ldots$, including f^*, is localized in a neighborhood of the point O of size d_M. Therefore, M homoclinic tangencies in the system f_M do not vanish under the passage to the subsequent systems $f_{M+1}, f_{M+2}, \ldots$ Thus, the limit system f^* has infinitely many homoclinic tangencies, which is the required result.

Conditions (4.15) and (4.17) gives an inductive rule for finding the appropriate values of κ_m (the scales to which our successive perturbations are localized). In order to find ν_m (the size of perturbations), we proceed by induction. We suppose that, for some $M \geq 1$, all the values ν_m, κ_m with $m \leq M-1$ are already determined. We choose $\kappa_M < \min(d_0, \ldots, d_{M-1})$ (i.e., in such a way that (4.15) and (4.17) are satisfied) and consider a one-parameter family of systems $f(\nu)$ that are different from f_{M-1} only in a neighborhood of the point O of size κ_M such that the mapping T_0 has the form

$$\begin{aligned} \bar{x} &= \lambda x + h(x,y), \\ \bar{y} &= (\gamma + \textstyle\sum_{m<M} \nu_m \delta_m \chi(\kappa_m^{-1} \cdot (x,y)) \\ &\quad + \nu \delta_M \chi(\kappa_M^{-1} \cdot (x,y)))y + g(x,y); \end{aligned} \tag{4.18}$$

here ν changes from 0 to κ_M^r (where r is the smoothness of system).

For $\nu = 0$, this system coincides with the system f_{M-1}, which has $M-1$ secondary homoclinic tangencies of the third class by the inductive assumption. Since the difference between this system and f_{M-1} for all ν is localized in a neighborhood of the point O of a sufficiently small size κ_M, these homoclinic tangencies persist for all ν and lie at finite distance from O. Moreover, θ varies continuously with ν:

$$\theta = -\frac{\ln \lambda}{\ln(\gamma + (\nu\delta_M + \sum_{m<M} \nu_m \delta_m)\chi(0))}.$$

By Theorem 3, we obtain that there exist values of ν arbitrarily close to $\nu = 0$ and such that the system has one more homoclinic tangency of the third class that is arbitrarily close to the original homoclinic trajectory Γ_0. Choosing such sufficiently small ν as the desired ν_M, we obtain the system f_M having already

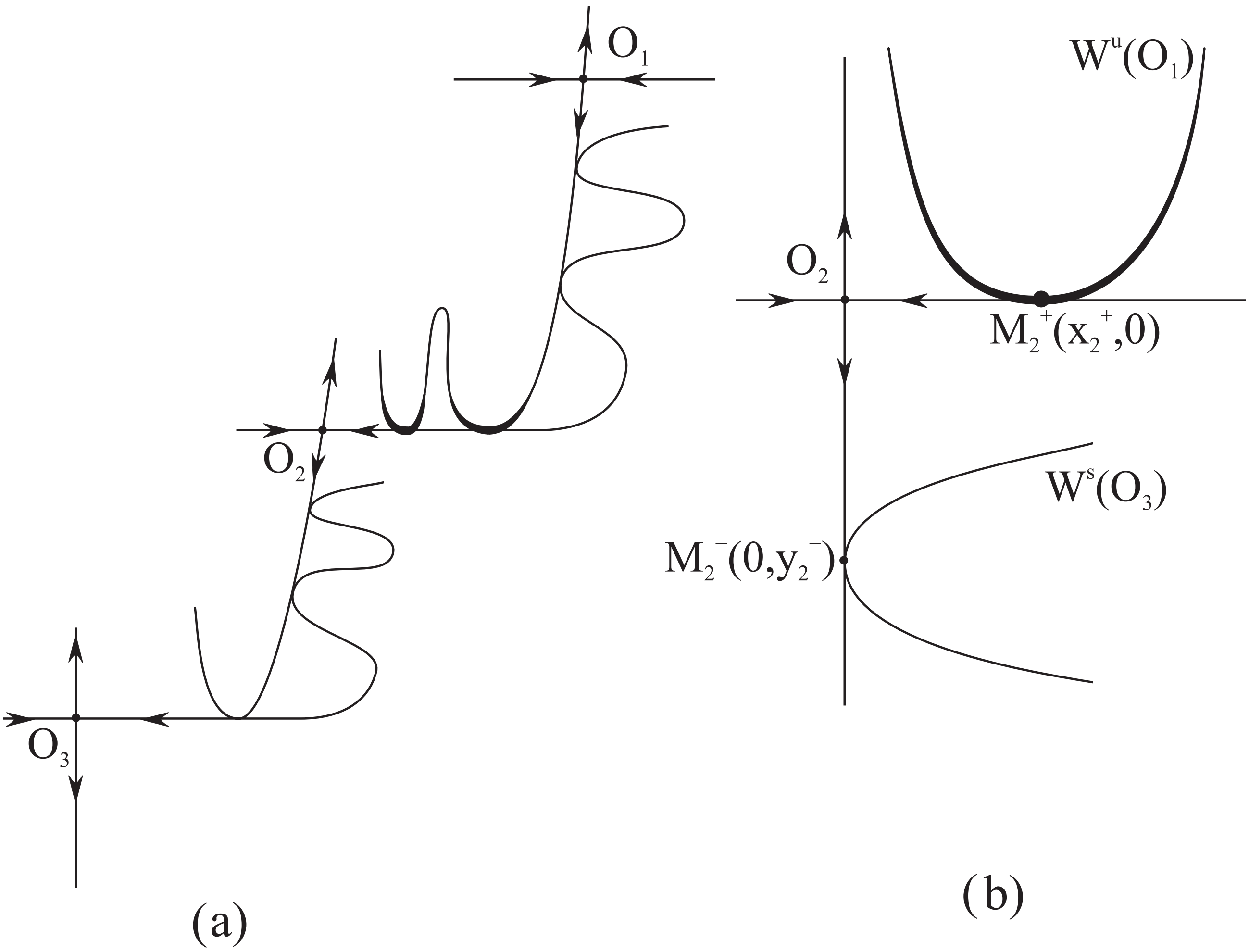

Fig. 21

M secondary homoclinic tangencies of the third class. Continuing this construction infinitely, we obtain the required system f^*, which lies in H and is C^r-smooth by construction. The theorem is proved.

5. Homoclinic Tangencies of an Arbitrarily High Order

In this section, we show that it is possible to obtain homoclinic tangencies of an arbitrarily high order by means of small perturbations of an infinite chain of coexisting homoclinic tangencies that was constructed in Theorem 4. Namely, the following result is valid.

Theorem 5. *Systems with homoclinic tangencies of any order are dense in the set of systems with homoclinic tangencies of the third class.*

Proof. We show that for any system $f_0 \in C^r$ with a quadratic homoclinic tangency of the third class, for any $\delta > 0$ and for any n, there exists a perturbation whose C^r-norm is not more than δ that does not split the original tangency and is such that the perturbed system has a trajectory with homoclinic tangency of order n. It is convenient to assume from the outset that $f_0 \in C^\infty$ (since we can always make the system f_0 more smooth by an arbitrarily small C^r-perturbation without splitting the original homoclinic tangency).

According to Theorem 4, in any neighborhood of f_0, in the C^r-topology on the bifurcation film H, there is a system f^* such that for some infinite sequence of subscripts $i_1 < i_2 < \dots$, each of the saddle fixed points $p_{i_m} \in \sigma^0_{i_m}$ of the mapping $\tilde{T}_{i_m}$ for the system f^* has nonrough homoclinic trajectory with a quadratic tangency. Moreover, if $\gamma < 0$ or if $\lambda < 0$, then all i_m are even (see the proof of Theorem 3).

Since $|\lambda\gamma| < 1$, it follows from Lemma 1 that for any sufficiently large i, the horseshoe $T_1\sigma^1_i$ intersects the strip σ^0_{i+1} tamely provided that $\gamma > 0$ and $\lambda > 0$. If γ or λ are negative, then in any case, for an even i, the horseshoe $T_1\sigma^1_i$ intersects the strip σ^0_{i+2} tamely. In any case, for any m, there exists a finite increasing sequence of subscripts $i_m \equiv i_{m0}, i_{m1}, \ldots, i_{ml} \equiv i_{m+1}$ such that the sequence $(\ldots i_m, i_m, i_{m1}, \ldots, i_{ml-1}, i_{m+1}, i_{m+1} \ldots)$ (which is asymptotic with respect to the periodic ones on both sides) is tame.

What this means is that we have the following pattern (Fig. 20a): the unstable manifold $W^u(p_{i_m})$ intersects the stable manifold $W^s(p_{i_{m1}})$ transversally, the unstable manifold $W^u(p_{i_{m1}})$ intersects the stable manifold $W^s(p_{i_{m2}})$ transversally,..., the unstable manifold $W^u(p_{i_{ml-1}})$ intersects the stable manifold $W^s(p_{i_{m+1}})$ transversally, and so on for any m. By Theorem 2, for any m, there exists a trajectory (with a coding

$$(\ldots i_m, i_m, i_{m1}, \ldots, i_{ml-1}, i_{m+1}, i_{m+1} \ldots))$$

with the transversal intersection of the manifolds $W^u(p_{i_m})$ and $W^s(p_{i_{m+1}})$.

Then it follows from the λ-lemma that the pieces of the stable manifold $W^s(p_{i_{m+1}})$ accumulate to the stable manifold $W^s(p_{i_m})$. Therefore, using an arbitrarily small C^r-smooth perturbation of the system that is localized in an arbitrarily small neighborhood of the point p_{i_m}, we can split the quadratic homoclinic tangency of the manifolds $W^u(p_{i_m})$ and $W^s(p_{i_m})$ in such a way that we would have the quadratic heteroclinic tangency of manifolds $W^u(p_{i_m})$ and $W^s(p_{i_{m+1}})$ near the former tangency.[15] Since these new perturbations are localized in arbitrarily small neighborhoods of points p_{i_m}, not one of them splits other homoclinic or heteroclinic tangencies. Thus, we can successively modify the mapping f^* in such a way that the system obtained in the limit would be still C^r-close to f_0, and, moreover, $W^u(p_{i_1})$ would be quadratically tangent to $W^s(p_{i_2})$ along some heteroclinic trajectory $\Gamma_{i_1 i_2}$, ..., $W^u(p_{i_m})$ would be quadratically tangent to $W^s(p_{i_{m+1}})$ along some heteroclinic trajectory $\Gamma_{i_m i_{m+1}}$, and so on to infinity (Fig. 20b).

We note that since $|\lambda\gamma| < 1$ and $i_m > i_1$ for $m > 1$ (and, in addition, all i_m are even for a negative γ or λ), the pair (i_m, i_1) is always tame, i.e., by Theorem 2, there always exists a trajectory with the transversal intersection of $W^u(p_{i_m})$ and $W^s(p_{i_1})$.

Now the assertion of the theorem follows from the lemma stated below.

Lemma 2. *Let L_1, L_2, and L_3 be saddle periodic trajectories of some C^∞-smooth system g. Let $W^u(L_1)$ and $W^s(L_2)$ have a tangency of order $n-1$ along some heteroclinic trajectory Γ_{12}, and let $W^u(L_2)$ and $W^s(L_3)$ have a quadratic tangency along some heteroclinic trajectory Γ_{23}. Then there exists an arbitrarily small (in the C^r-topology for an arbitrary r) perturbation localized in arbitrarily small neighborhoods of some heteroclinic points of the trajectories Γ_{12} and Γ_{23} such that the manifolds $W^u(L_1)$ and $W^s(L_3)$ of the new diffeomorphism $\tilde{g}$ have a tangency of order n at the points of some heteroclinic trajectory Γ_{13}.*

Namely, applying Lemma 2 to the contour $p_{i_1}, \Gamma_{i_1 i_2}, p_{i_2}, \Gamma_{i_2 i_3}, p_{i_3}$ (Fig 20b) consisting of the chain of heteroclinic tangencies that were constructed above, we obtain that, using a small perturbation localized in the neighborhoods of the trajectories $\Gamma_{i_1 i_2}$ and $\Gamma_{i_2 i_3}$, one can construct a heteroclinic trajectory $\Gamma_{i_1 i_3}$ that would correspond to the tangency of order two (i.e., to the cubic tangency) between the manifolds $W^u(p_{i_1})$ and $W^s(p_{i_3})$. The other heteroclinic tangencies are not split under such a perturbation, and, applying the lemma now to the contour $p_{i_1}, \Gamma_{i_1 i_3}, p_{i_3}, \Gamma_{i_3 i_4}, p_{i_4}$, we obtain a heteroclinic trajectory $\Gamma_{i_1 i_4}$ that would correspond to the tangency of order 3 between the manifolds $W^u(p_{i_1})$ and $W^s(p_{i_4})$, and so on, untill we obtain the tangency of order n between the manifolds $W^u(p_{i_1})$ and $W^s(p_{i_{n+1}})$. Since $W^u(p_{i_{n+1}})$ always has a trajectory along which it intersects $W^s(p_{i_1})$ transversally, it follows by virtue of the λ-lemma that the pieces of the stable manifold $W^s(p_{i_1})$ accumulate to the stable manifold $W^s(p_{i_{n+1}})$. Therefore, by one more arbitrarily small perturbation, we can split the heteroclinic tangency of order n between the manifolds $W^u(p_{i_1})$ and $W^s(p_{i_{n+1}})$ in order to obtain the required homoclinic tangency of order n (between the manifolds $W^u(p_{i_1})$ and $W^s(p_{i_1})$).

In order to complete the proof of the theorem, we have to prove Lemma 2.

Let O_2 be some point on a periodic trajectory L_2, and let the corresponding local mapping T_0 be reduced to the following form (see (2.1)):

$$\bar{x} = \lambda_2 x + h_2(x,y)x^2 y, \ \bar{y} = \gamma_2 y + g_2(x,y)xy^2, \tag{5.1}$$

where λ_2 and γ_2 are the multipliers of L_2 such that $0 < |\lambda_2| < 1 < |\gamma_2|$. Thus, the point O_2 coincides with the origin, $W^s_{\text{loc}}(O_2)$ is defined by the equation $y = 0$, and $W^s_{\text{loc}}(O_2)$ is defined by the equation $x = 0$. In the

[15] Here we can directly consider the nonrough heteroclinic contour of the type $C_{i_m j_{m+1}}$ (see the preceding section) for all types of quadratic homoclinic tangencies of the third class, except for the cases that correspond to the eighth and tenth columns of Table 1.

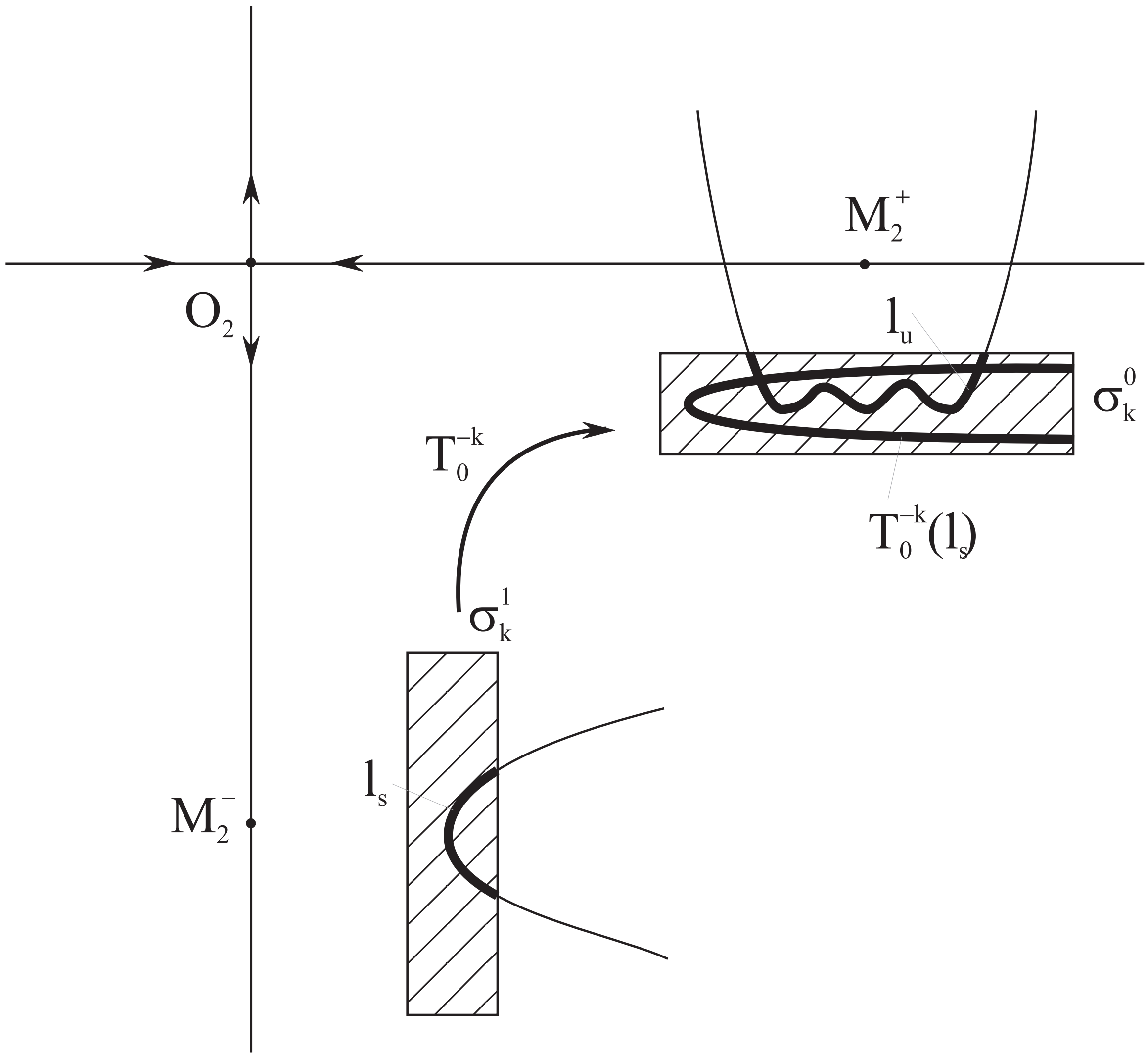

Fig. 22

neighborhood of the point O_2, we choose the following two heteroclinic points: the point $M_2^+(x_2^+, 0)$, which belongs to the trajectory Γ_{12}, and the point $M_2^-(0, y_2^-)$ which belongs to the trajectory Γ_{23} (Fig. 21).

Since Γ_{12} corresponds to the tangency of order $(n-1)$, the equation of the segment l_u of the unstable manifold of the trajectory L_1 near the point M_2^+ has the form

$$y = d_1(x - x_2^+)^n + o[(x - x_2^+)^n].$$

In turn, the equation of the segment l_s of the stable manifold of the trajectory L_3 near the point M_2^- can be written as

$$x = d_2(y - y_2^-)^2 + o[(y - y_2^-)^2].$$

The nonzero coefficients d_1 and d_2 in the above formulas can always be made equal to unity by a linear change of coordinates, and in what follows, we will assume that this was done.

We perturb the system g in such a way that the equation of the curve l_u becomes

$$\begin{aligned} y = \varepsilon_0 + \varepsilon_1(x - x_2^+) + \ldots + \varepsilon_{n-2}(x - x_2^+)^{n-2} \\ + \varepsilon_{n-1}(x - x_2^+)^{n-1} + (x - x_2^+)^n + o[(x - x_2^+)^n], \end{aligned} \tag{5.2}$$

while the equation of the curve l_s transforms into

$$x \;=\; \mu_0 + \mu_1(y - y_2^-) + (y - y_2^-)^2 + o[(y - y_2^-)^2]. \tag{5.3}$$

Here $\varepsilon_0, \ldots, \varepsilon_{n-2}, \varepsilon_{n-1}$, μ_0, and μ_1 are independent small parameters that should be determined. It is obvious that the corresponding perturbation can be localized in neighborhoods of two heteroclinic points (one on the trajectory Γ_{12} and the other on the trajectory Γ_{23}).

We show that for a sufficiently large k, the parameters $\varepsilon_0, ..., \varepsilon_{n-1}$ and μ_0, μ_1 can be chosen in such a way that the curves $T_0^{-k} l_s$ and l_u (Fig. 22) would have a tangency of order n near the point M_2^+; moreover, as $k \to +\infty$, the corresponding values of the parameters $\varepsilon_0, ..., \varepsilon_{n-1}$ and μ_0, μ_1 should tend to zero. For definitenes, in the case where $\lambda_2 < 0$, we take k even, so that always $\lambda_2^k > 0$.

According to (2.2), the curve $T_0^{-k} l_s$ becomes

$$\begin{aligned} &\lambda_2^k x + \lambda_2^k \xi_k(x, \tilde{y}) = \mu_0 + \mu_1(\tilde{y} - y_2^-) + (\tilde{y} - y_2^-)^2 + o[(\tilde{y} - y_2^-)^2], \\ &y = \gamma_2^{-k} \tilde{y} + \gamma_2^{-k} \eta_k(x, \tilde{y}), \end{aligned} \tag{5.4}$$

where $\tilde{y}$ is a parameter that runs over the values near $\tilde{y} = y_2^-$; the functions ξ_k and η_k, together with all their derivatives, tend to zero as $k \to +\infty$.

We introduce new variables $u = x - x_2^+$ and $w = \tilde{y} - y_2^-$. Then Eqs. (5.2) and (5.4) of the curves l_u and l_s are written in the form

$$y = \varepsilon_0 + \varepsilon_1 u + ... + \varepsilon_{n-1} u^{n-1} + u^n + o(u^n) \tag{5.5}$$

and

$$\begin{aligned} &\lambda_2^k u + \lambda_2^k \tilde{\xi}_k(u, w) = \tilde{\mu}_0 + \tilde{\mu}_1 w + w^2 + o(w^2), \\ &y - \gamma_2^{-k} y_2^- = \gamma_2^{-k} w + \gamma_2^{-k} \eta_k(u + x_2^+, w + y_2^-), \end{aligned} \tag{5.6}$$

respectively, where $\tilde{\mu}_0 = \mu_0 - \lambda_2^k(x_2^+ + \xi_k(x_2^+, y_2^-))$, $\tilde{\mu}_1 = \mu_1 - \lambda_2^k \dfrac{d}{d\tilde{y}} \xi_k(x_2^+, y_2^-)$, and the function $\tilde{\xi}_k$ is identically equal to zero for $u = 0$ (we have simply transferred the expression $\lambda_2^k \xi_k(x_2^+, y_2^- + w)$ to the right-hand side of the equation).

Now we perform the change of variable $y - \gamma_2^{-k} y_2^- - \gamma_2^{-k} \eta_k(u + x_2^+, y_2^-) = v$; after that, the second equation in (5.6) becomes

$$v = \gamma_2^{-k} w + \gamma_2^{-k} \tilde{\eta}_k(u, w), \tag{5.7}$$

where $\tilde{\eta}_k(u, w)$ is identically equal to zero for $w = 0$. Equation (5.5) of the curve l_u is rewritten in the form

$$v = \tilde{\varepsilon}_0 + \tilde{\varepsilon}_1 u + ... + \tilde{\varepsilon}_{n-1} u^{n-1} + u^n + o(u^n), \tag{5.8}$$

where the modified parameters $\tilde{\varepsilon}_j$ differ from the original parameters ε_j by quantities that tend to zero as $k \to +\infty$.

We fix the choice of μ_1 and ε_{n-1}, having required that $\tilde{\mu}_1 = 0$ and $\tilde{\varepsilon}_{n-1} = 0$. We also normalize the coordinates by performing the following substitutions in formulas (5.6)–(5.8):

$$u \mapsto \alpha_k u, \qquad v \mapsto \alpha_k^n v, \qquad w \mapsto w\sqrt{-\lambda_2^k \alpha_k}, \tag{5.9}$$

where the small factor α is given by the formula

$$\alpha_k^{2n-1} = -\lambda^k \gamma^{-2k}. \tag{5.10}$$

Equations (5.6) and (5.7) of the curve $T_0^{-k} l_s$ are rewritten in the new coordinates as follows:

$$u = \Delta - w^2 + p_{k1}(v), \quad v = w + p_{k2}(w),$$

where the functions p_{k1} and p_{k2}, together with their derivatives, tend to zero as $k \to +\infty$; therefore, this equation can be rewritten in an explicit form as follows:

$$u = \Delta - v^2 + p_k(v), \tag{5.11}$$

where p_k, together with its derivatives, tends to zero as $k \to +\infty$; Δ is the scaled parameter $\tilde{\mu}_0$ defined by

$$\Delta = \frac{\tilde{\mu}_0}{\alpha_k \lambda^k}. \tag{5.12}$$

Equation (5.5) of the curve l_u becomes

$$v = E_0 + E_1 u + ... + E_{n-2} u^{n-2} + u^n + q_k(u), \tag{5.13}$$

where q_k tends to zero, together with its derivatives, as $k \to +\infty$ and

$$E_j = \frac{\tilde{\varepsilon}_j}{\alpha_k^{n-j}}, \quad j = 0, ..., n-2. \tag{5.14}$$

We show that for any sufficiently large k, curves (5.11) and (5.13) have tangency of order n in the bounded range of values of u and v for bounded values of the parameters $\Delta, E_0, \dots, E_{n-2}$. Turning back to the unscaled variables u and v (formula (5.9)) and parameters $\tilde{\mu}_0$ and $\tilde{\varepsilon}_j$ (see (5.12) and (5.14)), we obtain that the tangency of the nth order between the curves $T_0^{-k} l_s$ and l_u takes place for small (tending to zero as $k \to +\infty$) values of $u, v, \tilde{\mu}$, and $\tilde{\varepsilon}$, i.e., in a small neighborhood of the point M_2^+ and for small values of the perturbation parameters μ and ε, as was required.

Thus, the problem is reduced to the question of the tangency of curves (5.11) and (5.13). In essence, our considerations are based on the following relatively simple algebraic problem.

Let the following two parabolas be given on the (u, v)-plane:

$$u = \Delta - v^2 \quad \text{and} \quad v = \sum_{j=0}^{n-2} E_j u^j + u^n; \tag{5.15}$$

one of them is of the second degree and the other one is of the nth degree. It is required to prove that there exist values of the parameters for which these parabolas have tangency of the nth order.

However, formally we do not reduce the problem to the question of the tangency of the polynomial curves (5.15), but directly take into account the small additional terms p_k and q_k in (5.11) and (5.13). We note that a similar problem was solved by us in [20], but, for completeness of presentation, we also carry out a detailed discussion in this paper.

Let

$$Q(u) = \sum_{j=0}^{n-2} E_j u^j + u^n. \tag{5.16}$$

The condition for the tangency of the nth order of curves (5.11) and (5.13) at some point (u^*, v^*) is written in the form

$$\begin{aligned}
&u^* = \Delta - Q^2(u^*) + \dots, \\
&1 = -2Q(u^*)Q'(u^*) + \dots, \\
&0 = -Q(u^*)Q''(u^*) - (Q'(u^*))^2 + \dots, \\
&\dots \\
&0 = -\sum_{i=0}^{j-1} C_{j-1}^i Q^{(i)}(u^*) Q^{(j-i)}(u^*) ..., \\
&0 = -\frac{d^n Q^2}{du^n}(u^*) + \dots,
\end{aligned} \tag{5.17}$$

where C_{j-1}^i are binomial coefficients, while the dots (here and in what follows) denote terms that tend to zero, together with their derivatives, as $k \to +\infty$. It immediately follows from (5.17) that $Q(u^*)$ should be different from zero and

$$\begin{aligned}
&Q'(u^*) = -\tfrac{1}{2} Q^{-1}(u^*) + \dots, \\
&Q''(u^*) = -\tfrac{1}{4} Q^{-3}(u^*) + \dots, \\
&\dots \\
&Q^{(j)}(u^*) = -\sigma_j Q(u^*)^{-(2j-1)} + \dots, \\
&\dots \\
&Q^{(n)}(u^*) = -\sigma_n Q(u^*)^{-(2n-1)} + \dots,
\end{aligned} \tag{5.18}$$

where σ_j are some constants, $\sigma_1 = 1/2, \sigma_2 = 1/4$, and

$$\sigma_j = \sum_{i=1}^{j-1} C_{j-1}^i \sigma_i \sigma_{j-i}.$$

It is seen from this formula that all σ_j are positive; what is important for us is that $\sigma_n \neq 0$.[16]

Since $Q(u)$ is a polynomial, we have

$$Q(u) = Q(u^*) + Q'(u^*)(u - u^*) + ... + \frac{Q^{(n)}}{n!}(u^*)(u - u^*)^n. \tag{5.19}$$

Since the coefficient of u^n in the polynomial Q should be equal to 1, while the coefficient of u^{n-1} should be equal to 0, it follows from (5.19) that

$$Q^{(n)}(u^*) = n! \tag{5.20}$$

$$Q^{(n-1)}(u^*) = u^* Q^{(n)}(u^*). \tag{5.21}$$

We find from the last equation in (5.18) and from the first equation in (5.21) that

$$Q(u^*) = -(\frac{\sigma_n}{n!})^{1/(2n-1)} + \dots,$$

and since $\sigma_n \neq 0$, we have that $Q(u^*)$ is bounded away from zero as $k \to +\infty$. Now we find from (5.18) the values of all other derivatives $Q^{(j)}(u^*)$ and calculate the coordinate u^* from the penultimate equation in (5.18) and from (5.21) as follows:

$$u^* = \frac{2}{2n-1}\left(\frac{\sigma_n}{n!}\right)^{2/(2n-1)} + \dots.$$

The coefficient Δ is found from the first equation in (5.17):

$$\Delta = \frac{2n+1}{2n-1}\left(\frac{\sigma_n}{n!}\right)^{2/(2n-1)} + \dots,$$

while the coefficients E_j are calculated from (5.19). (For instance, in the case $n = 2$, we have the following: $\Delta = 3/4 + \dots$, $E_0 = -3/4 + \dots$, $u^* = 1/2 + \dots$. Correspondingly, we obtain that the parabolas $u = 3/4 - v^2$ and $v = -3/4 + u^2$ have a cubic tangency at the point $v = -1/2$, $u = 1/2$ (Fig. 23)).

Since the obtained values of E_j, Δ, u^* and $v^* = Q(u^*) + \dots$ have finite limits as $k \to +\infty$, they remain bounded for all sufficiently large k, which is the required result. The lemma is proved.

We note that if the stable and unstable manifolds of some saddle periodic trajectory have a quadratic tangency of order $\geq r$, then, using a small perturbation of the system in the C^r-topology, one can always attain the coincidence of the stable and unstable manifolds on some interval, i.e., obtain a continuum (a one-parameter family) of nonrough homoclinic trajectories (we call such families *homoclinic bands*). Thus, Theorem 5 implies the following proposition.

Proposition 1. *Systems with homoclinic bands are dense in the set of systems with homoclinic tangencies of the third class.*

We note that, by Theorem 4, it is possible initially to perturb the original system of the third class in such a way that it would have countably many tangencies of the third class and then use the construction from Theorem 5 near each of these tangencies; we can do this independently, since the perturbation constructed in Theorem 5 is localized in a small neighborhood of individual, separately taken homoclinic trajectories. Thus, we can formally strengthen Theorem 5 in the following way:

Proposition 2. *Systems each of which has infinitely many homoclinic tangencies of every order $n = 1, \dots, \infty$, including infinitely many individual homoclinic bands, are dense in the set of systems with homoclinic tangencies of the third class.*

[16]The coefficients σ_j can be calculated explicitly: we note that for the function $\hat{Q}(u) = \sqrt{\Delta - u}$, relations (5.17) hold identically for any u^*; therefore, $\hat{Q}^{(j)}(u) \equiv -\sigma_j \hat{Q}^{-(2j-1)}(u)$ for any u (see (5.18)). Differentiating this identity, we have

$$\hat{Q}^{(j+1)}(u) = \sigma_j(2j-1)\hat{Q}^{-(2j)}(u)Q'(u) = -\sigma_j((2j-1)/2)\hat{Q}^{-(2j+1)},$$

whence $\sigma_{j+1} = \frac{2j-1}{2}\sigma_j$ and

$$\sigma_j = \frac{(2j-3)\cdot(2j-5)...3\cdot 1}{2^j}.$$

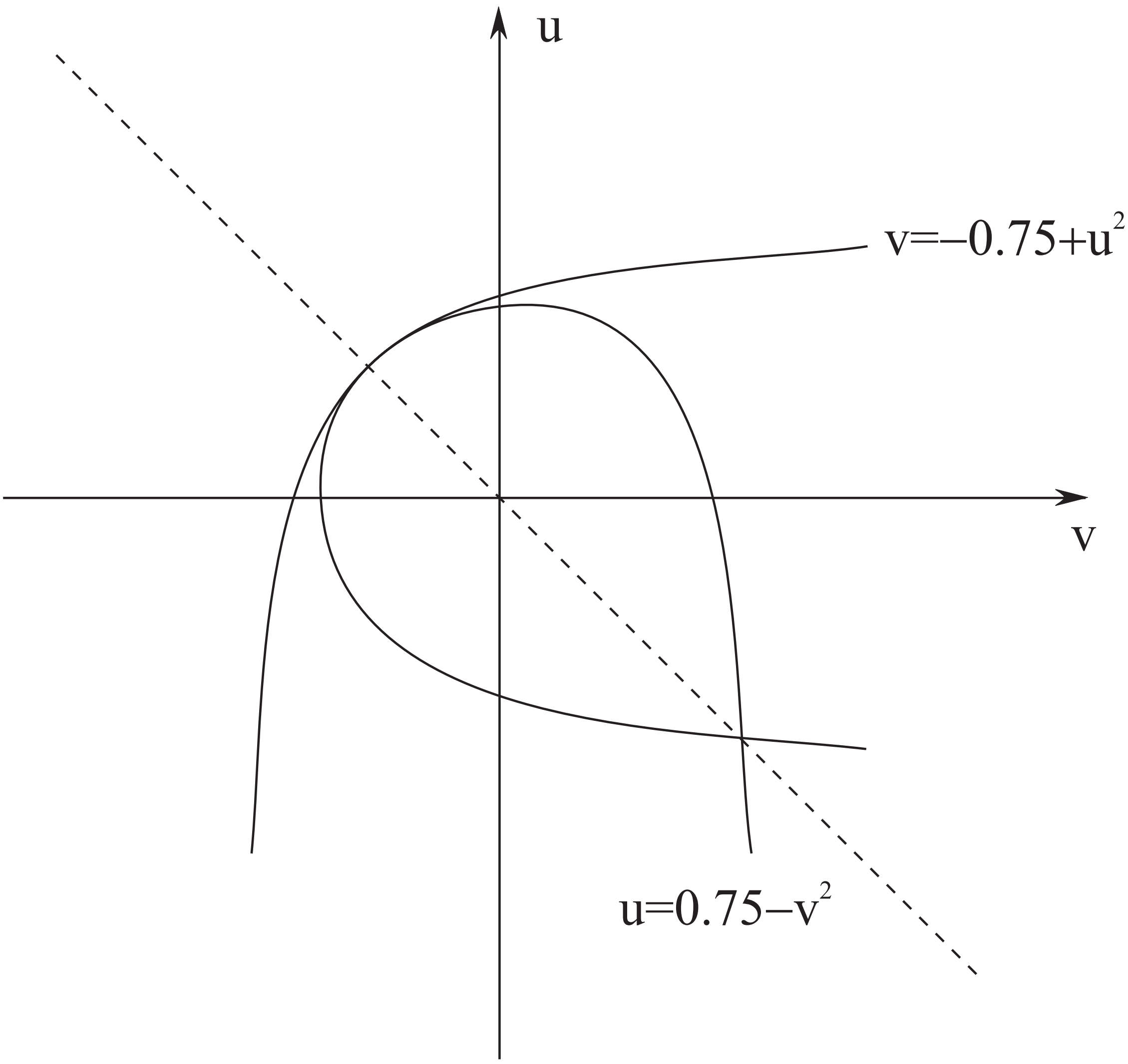

Fig. 23

6. Periodic Trajectories of High Orders of Degeneracy

In this section, in particular, we deal with degenerate periodic trajectories. We adhere to the following definition.

Definition 1. Let some C^r-smooth system g have a periodic trajectory L one of whose multipliers ν is equal to ± 1, while the second multiplier is different from unity in absolute value. Moreover, the restriction of the first-return mapping (the Poincaré mapping) near L to the one-dimensional central manifold is written either in the form

$$\bar{y} = y + l_n y^n + \ldots \quad \text{if} \;\; \nu = 1 \;\; , \bar{y} = -y - l_n y^{2n+1} + \ldots \quad \text{if} \;\; \nu = -1$$

(where $l_n \neq 0$ is the nth Lyapunov value; $1 \leq n \leq r-1$ for $\nu = 1$ and $1 \leq n \leq (r-1)/2$ for $\nu = -1$) or in the form

$$\bar{y} = \nu y + o(y^r)$$

if all Lyapunov values are equal to zero. In the first case, we say that this periodic trajectory has order of degeneracy $n-1$; in the second case, the periodic trajectory is said to have an infinite (uncertain) order of degeneracy.

Theorem 6. *Systems with periodic trajectories of any order of degeneracy (both with $\nu = 1$ and with $\nu = -1$) are dense in the set of systems with homoclinic tangencies of the third class.*

Remark. We note that, by Theorem 2, singly going around periodic trajectories near a nonrough homoclinic trajectory are always rough saddle ones (as long as we do not split a tangency). As far as the twicely going around trajectories are concerned, they can be nonrough; moreover, they have the first order of degeneracy under this procedure, and can be either with the multipler $+1$ or with the multiplier -1 (see [12, 23]). In the case of triply going around periodic trajectories, degeneracies of already higher order can occur. Thus, it is shown in [15] that systems having nonrough triply going around periodic trajectories of the second order of degeneracy are dense in the set of systems with homoclinic tangencies of the third class. Such periodic trajectories occur and are nonremovable in two-parameter families, in particular, in families in which the parameters θ and τ serve as controlling parameters. The study of periodic trajectories of the second order of degeneracy was based on the construction already with three horseshoes, in contrast to the construction with two horseshoes that was used in the proof of Theorem 3. If we involve more strips and horseshoes, then, in principle, we can obtain periodic trajectories of higher orders of degeneracy, but a direct study of fixed points of a mapping for many goings around along a neighborhood of a nonrough homoclinic trajectory is too difficult. Therefore, in this case, we proceed alternatively. We study bifurcations of singly going around periodic trajectories near the tangencies of high order, as stated in Theorem 5. Such tangencies can be obtained by a small perturbation near any quadratic homoclinic tangency of the third class; therefore, in order to prove Theorem 6, we take an order of a homoclinic tangency as high as may be required.

Proof of Theorem 6. Thus, consider a system $\tilde{f}$ which has a saddle periodic trajectory with a trajectory of a homoclinic tangency of some order n. As in the proof of Theorem 5, we assume that $\tilde{f} \in C^\infty$. The local mapping T_0 can still be written in the form (2.1).

The global mapping T_1 near the trajectory with homoclinic tangency of order n is written as

$$\begin{aligned} \bar{x} - x^+ &= ax + b(y - y^-) + ..., \\ \bar{y} &= cx + d(y - y^-)^{n+1} + ..., \end{aligned} \tag{6.1}$$

where the dots stand for the terms of second and higher orders in the first equation, while in the second equation the dots denote the terms of order $o(|x| + |y - y^-|^{n+1}) + O(|x| \cdot |y - y^-|)$.

In the generic n-parameter family $\tilde{f}_\varepsilon$ of systems that are close to $\tilde{f}$ ($\tilde{f}_0 \equiv \tilde{f}$), the parameters $\varepsilon = (\varepsilon_0, ..., \varepsilon_{n-1})$ can be introduced in such a way that the global mapping T_1 would take the following form:

$$\begin{aligned} &\bar{x} - x^+ = ax + b(y - y^-) + ..., \\ &\bar{y} = cx + \varepsilon_0 + \varepsilon_1(y - y^-) + ... + \varepsilon_{n-1}(y - y^-)^{n-1} + d(y - y^-)^{n+1} + ... \end{aligned} \tag{6.2}$$

The local mapping T_0 now depends on the parameters, i.e., formula (2.1) takes the form

$$\bar{x} = \lambda(\varepsilon)x + h(x, y, \varepsilon)x^2y, \ \bar{y} = \gamma(\varepsilon)y + g(x, y, \varepsilon)xy^2. \tag{6.3}$$

According to (2.2), we have the following representation for the mapping $T_0^k : (x_0, y_0) \mapsto (x_k, y_k)$ (see [13, 11]):

$$\begin{aligned} x_k &= \lambda(\varepsilon)^k x_0 + \lambda(\varepsilon)^k \gamma(\varepsilon)^{-k} \xi_k(x_0, y_k, \varepsilon), \\ y_0 &= \gamma(\varepsilon)^{-k} y_k + \gamma(\varepsilon)^{-2k} \eta_k(x_0, y_k, \varepsilon), \end{aligned} \tag{6.4}$$

where the functions ξ_k and η_k are uniformly bounded with respect to k, together with all their derivatives.

Now we consider the mapping $\tilde{T}_k(\varepsilon) \equiv T_1 T_0^k$ for one going around along the homoclinic trajectory. This mapping is defined on the strip σ_k^0, and the following lemma shows that the above mapping is close to a polynomial one-dimensional mapping.

Lemma 3. *Using linear transformations of coordinates and parameters, it is possible to reduce the mapping $\tilde{T}_k$ to the form*[17]

$$\begin{aligned} &\bar{X} = Y + O(|\lambda\gamma|^k + |\gamma|^{-k/n}), \\ &\bar{Y} = E_0 + E_1 Y + ... + E_{n-1} Y^{n-1} + Y^{n+1} + O(|\lambda\gamma|^k + |\gamma|^{-k/n}), \end{aligned} \tag{6.5}$$

where the range of values of the new variables X and Y and parameters $E_0, \ldots, E_{n-1}$ is unboundedly increasing with increase in k and covers all finite values in the limit.

[17] We recall that $|\lambda\gamma| < 1$; therefore, the $O(\cdot)$-terms in (6.5) tend to zero as $k \to +\infty$.

Proof. Using (6.4) and (6.2), we can write the mapping $\tilde{T}_k$ in the following form:

$$\begin{array}{l} \bar{x} - x^+ = a\lambda^k(x + ...) + b(y - y^-) + ... \\ \gamma^{-k}(\bar{y} + \gamma^{-k}\eta_k(\bar{x}, \bar{y})) = c\lambda^k(x + ...) + \\ \quad +\varepsilon_0 + \varepsilon_1(y - y^-) + ... + \varepsilon_{n-1}(y - y^-)^{n-1} + d(y - y^-)^{n+1} + ... \end{array} \tag{6.6}$$

We note that the coordinates that are used on the strip σ_0^k coincide with the coordinates (x', y') defined in Sec. 3: x' coincides with the original coordinate x, while $y' = y_k$ is a y coordinate of a kth iteration of a point under the action of the mapping T_0.

We shift the origin in the following way: $x \to x + x^+$, $y \to y + y^-$. Then mapping (6.6) is reduced to the form

$$\begin{array}{l} \bar{x} = by + O(\lambda^k) + O(y^2), \\ \gamma^{-k}\bar{y} + \gamma^{-2k}O(|\bar{y}|) = (\varepsilon_0 - \gamma^{-k}y^- + c\lambda^k x^+ + ...) + \\ \quad +\varepsilon_1 y + ... + \varepsilon_{n-1}y^{n-1} + dy^{n+1} \\ \quad +O(y^{n+2}) + \lambda^k O(|x| + |y|). \end{array}$$

Now, if we normalize the coordinates and the parameters in the following way:

$$x = bd^{-1/n}\gamma^{-k/n}X, \; y = d^{-1/n}\gamma^{-k/n}Y,$$

$$(\varepsilon_0 - \gamma^{-k}y^- + c\lambda^k x^+ + ...) = d^{-1/n}\gamma^{-k(1+1/n)}E_0,$$

$$\varepsilon_j = d^{-1/n}\gamma^{-k}\gamma^{\frac{k}{n}(j-1)}E_j,$$

the above mapping is reduced to the form (6.5), which completes the proof of the lemma.

According to Lemma 3, any bifurcations occurring in mapping (6.5) for arbitrary finite values of $E_0, \ldots, E_{n-1}$ in the finite domain (X, Y) occur near a homoclinic tangency of order n. Thus, this mapping has a fixed point with multiplier $\nu = \pm 1$ when

$$(X, Y) = 0 + \ldots, \quad E_0 = 0 + \ldots, \quad E_1 = \nu + \ldots, \tag{6.7}$$

where the dots denote the terms that tend to zero as $k \to +\infty$. The central manifold of this point is tangent at this very point to the eigenvector of the linearization matrix that corresponds to the multiplier ν, i.e., this manifold has the form $X = \nu^{-1}Y + O(Y^2)$. The restriction of mapping (6.5) to this manifold has the form

$$\bar{Y} = \sum_{j=0}^{n-1} E_j Y^j + Y^{n+1} + \ldots,$$

and we can always choose

$$E_0 = 0 + \ldots, E_1 = \pm 1 + \ldots, E_j = 0 + \ldots \quad (2 \le j \le n - 1)$$

in such a way that we would obtain a fixed point of the nth order of degeneracy with the multiplier $\nu = 1$ or a fixed point of the $([n/2] - 1)$th order of degeneracy with the multiplier $\nu = -1$. In the case where $n \ge r$, the corresponding fixed point has infinite (uncertain) order of degeneracy: if the values of the parameters are as defined by (6.7), then the restriction of mapping (6.5) to the central manifold has the form $\bar{Y} = \nu Y + o(Y^r)$. Theorem 4 then implies Theorem 6.

It is clear that in the case of uncertain order of degeneracy of a periodic trajectory, we can arbitrarily slightly locally C^r-smoothly perturb the original system in such a way that the restriction of mapping (6.5) to the central manifold in some sufficiently small neighborhood of zero would already be of the form $\bar{Y} = Y$ for $\nu = 1$ and of the form $\bar{Y} = -Y$ for $\nu = -1$. Thus, we obtain the continuum (the one-parameter family) of nonrough periodic points. We call such families *periodic bands.* Therefore, Theorem 6 implies the following proposition.

Proposition 3. *Systems with periodic bands are dense in the set of systems with homoclinic tangencies of the third class.*

We note that Lemma 3 contains more information than is necessary for the proof of Theorem 6. The lemma shows that, near the homoclinic tangency of order n, the first-return mapping is close (in suitable coordinates) to a one-dimensional polynomial mapping *with arbitrary finite coefficients.* Moreover, a stronger assertion can be made:

Somewhere near a quadratic tangency, we can find a multiply going around Poincaré mapping that (in some rescaled coordinates) is C^r-close to the mapping $\bar{X} = Y$, $\bar{Y} = \Phi(Y)$ with any preassigned C^r-smooth function $\Phi(Y)$ (defined on the closed interval $[-1, 1]$).

This result is based on the study of first-return mappings in the neighborhood of a homoclinic tangency that corresponds to the local coincidence of the stable and unstable manifolds of a saddle periodic trajectory (by Proposition 1, systems with tangencies of this kind are dense in H). Let $\hat{f}$ be precisely such a diffeomorphism. We are assuming that $\hat{f} \in C^\infty$ and that the product of multipliers λ and γ of a saddle periodic trajectory having a homoclinic band is less than unity in absolute value. The mapping T_0 for the diffeomorphism $\hat{f}$ is given by formula (2.1), and the global mapping T_1 near the homoclinic band has the form

$$\begin{aligned} &\bar{x} - x^+ = ax + b(y - y^-) + O[(|x| + |y - y^-|)^2], \\ &\bar{y} = \beta(x, y), \end{aligned} \tag{6.8}$$

where

$$\beta(0, y) \equiv 0 \quad \text{for} \quad |y - y^-| \le \rho \tag{6.9}$$

for some sufficiently small $\rho > 0$. The mapping T_1 acts from a neighborhood (of size ρ) of some homoclinic point $M^-(0, y^-)$ into a neighborhood of the homoclinic point $M^+(x^+, 0)$. Now we perturb the diffeomorphism $\hat{f}$ in a special way, namely, we assume that the mapping T_1 is given by the formula

$$\begin{aligned} &\bar{x} - x^+ = ax + b(y - y^-) + ..., \\ &\bar{y} = \tilde{\phi}(y) \cdot \chi(x, y) + \beta(x, y), \end{aligned} \tag{6.10}$$

where χ is identically equal to unity in a neighborhood of the point M^- of size ρ and is identically equal to zero outside the neighborhood of the point M of size 2ρ. This corresponds to the consideration of some functional family $\hat{f}_\phi$ of diffeomorphisms that coincide with $\hat{f}$ everywhere, except for a neighborhood of the point M^-, and are defined by formula (6.10) near the point M^-. It is clear that if ρ is chosen sufficiently small, then the mapping T_0 remains the same as it is for $\hat{f}$.

Let X_K be the set of C^r-smooth functions defined on the closed interval $[-1, 1]$ and bounded in the C^r-norm by a constant $K > 0$. Let $\Phi \in X_K$. Then we specify $\tilde{\phi}$ in (6.10) by the formula

$$\tilde{\phi}(y) = (\rho\Phi(\frac{y - y^-}{\rho}) + y^-)\gamma^{-k}. \tag{6.11}$$

Combining the formula (2.2) for T_0^k with (6.10), we obtain that the mapping $T_1 T_0^k$ has the following form on the strip σ_0^k for $|y - y^-| \le \rho$ (in the same coordinates as in Lemma 3):

$$\begin{aligned} &\bar{x} - x^+ = a\lambda^k(x + ...) + b(y - y^-) + O(|y - y^-|^2), \\ &\gamma^{-k}(\bar{y} + \gamma^{-k}\eta_k(\bar{x}, \bar{y})) = \tilde{\phi}(y - y^-) + \beta(\lambda^k(x + ...), y - y^-). \end{aligned} \tag{6.12}$$

We note that $\beta = O(|\lambda|^k)$ by (6.9); therefore, $\gamma^k\beta$ tends to zero as $k \to +\infty$ (because $|\lambda\gamma| < 1$). Now, by the change of coordinates

$$\frac{(x - x^+)}{b\rho} + \cdots = X, \quad \frac{(y - y^-)}{\rho} = Y,$$

we reduce the mapping $T_1 T_0^k$ to the form

$$\bar{X} = Y + \rho O(Y^2) + O(\lambda^k), \quad \bar{Y} = \Phi(y) + O(\lambda^k\gamma^k),$$

which was required.

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