Exponential Fermi acceleration in adiabatically perturbed Hamiltonian systems

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Summary. We show that a slow periodic change in parameters of a homogeneous Hamiltonian system should universally lead to an exponential growth of energy, provided the system shows a chaotic behaviour which is not ergodic for a certain range of parameter values.

Historically, the Fermi acceleration was proposed to explain the anomalously high energies of cosmic rays' particles: by [1], a charged particle can accelerate by hitting inhomogeneities of a magnetic field that move with random velocities. In a more abstract setting, the Fermi acceleration can be defined as a process of energy transfer from heavy objects to light ones (e.g., in the cosmic ray problem, from magnetic field inhomogeneities to charged particles). In other words, we have a Hamiltonian system where the inertia of certain degrees of freedom is high, i.e. the effect on them of the other degrees of freedom can be neglected. This means we can regard the action of these "heavy" degrees of freedom as an external forcing applied to the rest of the system, i.e. the motion of the rest of the degrees of freedom is described by a Hamiltonian system whose parameters are time-dependent.

We will further assume the dependence of the Hamiltonian $H(q,p,\tau)$ on time τ is periodic. Thus, we depart from the original Fermi's setting where the external forcing is random; instead, the randomness necessary for the acceleration will be generated by chaotic motion of the internal degrees of freedom (cf. [2]). We are interested in the question of how these periodic changes of the parameters of the Hamiltonian system can lead to a sustained growth of energy. As we must operate in the high energy limit, it is natural to assume that such limit exists, i.e. that at large energies the system becomes, to the main order, energy independent in properly chosen coordinates. For example, the high-energy limit of a particle motion in a potential which is infinite outside a certain region D is a billiard in D, see [3]. For the particle in a polynomial potential, the highest order monomials give the greatest contribution at high energies, so the high-energy limit corresponds to the motion in a homogeneous polynomial potential. We will further neglect the contribution of lower order terms and will assume from the very beginning that our Hamiltonian $H(q,p,\tau)$ is *homogeneous* at every frozen value of τ. This means that if we consider the frozen system, i.e. the autonomous system defined by the Hamiltonian H at τ fixed, then for every τ and every $E>0$ there exists a coordinate transformation that keeps the system the same, sends the energy level $H=1$ to $H=E$, and has a constant Jacobian $J(E)=E^\alpha$, $\alpha>0$. We will assume that the positive energy levels are compact, so $J(E)=V(E)/V(1)$ where $V(E)$ is the volume of the (q,p)-space between the energy levels $H=E$ and $H=0$. Thus, we can label the points in the phase space (q,p) by the coordinates (x,E) where E is the energy and x is a projection to the energy level $H=1$, so that the evolution of the coordinates x in the frozen system will be E-independent. The examples are given by a billiard in a τ-dependent region D_τ, a particle in a homogeneous polynomial potential with τ-dependent coefficients, a geodesic flow on a space with a τ-dependent metric.

When τ changes with time the energy $E=H(p,q,\tau)$ will be no longer preserved by the system: $\dfrac{dE}{dt}=\dfrac{\partial H}{\partial\tau}\dot\tau$. We assume that at large energies the motion in the frozen system is fast (e.g. in the above mentioned examples the velocity $\dot q$ grows as a square root of the kinetic energy). Therefore, in the system with τ varying with time the variables x change, at large energies, much faster than τ does. We also assume that $\partial H/\partial\tau$ has the same order as H, so the speed of change of $\ln E$ will be comparable with $\dot\tau$. Thus, we have a slow-fast system, with fast variables x and slow variables τ and $\ln E$. Then, a fundamental theorem by Anosov [4] is applied. Namely, if the frozen system is ergodic on every energy level with respect to the Liouville measure $\mu=\delta(E-H(p,q,\tau))dpdq$, we are guaranteed that averaging over this measure gives a good approximation of the slow evolution of the energy for a large set of initial conditions (see [5]):

$$\dot E=\frac{\displaystyle\int\frac{\partial H}{\partial\tau}(p,q,\tau)\delta(E-H(p,q,\tau))dpdq}{\displaystyle\int\delta(E-H(p,q,\tau))dpdq}\,\dot\tau. \tag{1}$$

We note that no mixing is required, i.e. one should not think of a fast relaxation to the Liouville measure at every moment of the slow time. Just the measure of the set of initial conditions for which the evolution of energy deviates noticeably from that given by this averaged equation is small.

By analogy, in the general case when the frozen system is *not ergodic* we may assume that the slow evolution of the energy is given by an averaged equation

$$\dot E=\int\frac{\partial H}{\partial\tau}\delta(E-H)\mu_\tau(dx)\;\dot\tau \tag{2}$$

where μ_τ is a certain ergodic measure on the space of fast variables. This measure can depend on τ and be different for different initial conditions. Choose some family μ_τ of these measures and let M be a set of initial x values for which the evolution of the energy over the period of τ is given by the averaged equation (2) with this particular measure for all τ. Let E_0 and E_1 be two sufficiently large values of energy. If the points with initial conditions $E=E_0$, $x\in M$ move to the energy level $E=\bar E_0=e^\lambda E_0$ after the period of τ, then the points with initial conditions $E=E_1$, $x\in M$ move

to the level $E = \bar{E}_1 = e^\lambda E_1$, by the homogeneity of (2). Since the original, non-averaged system preserves volume in the (q, p)-space, it follows that the volume occupied by the points with $x \in M$ between the levels $E = E_0$ and $E = E_1$ equals to the volume occupied by the points with $x \in \bar{M}$ between the levels $E = \bar{E}_0$ and $E = \bar{E}_1$, where $\bar{M}$ denotes the image of the set M by the flow of the full system after the period of τ. This gives us

$$\alpha\lambda = \ln(v(M)/v(\bar{M})) \tag{3}$$

where v is the volume in the x-space (i.e. in the energy level $H = 1$).

Let at least at some value of τ the frozen system be chaotic in a sufficiently strong sense. Namely, we assume that the system relaxes to the Liouville measure on each energy level. This does not fix distribution of energies, which can be arbitrary. We can, at this value of τ, define the entropy of the system as an averaged value of the $\ln(V(E)/V(1)) = \ln J(E)$, i.e.

$$S = \alpha \int < \ln E >_x \, dx, \tag{4}$$

where the integral is taken over the x-space (we assume its volume is scaled to 1), and $< \ln E >_x$ in this integral is the value of $\ln E$ averaged over initial conditions that get to the volume dx around the point x at the time τ. Let us label all possible sets M (each corresponding to each own family of measures μ_τ) by some index k. By (3), the change of the entropy over the period of τ is

$$\Delta S = \sum \ln \left[\frac{v(M_k)}{v(\bar{M}_k)} \right] v(M_k). \tag{5}$$

As $\sum v(M_k) = \sum v(\bar{M}_k) = 1 =$ the total volume of the $x-$space, it follows that $\Delta S \geq 0$ (to see this, denote $v(M_k) = v_k$, $v(\bar{M}_k) = \rho_k v_k$; we have $\sum \rho_k v_k = \sum v_k = 1 \Longrightarrow \Delta S = -\ln(\prod \rho_k^{v_k}) \geq -\ln(\sum \rho_k v_k) = 0$). Thus, the entropy (evaluated at the beginning of each period) is a non-decreasing function of time. Note that in the case the frozen system is ergodic for each τ, formula (1) gives $\Delta S = 0$ (to see this, write $V(E, \tau) = \int_{H(p,q,\tau) \leq E} dpdq = \int \theta(E - H(p, q, \tau))dpdq$ where θ is the Heaviside function; since $\theta' = \delta$, it follows that $\frac{\partial V}{\partial \tau} = -\int \frac{\partial H}{\partial \tau}(p, q, \tau)\delta(E - H(p, q, \tau))dpdq$, hence $\frac{\partial V}{\partial \tau} = \int \delta(E - H(p, q, \tau))dpdq$, so $\frac{d}{dt}V(E, \tau) = 0$ by (1)).

It is rare that a chaotic Hamiltonian system is ergodic. Therefore, in the general case one should not assume the ergodicity for all τ, so there is no restrictions on the growth of entropy, and we should expect

$$\Delta S > 0$$

in (5). As there is no dependence on energy in the right-hand side of (5), we will get the same increment in entropy over each period of τ, so S will grow linearly in time. By (4), this corresponds to an *exponential growth of energy*, with the rate $\Delta S/\alpha$ for a typical initial condition.

We can view our system as a gas of non-interacting particles (different particles correspond to different initial conditions). As there is no interaction, there is no equilibrium distribution in energies. However, in the ergodic case we still recover the entropy conservation at the adiabatic (i.e. slow) change of parameters. In the non-ergodic case we can think of particles as being, at each value of the parameter τ, in different states which correspond to different ergodic measures μ_τ over which the averaging is performed. Thus, our gas can be considered as a mixture of different phases or fractions; the adiabatic change of parameter can lead to particles changing their states, so the relative densities of each fraction in the gas can vary, and this naturally leads to the entropy growth.

Note that these conclusions are true only when the system indeed follows the models (2) or (1). These models are approximate, and do not need to be followed for all initial conditions. When the initial energy is taken higher, we expect the accuracy to increase and the measure of the set of initial conditions, for which this approximation is not valid, to decrease. Still, the (approximate) conservation of entropy and energy in the ergodic case can be valid only for a finite number of periods. On the other hand, the steady growth of entropy and energy in the non-ergodic case has to be a much more robust phenomenon. These conclusions were numerically confirmed for various types of chaotic billiards with periodically moving boundaries [6, 7] and for a particle in a polynomial potential of degree 4 with periodically changing coefficients at the terms of the highest order [8].

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