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MATHEMATICS

Pseudohyperbolicity and the Problem on Periodic Perturbations of Lorenz-Type Attractors

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In this paper, we construct a theory of pseudohypebolic attractors, which include, in particular, Lorenz attractors, wild spiral attractors, and their periodic perturbations.

Consider a C^r-smooth ($r \ge 1$) flow X_t on $\mathbf{R}^n$. An open bounded domain $\mathcal{D}$ is absorbing for X_t if $X_t(\overline{\mathcal{D}}) \subset \mathcal{D}$ for all t larger than some $T > 0$. An effective criterion for dynamical chaos in $\mathcal{D}$ is the following pseudohyperbolicity condition: for each point of $\mathcal{D}$, there exists a pair of transversal subspaces N_1 and N_2 continuously depending on the point (with $\dim N_2 = k \ge 1$ and $\dim N_1 = n - k$) such that the families of subspaces thus obtained are invariant with respect to the linearized flow $DX_t = \dfrac{\partial X_t(x)}{\partial x}$ (i.e., $DX_tN_1(x) = N_1(X_t(x))$ and $DX_tN_2(x) = N_2(X_t(x))$ for $t \ge T$); moreover, for each orbit $x(t) = X_t(x_0)$, the maximal Lyapunov exponent corresponding to the subspace N_1 is strictly smaller than the minimal Lyapunov exponent corresponding to N_2:

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\sup_{\substack{u\in N_1(x_0),\\ \|u\|=1}}\|DX_t(x_0)u\|$$

$$<\liminf_{t\to+\infty}\frac{1}{t}\ln\inf_{\substack{v\in N_2(x_0),\\ \|v\|=1}}\|DX_t(x_0)v\|;$$

the restriction of the linearized flow to N_1 is contractive:

$$\|DX_t|_{N_1}\| \le Ce^{-\alpha t} \quad (\alpha > 0, C > 0);$$

and the linearized flow exponentially expands volume in N_2:

$$\det(DX_t|_{N_2}) \ge Ce^{\sigma t} \quad (\sigma > 0, C > 0).$$

This definition has the following immediate consequence.

Property 1. *All orbits in $\mathcal{D}$ are unstable: each of them has positive maximal Lyapunov exponent* $\lambda_{\max}(x) = \limsup\limits_{t\to+\infty}\frac{1}{t}\ln\|DX_t(x)\| > 0$.

The following assertions are proved in a standard way (cf. [1]).

Property 2. *The pseudohyperbolicity conditions are not violated under small smooth perturbations of the system. Moreover, the subspaces N_1 and N_2 change continuously under such perturbations.*

Property 3. *There exists a unique invariant absolutely continuous contractive foliation with C^r-smooth leaves tangent to N_1 on $\mathcal{D}$.*

Pseudohypebolic attractors were introduced in [14]. They include hyperbolic attractors and Lorenz-type attractors. New examples are constructed by applying the following two useful theorems, which are implied by Property 2.

Theorem 1. *If a system of the form $\dot{y} = Y(y)$ is pseudohypebolic on an absorbing domain $\mathcal{D}$ and a function $p(y, \theta)$ is periodic in θ and small together with its derivatives, then the system*

$$\dot{y} = Y(y) + p(y, \theta), \quad \dot{\theta} = 1$$

is pseudohypebolic on $\mathcal{D} \times S^1$.

Theorem 2. *If a system of the form $\dot{y} = Y(y)$ is pseudohypebolic on an absorbing domain $\mathcal{D}$ and functions $p(y, z)$ and $q(y, z)$ are small together with their derivatives, then the system*

$$\dot{y} = Y(y) + p(y, z), \quad \dot{z} = Y(z) + q(y, z)$$

is pseudohypebolic on $\mathcal{D} \times \mathcal{D}$.

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An analogue of Theorem 1 was also proved in [12].

An absorbing domain $\mathcal{D}$ always contains a maximal attractor (a set of orbits contained in $\mathcal{D}$ entirely, for all $t \in (-\infty, +\infty)$), but it may contain also nonattracting limit sets; the domain from which these sets are removed decomposes into finitely or even infinitely many strange attractors. Strange attractors are usually defined as compact asymptotically stable transitive sets. However, this definition involves a problem, which is a matter of principle: not all systems have transitive asymptotically stable attractors (e.g., such an attractor may be absent in a Lorenz geometric model [2]). For this reason, following [11, 14], we somewhat relax the conditions and call a compact invariant set A an attractor if it is chain-transitive and epsilon-stable, i.e., if, for any ε, every two points of A can be joined by an ε-trajectory contained entirely in A, and for any neighborhood U of A, there exists an ε such that the ε-rajectories starting from A never leave U (the epsilon-stability property can also be formulated as follows: A is an intersection of embedded absorbing domains). Importantly, any absorbing domain of any dynamical system contains at least one chain-transitive epsilon-stable set.

The last assertion is based on the following observation: for any point P, the set of points attainable by ε-trajectories starting at P for any ε is epsilon-stable.

This also implies the following theorem.

Theorem 3. *Suppose that $\mathcal{D}$ contains a finite set of points $Q_1, Q_2, \ldots, Q_m$ such that, for any point $P \in D$, some of the points Q_j is accessible from P* (*i.e., for any* $\varepsilon > 0$, *there exists an ε-trajectory joining P to Q_j*), *but Q_j is not accessible from Q_i for any $i \neq j$.*

Then, *$\mathcal{D}$ contains precisely m attractors $A_1, A_2, \ldots, A_m$, where each A_j is precisely the set of points accessible from Q_j.*

This theorem is general, but in the case of pseudo-hypebolic flows, it becomes a fairly constructive criterion for the finiteness of the number of attractors. Namely, suppose that a flow in $\mathcal{D}$ has a secant, that is, a finite number of smooth $(n-1)$-surfaces $\Pi_1, \Pi_2, \ldots, \Pi_l$ transversal to the flow and such that the positive semitrajectory of any point from $\mathcal{D}$ (except, possibly, finitely many equilibrium states or periodic orbits) attains one of the surfaces $\Pi_1, \Pi_2, \ldots, \Pi_l$ at some moment. This means that, on $\Pi_1 \cup \Pi_2 \cup \ldots \cup \Pi_l$, a Poincaré mapping is defined. Since the flow is contractive on N_2, the phase velocity of the flow does not belong to N_2; therefore, the invariant absolutely continuous contractive foliation tangent to N_2 for this flow (see Property 3) determines an invariant absolutely continuous contractive foliation for the Poincaré mapping on the secant. The corresponding quotient mapping expands $(k-1)$-volume. Now, suppose that $\mathcal{D}$ contains finitely many equilibrium states $O_1, O_2, \ldots, O_m$ and/or periodic orbits $L_1, L_2, \ldots, L_p$ such that the secant minus finitely many pieces of the stable manifolds of the orbits $O_1, O_2, \ldots, O_m$ and $L_1, L_2, \ldots, L_p$ decomposes into connected components, on each of which the quotient mapping is injective. Since the quotient mapping expands volume, it follows immediately that some iteration of any open domain under the action the Poincaré mapping must intersect the stable manifold of one of the orbits $O_1, O_2, \ldots, O_m$ and $L_1, L_2, \ldots, L_p$. Thus, we obtain the following property.

Property 4. The union of stable manifolds of the equilibrium states $O_1, O_2, \ldots, O_m$ and the periodic orbits $L_1, L_2, \ldots, L_p$ is dense in $\mathcal{D}$.

This means, in particular, that, from any point of the domain $\mathcal{D}$, at least one of these orbits is accessible. Now, Theorem 3 implies the following assertion.

Theorem 4. *The domain $\mathcal{D}$ contains finitely many attractors, and each of them is the set of points attainable from one of the points O_j or from one of the periodic orbits L_j.*

In what follows, we separate out a class of pseudo-hypebolic systems with one equilibrium state, a saddle with one-dimensional unstable manifold. This class includes the Lorenz geometric model [2] and the model with wild spiral attractor [14]. We recall the main properties of systems from this class and consider their periodic perturbations.

Suppose that the saddle equilibrium state O has characteristic exponents $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \gamma$, where $\gamma > 0$ and $\mathrm{Re}\lambda_j < 0$ for $j = 1, 2, \ldots, n-1$. Suppose also that $\mathrm{Re}\lambda_1 \geq \mathrm{Re}\lambda_2 \geq \ldots \geq \mathrm{Re}\lambda_m > \mathrm{Re}\lambda_{m+1} \geq \ldots \geq \mathrm{Re}\lambda_{n-1}$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_m + \gamma > 0$ for some $m \geq 1$ (if λ_1 is complex, then necessarily $m \geq 2$).

Let us introduce coordinates (x, y, z) (where $x \in \mathbf{R}^1$, $y \in \mathbf{R}^m$, $z \in \mathbf{R}^{n-m-1}$) with origin at the equilibrium state O so that the unstable 1-manifold W^u of the equilibrium state is tangent to the x axis and the stable $(n-1)$-manifold W^s is tangent to the plane $x = 0$. The y coordinates correspond to the exponents $\lambda_1, \lambda_2, \ldots, \lambda_m$ and the z coordinate corresponds to the remaining exponents λ. Suppose that the flow has the secant Π: $\{\|y\| = 1, \|z\| \leq 1, |x| \leq 1\}$.

Since W^s is tangent to the plane $x = 0$ at O, it follows that, locally, it is determined by the equation $x = h^s(y, z)$, where h^s is a smooth function. Changing coordinates, we can achieve the condition $h^s = 0$ locally; we assume this representation to be valid for $\|y\| \leq 1$ and $\|z\| \leq 1$. Thus, Π is a secant for W^s. The intersection surface $\Pi_0 = \Pi \cap W^s_{\mathrm{loc}} = \{x = 0\}$ divides Π into two parts, Π_+: $\{x > 0\}$ and Π_-: $\{x < 0\}$. We suppose that all trajectories starting on $\Pi\backslash\Pi_0$ return to $\mathrm{int}(\Pi)$, thereby determining Poincaré mappings T_+: $\Pi_+ \to \Pi$ and T_-: $\Pi_- \to \Pi$. Obviously,

$$\lim_{x \to -0} T_-(x, y, z) = P_-, \quad \lim_{x \to +0} T_+(x, y, z) = P_+,$$

where P_- and P_+ are the intersection points of the one-dimensional separatrices of O with Π. Consider the set formed by the trajectories starting at Π plus the point O and its separatrices. A sufficiently small neighborhood $\mathcal{D}$ of this set is an absorbing domain. We assume that the semiflow on $\mathcal{D}$ satisfies the pseudohyperbolicity conditions, the $(n-m-1)$-spaces N_1 are uniquely projected onto the z axis, and the $(m+1)$-spaces N_2 are uniquely projected onto the (x, y) plane (at the point O, the space N_1 coincides with the z space and N_2 coincides with the (x, y) space).

The projection along the leaves of the contractive invariant foliation determines a volume-expanding Poincaré quotient mapping $\tilde{T}: (x, y) \mapsto (\bar{x}, \bar{y})$. Moreover, the restrictions of $\tilde{T}$ to the domains $\{x > 0\}$ and $\{x < 0\}$ are injective. Therefore, it follows directly from Theorem 4 that, under the above conditions, the system has a unique epsilon-stable chain-transitive attractor A in the domain $\mathcal{D}$, namely, the set of points accessible from O.

Note that A contains, in particular, the point O, its unstable separatrices, and their closures. The manifold W^s is everywhere dense in $\mathcal{D}$ (see Property 4).

There are verifiable conditions on the Poincaré mapping that ensure pseudohyperbolicity. Let $\varphi \in S^{m-1}$ be coordinates on $y = 1$. Let us write the mapping $T: \Pi\backslash\Pi_0 \to \Pi$ in the form

$$(\bar{x}, \bar{\varphi}) = g(x, \varphi, z), \quad \bar{z} = f(x, \varphi, z),$$

where f and g are functions smooth at $x \neq 0$ and discontinuous at $x = 0$.

Theorem 5. *Suppose that* $\det\frac{\partial g}{\partial(x, \varphi)} \neq 0$ *and*

$$\lim_{x \to 0} C = 0, \quad \lim_{x \to 0} \|A\| \cdot \|D\| = 0,$$

$$\sup_{P \in \Pi\backslash\Pi_0} \sqrt{\|A\| \cdot \|D\|} + \sqrt{\sup_{P \in \Pi\backslash\Pi_0} \|B\| \cdot \sup_{P \in \Pi\backslash\Pi_0} \|C\|} < 1,$$

where

$$A = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial(x, \varphi)} - \left(\frac{\partial g}{\partial(x, \varphi)}\right)^{-1} \frac{\partial g}{\partial z},$$

$$B = \frac{\partial f}{\partial(x, \varphi)}\left(\frac{\partial g}{\partial(x, \varphi)}\right)^{-1}, \quad C = \left(\frac{\partial g}{\partial(x, \varphi)}\right)^{-1} \frac{\partial g}{\partial z},$$

$$D = \left(\frac{\partial g}{\partial(x, \varphi)}\right)^{-1}.$$

Then, the mapping admits a continuous invariant foliation with leaves of the form $(x, \varphi) = h(z)|_{\|z\| \le 1}$, *where the derivative* $h'(z)$ *is uniformly bounded. If, in addition*,

$$\sup_{P \in \Pi\backslash\Pi_0} \|A\| + \sqrt{\sup_{P \in \Pi\backslash\Pi_0} \|B\| \cdot \sup_{P \in \Pi\backslash\Pi_0} \|C\|} < 1,$$

then the foliation is contractive; moreover, it is absolutely continuous under the additional condition that, for some $\beta > 0$, *the functions* $A|x|^{-\beta}$, $D|x|^{\beta}$, B, *and* C *are uniformly bounded and Hölder continuous and a* $\frac{\partial \ln \det D}{\partial z}$ *and* $\frac{\partial \ln \det D}{\partial(x, \varphi)}|x|^{\beta}$ *are uniformly bounded. If, in addition*,

$$\frac{1 - \sqrt{\sup_{P \in \Pi\backslash\Pi_0} \|B\| \cdot \sup_{P \in \Pi\backslash\Pi_0} \|C\|}}{\sup_{P \in \Pi\backslash\Pi_0} \sqrt{\det D}} = q > 1,$$

then the quotient mapping $\tilde{T}$ *expands m-volume.*

The case of $m = 1$ corresponds to the Lorenz geometric model [2]. In this case, if $g > \frac{1+\sqrt{5}}{2}$ (or if $q > \sqrt{2}$, provided that the mapping is symmetric with respect to the change $x \to -x$), then A is also a maximal attractor. Moreover, the intersection $A \cap \Pi$ coincides with the nonwandering set Ω of the Poincaré mapping, it is one-dimensional and transitive, and the saddle periodic points and structurally stable homoclinic points are dense in it.

For an arbitrary q, Ω may contain a nonempty zero-dimensional invariant hyperbolic set Σ_0 on which the mapping T is conjugate to a TMC with finitely many symbols. The closure $\Sigma_1 = \mathrm{cl}(\Omega\backslash\Sigma_0)$ is one-dimensional and transitive, and the saddle periodic and structurally stable homoclinic points are dense in it. If $\Sigma_1 \cap \Sigma_0 = \emptyset$, then $A \cap \Pi = \Sigma_1$ and A is an asymptotically stable transitive attractor. However, it may happen that $\Sigma_1 \cap \Sigma_0 \neq \emptyset$ [2]; then, Σ_1 is unstable, and the system has no stable transitive sets at all. Thus, the chain-transitive attractor A is nontransitive in this case: in addition to $A \cap \Pi$ contains the set of points accessible from Σ_1.

As mentioned above, the stable manifold of the point O is dense in $\mathcal{D}$. Thus, it is natural to expect that if the one-dimensional separatrices of the initial system do not return to O, then they can be closed by a small perturbation of the vector field. Let r be the degree of smoothness of the system, and let $r_0 \geq 1$ be the maximum integer strictly less than $\frac{\gamma}{|\lambda_1 + \lambda_2 + \ldots + \lambda_m|}$.

Theorem 6. *If* $s = r$ *for* $m = 1$ *and* $s = \min(r, r_0)$ *for* $m \geq 2$, *then both separatrices of the point O can be closed by an arbitrarily small* (*in the* C^s-*norm*) *perturbation of the vector field.*

For the Lorenz geometric model, the C^1-version of this theorem was proved in [2].

In the case where O is a saddle-focus ($m \geq 2$), it was shown in [14] that, if a C^r-smooth ($r \geq 2$) pseudohypebolic system of the form described above has a homoclinic loop, then any neighborhood of this system in the C^r-topology contains C^r-open domains such that the attractor A is wild (i.e., contains a wild hyperbolic set together with its unstable manifold) for systems from these domains. Recall that a transitive hyperbolic set is said to be wild [9] if its stable and unstable manifolds have a tangency which cannot be removed by a small perturbation. Thus, in the domains where the attractor is wild, it contains nontransversal intersection points of the stable and unstable manifolds of some orbits of a transitive hyperbolic set Λ, and for a C^r-dense set of systems from these domains, it contains points of homoclinic tangency of the stable and unstable manifolds of saddle periodic orbits from Λ. This, in fact, implies (see [3, 5, 6, 14]) that, for a C^r-dense set of systems from these domains, the wild attractor A contains homoclinic tangencies of infinite order and non–structurally stable periodic orbits with infinite-order degeneracy. Thus, in contrast to the Lorenz geometric model, a complete description of the dynamics and bifurcations of wild pseudohypebolic attractors is impossible in principle.

A new class of wild attractors arises under a periodic perturbation of pseudohypebolic systems with an equilibrium state. We set $u = (x, y, z)$ and assume that the system

$$\dot{u} = U(u) \tag{1}$$

satisfies the above conditions in the absorbing domain $\mathcal{D}$, namely, that it has a secant Π, which is separated by the stable manifold of a structurally stable equilibrium state O into two components Π_+ and Π_-; the Poincaré-mapping T: $\Pi_+ \cup \Pi_- \to \Pi$ admits an absolutely continuous contractive invariant foliation; and the corresponding quotient mapping expands volume and is injective on Π_+ and Π_-. Let $p(u, \theta, \mu)$ be a function periodic in θ and bounded together with its derivatives. For small μ, in the periodically perturbed system

$$\dot{u} = U(u) + \mu p(u, \theta, \mu), \quad \dot{\theta} = 1 \tag{2}$$

the equilibrium state O system (1) corresponds to a saddle periodic orbit L_μ with unstable 2-manifold and stable n-manifolds.

Theorem 7. *For all small* μ, *system* (2) *has a unique chain-transitive attractor A in the absorbing domain* $\mathcal{D} \times S^1$, *which coincides with the set of points attainable from* L_μ (*the attractor A contains, in particular, the unstable* 2-*manifold* $W^u(L_\mu)$ *and its closure*). *If, in the unperturbed system* (1), *the point O has a homoclinic loop, then the addition p can be chosen so that the one-parameter family* (2) *contains a countable set of intervals of values of* μ *accumulating to* $\mu = 0$ *at which* L_μ *is contained in a wild hyperbolic set and the attractor A is wild.*

Remark. A similar situation occurs in the general case, where the system has a saddle periodic orbit L in the absorbing domain. Suppose that the positive semi-trajectories of all points in the absorbing domain, except of those belonging to L and $W^s_{\text{loc}}(L)$, reach some secant Π, which is separated by the manifold $W^s_{\text{loc}}(L)$ into two components, Π_+ and Π_-. If the Poincaré mapping T: $\Pi_+ \cup \Pi_- \to \Pi$ admits an absolutely continuous contractive invariant foliation and the corresponding quotient mapping expands volume and is injective on Π_+ and Π_-, then the chain-transitive attractor A is unique and coincides with the set of points accessible from L. The attractor A is wild if the system belongs to a Newhouse domain, i.e., a C^r-open set of systems for which L is contained in a wild hyperbolic set (such domains exist in any C^r-neighborhood of any system for which the manifolds $W^u(L)$ and $W^s(L)$ have homoclinic tangency [4, 9, 10]). In the same way, wild attractors for diffeomorphisms are constructed: it suffices to verify that the suspension of the diffeomorphism under consideration satisfies the conditions specified above.

If the maximal (in absolute value) stable multiplier of the periodic orbit L is real, then, on the domains where the attractor is wild, systems having two countable families of periodic orbits with unstable manifolds of dimensions 2 and 3, respectively, are dense, and if the maximal multipliers form a pair of complex conjugate multipliers, then the systems having three countable families of periodic orbits with unstable manifolds of dimensions 2, 3, and 4 are dense in these domains [5, 7]. As the parameters change, these orbits bifurcate; as mentioned above, the order of degeneracy of non–structurally stable periodic orbits can be arbitrarily high.

The problem about periodic perturbations of Lorenz-type attractors naturally arises in studying local bifurcations of periodic orbits. For example, as shown in [13], the normal form for the bifurcation of codimension 3 corresponding to the multipliers (–1, –1, 1) coincides up to terms of arbitrary order of smallness with a shift of the autonomous three-dimensional flow, which may have a Lorenz-type attractor for certain parameter values; taking into account the small terms neglected is equivalent to a small periodic action on the system. Periodically perturbed Lorenz-type attractors near bifurcations of periodic points of codimension 3 were discovered numerically in the three-dimensional Henon map [8].

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