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ON BIFURCATIONS OF A HOMOCLINIC "FIGURE EIGHT" FOR A SADDLE WITH A NEGATIVE SADDLE VALUE

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Consider a dynamical system X_0 of class C^2 given on a C^2-smooth m-dimensional manifold. Suppose that X_0 has an equilibrium state O of saddle type. Let λ_i, $i = 1, \dots, m$, be the roots of the characteristic equation of the system at O. We suppose that $\lambda_m > 0$ and $\operatorname{Re} \lambda_i < 0$, $i = 1, \dots, m-1$. In this case the stable manifold W^s of the saddle is $(m-1)$-dimensional and the unstable manifold W^u is one-dimensional. Suppose that both the orbits Γ_0^+ and Γ_0^- leaving the saddle return to the saddle as $t \to +\infty$. Suppose also that the saddle value $\sigma = \lambda_m + \max_{i=1,\dots,m-1} \operatorname{Re} \lambda_i$ is negative.

THEOREM 1. *There is a neighborhood V of the contour $\Gamma_0^+ \cup \Gamma_0^- \cup O$ such that for any system X sufficiently close to X_0 in the C^2-topology the set of nonwandering orbits of X in V consists of the point O and one of the following:*

1) *one or two stable limit cycles, or*
2) *one or two homoclinic saddle curves, or*
3) *a stable limit cycle and a homoclinic saddle curve, or*
4) *a quasiminimal set.*

In the last case both the separatrices leaving the saddle will be Poisson P^+-stable.

Denote by $W^s(X)$ and $W^u(X)$ the stable and unstable manifolds of the saddle point $O(X)$ for the system X close to X_0, and let $\Gamma^+(X)$ and $\Gamma^-(X)$ be the orbits leaving the saddle point. We introduce local coordinates x_i, $i = 1, \dots, m$, in a neighborhood of $O(X)$ so that the equation of $W^u(X)$ is $x_1 = 0, \dots, x_{m-1} = 0$, and the equation of $W^s(X)$ is $x_m = 0$.

We shall suppose that $\Gamma^+(X)$ exits into the region $x_m > 0$, and $\Gamma^-(X)$ into the region $x_m < 0$. Clearly, for r and d sufficiently small the surface $\pi\colon x_1^2 + \cdots + x_{m-1}^2 = r^2$, $|x_m| \le d$ will be transverse to the flow. Assume that $\pi_0 = \{M(x_1, \dots, x_m) \in \pi | x_m = 0\}$ separates π into two components $\pi_+ = \{M(x_1, \dots, x_m) \in \pi | x_m > 0\}$ and $\pi_- = \{M(x_1, \dots, x_m) \in \pi | x_m < 0\}$. According to [1], for X sufficiently close to X_0 and for $r^2 + d^2$ sufficiently small it is possible to define maps $T_+ : \pi_+ \to \pi$ and $T_- : \pi_- \to \pi$ such that $T_+(M) \to M^+$ as $x_m \to^+ 0$ and $T_-(M) \to M^-$ as $x_m \to^- 0$. Here $M^+(a_1^+, \dots, a_m^+)$ and $M^-(a_1^-, \dots, a_m^-)$ denote the points of first intersection of $\Gamma^+(X)$ and $\Gamma^-(X)$ with π. In the case of X_0 the points M^+ and M^- belong to π_0, and so $a_m^+ = a_m^- = 0$.

Consider the sequences of points $\{M_i^+\}$ and $\{M_i^-\}$ in which $\Gamma^+(X)$ and $\Gamma^-(X)$ intersect π. Define sequences $S^+ = \{S_i^+\}_{i=0}^{+\infty}$ and $S^- = \{S_i^-\}_{i=0}^{+\infty}$ of symbols from the alphabet $\{-1, 0, 1\}$ as follows: $S_0^+ = 1$, $S_0^- = -1$, while $S_i^* = 1$ if $M_i^* \in \pi_+$, $S_i^* = -1$ if $M_i^* \in \pi_-$, and $S_i^* = 0$ if $M_j^* \in \pi_0$ for some $j \le i$ (where $* \in \{+, -\}$). Clearly if $\Gamma^*(X)$ returns to the saddle as $t \to +\infty$ then there is some j such that $S_i^* = 0$ for all $i \ge j$. We call such a homoclinic orbit an *orbit of type* $s = \{S_0^* \cdots S_{j-1}^*\}$. A periodic orbit of X that lies in V and is homotopic in V to a homoclinic orbit of type s is called a *cycle of type* s, with s being some finite word in the alphabet $\{-1, 1\}$.

Next we construct a binary tree as follows: at the first vertex we place the symbol pair $(1, -1)$; from this two arrows go to the vertices $(1, -1, 1)$ and $(1, -1, -1)$, and so

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on by the rule that from a vertex (p,s) there are arrows going to the vertices (p,sp) and (ps,s), where p and s denote finite words in the alphabet $\{-1,1\}$. We call a finite word-pair (p,s) in the alphabet $\{-1,1\}$ *admissible* if it is at one of the vertices of the tree so constructed. An infinite word-pair (p,s) is called *admissible* if there is a sequence of admissible finite pairs (p_i,s_i) such that each p_i is a prefix of p, each s_i is a prefix of s, the lengths of p_i and s_i approach infinity as $i \to \infty$, and for every i the pair (p_{i+1}, s_{i+1}) belongs to the subtree with initial vertex (p_i, s_i). A word s is *admissible* if it is an element of an admissible pair.

Let us note some properties of the tree of admissible pairs:

1) Each finite admissible pair is encountered in the tree only once.

2) If a finite admissible word appears as an element of two admissible pairs, then one of these pairs belongs to the subtree starting at the other pair.

3) Each infinite admissible word appears as an element of only one admissible pair.

THEOREM 2. *There exists a neighborhood V of the contour $\Gamma_0^+ \cup \Gamma_0^- \cup O$ such that for any system X sufficiently close to X_0 in the C^2-topology the limit cycles and homoclinic orbits that lie in V can have only admissible types. In the case of two limit cycles or two homoclinic saddle curves or a pair consisting of a limit cycle and a homoclinic saddle curve the types of these orbits are an admissible pair. If $\Gamma^+(X)$ and $\Gamma^-(X)$ are P^+-stable, then the pair (S^+, S^-) is admissible.*

In the general case systems with two homoclinic saddle curves constitute sets of codimension two, distinguished by the following conditions: 1′) λ_1 is complex and $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > \operatorname{Re}\lambda_i$, $i = 3,4,\ldots,m-1$, or 1″) λ_1 is real and $\lambda_1 > \operatorname{Re}\lambda_i$, $i = 2,3,\ldots,m-1$; 2) Γ_0^+ and Γ_0^- do not lie in a lower-dimensional submanifold of W^s; and 3) the separatrix parameters (the quantity δ in the notation of [2] and A in the notation of [3]) are different from zero. In view of this we shall now consider two-parameter families X_μ of C^2-smooth dynamical systems that contain X_0 and depend smoothly on $\mu = (\mu_+, \mu_-)$. The coordinates a_m^+ and a_m^- of the points M^+ and M^- will be smooth functions of μ with $a_m^+(0) = a_m^-(0) = 0$. We shall suppose that $\mu_+ = a_m^+(\mu)$ and $\mu_- = a_m^-(\mu)$.

Let C_s denote the bifurcation set in the (μ_+, μ_-)-plane that corresponds to the existence of a homoclinic orbit of type s. This C_s is a curve, given either by an equation $\mu_+ = h_s(\mu_-)$ or by $\mu_- = h_s(\mu_+)$, where h_s is a function with Lipschitz constant less than 1, defined on some open subset of R^1.

In the case when λ_1 is complex all possibilities for the family X_μ allowed by Theorems 1 and 2 are realized. The bifurcation diagram for this case is shown in Figure 1. The configuration of bifurcation curves along which homoclinic saddle curves exist is as follows: $\mu_+ = 0$ is the curve C_1, $\mu_- = 0$ is the curve C_{-1}, while two further curves $C_{1\ -1}$ and $C_{-1\ 1}$ also leave the point $(0,0)$; the curves $C_{1\ -1}$ and $C_{-1\ 1}$ intersect C_{-1} and C_1 respectively at infinitely many points. Continue according to this rule: let $(p,s) \neq (1,-1)$ be an admissible pair; then C_p and C_s intersect at infinitely many points, and if P and Q are neighboring points of intersection of C_p and C_s such that $h_p > h_s$ on the interval between P and Q, then P and Q are further connected by curves C_{ps} and C_{sp}. The region bounded by C_{ps} and C_{sp} corresponds to the existence of a cycle of type ps; if $h_p < h_s$ on the interval between P and Q then the region bounded by the segments of the curves C_p and C_s from P to Q corresponds to the existence of a pair of cycles of types p and s. To the points of the set $F = \operatorname{cl}(\bigcup_s C_s) \setminus \bigcup_s C_s$ (the union taken over all finite admissible words s) correspond quasiminimal sets.

In the case of real λ_1 the bifurcation diagram depends on the separatrix parameters A_+ and A_- and on the relative position of the orbits Γ_0^+ and Γ_0^-.

Below we consider the case where Γ_0^+ and Γ_0^- arrive at the saddle tangent to each other. Note that this case was considered in [4], where sufficient conditions were given for the existence of cycles of types $\{1\}$, $\{-1\}$, and $\{1\ -1\}$.

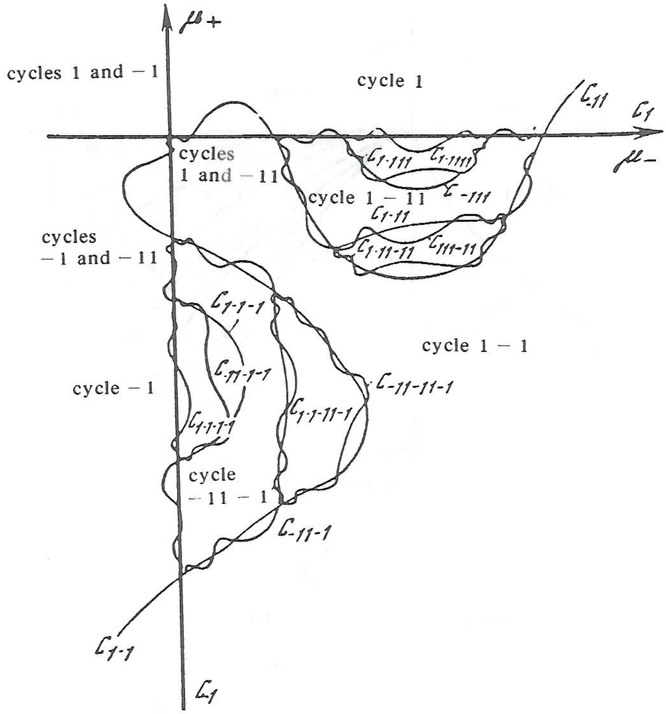

FIGURE 1

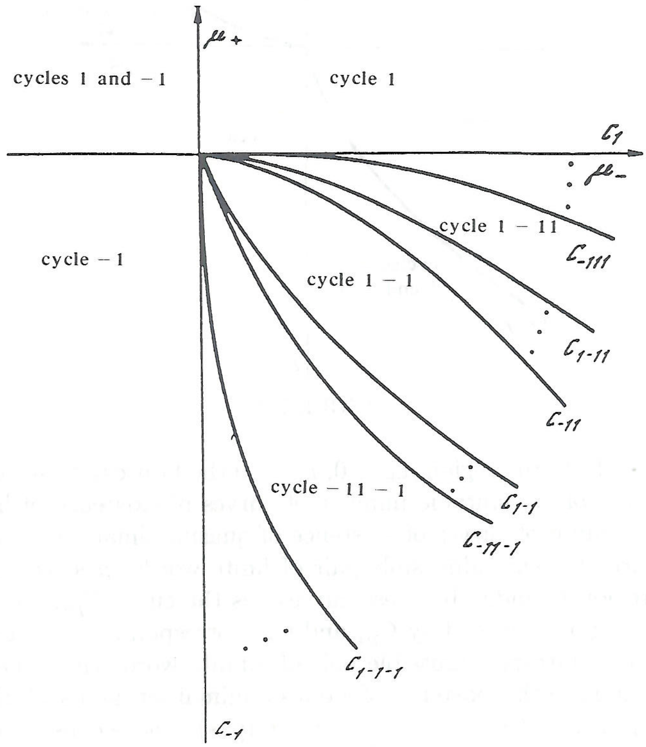

FIGURE 2

In the case $A_+ > 0$, $A_- > 0$ (see Figure 2) the region $\mu_+ > 0$, $\mu_- < 0$ corresponds to the existence of a pair of cycles of types $\{1\}$ and $\{-1\}$, the region $\mu_+ > 0$, $\mu_- > 0$ to the existence of a unique cycle of type $\{1\}$, and the region $\mu_+ < 0$, $\mu_- < 0$ to the existence of

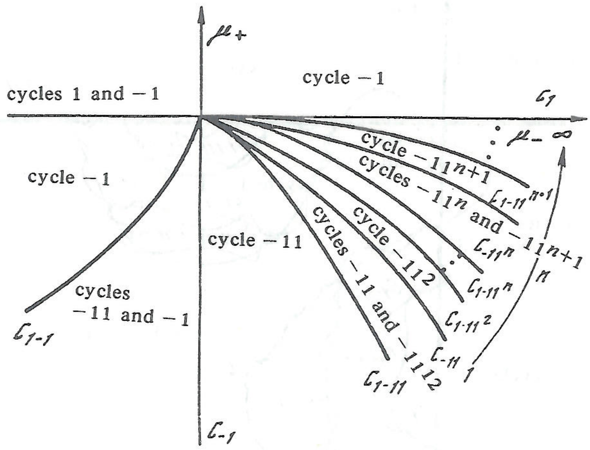

FIGURE 3

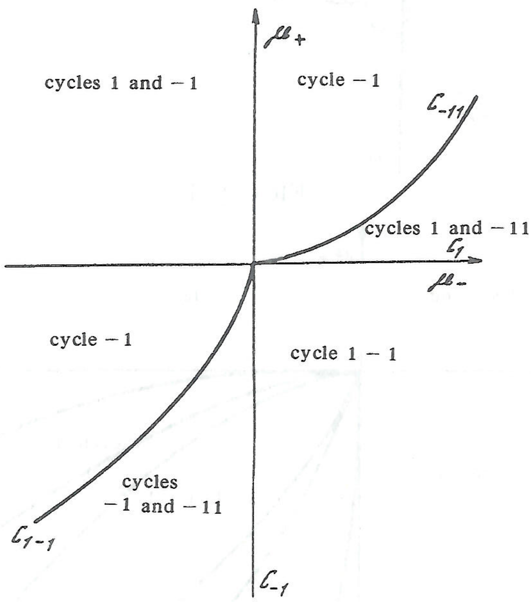

FIGURE 4

a cycle of type $\{-1\}$. In the region $\mu_+ < 0$, $\mu_- > 0$ the bifurcation set is a Cantor pencil of curves consisting of a countable number of curves of existence of homoclinic saddle curves and a continuum of curves of existence of quasiminimal sets. The region has the following structure: for any admissible pair of finite words (p, s) there exist curves C_p and C_s; in the region bounded by these curves lies the curve C_{ps}, and between C_p and C_{ps} lies C_{sp}; the region bounded by C_{ps} and C_{sp} corresponds to the existence of cycles of type ps. For an arbitrary admissible pair of infinite words (p, s) there exists a curve $C(p, s)$ corresponding to the existence of a quasiminimal set, and such that if $\mu \in C(p, s)$ then $S^+(X_\mu) = p$ and $S^-(X_\mu) = s$. The curve $C(p, s)$ can be constructed as

$$\lim_{i\to\infty} C_{s_i} = \lim_{i\to\infty} C_{p_i},$$

where (p_i, s_i) is a sequence of admissible pairs approximating (p, s).

In the case $A_+ > 0$, $A_- < 0$ (see Figure 3) there exist cycles only of types $\{1\}$, $\{-1\}$, and $\{-1\ 1^n\}$, $n = 1, 2, \dots$ (here 1^n means the word consisting of n ones), and the parameter plane is partitioned into a countable number of regions by the curves C_1, C_{-1}, $C_{1\ -1}$, $C_{-1\ 1^n}$, and $C_{1\ -1\ 1^n}$, $n = 1, \dots, \infty$.

In the case $A_+ < 0$, $A_- < 0$ (see Figure 4) there exist cycles only of types $\{1\}$, $\{-1\}$, and $\{1\ \ -1\}$; the parameter plane here is partitioned into 6 regions.

The case when Γ_0^+ and Γ_0^- arrive at the saddle from opposite directions has been studied in [5]. In this case the system X_μ has no Poisson stable nonclosed orbits, and the cycles are only of types $\{1\}$, $\{-1\}$, $\{1\ \ -1\}$, $\{1\ (-1\ \ 1)^n\}$, and $\{-1\ (1\ \ -1)^n\}$, $n = 1, \dots, \infty$. In the cases $A_+ > 0$, $A_- > 0$ and $A_+ < 0$, $A_- > 0$ the bifurcation set consists of a finite number of curves, and in the case $A_+ < 0$, $A_- < 0$ it consists of countably many curves.

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