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DYNAMICAL PHENOMENA IN MULTIDIMENSIONAL SYSTEMS WITH A STRUCTURALLY UNSTABLE HOMOCLINIC POINCARÉ CURVE

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In [1] and [2] the impossibility of giving a complete description of the dynamics and bifurcations in three-dimensional models with a structurally unstable homoclinic Poincaré curve was established. Here this result is generalized to the higher-dimensional case. In addition, we study new bifurcation phenomena in Newhouse domains.

We consider a C^r-smooth ($r \geq 3$) dynamical system X on an $(m+n+1)$-dimensional manifold, having a saddle periodic orbit L with an $(m+1)$-dimensional stable manifold $\mathscr{W}^s$ and an $(n+1)$-dimensional unstable manifold $\mathscr{W}^u$. We assume that $\mathscr{W}^s$ and $\mathscr{W}^u$ are tangent with respect to some structurally unstable homoclinic orbit Γ.

Let S be a smooth section to L. On S there is a mapping T_0 on the orbits of the system X that are close to L. The point $O = L \cap S$ is a saddle fixed point, and its invariant manifolds are denoted W^s and W^u. Let $M^+ \in W^s_{\text{loc}}$ and $M^- \in W^u_{\text{loc}}$ be two points of the intersection of Γ with S. Then a mapping T_1 is defined from a neighborhood of the point M^- into a neighborhood of M^+ over orbits of X lying close to Γ. By hypothesis $T_1(W^u_{\text{loc}})$ is tangent to W^s_{loc} in M^+. We denote by $E_M W$ the tangent space to W at M. We assume that

(A) $\dim(E_{M^+} W^s \cap E_{M^+} T_1 W^u_{\text{loc}}) = 1$.

(B) $T_1(W^u_{\text{loc}})$ and W^s_{loc} have a quadratic tangency at M^+.

Let λ_i and γ_j be the multipliers of L, with $|\gamma_n| \geq \cdots \geq |\gamma_1| > 1 > |\lambda_1| \geq \cdots \geq |\lambda_m|$. We let $\lambda = |\lambda_1|$ and $\gamma = |\gamma_1|$. The multipliers λ_i and γ_j such that $|\lambda_i| = \lambda$ and $|\gamma_j| = \gamma$ are said to be *leading*, and the remainder are *nonleading*. Let p_s denote the number of leading λ_i and let p_u denote the number of leading γ_j. We make another assumption:

(C) The leading multipliers are not multiple, and either $p_s = 1$ and λ_1 is real, or $p_s = 2$ and $\lambda_1 = \bar{\lambda}_2 = \lambda e^{i\varphi}$ $(\varphi \neq 0, \pi)$; analogously, either $p_u = 1$ and γ_1 is real, or $p_u = 2$ and $\gamma_1 = \bar{\gamma}_2 = \gamma e^{i\psi}$ $(\psi \neq 0, \pi)$.

We shall say that X *has type* (p_s, p_u). Our next assumption is

(D) The saddle quantity $\sigma = \lambda\gamma \neq 1$.

The eigensubspace of the linear part of the mapping T_0 corresponding to the multipliers $\lambda_1, \dots, \lambda_m$ is denoted by $\mathscr{E}^s$, the eigensubspace corresponding to the leading λ_i is denoted by $\mathscr{E}^{s+}$, and the eigensubspace corresponding to the nonleading ones by $\mathscr{E}^{ss}$. Analogously we can define the subspaces $\mathscr{E}^u$, $\mathscr{E}^{u+}$, and $\mathscr{E}^{uu}$ corresponding to the multipliers γ_j. On W^s there is a nonleading invariant C^r-manifold W^{ss} tangent to $\mathscr{E}^{ss}$; on W^u there is a nonleading manifold W^{uu}. We make the following assumption:

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(E) $M^+ \notin W^{ss}$ and $M^- \notin W^{uu}$.

One can show that W^{ss} and W^{uu} are uniquely embedded in the invariant C^{r-1}-foliations F^{ss} and F^{uu} on W^s_{loc} and W^u_{loc}, and that there exist invariant C^1-manifolds H_u and H_s tangent to $\mathscr{E}^u \oplus \mathscr{E}^{s+}$ and $\mathscr{E}^s \oplus \mathscr{E}^{u+}$. H_u and H_s are not unique, but $H_u \supseteq W^u_{\mathrm{loc}}$ and $H_s \supseteq W^s_{\mathrm{loc}}$, and $E_{M^-}H_u$ and $E_{M^+}H_s$ are uniquely determined. We make our last assumption:

(F) $T_1(H_u)$ is transversal to F^{ss} at M^+, and $T_1^{-1}(H_s)$ is transversal to F^{uu} at M^-.

Flows close to X with a structurally unstable homoclinic curve close to Γ satisfy (A)–(F) and form a surface $\mathscr{H}$ of codimension one in the space of dynamical systems.

One can show that on S there exist C^{r-1}-coordinates in which T_0 has the form

$$(\bar{x}, \bar{u}) = A(x, u) + (f_{11}, f_{21})x + (f_{12}, f_{22})u,$$
$$(\bar{y}, \bar{v}) = B(y, v) + (g_{11}, g_{21})y + (g_{12}, g_{22})v,$$

where A and B are respectively $m \times m$ and $n \times n$ matrices, x and y are leading coordinates, and u and v are nonleading coordinates.

$$f_{ij}(x, u, y, v), \quad g_{ij}(x, u, y, v) \subset C^{r-1}, \quad f_{ij}(x, u, 0, 0) \equiv 0,$$
$$f_{i1}(0, u, y, v) \equiv 0, \quad g_{1j}(0, 0, y, v) \equiv 0, \quad \text{and} \quad g_{i1}(x, u, 0, v) \equiv 0.$$

In these coordinates the equation of W^s_{loc} is $(y, v) = 0$, the equation of W^{ss}_{loc} is $(y, v, x) = 0$, the equation of W^u_{loc} is $(x, u) = 0$, and the equation of W^{uu}_{loc} is $(x, u, y) = 0$; the foliations F^{ss} and F^{uu} have the form $\{x = \mathrm{const}, y = 0, v = 0\}$ and $\{y = \mathrm{const}, x = 0, u = 0\}$; the manifold H_u is the tangent on W^u_{loc} of the space $\{v = 0\}$, and H_s is the tangent on W^s_{loc} of $\{u = 0\}$. From this by conditions (A), (B), (E), and (F) one can obtain that the mapping $T_1\colon (x, u, y, v) \mapsto (\bar{x}, \bar{u}, \bar{y}, \bar{v})$ is written in the form [1]

$$\begin{aligned}
\bar{x}_1 - x_1^+ &= a_{11}x_1 + a_{12}x_2 - b_{11}(y_1 - y_1^-) + b_{12}(y_2 - y_2^-) + a_{13}u + b_{13}\bar{v} + \cdots, \\
\bar{x}_2 - x_2^+ &= a_{21}x_1 + a_{22}x_2 + b_{21}(y_1 - y_1^-) + b_{22}(y_2 - y_2^-) + a_{23}u + b_{23}\bar{v} + \cdots, \\
\bar{y}_1 &= c_{11}x_1 + c_{12}x_2 + D_1(y_1 - y_1^-)^2 + D_2(y_1 - y_1^-)(y_2 - y_2^-) \\
&\quad + D_3(y_2 - y_2^-)^2 + c_{13}u + d_{13}\bar{v} + \cdots, \\
\bar{y}_2 &= c_{21}x_1 + c_{22}x_2 + d_{21}(y_1 - y_1^-) + d_{22}(y_2 - y_2^-) + c_{23}u + d_{23}\bar{v} + \cdots, \\
\bar{u} - u^+ &= a_{31}x_1 + a_{32}x_2 + b_{31}(y_1 - y_1^-) + b_{32}(y_2 - y_2^-) + a_{33}u + b_{33}\bar{v} + \cdots, \\
v - v^- &= c_{31}x_1 + c_{32}x_2 + d_{31}(y_1 - y_1^-) + d_{32}(y_2 - y_2^-) + c_{33}u + d_{33}\bar{v} + \cdots,
\end{aligned}$$

where $(x^+, u^+, 0, 0)$ and $(0, 0, y^-, v^-)$ are the coordinates of the points M^+ and M^- respectively. We shall set $x_1^+ > 0$ and $y_1^- > 0$. By (E) and (F) we have

$$\det\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix} \neq 0.$$

In particular, if λ_1 is real, then $c \equiv c_{11} \neq 0$. Moreover, it follows from (A) that if γ_1 is complex, then $d_{21}^2 + d_{22}^2 \neq 0$; in this case by choosing the homoclinic points M^+ and M^- on S we can arrange that $d_{22} \neq 0$. We write $d = D_1$ if γ_1 is real,

[1] Here if the multiplier λ_1 (resp. γ_1) is complex, then $x = (x_1, x_2)$ (resp. $y = (y_1, y_2)$), and if λ_1 (resp. γ_1) is real, then $x \equiv x_1$ (resp. $y \equiv y_1$) and the terms with x_2 (resp. y_2) are to be omitted.

and $d = D_1 - D_2(d_{21}/d_{22}) + D_3(d_{21}/d_{22})^2$ if γ_1 is complex. Condition (B) means that $d \neq 0$.

Following [3]–[5], we partition the systems satisfying (A)–(F) into three classes. The systems of the first classes are defined by the condition that if $\sigma < 1$, then γ_1 is real and positive and $d < 0$, and if $\sigma > 1$, then λ_1 is real and positive and $dc > 0$. The second class consists of systems with real and positive γ_1 and λ_1 such that $d > 0$ and $c < 0$. All the rest go into the third class.

We consider a small neighborhood U of the contour $L \cup \Gamma$. Let N be the set of orbits of the system X that lie entirely within U.

Theorem 1. *For systems of the first class, $N = \{L, \Gamma\}$. For systems of the second class N is equivalent to the suspension over the Bernoulli shift on three symbols $\{0, 1, 2\}$ in which the orbits $(\cdots 0 \cdots 010 \cdots 0 \cdots)$ and $(\cdots 0 \cdots 020 \cdots 0 \cdots)$ are identified.*

For systems of the third class the set N also has nontrivial hyperbolic subsets, but they do not in general exhaust all of N. Moreover, under a motion along the film $\mathscr{H}_3$ of systems of the third class the structure of the set N varies continuously. The main reason for this is the presence of moduli (i.e. continuous invariants) of Ω-equivalence. We set $\theta = -\ln \lambda / \ln \gamma$.

Theorem 2. *Let $X_1, X_2 \in \mathscr{H}_3$. Then for the Ω-equivalence*[2] *of X_1 and X_2 it is necessary that $\theta_1 = \theta_2$ in the case of systems of type $(1, 1)$; that $\theta_1 = \theta_2$ and $\varphi_1 = \varphi_2$ for systems of type $(2, 1)$; $\theta_1 = \theta_2$ and $\psi_1 = \psi_2$ for systems of type $(2, 1)$; and, for type $(2, 2)$, $\varphi_1 = \varphi_2$, $\psi_1 = \psi_2$, and also $\theta_1 = \theta_2$, except perhaps for the case $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 \in \{2\pi/3, \pi/2, 2\pi/5, \pi/3\}$.*

The quantities ψ, φ, and θ are the fundamental moduli in the sense that they are defined everywhere on $\mathscr{H}_3$. If $\psi/2\pi$, $\varphi/2\pi$, or θ is irrational, then there also exist other moduli, for example, analogous to the quantity τ introduced in [5]. Moreover, we have

Theorem 3. *In $\mathscr{H}_3$ there is a dense set $\mathscr{B}^*$ of systems that have a countable number of saddle periodic orbits with structurally unstable homoclinic curves of the third class.*

For systems from $\mathscr{B}^*$ the quantities θ computed for the corresponding saddle periodic orbits are moduli of Ω-equivalence (Theorem 2). Thus, we have a theorem which generalizes a result from [1] and [2] to the higher-dimensional case.

Theorem 4. *In $\mathscr{H}_3$ the systems with a countable number of moduli of Ω-equivalence are dense.*

We note that if the moduli of Ω-equivalence are chosen as parameters, then, as they change, bifurcations of the nonwandering orbits must occur—periodic orbits, homoclinic orbits, etc. Here the presence of a countable number of moduli leads to infinitely degenerate bifurcations. Thus, we have

Theorem 5. *In $\mathscr{H}_3$ the systems with a countable number of structurally unstable homoclinic curves of arbitrary orders of tangency and with a countable number of structurally unstable periodic orbits, both with multiplier $\rho = 1$ and with $\rho = -1$, of arbitrary orders of degeneration are dense.*

Let X be a system from $\mathscr{H}$ (not necessarily of third class). We consider a C^r-smooth finite-parameter family X_ε, transversally intersecting $\mathscr{H}$ at X for $\varepsilon = 0$. We shall call open sets in the space of dynamical systems, in which the systems

[2] Analogously to [1] and [2] we talk about Ω-equivalence via a homeomorphism that is homotopic to the identity in U.

with structurally unstable homoclinic curves are dense, *Newhouse domains*. From the results of [6] we get

Theorem 6. *There exists a sequence of open sets Δ_i accumulating to $\varepsilon = 0$ such that the X_ε for $\varepsilon \in \Delta_i$ lie in a Newhouse domain, and the values of ε for which L_ε has a structurally unstable homoclinic curve of the third class are dense in Δ_i.*

From Theorems 5 and 6 we get

Theorem 7. *In Newhouse domains the systems with infinitely degenerate periodic and homoclinic orbits and systems with a countable number of moduli of Ω-equivalence are dense.*

Thus, the models with structurally unstable homoclinic Poincaré curves are not good in the sense of [1] and [2]. Here a complete description of all the bifurcations is not possible. However, one can indicate restrictions on the nature of the possible motions. Thus, the following theorem shows that the dynamics of the system X and all nearby ones is effectively determined only by the leading coordinates, and the nonleading coordinates lead to trivial dynamics.

Theorem 8. *For all systems close to X, the set N is contained in an invariant $(p_s + p_u + 1)$-dimensional C^1-manifold $\mathfrak{M}_C$, depending continuously on the system. Any orbit $\mathscr{L} \subset N$ has $(m - p_s + 1)$-dimensional strong stable and $(n - p_u + 1)$-dimensional strong unstable invariant C^r-manifolds $\mathscr{W}^{ss}(\mathscr{L})$ and $\mathscr{W}^{uu}(\mathscr{L})$ respectively, such that the orbits belonging to $\mathscr{W}^{ss}(\mathscr{L})$ (resp. $\mathscr{W}^{uu}(\mathscr{L})$) tend exponentially to $\mathscr{L}$ as $t \to +\infty$ (resp. as $t \to -\infty$) with exponent not less in modulus than $|\ln \hat{\lambda}|/T$ (resp. $\ln \hat{\gamma}/T$), where T is the period of the orbit L, $0 < \hat{\lambda} < \lambda$, and $\hat{\gamma} > \gamma$. The intersection $\mathfrak{M}_C \cap S$ is defined by an equation of the form $(u, v) = f_C(x, y)$, where $f_C(0, 0) = 0$ and $\partial f_C(0, 0)/\partial(x, y) = 0$. The manifolds $\mathscr{W}^{ss}(\mathscr{L})$ and $\mathscr{W}^{uu}(\mathscr{L})$ in the intersection with S are collections of leaves transversal to H^u and H^s respectively and having the form $(x, y, v) = f_{ss}(u)$ and $(x, y, u) = f_{uu}(v)$.*

The dynamics on $\mathfrak{M}_C$ essentially depends on the modulus of the product of the leading multipliers, i.e., on $\tilde{\sigma} = \lambda^{p_s}\gamma^{p_u}$. We note the connection between $\tilde{\sigma}$ and $\theta : \ln \tilde{\sigma} / \ln \gamma = p_u - p_s\theta$.

Theorem. 9. *The orbits of the restriction to $\mathfrak{M}_C$ of any system close to X have for $\tilde{\sigma} < 1$ not less than $(p_s - [p_u/\theta]) \geq 1$ negative Lyapunov exponents, and for $\tilde{\sigma} > 1$ they have not less than $(p_u - [p_s\theta]) \geq 1$ positive Lyapunov exponents.*

For $\tilde{\sigma} < 1$ we distinguish three cases:

(1^+) $(p_s, p_u) = (1, 1)$ or $(p_s, p_u) = (2, 1)$ for $\lambda\gamma < 1$.

(2^+) $(p_s, p_u) = (2, 1)$ for $\lambda\gamma > 1$, and $(p_s, p_u) = (1, 2)$ or $(p_s, p_u) = (2, 2)$ for $\lambda\gamma^2 < 1$.

(3^+) $(p_s, p_u) = (2, 2)$ for $\lambda\gamma^2 > 1$.

For $\tilde{\sigma} > 1$ we distinguish the three cases (1^-), (2^-), and (3^-), which are obtained respectively from (1^+)–(3^+) by replacing t by $-t$. It follows from Theorem 9 that in case (1^-) (resp. (1^+)) the restriction to $\mathfrak{M}_C$ of any system close to X cannot have periodic orbits with more than l stable (resp. unstable) multipliers. From this and from Theorem 8 we get

Theorem 10. *If $\tilde{\sigma} > 1$ or L has nonleading unstable multipliers, then all the orbits of both X and any system sufficiently close to it are unstable.*

Let μ be a functional on the space of systems such that $\mathscr{H}$ close to X is defined by the condition $\mu = 0$. We include X in a family X_ε transversal to $\mathscr{H}$: $\varepsilon = \mu$ in cases (1^+) and (1^-); $\varepsilon = (\mu, \varphi)$ in cases (2^+) for systems of type (2, 1) and (2^-) for systems of types (2, 1) and (2, 2); $\varepsilon = (\mu, \psi)$ in cases (2^-) for systems of type (1, 2) and (2^+) for systems of types (1, 2) and (2, 2); and $\varepsilon = (\mu, \varphi, \psi)$ in cases (3^+) and (3^-).

Theorem 11. *In case (l^+) (resp. (l^-)) in the family (X_ε) the values of ε for which $X_{\varepsilon|\mathfrak{M}_C}$ has a countable number of periodic orbits with j positive (resp. negative) Lyapunov exponents for any $j \in \{0, \dots, l\}$ are dense in the Newhouse domains Δ_i.*

Theorem 12. *If L has no nonleading unstable multipliers and $\tilde{\sigma} < 1$, then the systems with a countable number of stable periodic orbits are dense in the family X_ε in the domains Δ_i.*

Theorem 13. *In cases (2^+) and (2^-) the systems having structurally unstable periodic orbits with multipliers $(\rho_1, \rho_2) \in \{(1, 1), (-1, 1), (-1, -1), (e^{i\omega}, e^{-i\omega})\}$ are dense in the domains Δ_i, and in cases (3^+), (3^-) those systems with multipliers $(\rho_1, \rho_2, \rho_3) \in \{(1, 1, 1), (1, -1, 1), (1, -1, -1), (-1, -1, -1), (1, e^{\pm i\omega}), (-1, e^{\pm i\omega})\}$ are dense in Δ_i.*

Corollary. *In cases (2^+), (2^-), (3^+), and (3^-) the systems with a countable number of two-dimensional invariant tori are dense in Newhouse domains.*

The proof of Theorems 11–13 is based on the study of the mappings $T_k \equiv T_1(\varepsilon)T_0^k(\varepsilon)$. If L has no nonleading multipliers (their contribution is trivial), then for $\tilde{\sigma} < 1$ the mapping T_k in normalized variables is close for $p_u = 1$ to the following mappings: in case (1^+)

$$\bar{x}_2 = 0, \qquad \bar{x}_1 = y, \qquad \bar{y} = M - y^2,$$

in case (2^+)

$$\bar{x}_2 = x_1, \qquad \bar{x}_1 = y, \qquad \bar{y} = M - y^2 + Bx_1,$$

and for $p_u = 2$ it is close to the mappings: in case (2^+)

$$\bar{x}_2 = 0, \quad \bar{x}_1 = y_1, \quad \bar{y}_1 = y_2, \quad \bar{y}_2 = M - y_1^2 + Cy_2,$$

in case (3^+)

$$\bar{x}_2 = 0, \quad \bar{x}_1 = y_1, \quad \bar{y}_1 = y_2, \quad \bar{y}_2 = M - y_1^2 + Cy_2 + Bx_1,$$

where

$$B \sim (\tilde{\sigma}/\lambda)^k(\hat{b}_1 \cos k\varphi + \hat{b}_2 \sin k\varphi),$$
$$C \sim \gamma^k(\hat{c}_1 \cos k\psi + \hat{c}_2 \sin k\psi)$$

for some $\hat{b}_i$ and $\hat{c}_i$ such that $\hat{b}_1^2 + \hat{b}_2^2 \neq 0$ and $\hat{c}_1^2 + \hat{c}_2^2 \neq 0$. For $p_u = 1$ the quantity $M \sim \mu\gamma^{2k}$, and for $p_u = 2$ the quantity $M \sim \mu\gamma^{3k}$. It is clear that one can always choose k so large that by a small perturbation of μ, φ, and ψ the quantities M, B, and C can assume arbitrary finite values, in particular, those for which there is a fixed point with an arbitrary set of multipliers permitted by Theorem 9.

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