On the splitting of invariant curves

Dmitry Turaev[*]

[*]*Mathematics Department, Imperial College, London, UK*

Summary. Let a real-analytic Hamiltonian system have a normally-hyperbolic cylinder such that the Poincare map on the cylinder has a twist property. Let the stable and unstable manifolds of the cylinder intersect transversely. The homoclinic channel is a small neighbourhood of the union of the cylinder and the homoclinic. We show that generically (in the real-analytic category) in the channel there always exist orbits which move without a bound. This opens a way of showing that given an integrable system with 4 or more degrees of freedom, for arbitrarily small generic Hamiltonian perturbations the change of action variables along resonances is bounded away from zero.

The question of stability/instability of near-integrable systems is a central question of the Hamiltonian mechanics since its inception. By KAM-theory, we have stability for most of the initial conditions. For near-integrable systems with 2 degrees of freedom this implies that action variables cannot significantly deviate from their initial values. This is not, however, true for 3 and more degrees of freedom: as follows from an example of Arnold [1] one can always add an arbitrarily small perturbation to the Hamiltonian of an integrable system such that the action variables in the perturbed system would deviate from the initial values to a distance of order 1. How generic this instability (the Arnold diffusion) is, this question is under discussion for 4 decades by now. The difficulty is due to Nekhoroshev bounds [2] on the possible velocity of the action variables drift: it is so slow that it cannot be detected by any finite order expansion in the magnitude of perturbation.

It was announced in [3] that the Arnold diffusion can indeed be proven for generic near-integrable systems with 3 degrees of freedom. In [4], the genericity of the Arnold diffusion instability was shown for smooth near-integrable systems with any number of degrees of freedom. The analytic case remains open. In this talk I discuss how the bounded away from zero drift of the action variables can be obtained by a generic analytic perturbation of an integrable system with any number of degrees of freedom greater than 3. The main construction is similar to that developed by many authors since early 90s (see a survey in [5]). Namely, the resonant tori of the integrable system give rise to normally-hyperbolic cylinders of the perturbed system, moreover these cylinders acquire homoclinics, which correspond to transverse intersections of their stable and unstable manifolds. In fact, the stable manifold of such a cylinder is foliated by strong-stable leaves, and the unstable manifold is foliated by strong-unstable leaves. We consider a model (a joint work with Gelfreich) where the homoclinics obtained correspond to a transverse intersection of the unstable manifold to the strong-stable leaves and of the stable manifold to the strong-unstable leaves. Following [6], this strong transversality property allows us to define the homoclinic map $F : A \to A$ along the leaves of the foliation (A denotes the two-dimensional normally-hyperbolic cylinder of the Poincare map G on a cross-section). We prove the following fact: the sought unbounded drift along the "homoclinic channel" formed by A and the homoclinic is possible when on A there is an unbounded orbit of the pair of maps (F, G). Since G has a twist property here, it follows from the Birkhoff theory of twist maps that the latter orbit exists if and only if the maps F and G have no common invariant curve which divides the cylinder. In this way the problem of establishing a bounded away from zero drift of the action variables is reduced to the problem of destroying joint invariant curves of the pair of two maps, F and G, defined above.

While the Hamiltonian is analytic, the maps F and G have only finite smoothness, and the difference between them is small beyond any order in the perturbation parameter. It makes it therefore difficult to control what happens with these maps under small real-analytic perturbations. Another problem is that the number of invariant curves of the maps F and G is uncountable, and one has to control the splitting of all of them simultaneously, including those curves which are not KAM and not smooth. In the talk, I will show how these difficulties are bypassed and the following result is proved:

for a generic real-analytic Hamiltonian system which possesses a normally-hyperbolic invariant cylinder with a strongly transverse homoclinic the pair of maps G and F (the Poincare map on the cylinder and the homoclinic map) has no common invariant curves.

This result can be a cricial tool in proving the genericity of the Arnold diffusion type instability of the action variables for small analytic perturbations of integrable systems.

References

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