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ON THE EXISTENCE OF NEWHOUSE DOMAINS IN A NEIGHBORHOOD OF SYSTEMS WITH A STRUCTURALLY UNSTABLE POINCARÉ HOMOCLINIC CURVE (THE HIGHER-DIMENSIONAL CASE)

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In [1] Newhouse established the following remarkable fact: *in any neighborhood of a two-dimensional C^r-diffeomorphism* $(r \geq 2)$, *having a saddle periodic point with a structurally unstable homoclinic trajectory, there exist domains* (*so-called Newhouse domains*) *in which systems with structurally unstable Poincaré homoclinic trajectories are dense.* In the present paper we generalize this result to the higher-dimensional case.

Theorem 1. *In the space of C^r-smooth $(r \geq 2)$ flows on an arbitrary finite-dimensional manifold, in any neighborhood of a flow which has a saddle periodic trajectory whose invariant manifolds are tangent along some structurally unstable homoclinic curve, Newhouse domains exist.*

Theorem 1 follows from a more general assertion concerning one-parameter families of flows X_μ of class C^r $(r \geq 3)$, having a saddle periodic trajectory L_0 for $\mu = 0$ with structurally unstable homoclinic curve Γ_0.

Let S be a smooth secant to L_0. For small μ a mapping T_0 is defined on S along the trajectories of X_μ close to L_μ. The point $O = L_\mu \cap S$ is a saddle fixed point, and its invariant manifolds are denoted by $W^s(O)$ and $W^u(O)$. Let $M^+ \in W^s_{\mathrm{loc}}$ and $M^- \in W^u_{\mathrm{loc}}$ be any two intersections of Γ_0 with S. For small μ a mapping T_1 is defined from a neighborhood Π_1 of M^- to a neighborhood Π_0 of M^+ along the trajectories of the flow X_μ that lie close to Γ_0. By hypothesis $T_1(W^u_{\mathrm{loc}})$ is tangent to W^s_{loc} at M^+ for $\mu = 0$. We write: $E_M W$ is the tangent space to W at M. We make the following assumptions:

(A) $\dim(E_{M^+} W^s \cap E_{M^+} T_1 W^u_{\mathrm{loc}}) = 1$.

(B) The tangency of $T_1(W^u_{\mathrm{loc}})$ and W^s_{loc} is quadratic.

Let λ_i and γ_j be the multipliers of the trajectory L_0, $|\gamma_n| \geq \cdots \geq |\gamma_1| > 1 > |\lambda_1| \geq \cdots \geq |\lambda_m|$; here $n = \dim W^u$ and $m = \dim W^s$. We write $\lambda = |\lambda_1|$ and $\gamma = |\gamma_1|$. We make some more assumptions:

(C) One of the following four cases occurs:

(1.1) λ_1 and γ_1 are real, and $\lambda > |\lambda_2|$, $\gamma < |\gamma_2|$.

(2.1) $\lambda_1 = \overline{\lambda}_2 = \lambda e^{i\varphi}$ $(\varphi \neq 0, \pi)$, γ_1 is real, and $\lambda > |\lambda_3|$, $\gamma < |\gamma_2|$;

(1.2) λ_1 is real, $\gamma_1 = \overline{\gamma}_2 = \gamma e^{i\psi}$ $(\psi \neq 0, \pi)$, and $\lambda > |\lambda_2|$, $\gamma < |\gamma_3|$;

(2.2) $\lambda_1 = \overline{\lambda}_2 = \lambda e^{i\varphi}$, $\gamma_1 = \overline{\gamma}_2 = \gamma e^{i\psi}$ $(\varphi, \psi \neq 0, \pi)$, and $\lambda > |\lambda_3|$, $\gamma < |\gamma_3|$.

(D) The saddle quantity $\sigma = \lambda\gamma < 1$.

Multipliers λ_i and γ_j such that $|\lambda_i| < \lambda$ and $\gamma < |\gamma_j|$ are said to be *nonleading*, while the remainder are termed *leading*. The eigensubspace of the linear part of T_0

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corresponding to the multipliers $\lambda_1, \dots, \lambda_m$ is denoted $\mathscr{E}^s$, the one corresponding to the leading λ_i is denoted $\mathscr{E}^{s+}$, and the one corresponding to the nonleading λ_i is $\mathscr{E}^{ss}$. The subspaces $\mathscr{E}^u$, $\mathscr{E}^{u+}$, $\mathscr{E}^{uu}$ are defined analogously; these correspond to the multipliers γ_j. On W^s there is an invariant C^r-manifold W^{ss}, tangent to $\mathscr{E}^{ss}$; in just the same way on W^u there is a nonleading manifold W^{uu}. We assume that

(E) $M^+ \notin W^{ss}$ and $M^- \notin W^{uu}$.

One can show that W^{ss} and W^{uu} are uniquely included in invariant C^{r-1}-foliations F^{ss} and F^{uu} on W^s and W^u respectively, and that there exist invariant C^1-manifolds H_u and H_s, tangent to $\mathscr{E}^u \oplus \mathscr{E}^{s+}$ and $\mathscr{E}^s \oplus \mathscr{E}^{u+}$. These manifolds H_u and H_s are not unique, but $E_{M^-}H_u$ and $E_{M^+}H_s$ are uniquely defined. We assume that

(F) $T_1(H_u)$ is transversal to F^{ss} at M^+ and $T_1^{-1}(H_s)$ is transversal to F^{uu} at M^-.

The flows close to X_0 with a structurally unstable homoclinic curve close to Γ_0, and satisfying (A)–(F), form a surface H of codimension one in the space of dynamical systems.

Theorem 2. *Let X_μ be transversal to H. Then there exists a sequence of intervals Δ_i accumulating to $\mu = 0$ such that for $\mu \in \Delta_i$ the system X_μ lies in a Newhouse domain. Moreover, the values of μ for which L_μ has a structurally unstable homoclinic curve are dense on Δ_i.*(1)

Theorem 1 is an obvious consequence of Theorem 2 (although $X_\mu \in C^3$, the Newhouse domains are open even in the C^2-topology and Theorem 1 remains valid for $r = 2$). We shall give a sketch of the proof of Theorem 2. We assume that:

(G) In any neighborhood of $\Gamma_0 \cup L_0$ there exist structurally stable homoclinic curves of the trajectory L_0.

Condition (G) is equivalent to conditions (*) and (**) (see below), and may not hold in general (for details see [2]–[4]). However the following result is valid.

Lemma 1. *For the family X_μ of Theorem* 1 *there exists a sequence $\mu_j \to 0$ such that $L_{\mu j}$ has a homoclinic curve satisfying* (A)–(G), *and the family X_μ is transversal to the corresponding bifurcation surface for $\mu = \mu_j$.*

According to [5], for some sufficiently large $\overline{k}$, on Π_0 there is a sequence of "strips" σ_k^0, $k = \overline{k}, \overline{k}+1, \dots$, accumulating to W^s_{loc}, on which the mappings $T_k \equiv T_1 T_0^k$ are defined. In order to obtain analytic expressions for T_k it is convenient to choose special coordinates on S.

Lemma 2. *In some C^{r-1}-coordinates T_0 has the form*

$$(\overline{x}, \overline{u}) = A(\mu)(x, u) + (f_{11}, f_{21})x + (f_{12}, f_{22})u,$$
$$(\overline{y}, \overline{v}) = B(\mu)(y, v) + (g_{11}, g_{21})y + (g_{12}, g_{22})v,$$

where x, y are leading coordinates, u, v are nonleading coordinates,(2) *and*

$$f_{ij}(x, u, y, v, \mu), g_{ij}(x, u, y, v, \mu) \in C^{r-1},$$
$$f_{1j}(x, u, 0, 0, \mu) \equiv 0, \qquad f_{i1}(0, u, y, v, \mu) \equiv 0,$$
$$g_{1j}(0, 0, y, v, \mu) \equiv 0, \qquad g_{i1}(x, u, 0, v, \mu) \equiv 0.$$

(1)In case (1.1) this theorem for $\ln \lambda_2 / \ln \lambda_1 > 3$ and $\ln \gamma_2 / \ln \gamma_1 > 3$ can be reduced to [1] by reduction to a two-dimensional invariant C^3-manifold tangent to $\mathscr{E}^{s+} \oplus \mathscr{E}^{u+}$.

(2)It would be very convenient to have coordinates in which T_0 is linear, but in order for a smooth linearization to be possible it is necessary to impose some additional restrictions on the system, such as the absence of principal resonances, which is not related to the essence of the problem.

Lemma 3. *The mapping* $T_0^k\colon (x, u, y, v) \mapsto (\overline{x}, \overline{u}, \overline{y}, \overline{v})$ *for sufficiently small* μ *and large* k *can be written in the form*

$$(1)\qquad \begin{aligned}(\overline{x}, \overline{u}) &= A^k(\mu)(x, y) + (\varphi_{k1}, \varphi_{k2}),\\ (y, v) &= B^{-k}(\mu)(\overline{y}, \overline{v}) + (\psi_{k1}, \psi_{k2}),\end{aligned}$$

where

$$\|\partial^i\varphi/\partial(x, u, \overline{y}, \overline{v}, \mu)^i\| \le \hat{\lambda}^k, \qquad \|\partial^i\psi/\partial(x, u, \overline{y}, \overline{v}, \mu)^i\| \le \hat{\gamma}^{-k},$$

for $i = 0, \dots, (r-2)$ *and*

$$\begin{gathered}\|\partial^{r-1}(\varphi, \psi)/\partial(x, u, \overline{y}, \overline{v}, \mu)^{r-1}\| \le \varepsilon_k, \qquad \varepsilon_k \to 0,\\ \lambda > \hat{\lambda} > \max\{|\lambda_i|, \lambda/\gamma\}, \qquad \gamma < \hat{\gamma} < \min\{|\gamma_j|, \gamma^j\}.\end{gathered}$$

Thus, in these coordinates the mapping T_0^k is linear in principal order. Conditions (A), (B), (E), and (F) for the mapping $T_1\colon (x, u, y, v) \mapsto (\overline{x}, \overline{u}, \overline{y}, \overline{v})$ allow us to write down the following expression (here if the multiplier λ_1 (resp. γ_1) is complex, then $x = (x_1, x_2)$ (resp. $y = (y_1, y_2)$), and if λ_1 (resp. γ_1) is real, then $x \equiv x_1$ (resp. $y \equiv y_1$) and the terms with x_2 (resp. y_2) are to be omitted):
(2)

$$\begin{aligned}\overline{x}_1 = x_1^+ &= b_0(y_1 - y_1^-) + b_1(\overline{y}_2, \overline{v}) + a_{11}x + a_{12}u + \cdots,\\ \overline{y}_1 - \beta(\mu)\overline{v} &= \mu + d_0(y_1 - y_1^-)^2 + c_1x_1 + c_2x_2 + C_1u + \cdots,\\ (\overline{x}_2 - x_2^+, \overline{u} - u^+ - \alpha(\mu)(\overline{x}_1 - x_1^+)) &= b_2(\overline{y}_2, \overline{v}) + a_{21}x + a_{22}u + \cdots,\\ (y_2 - y_2^- - D(y_1 - y^-), v - v^- - D(y_1 - y_1^-)) &= D_2\overline{y}_2 + D_3\overline{v} + C_2x + C_3u + \cdots.\end{aligned}$$

By (E) and (F) the coefficients $b_0 \ne 0$ and $c_1^2 + c_2^2 \ne 0$; by (B) we have $d_0 \ne 0$.

We see from (1) and (2) that by translating the origin and normalizing $(X, Y) = (x_1, y_1)$ by γ^{-k} and $\xi = (x_2, u)$ and $\eta = (y_2, v)$ by $\delta_k\gamma^{-k}$, where δ_k tends exponentially to zero as $k \to \infty$, the mapping T_k can be reduced to the form

$$(3)\qquad \begin{gathered}\overline{Y} = a - dy^2 + b(\lambda\gamma)^kX + \chi_k^2(X, Y, \xi, \overline{\eta}, \mu),\\ \overline{X} = Y + \chi_k^1(X, Y, \xi, \overline{\eta}, \mu),\\ \eta = \chi_k^3(X, Y, \xi, \overline{\eta}, \mu), \qquad \xi = \chi_k^4(X, Y, \xi, \overline{\eta}, \mu),\end{gathered}$$

where $\chi_k^1 = O(\gamma^{-k})$, $\chi_k^2 = O(\hat{\lambda}^k\gamma^k\|(X, \xi)\| + \gamma^{-k}\|(Y, \overline{\eta})\|)$, $\chi_k^3 = O(\gamma'^{-k})$, $\chi_k^4 = O(\lambda'^k\|(X, \xi)\| + \gamma'^{-k}\|(Y, \overline{\eta})\|)$, λ' is close to λ, γ' is close to γ, and

$$(4)\qquad \begin{aligned}a = {}&\gamma^{2k}(\mu - \gamma^{-k}y_1^-\cos(k\psi) + y_2^-\sin(k\psi))\\ &+ \lambda^k((c_1x_1^+ + c_2x_2^+)\cos(k\varphi) + (c_1x_2^+ - c_2x_1^+)\sin(k\varphi) + \cdots),\end{aligned}$$

$$(5)\qquad b \sim b_0(c_1\cos(k\varphi) + c_2\sin(k\varphi))/(\cos(k\psi) + D\sin(k\psi)).$$

$$(6)\qquad d = -d_0/(\cos(k\psi) + D\sin(k\psi))^2.$$

For real positive γ_1 (resp. λ_1) the formulas for a, b, d are obtained if we set $\psi = 0$ (resp. $\varphi = 0$) in (4)–(6), and for negative γ_1 (resp. λ_1) we set $\psi = \pi$ (resp. $\varphi = \pi$).

Lemma 4. *There exists an arbitrarily large* k *such that* T_k *for all small* μ *has a hyperbolic set* Ω_u *equivalent to the Bernoulli scheme* $\mathfrak{B}_2$ *on two symbols.* $W^u(\Omega_u)$ *lies in an invariant* $C^{1+\varepsilon}$*-manifold* $\mathfrak{M}^u$ *of the form* $\xi = \rho_k^u(X, Y, \eta, \mu)$, *where* $\rho_k^u \to 0$ *as* $k \to \infty$. *Here* $W^u(\Omega_u)$ *is included in an invariant* C^1*-foliation* $\mathfrak{F}_u$ *of codimension one on* $\mathfrak{M}^u$.

The following two conditions are necessary and sufficient for (G) to hold:

(*) Either γ_1 is complex, or $\gamma_1 < 0$, or $\gamma_1 > 0$ and $d_0 y_1^- > 0$
(**) Either λ_1 is complex, or $\lambda_1 < 0$, or $\lambda_1 > 0$ and $c_1 x_1^+ < 0$

It follows from (4)–(6) and (*) that for $\mu = 0$ there exists a countable set of values k for which b and d in (3) are different from zero and infinity, $ad < 0$ and $a \sim \gamma^k$. We fix a sufficiently large such k and in (3) we set $\xi = 0$, $\overline{\eta} = 0$. We obtain a mapping $\widehat{T}_k\colon (x, y) \mapsto (\overline{x}, \overline{y})$, close to the Hénon mapping with Jacobian $J_k \sim (\lambda\gamma)^k$. Since a is large, $\widehat{T}_k$ has a hyperbolic nonwandering set $\widehat{\Omega}$ equivalent to $\mathfrak{B}_2$. The expansion on $\widehat{\Omega}$ has order $\sqrt{a} \sim \gamma^{k/2}$, and contraction has order $J_k/\sqrt{a} \sim \lambda^k \gamma^{k/2}$. For the original mapping (3) the contraction in ξ is stronger (of order not greater than λ'^k), which via the standard technique connected with normal hyperbolicity [6] yields the lemma.

We take sequences $\mu_i \to 0$ and $k_i \to \infty$ such that $ad \doteq 2$ and b and d are different from zero and infinity in (3). The change of variables $Y = -2\cos(\zeta)/d$, $X = X$, $\xi = \xi$, $\eta = \eta$ brings the mapping T_{k_i} to the form $\overline{\zeta} = 2\min(\zeta, \pi - \zeta) + \Phi_k$, $\overline{X} = -2\cos(\zeta)/d + \chi_k^1$, $\eta = \chi_k^3$, $\overline{\xi} = \chi_k^4$, where $\Phi_k \sim [(\lambda\gamma)^k X + \chi_k^2]/\sin(2\zeta)$. In the limit $k_i = \infty$ the mapping T_{k_i} degenerates to the one-dimensional mapping $T_\infty\colon \overline{\zeta} = 2\min(\zeta, \pi - \zeta)$. We consider a hyperbolic set Ω_N consisting of trajectories of the mapping T_∞ that lie entirely within $[4\nu_N, \pi/2 - \nu_N] \cup [\pi/2 + \nu_N, \pi - 2\nu_N]$, where $4\nu_N = 2\pi(2^{N+2} - 1)^{-1}$ is a point of period $N + 2$ for T_∞. For large k_i the mapping T_{k_i} close to Ω_N has an expansion of order 2 in the ζ-coordinate, and is significantly stronger in the η-coordinate, of order not less than γ'^k, which allows us to obtain the following result.

Lemma 5. *Close to Ω_N, T_{k_i} has an invariant hyperbolic set Ω_s such that $W^s(\Omega_s)$ lies in a smooth*(3) *invariant manifold $\mathfrak{M}_s$ of the form $\eta = \rho_k^s(X, Y, \xi, \mu)$, $\rho_k^s \to 0$ as $k_i \to \infty$. Here $W^s(\Omega_s)$ is included in a smooth invariant foliation $\mathcal{F}_s$ of codimension one on $\mathfrak{M}_s$.*

For large k_i close to μ there exist values of μ for which the invariant manifolds W_k^s and W_k^u of the saddle fixed point O_k of the mapping T_{k_i} have a quadratic tangency, and the family X_μ is transversal to the corresponding bifurcation surface. One can show that here W_k^s transversally intersects $W^u(\Omega_u)$, and W_k^u transversally intersects $W^s(\Omega_s)$ (see Figure 1 on the next page). From this it follows that by an additional small change of μ we can add a nondegenerate heteroclinic tangency of $W^u(\Omega_u)$ with $W^s(\Omega_s)$.

Lemma 6. *The manifolds $\mathfrak{M}_u$ and $\mathfrak{M}_s$ intersect transversally close to a point of tangency along a smooth two-dimensional manifold on which the foliations $\mathfrak{F}_u$ and $\mathfrak{F}_s$ are tangent at the points of some smooth curve h transversal to the leaves.*

The Cantor sets $K_u = W^u(\Omega_u) \cap h$ and $K_s = W^s(\Omega_s) \cap h$ intersect at the points of tangency of $W^u(\Omega_u)$ and $W^s(\Omega_s)$. It is known [1] that if the product of the thicknesses $\tau(K_u)\tau(K_s) > 1$, then the intersection of K_u and K_s cannot be removed by a small perturbation. One can show that the factorization of the mapping $T_{k_i}|_{\mathfrak{M}_s}$ along the leaves of the foliation $\mathfrak{F}_s$ is an expanding mapping and satisfies the "bounded distortion" property [7], from which it follows analogously [1], [7] that the thickness of the intersection of $W^s(\Omega_s)$ with any transversal smooth curve does not depend on the curve, on the point on $W^s(\Omega_s)$, is different from zero and infinity, and depends continuously on the mapping T_{k_i} in the C^2-topology. The very same result is valid

(3)The smoothness of $\mathfrak{M}_s$ is estimated by the ratio of the logarithms of the exponents of expansion in η and ζ, and has order $k_i \ln\gamma/\ln 2$.

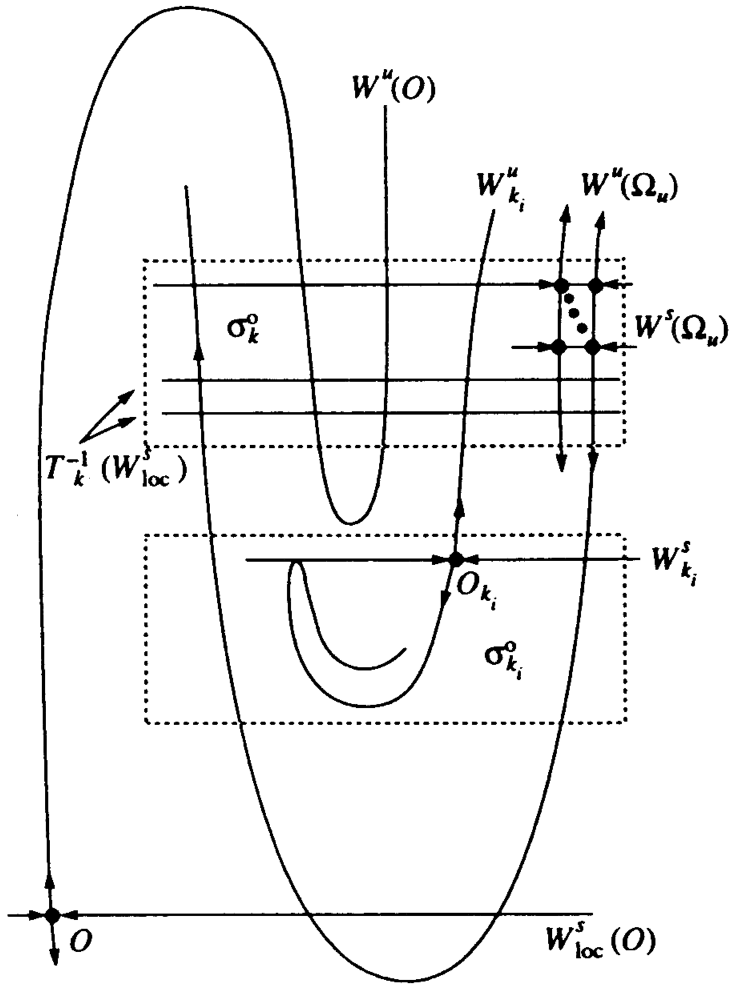

FIGURE 1

for $W^u(\Omega_u)$, and so $\tau(K_u)$ is different from zero, and $\tau(K_s)$ for large k_i can be made arbitrarily close to $\tau(\Omega_N)$.

Lemma 7. *Ω_N is a Cantor set of thickness $\tau(\Omega_N) = 2^N - 1$.*

It follows from the lemma that, by increasing N, we can always arrange it so that $\tau(K_u)\tau(K_s) > 1$, and so $W^s(\Omega_s)$ and $W^u(\Omega_u)$ have an unremovable tangency for the indicated values of $\mu = \mu_i^k$.

From (G) it follows that the closure of $W^u(O)$ contains $W^s(\Omega_s)$ and the closure of $W^s(O)$ contains $W^u(\Omega_u)$ (see Figure 1). Therefore the flows that have structurally unstable curves of the trajectory L are dense in the space of smooth flows in a small neighborhood of $X_{\mu_i^*}$. Moreover, since the tangencies of $W^s(O)$ and $W^u(O)$ corresponding to these curves lie close to the tangency of the manifolds W_k^s and W_k^u, the family X_μ is transversal to the corresponding bifurcation surfaces and, hence, the values of μ corresponding to homoclinic tangencies are dense in a small interval Δ_i containing μ_i^*, which completes the proof of Theorem 2.

After this paper was completed we learned that the case $\dim W^u = 1$ has been considered by J. Palis and M. Viana [8]. Palis also informed us that a closely related problem was considered by N. Romero. We thank Ya. G. Sinaĭ, K. M. Khanin and J. Palis for having attracted our attention to these results.

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