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ON HAMILTONIAN SYSTEMS WITH HOMOCLINIC SADDLE CURVES

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As we know, for Hamiltonian systems the existence of a structurally stable homoclinic curve of an equilibrium state of saddle type is a typical phenomenon. This is connected with the fact that since the equilibrium state and its stable (W^s) and unstable (W^u) manifolds lie on one energy level then W^s and W^u may intersect transversally along homoclinic curves. Hence one may expect that the set of all trajectories of a Hamiltonian system lying entirely in a neighborhood of a homoclinic curve or a bouquet of homoclinic curves of an equilibrium state of saddle type has a reasonable, if not complete, description. For the case of a saddle-focus homoclinic curve this case was considered by Devaney [1]. He established that the set of trajectories lying in an energy level of a saddle-focus has a description in terms of symbolic dynamics with countably many symbols. It is interesting (see [2]) that this description is completely analogous to the description of the structure of a neighborhood of a structurally stable Poincaré homoclinic curve [3].([1]) In this note we consider the case when the equilibrium state is a saddle.

Assume that a system X with a Hamiltonian $H \in C^3$ in a domain $D \subseteq R^{2n}$, $n \geq 2$, has an equilibrium state O. Let $\pm\lambda_1, \dots, \pm\lambda_n$ be the roots of the characteristic equation of O. Assume that O is a saddle, i.e., $0 < \lambda_1 < \operatorname{Re}\lambda_i$, $i = 2, \dots, n$. Near the saddle the vector field is written in the form $\dot{x} = -\lambda_1 x + \cdots$, $\dot{y} = -Ay + \cdots$, $\dot{u} = \lambda_1 u + \cdots$, $\dot{v} = A^{\mathrm{T}} v + \cdots$, where $x \in R^1$, $y \in R^{n-1}$, $u \in R^1$, $v \in R^{n-1}$,

$\operatorname{Spec} A = \{\lambda_2, \dots, \lambda_n\}$, and the dots denote terms of order higher than one. W^u is tangent at O to the plane $u = 0$, $v = 0$, and W^u is tangent to the plane $x = 0$, $y = 0$. We denote by W^{ss} (W^{uu}) the stable (unstable) nonleading $(n-1)$-dimensional saddle manifold. W^{ss} is tangent to the y-axis, and W^{uu} is tangent to the v-axis. W^{ss} divides W^s into two parts: W^s_+ and W^s_-. Similarly, we have $W^u = W^{uu} \cup W^u_+ \cup W^u_-$. We shall assume that W^u_+ approaches W^{uu} from the domain $u > 0$, and W^s_+ approaches W^{ss} from the domain $x > 0$. Let us assume that W^u and W^s intersect transversally along m homoclinic trajectories $\Gamma_1, \dots, \Gamma_m$ not lying in W^{uu} and W^{ss}. The latter means that the Γ_i enter the saddle and leave it tangentially to the leading directions, the x and u axes respectively. Let us number the Γ_i so that

$$\bigcup_{i=1}^{m_1} \Gamma_i \subseteq W^s_+ \cap W^u_+, \qquad \bigcup_{i=m_1+1}^{m_1+m_2} \Gamma_i \subseteq W^s_+ \cap W^u_-,$$

$$\bigcup_{i=m_1+m_2+1}^{m_1+m_2+m_3} \Gamma_i \subseteq W^s_- \cap W^u_-; \qquad \bigcup_{i=m_1+m_2+m_3+1}^{m_1+m_2+m_3+m_4} \Gamma_i \subseteq W^s_- \cap W^u_+,$$

$$m_1 + m_2 + m_3 + m_4 = m.$$

We shall assume that $m_1 \neq 0$.

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([1])In the case of general smooth dynamical systems with a homoclinic saddle-focus curve we have also a complicated structure in the behavior of the trajectories [4], and on the bifurcation surface of such systems there is a dense structural instability.

Let $H = 0$ be the level containing O. We denote by X_h the restriction of the system to the level $H = h$. Let V be a small neighborhood of the bouquet $\Gamma_1 \cup \cdots \cup \Gamma_m \cup O$. We denote by Ω_h the set of trajectories of X_h lying entirely in V.

THEOREM 1. *For a sufficiently small V and a sufficiently small $h_0 > 0$ depending on V, the following assertions are true:*

1) $\Omega_0 = \{\Gamma_1, \dots, \Gamma_m, O\}$.

2) *If $m_3 = m_4 = 0$, then $X_h|_{\Omega_h}$ for $h \in (0, h_0)$ is topologically equivalent to a suspension of a Bernoulli scheme of m_1 symbols (if $m_1 = 1$ then Ω_h consists of one saddle cycle); and for $h \in (-h_0, 0)$, if $m_2 > 0$ it is equivalent to a suspension of a Bernoulli scheme of m_2 symbols, while if $m_2 = 0$ then $\Omega_h = \varnothing$.*

3) *If $m_3 \neq 0$ or $m_4 \neq 0$ then $X_h|_{\Omega_h}$ for $h \in (-h_0, h_0)\backslash\{0\}$ is topologically equivalent to a suspension of a topological Markov chain (TMC) given in the case $h > 0$ by the transition matrix*

$$\begin{pmatrix} m_1 & m_2 \\ m_4 & m_3 \end{pmatrix},$$

and in the case $h < 0$ by

$$\begin{pmatrix} m_2 & m_1 \\ m_3 & m_4 \end{pmatrix}.^{(2)}$$

REMARK 1. Clearly, for $m \geq 3$ at least one of the graphs given by the matrices has a vertex belonging to at least two cycles. Therefore, for $m \geq 3$ the system X has a complicated structure.

REMARK 2. It is possible to number the edges of the graphs so that a periodic trajectory of the TMC $\{[i_1, \dots, i_k]\}$, $i_j \in \{1, \dots, m\}$, corresponds to a periodic trajectory of the system X_h homotopic in V to the product $\Gamma_{i_1} \cdots \Gamma_{i_k}$.

We consider below a simple case of a bouquet of countably many homoclinic curves. Assume that X has a saddle periodic motion L in the energy level of a saddle. Then, as we know, X has a one-parameter family L_k of saddle periodic motions, with $L_0 = L$. Let us assume that $W^u(O)$ and $W^s(L)$ intersect transversally along a trajectory Γ_1, and $W^u(L)$ and $W^s(O)$ also intersect transversally along a trajectory Γ_2. Assume that $\Gamma_1 \not\subset W^{uu}(O)$ and $\Gamma_2 \not\subset W^{ss}(O)$ ($\Gamma_1 \subset W^u_+$ and $\Gamma_2 \subset W^s_+$). Let us take a small neighborhood V of the contour $\Gamma_1 \cup \Gamma_2 \cup L \cup O$. Its fundamental group has two generators: we choose L as one of the generators, and we choose the second arbitrarily and denote it by S. Let us denote by Ω_h the set of trajectories of X_h lying entirely in V.

THEOREM 2. *For a sufficiently small V and a small $h_0 > 0$ depending on V, the following assertions are true:*

1) $\Omega_0 = \{\Gamma_1, \Gamma_2, L, O\} \cup (\bigcup_{i \geq i_0} \{\gamma_i\})$, *where i_0 is an integer and γ_i is a trajectory homoclinic to O, homotopic to SL^i in V.*

2) *If $h \in (-h_0, 0)$, then $\Omega_h = \{L_h\}$.*

3) *If $h \in (0, h_0)$, then $X_h|_{\Omega_h}$ is topologically equivalent to a suspension of a Bernoulli scheme of two symbols L and S; moreover, a periodic trajectory of the Bernoulli scheme $\{[i_1 \cdots i_k]\}$ corresponds to a periodic trajectory of the system X_h homotopic in V to the product $i_1 \cdots i_k$.*

Let us consider now the case when there are two saddles O_1 and O_2 in the level $H = 0$. We assume that $W^u(O_1) \cup W^u(O_2)$ intersects transversally in the level $H = 0$ with $W^s(O_1) \cup W^s(O_2)$ along m trajectories $\Gamma_1, \dots, \Gamma_m$ not lying in $W^{ss}(O_1) \cup W^{ss}(O_2) \cup W^{uu}(O_1) \cup W^{uu}(O_2)$. We set $W_1^{s(u)} = W_+^{s(u)}(O_1)$, $W_2^{s(u)} = W_-^{s(u)}(O_1)$,

(2)See [6]–[8] for representation of a TMC by a multigraph and a transition matrix.

$W_3^{s(u)} = W_+^{s(u)}(O_2)$, and $W_4^{s(u)} = W_-^{s(u)}(O_2)$. Let $m_{i,j}$ be the number of trajectories from the array $\Gamma_1, \dots, \Gamma_m$ lying in $W_i^u \cap W_j^s$, $i, j \in \{1,2,3,4\}$, and put $\sum_{i,j} m_{i,j} = m$. We denote by Ω_h the set of trajectories of the system X_h lying entirely in a small neighborhood V of the contour $\Gamma_1 \cup \dots \cup \Gamma_m \cup O_1 \cup O_2$.

Let us consider an arbitrary integer square matrix Q. If all the entries of some row of the matrix are zero we remove from Q this row and the column with the same index. We repeat this process until we obtain a matrix having a nonzero element in each row. We denote this matrix by $\tilde{Q}$.

THEOREM 3. *For a sufficiently small V and a small $h_0 > 0$ depending on V, the following assertions are true:*

1) $\Omega_0 = \{\Gamma_1, \dots, \Gamma_m, O_1, O_2\}$.

2) *If $h \in (-h_0, h_0)\backslash\{0\}$, then $X_h|_{\Omega_h}$ is topologically equivalent to a suspension of a TMC given for $h > 0$ by the matrix $\tilde{Q}_1$, and for $h < 0$ by the matrix $\tilde{Q}_2$, where Q_1 and Q_2 are respectively*

$$\begin{pmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m_{21} & m_{11} & m_{41} & m_{31} \\ m_{22} & m_{12} & m_{42} & m_{32} \\ m_{23} & m_{13} & m_{43} & m_{33} \\ m_{24} & m_{14} & m_{44} & m_{34} \end{pmatrix}.$$

This situation is not structurally stable. In this connection we consider a one-parameter family X_μ of dynamical systems with Hamiltonian $H_\mu \in C^3$. We assume that $H_\mu(O_1) = \mu$ and $H_\mu(O_2) = -\mu$. Let us denote by $\Omega_{h\mu}$ the set of trajectories of $X_\mu|_{H_\mu=h}$ lying entirely in V. By the symmetry of the problem we may restrict ourselves to the case $\mu > 0$.

THEOREM 4. *For a sufficiently small V and small $h_0 > 0$ and $\mu_0 \in (0, h_0)$, depending on V, if $\mu \in (0, \mu_0)$, $|h| < h_0$, and $|h| \neq \mu$, then $X_h|_{\Omega_{h\mu}}$ is topologically equivalent to a suspension of a TMC given for $h \in (\mu, h_0)$ by the matrix $\tilde{Q}_1$, for $h \in (-h_0, -\mu)$ by the matrix $\tilde{Q}_2$, and for $h \in (-\mu, \mu)$ by the matrix $\tilde{Q}_3$, where*

$$Q_3 = \begin{pmatrix} m_{21} & m_{11} & m_{31} & m_{41} \\ m_{22} & m_{12} & m_{32} & m_{42} \\ m_{23} & m_{13} & m_{33} & m_{43} \\ m_{24} & m_{14} & m_{34} & m_{44} \end{pmatrix}.$$

For $h = \mu$ ($h = -\mu$), $\Omega_{h\mu}$ contains a bouquet of homoclinic curves of the saddle O_1 (O_2). $X_h|_{\Omega_{h\mu}}$ is equivalent to a suspension of a TMC given for $h = \mu$ by the matrix $\tilde{Q}_4$, and for $h = -\mu$ by the matrix $\tilde{Q}_5$, where we have identified two trajectories:

$$\cdots (m+1)(m+1)(m+1) \cdots \quad \text{and} \quad (m+2)(m+2)(m+2) \cdots .$$

Here Q_4 and Q_5 are respectively

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & m_{31} + m_{32} & m_{41} + m_{42} \\ m_{13} + m_{23} & 0 & m_{33} & m_{43} \\ m_{14} + m_{24} & 0 & m_{34} & m_{44} \end{pmatrix}$$

and

$$\begin{pmatrix} m_{21} & m_{11} & 0 & m_{31} + m_{41} \\ m_{22} & m_{12} & 0 & m_{32} + m_{42} \\ m_{23} + m_{24} & m_{23} + m_{14} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

in the case $h = \mu$ ($h = -\mu$) we have denoted by $(m+1)$ and $(m+2)$ the edges from the vertex 1 (the vertex 4) into itself and from the vertex 2 (the vertex 3) into itself.

To the pair of identified trajectories in the suspension there correspond the saddle O_1 for $h = \mu$ and the saddle O_2 for $h = -\mu$.(3) *To the trajectory $\{i_j\}_{-\infty}^{+\infty}$ of the TMC, where $i_j = m+1$ for $j < 0$, $i_j \in \{1, \dots, m\}$ for $j = 1, \dots, k$, and $i_j = m+2$ for $j > k$, there corresponds a homoclinic trajectory to the saddle, homotopic in V to the product $\Gamma_{i_1} \cdots \Gamma_{i_k}$.*

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(3)See [6] for suspensions including an equilibrium state.