Ergod. Th. & Dynam. Sys. (2011), **31**, 563–569
doi:10.1017/S014338570900114X

Diffeomorphisms which cannot be topologically conjugate to diffeomorphisms of a higher smoothness

DMITRY TURAEV

Imperial College, London SW7 2AZ, UK
(e-mail: dturaev@imperial.ac.uk)

(*Received* 9 *June* 2009 *and accepted in revised form* 2 *December* 2009)

Abstract. We provide examples of two-dimensional diffeomorphisms which cannot be topologically conjugate to any diffeomorphism of a higher smoothness.

In this paper we construct new examples of two-dimensional C^k-diffeomorphisms ($k > 1$) which are not topologically conjugate to any diffeomorphism of a higher smoothness $p > k$. We note that examples of such 'unsmoothable' diffeomorphisms have been known since the mid-1970s [**5**], for any dimension of the phase space larger than one [**4, 6**]. Our constructions use the dimension-two condition explicitly at a certain point. However, the idea is completely different from that of [**4–6**]. Moreover, we show that the set of unsmoothable diffeomorphisms is sufficiently large—it is dense in an open subset of the space of C^k-diffeomorphisms. The construction combines results by de Melo [**12**] on the rigidity of topological conjugacy for diffeomorphisms with a heteroclinic tangency, Belitsky theory of functional moduli of smooth conjugacy for one-dimensional diffeomorphisms [**1**] and Katok results [**11**] on the approximation of non-uniformly hyperbolic invariant sets by uniformly hyperbolic ones. The paper was inspired by the Downarowicz–Newhouse theorem [**2**] which claims that C^k-diffeomorphisms ($1 < k < \infty$) that are not topologically conjugate to any C^∞-diffeomorphism are locally generic; that is, such diffeomorphisms form a residual set in the so-called Newhouse domain in the space of C^k-smooth diffeomorphisms. Similarly, although by a different method, we prove here the following result.

THEOREM. *Given any $k > 1$, in the Newhouse domain of the space of C^k-diffeomorphisms of a two-dimensional disk there is a dense subset such that none of the diffeomorphisms that belong to this subset is topologically conjugate to any C^p-diffeomorphism with $p > k$.*

Before starting the proof, we clarify our notation. Throughout the paper we write C^k instead of $C^{[k],\{k\}}$ where $[k]$ is the integer part of k and $\{k\} = k - [k]$, i.e. we deal with $C^{[k]}$-functions whose $[k]$th derivatives are Hölder continuous with exponent $\{k\}$. We also recall that the Newhouse domain is the open region in the space of C^k-diffeomorphisms ($k > 1$) where diffeomorphisms with homoclinic tangencies are dense. The (quite unexpected) fact that this region is non-empty and 'large' (in the sense that its closure contains all C^k-diffeomorphisms with homoclinic tangencies) is proved in [**13**]. A homoclinic tangency is an orbit at the points of which the stable and unstable invariant manifolds of a hyperbolic periodic point have a tangency. Obviously, any given homoclinic tangency can be removed by an arbitrarily small (in C^k-metric) perturbation of the diffeomorphism, so every diffeomorphism in the Newhouse domain is *structurally unstable*—its dynamics may change by an arbitrarily small perturbation. In fact, it was shown in [**7**] that diffeomorphisms with homoclinic tangencies of *arbitrarily high orders* form a dense subset of the Newhouse domain. This theorem shows an extreme sensitivity of the dynamics to perturbations of the diffeomorphism. As our result here and the above-mentioned result from [**2**] show, the dynamics of the diffeomorphisms that belong to the Newhouse domain is sensitive even to an increase in smoothness (both theorems use the theorem from [**7**] as an ingredient).

Proof of the theorem. We need a stronger version of the result of [**7**] (see [**10**]): in the Newhouse domain of the space of C^k-diffeomorphisms of a two-dimensional disk there is a dense subset such that each diffeomorphism that belongs to this subset has a non-trivial basic set Λ, each periodic orbit in which has an orbit of a flat homoclinic tangency.

We call the tangency between two C^k-curves flat if the distance between the curves decays faster than the kth power of the distance to the point of tangency. Obviously, if a C^k-diffeomorphism has a homoclinic point M at which the stable and unstable manifolds W^s and W^u of some periodic point O have a flat tangency, then by adding to f a perturbation which is arbitrarily small in the C^k-metric and localized in an arbitrarily small neighborhood of M we can make the manifolds W^s and W^u coincide locally along a small interval passing through M (and along each image of this interval by the iterations of f). The orbit of every point in this interval is homoclinic to the same periodic point O (it tends to the orbit of O both at forward and backward iterations of f, as it belongs both to the stable and unstable manifold of O). Therefore, we say that these points form a *homoclinic band*.

We will need a certain refinement of the above-mentioned result from [**10**], so we repeat the construction which led to it. According to [**13**], every diffeomorphism that belongs to the Newhouse domain possesses a non-trivial basic set (i.e. a compact, invariant, locally maximal, uniformly hyperbolic, transitive set with positive topological entropy) Λ' such that one of its unstable leaves has a tangency with one of its stable leaves. Every stable leaf is approximated by the stable manifold of any periodic point from Λ' and every unstable leaf is approximated by the unstable manifold of any periodic point from Λ', therefore we can choose a pair of periodic points $P \in \Lambda'$ and $Q \in \Lambda'$ (we need them to be different) and create, by an arbitrarily small perturbation of the diffeomorphism, a tangency between the unstable manifold $W^u(Q)$ and the stable manifold $W^s(P)$. As P and Q belong to the same basic set, we also have a transverse intersection between $W^u(P)$ and $W^s(Q)$. The four orbits (periodic orbits of the points P and Q and the heteroclinic orbits which correspond to the tangency between $W^u(Q)$ and $W^s(P)$ and to the transverse intersection of $W^u(P)$

and $W^s(Q)$) comprise a heteroclinic cycle. Any further small perturbations will not destroy this cycle.

Choose a non-trivial basic subset $\Lambda \subset \Lambda'$ which does not include the points P and Q (by compactness, Λ lies at a finite distance from the heteroclinic cycle). Given any periodic point $S \in \Lambda$, its stable and unstable manifolds intersect transversely with $W^u(P)$ and $W^s(Q)$ respectively, hence $W^u(S)$ accumulates on $W^u(Q)$ and $W^s(S)$ accumulates on $W^s(P)$ (by the lambda lemma). Hence, as $W^u(Q)$ has a tangency to $W^s(P)$, it is easy to create a tangency between $W^u(S)$ and $W^s(S)$ by an arbitrarily small perturbation of the diffeomorphism. In fact, it is shown in [**10**] (based on a similar result in [**8**]) that such tangency can indeed be created by adding to the diffeomorphism under consideration an arbitrarily small (in C^k) perturbation localized in an arbitrarily small neighborhood of the point P; moreover, the perturbation is such that the tangency between $W^u(Q)$ and $W^s(P)$ does not disappear.

We now enumerate all the periodic orbits in Λ: $S_1, S_2, \ldots$. We will add to the diffeomorphism a countable sequence of perturbations of exponentially decreasing magnitude (so that their sum will converge). After the mth step we will have certain orbits of homoclinic tangency to the periodic orbits $S_1, \ldots, S_m$; each of the further perturbations will be localized at a non-zero distance from these orbits of the homoclinic tangency, so the orbits created at the first m steps will not be affected at the further steps of the perturbation algorithm. The single step (number $m+1$) is described as follows: once the homoclinic tangencies between $W^u(S_j)$ and $W^s(S_j)$ are created for each $j = 1, \ldots, m$, we take the periodic orbit S_{m+1} and create k new orbits of homoclinic tangency between $W^u(S_{m+1})$ and $W^s(S_{m+1})$ in the way described above. Since the perturbations can all be localized in an arbitrarily small neighborhood of the periodic point P, we can create the new tangencies without affecting the ones previously obtained (the heteroclinic tangency between $W^u(Q)$ and $W^s(P)$ persists too, as mentioned). Next, as shown in [**9, 10**], one can perturb the k newly obtained homoclinic tangencies in such a way that a homoclinic tangency of order k (i.e. a flat tangency) between $W^u(S_{m+1})$ and $W^s(S_{m+1})$ will appear. Such perturbations can be taken arbitrarily small in C^k for any k and localized in an arbitrarily small neighborhood of S_{m+1}, i.e. neither the previously obtained tangencies between $W^u(S_j)$ and $W^s(S_j)$ ($j = 1, \ldots, m$) nor the tangency between $W^u(Q)$ and $W^s(P)$ are affected. After the flat tangency is obtained, we make an additional small and localized perturbation to create a homoclinic band. Next, we make irrational the ratio θ of the logarithms of the absolute values of the multipliers of the periodic point S_{m+1}—this is also made by an arbitrarily small and localized (in a small neighborhood of S_{m+1}) perturbation which does not destroy the homoclinic band (see [**9, 10**]). Then, by a C^k-small perturbation localized in a small neighborhood of a point in the homoclinic band, we make the *Belitsky invariant* related to the band C^k-generic (the Belitsky invariant, which we will discuss in more detail below, is a certain C^k-smooth function on a straight line; we will call it C^k-generic if it is not C^p for any $p > k$). Thus, at the end of the $(m+1)$th step we will have a homoclinic band for each of the periodic orbits $S_1, \ldots, S_{m+1}$, the Belitsky invariant for each of these bands will be C^k-generic, and the values of $\theta(S_j)$ ($j = 1, \ldots, m+1$) will be irrational. By continuing this to infinity, we obtain, that arbitrarily C^k-close to any given C^k-diffeomorphism that belongs to the Newhouse domain there exists a diffeomorphism which has a non-trivial basic set Λ, each periodic point in which has a homoclinic band with a C^k-generic Belitsky invariant, and the ratio θ of the logarithms of the absolute values of the multipliers is irrational for each of the periodic orbits.

We denote the set of such diffeomorphisms by $\mathcal{N}^*$. As we have just shown, this set is dense in the Newhouse domain. As mentioned, the same construction was performed in [**10**]; our enhancement here is that in addition to the existence of homoclinic bands for all periodic orbits we ensure the C^k-genericity of the corresponding Belitsky invariants and the irrationality of the θs.

Let us now define the Belitsky invariant for a homoclinic band. This is similar to the functional invariants of the smooth conjugacy introduced in [**1**] for one-dimensional maps. Let O be a saddle periodic point of a C^k-diffeomorphism f of a plane. Let n be the period of O (so that $f^n(O) = O$), and let $\gamma(O)$ and $\lambda(O)$ be its multipliers (the eigenvalues of the derivative of f^n at O). We assume that $|\gamma| > 1$ and $|\lambda| < 1$. Two C^k-smooth curves, $W^u(O)$ and $W^s(O)$, pass through O, they are both invariant with respect to the map f^n, and the restriction of f^n onto $W^u(O)$ is given by

$$x \mapsto \gamma x + o(x),$$

while the restriction of f^n onto $W^s(O)$ is given by

$$x \mapsto \lambda x + o(x)$$

(the point O is zero in both these formulas). The point O divides $W^u(O)$ into two halves, and we denote one of the halves by w^u (we include in w^u the point O as well). Analogously, we denote by w^s one half of $W^s(O)$. Denote by T_u the restriction onto w^u of the map f^n if $\gamma > 0$, and of the map f^{2n} if $\gamma < 0$. Analogously, T_s will denote the restriction onto w^s of the map f^n if $\lambda > 0$, and of the map f^{2n} if $\lambda < 0$. As we can see, $T_u w^u = w^u$ and $T_s w^s = w^s$. As the zero fixed point of the one-dimensional map T_s is hyperbolic (i.e. $T_s'(0) \neq 1$), there exists a uniquely defined C^k-linearization of T_s (this is well known to be true for any $k > 1$). This means that there exists a uniquely defined C^k-smooth function $h_s : R_+ \to w^s$ such that

$$T_s(h_s(x)) = h_s(\lambda_+ x) \tag{1}$$

for all $x \geq 0$ (here $\lambda_+ = \lambda$ if $\lambda > 0$, and $\lambda_+ = \lambda^2$ if $\lambda < 0$). Analogously, there exists a uniquely defined C^k-smooth function $h_u : R_+ \to w^u$ such that

$$T_u(h_u(x)) = h_u(\gamma_+ x) \tag{2}$$

for all $x \geq 0$ (where $\gamma_+ = \gamma$ if $\gamma > 0$, and $\gamma_+ = \gamma^2$ if $\gamma < 0$). Let the point O have a homoclinic band in the intersection of w^s and w^u, i.e. w^s and w^u have a common closed segment. Let it be a segment between the points $h_s(a_1)$ and $h_s(a_2)$ for some $0 < a_1 < a_2$ (we will assume that a_2 is sufficiently close to a_1, i.e. the homoclinic band is small—if it happens to be large, we will just take a small part of it). Then the function $b : [a_1, a_2] \to R_+$ defined as

$$b = h_u^{-1} \circ h_s$$

will be called the Belitsky invariant of the homoclinic band. By construction, $b \in C^k$.

Note that the linearizing functions h_s and h_u are uniquely defined near zero by the maps T_s and T_u in a small neighborhood of O, then they are continued onto the whole of R_+ by iteration of formulas (1) and (2). Therefore, if we take any $a \in (a_1, a_2)$ and let $M = h_s(a)$, then by adding to the map f a small perturbation localized near M we can make arbitrary (sufficiently small) changes in h_s near a, while keeping h_u unchanged near $h_u^{-1}(M)$. Thus, by C^k-small perturbations of f which are localized in an arbitrarily small neighborhood of a single point in the homoclinic band, we can add arbitrary C^k-small local perturbations to the Belitsky function b, i.e. we can indeed make it C^k-generic, as claimed above.

Let us now prove that no C^k-diffeomorphism that belongs to $\mathcal{N}^*$ is topologically conjugate to a diffeomorphism of higher smoothness. Indeed, let $f \in \mathcal{N}^*$ and let a C^p-diffeomorphism g be topologically conjugate to f, i.e. $g = \xi \circ f \circ \xi^{-1}$, where ξ is a homeomorphism. We assume that $p \geq k$, and, in order to prove the theorem, we must show that $p = k$.

Since the set Λ is a basic set for f, it follows that the set $\xi(\Lambda)$ is a closed, locally maximal, invariant set of g (these properties, as opposed to uniform hyperbolicity, are preserved by the topological conjugacy). The topological entropy of $g|_{\xi(\Lambda)}$ is positive (since it is equal to the entropy of $f|_\Lambda$). Therefore, according to a theorem by Katok [**11**], the map g has a hyperbolic periodic point in $\xi(\Lambda)$. Denote this point by G. By construction, $G = \xi(O)$ where $O \in \Lambda$ is a periodic point of the map f. Since all periodic points in Λ are hyperbolic, the point O is hyperbolic. The homeomorphism ξ takes the unstable manifold of O to the unstable manifold of G, and the stable manifold of O to the stable manifold of G. Thus, the point G has a homoclinic band, like O does.

Choose a half $w^s(O)$ of $W^s(O)$ and a half $w^u(O)$ of $W^u(O)$ such that the intersection of $w^s(O)$ and $w^u(O)$ contains a segment I of the homoclinic band. Write $w^s(G) = \xi w^s(O)$ and $w^u(G) = \xi w^u(O)$; the segment ξI corresponds to the homoclinic band for the map G. Denote by γ_O, λ_O and γ_G, λ_G the multipliers of O and G such that $|\gamma| > 1$, $|\lambda| < 1$. It has been known since [**12, 14, 15**] that if two diffeomorphisms, each having a hyperbolic periodic orbit and an orbit of a homoclinic tangency, are topologically conjugate, then the conjugating homeomorphism possesses certain 'smooth' properties. Thus, according to a theorem by de Melo [**12**], the following three facts imply that the restriction of the homeomorphism ξ on $w^s(O)$ and $w^u(O)$ is defined uniquely, up to multiplication to a constant.

(1) Both the point O and its image G by the conjugating homeomorphism ξ are hyperbolic.

(2) O has a homoclinic band.

(3) The value $\theta(O) = -\ln|\gamma_O|/\ln|\lambda_O|$ is irrational (recall that θ is irrational for every periodic point in Λ, by construction).

The precise statement is that if the coordinates on $w^u(O)$ and $w^u(G)$ are chosen such that the corresponding maps T_u are linear, then

$$\xi|_{w^u(O)}(x) = C_u x^\nu$$

where $\nu = \ln|\gamma_G|/\ln|\gamma_O|$ and $C_u > 0$ is a constant; analogously, if the coordinates on $w^s(O)$ and $w^s(G)$ are chosen such that the corresponding maps T_s are linear, then

$$\xi|_{w^s(O)}(x) = C_s x^\nu$$

where $C_s > 0$ is a constant and ν is the same (note that conditions (1) and (2) imply, according to [**14**], that $\theta(G) = \theta(O)$, hence $\ln|\gamma_G|/\ln|\gamma_O| = \ln|\lambda_G|/\ln|\lambda_O|$). The general form of a continuous conjugacy between two one-dimensional linear maps is $x \mapsto x^\nu \eta(\ln x)$ with a certain periodic function η. The de Melo theorem thus shows that when the conjugacy is induced by a conjugacy of two-dimensional diffeomorphisms satisfying the global condition (2) (along with the local condition (3)), the function η must be constant, both for the restriction onto the unstable manifold and for the restriction onto the stable manifold.

Note also that condition (1) is crucial in the de Melo theorem (if G is not hyperbolic and, say, both its multipliers are equal to 1, then neither the mere smoothness of $w^s(G)$ and $w^u(G)$ nor the existence of smooth linearizing coordinates on $w^s(G)$ and $w^u(G)$ is obvious). We avoid this problem by using the Katok theory which ensures the hyperbolicity of G (the price we pay is that we have to restrict ourselves here to the two-dimensional case only, as the Katok theorem in the way we use it—'positive entropy implies a hyperbolic periodic point'—is essentially two-dimensional).

If we denote by $h_{s,O}$ and $h_{u,O}$ the linearizing transformations for the maps T_s and T_u on, respectively, $w^s(O)$ and $w^u(O)$ and by $h_{s,G}$ and $h_{u,G}$ the linearizing transformations for the maps T_s and T_u on, respectively, $w^s(G)$ and $w^u(G)$, then the above formulas for $\xi|_{w^u(O)}$ and $\xi|_{w^s(O)}$ can be rewritten as

$$\xi(h_{s,O}(x)) = h_{s,G}(C_s x^\nu), \quad \xi(h_{u,O}(x)) = h_{u,G}(C_u x^\nu).$$

Thus, the Belitsky invariant $b_f = h_{u,O}^{-1} \circ h_{s,O}$ for the homoclinic band segment I and the Belitsky invariant $b_g = h_{u,G}^{-1} \circ h_{s,G}$ for the homoclinic band segment ξI are related:

$$b_f(x) = \left(\frac{b_g(C_s x^\nu)}{C_u}\right)^{1/\nu}.$$

As mentioned, the Belitsky invariant inherits the smoothness of the original map, so b_g must be at least C^p. Therefore, b_f must be C^p as well. On the other hand, since $f \in \mathcal{N}^*$, b_f is C^k-generic by construction, i.e. $p = k$. □

It is shown in [**3, 16**] that the groups of C^k-diffeomorphisms of R^n and C^p-diffeomorphisms of R^n are not isomorphic for $p \neq k$. According to [**3, 16**], our theorem opens up one more way to prove this result—for the two-dimensional case.

Acknowledgements. The author is grateful to M. Rubin, G. Belitsky and S. Gonchenko for useful discussions. Support by a Leverhulme Trust grant is gratefully acknowledged.

REFERENCES

[1] G. Belitskii. Smooth classification of one-dimensional diffeomorphisms with hyperbolic fixed points. *Sib. Math. J.* **27** (1986), 801–804.

[2] T. Downarowicz and S. Newhouse. Symbolic extensions and smooth dynamical systems. *Invent. Math.* **160** (2005), 453–499.

[3] R. Filipkiewicz. Isomorphisms between diffeomorphism groups. *Ergod. Th. & Dynam. Sys.* **2** (1982), 159–171.

[4] M. H. Freedman and Z.-X. He. A remark on inherent differentiability. *Proc. Amer. Math. Soc.* **104** (1988), 1305–1310.

[5] J. Harrison. Unsmoothable diffeomorphisms. *Ann. of Math.* (2) **102** (1975), 85–94.

[6] J. Harrison. Unsmoothable diffeomorphisms on higher dimensional manifolds. *Proc. Amer. Math. Soc.* **73** (1979), 249–255.

[7] S. Gonchenko, D. Turaev and L. Shilnikov. On models with structurally unstable Poincaré homoclinic curve. *Soviet Math. Docl.* **44** (1992), 422–425.

[8] S. Gonchenko, D. Turaev and L. Shilnikov. On Newhouse regions of two-dimensional diffeomorphisms close to a diffeomorphism with a nontransversal heteroclinic cycle. *Proc. Steklov Inst. Math.* **216** (1997), 70–118.

[9] S. Gonchenko, D. Turaev and L. Shilnikov. Homoclinic tangencies of any order in Newhouse regions. *J. Math. Sci.* **105** (2001), 1738–1778.

[10] S. Gonchenko, L. Shilnikov and D. Turaev. Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps. *Nonlinearity* **20** (2007), 241–275.

[11] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980), 137–173.

[12] W. de Melo. Moduli of stability of two-dimensional diffeomorphisms. *Topology* **19** (1980), 9–21.

[13] S. Newhouse. The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 101–151.

[14] J. Palis. A differentiable invariant of topological conjugacies and moduli of stability. *Astérisque* **51** (1978), 335–346.

[15] W. de Melo, J. Palis and S. van Strien. Characterising diffeomorphisms with modulus of stability one. *Dynamical Systems and Turbulence, Warwick 1980 (Lectures Notes in Mathematics, 898)*. Eds. D. A. Rand and L.-S. Young. Springer, Berlin, 1981, pp. 266–285.

[16] M. Rubin. On the reconstruction of topological spaces from their groups of homeomorphisms. *Trans. Amer. Math. Soc.* **312** (1989), 487–538.