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MATHEMATICS

# Attractors and Repellers Near Generic Elliptic Points of Reversible Maps¶

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Let $M$ be a two-dimensional manifold. Let $g$: $M \to M$ be an involution, so $g \circ g = \mathrm{id}$. A map $f$: $M \to M$ is called reversible if it is conjugate by the involution $g$ to its own inverse, i.e. $f^{-1} = g \circ f \circ g$.

Let $\mathcal{R}_g^r$ denote the set of the reversible maps, endowed with the $C^r$-topology, $r = 1, 2, \ldots, \infty, \omega$. In the analytic case, $r = \omega$, we choose, first, some fixed complex neighbourhood $Q$ of $M$, and $\mathcal{R}_g^\omega = \mathcal{R}_{g,Q}^\omega$ is then the space of reversible maps which are real on $M$ and analytic on $Q$, with the topology of uniform convergence on compacta in $Q$. We also assume that the involution $g$ itself is $C^r$-smooth or, in the case of $\mathcal{R}_g^\omega$, real analytic.

A subset of $\mathcal{R}$ is called residual if it is an intersection of a countable sequence of open and dense sets; a property is called generic if it holds for maps which comprise a residual set.

An orbit of a reversible map $f$ is called symmetric if it is invariant with respect to $g$; namely, $g(x_0) = f^j(x_0)$ for some $j$ (then $gf(x_0) = f^{j-1}(x_0)$, $gf^2(x_0) = f^{j-2}(x_0)$, $\ldots$). It is easy to see that for a symmetric periodic orbit at least one of its points is either a fixed point of the involution $g$, or a fixed point of $f \circ g$. Such point will be called a symmetric periodic point.

Dynamics in a small neighbourhood of a non-symmetric periodic orbit can be arbitrary. However, the image $gL$ of such orbit by the involution $g$ is also a periodic orbit, and if $L$ has multipliers $\lambda$ and $\gamma$, then the multipliers of $gL$ will be $\lambda^{-1}$ and $\gamma^{-1}$. Therefore, in reversible maps, asymptotically stable periodic orbits (sinks) and completely unstable ones (sources) always exist in pairs.

For a symmetric periodic orbit, if $\lambda$ is a multiplier, then $\lambda^{-1}$ is also a multiplier. Therefore, elliptic symmetric periodic orbits (i.e., those with multipliers $e^{\pm i\psi}$ where $\psi \neq 0, \pi$) persist at $C^1$-small perturbations within the space of reversible maps. It is well-known that the dynamics near a generic elliptic orbit of a reversible map appears pretty much conservative. Namely, every point of such orbit is surrounded by invariant KAM-curves. Moreover, the KAM-curves occupy most of the neighbourhood of the elliptic point [1].

However, between the KAM-curves there exist resonant zones. We show here that the generic dynamics in the resonant zones is not conservative and has a mixed nature (in the sense of [2, 3]). Namely, our main result is the following

**Theorem 1.** *In the space* $\mathcal{R}_g^r$, $r = 1, 2, \ldots, \infty, \omega$, *there is a residual subset* $\mathcal{R}^*$ *such that for every* $f \in \mathcal{R}^*$ *every point of each symmetric elliptic periodic orbit is a limit of periodic sinks, sources, saddles, and other elliptic points.*

Let $\Lambda$ be a non-trivial, transitive, zero-dimensional, compact, locally-maximal uniformly-hyperbolic set. Following [4], we simply call such sets basic.[1] A basic set $\Lambda$ is called a wild hyperbolic set if its stable and unstable manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ have a nondegenerate tangency and if this property persists at all perturbations sufficiently small in $C^2$. According to [4], if a $C^r$-map ($r = 2, 3, \ldots, \omega$) has a basic set $\Lambda$ such that $W^s(\Lambda)$ and $W^u(\Lambda)$ have a tangency, then an arbitrarily small $C^r$-neighborhood of this map intersects a $C^2$-open set $\mathcal{N}$ (the Newhouse region) such that for every map from $\mathcal{N}$ the hyperbolic set $\Lambda$ is wild. A basic set $\Lambda$ of a reversible map is called symmetric if $g\Lambda = \Lambda$.

**Lemma 1.** *Let $f$ be a reversible map with a symmetric wild hyperbolic set $\Lambda$. Let $\mathcal{N}$ be the corresponding Newhouse region, and let $\mathcal{U}_{\mathrm{rev}}$ be a neighbourhood of $f$ in the*

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[1] The restriction of the map on a basic set is topologically conjugate to a finite Markov chain. Main examples are given by the Smale horseshoe, the Shilnikov hyperbolic set in a neighbourhood of a transverse homoclinic [5], etc.

*space of reversible maps. Then, for any map from $\mathcal{N}_{\text{rev}} = \mathcal{N} \cap \mathfrak{U}_{\text{rev}}$ the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ have a pair of non-symmetric quadratic tangencies. Moreover, in $\mathcal{N}_{\text{rev}}$, for each $k \geq 2$ there exists a dense subset $\mathcal{N}_{\text{sym},\,k}$ such that every map from $\mathcal{N}_{\text{sym},\,k}$ has a symmetric orbit at the points of which $W^s(\Lambda)$ and $W^u(\Lambda)$ have a tangency of the order k.*[2]

The proof is based on the fact that near non-symmetric orbits of tangency of the stable and unstable manifolds the dynamic behaviour of a reversible map does not differ from that for the general case, therefore the original Newhouse arguments [4] are applicable. The part concerning symmetric orbits is proven by showing, using the methods of [6–8], that the symmetric tangencies (of any order) can be obtained by a perturbation of a pair of non-symmetric ones.

Since periodic orbits are dense in $\Lambda$ and their stable and unstable manifolds are dense, respectively, in $W^s(\Lambda)$ and $W^u(\Lambda)$, we derive the following result from Lemma 1.

**Lemma 2.** *In $\mathcal{N}_{\text{rev}}$, reversible maps which possess the following properties are dense:*

1. *The set $\Lambda$ contains a pair of saddle periodic orbits $P_1$ and $P_2$ such that $P_2 = gP_1$, and $W^u(P_1)$ has a transverse intersection with $W^s(P_2)$ while $W^u(P_2)$ has a quadratic tangency with $W^s(P_1)$ at the points of symmetric heteroclinic orbits $\Gamma_{12}$ and, respectively, $\Gamma_{21}$.*

2. *The set $\Lambda$ contains a pair of symmetric periodic orbits $Q_1$ and $Q_2$ such that the manifold $W^u(Q_1)$ has a quadratic tangency with $W^s(Q_2)$ while $W^u(Q_2)$ has a quadratic tangency with $W^s(Q_1)$ at the points of heteroclinic orbits $\Gamma_{12}$ and, respectively, $\Gamma_{21}$, where $\Gamma_{21} = g\Gamma_{12}$.*

Since the orbit $P_1$ is not symmetric, the Jacobian $J(P_1)$ of the Poincare map for this orbit does not need to be equal to $\pm 1$, and in any case one can always apply an arbitrarily small perturbation and make $|J(P_1)| \neq 1$. Then, due to the reversibility, $J(P_2) = J(P_1)^{-1} \neq \pm 1$. It was shown in [9] that under this condition arbitrarily small (and not leading out of the class of reversible maps) perturbations of the heteroclinic cycle described in item 1 of Lemma 2 lead to the birth of non-symmetric periodic sinks and sources along with symmetric elliptic periodic orbits (see also [2]). By [10], the same conclusion holds for perturbations of the heteroclinic cycle from item 2 of Lemma 2 (eventhough the bifurcation scenario is different here as $|J(Q_1)| = |J(Q_2)| \equiv 1$ for all small perturbations within the class of reversible maps). Using these results, we infer the following theorem from Lemma 2.

**Theorem 2.** *In the Newhouse domain $\mathcal{N}_{\text{rev}}$ a residual subset $\mathfrak{B}$ is comprised by reversible maps each having infinitely many sinks, sources and symmetric elliptic points. Moreover, within $\mathfrak{B}$ there is a residual subset $\mathfrak{B}^*$ which is comprised by the maps such that for each of them the closure of sinks, the closure of sources and the closure of elliptic points have a non-empty intersection that contains the wild set $\Lambda$.*

Theorem 2 is, in a sense, analogous to Newhouse theorem [11] on the existence of open regions (in the space of dissipative systems) where systems with infinitely many periodic sinks are generic. This result was generalized to the non-dissipative case in [2]: in the space of two-dimensional maps that have periodic saddles with Jacobians both greater and less that 1 there exist Newhouse regions where a generic map has infinitely many sinks and sources.[3]

It is well known that the dynamics in resonant zones near elliptic periodic points is typically chaotic. In the conservative case the existence of transverse and non-transverse homoclinic orbits near elliptic points was shown in [12–14]. In the reversible case an analogous statement holds true.

**Theorem 3.** *For a generic map from $\mathcal{R}_g^r$ ($r = 2, 3, \ldots, \omega$) each symmetric elliptic periodic orbit is a limit of symmetric wild-hyperbolic sets.*

Obviously, Theorem 3 implies Theorem 1 by virtue of Theorem 2.

In fact, Theorem 1 implies another instance of coexistence of conservative and dissipative dynamics, as given by Theorem 4 below. Let $C$ be a closed invariant curve of a reversible $C^r$-map $f$ ($r = 4, 5, \ldots, \infty, \omega$). In analogy with the conservative case we will call the invariant curve $C$ a KAM-curve if the following conditions hold: (1) $gC = C$ (i.e., $C$ is a symmetric invariant curve); (2) there exist $C^4$-coordinates $(\rho, \theta)$ near $C$ (where $\theta$ is an angular variable and $\rho$ runs a small interval around zero) such that $C$ is given by the equation $\rho = 0$, and the map $f$ takes the form $\bar{\rho} = q(\rho, \theta)$, $\bar{\theta} = \theta + \psi(\rho, \theta)$, where $q$, $\psi$ are functions, 1-periodic in $\theta$, such that $q(0, \theta) \equiv 0$ and $\psi(\rho, \theta) = \psi_0 + \psi_1\rho + O(\rho^2)$, where $\psi_0$ and $\psi_1$ are some constants; (3) the twist condition is satisfied: $\psi_1 \neq 0$; (4) the rotation number $\psi_0$ is Diophantine: $|k\psi_0 + p| \geq K|k|^{-\alpha}$ for all integer $k \neq 0$ and $p$, and for some fixed $K > 0$ and $\alpha > 0$.

**Theorem 4.** *In the space $\mathcal{R}_g^r$ there is a residual subset $\mathcal{R}^{**}$ such that for every $f \in \mathcal{R}^{**}$ every point of each* KAM-*curve is a limit of sinks and sources.*[4]

In the final part of this letter we describe one more mechanism of the destruction of conservativity near elliptic points, which is not related to homoclinic tangencies. As we show, a degenerate resonant elliptic

[2] Quadratic and cubic tangencies ($k = 2, 3$) appear in generic one-parameter families of systems from $\mathcal{N}_{\text{rev}}$.

[3] This type of the Newhouse phenomenon was later called *mixed dynamics* [3]. It is characterized by the coexistence of infinitely many periodic orbits of all possible structurally-stable topological types, and by their inseparability from each other (the closures of the sets of the periodic orbits of different topological types have a non-empty intersection).

[4] In the analytic case we were unable to prove that every KAM-curve is a limit of elliptic points. In the $C^\infty$-case this claim is easily proved by reducing the map to an integrable normal form, similar to that discussed below.

point can itself undergo a symmetry-breaking bifurcation that produces a sink and a source. Let us have a symmetric elliptic periodic orbit with the multipliers $e^{\pm i\psi}$ where $\psi = \frac{2\pi p}{q} + \mu$, where $\mu$ is a small parameter. Then for all small $\mu$ the Poincare map $T$ near this orbit can be brought to the normal form: $T = R_{2\pi p/q} \circ F + o(\rho^{q+1})$ where $F$ is the time-1 map of the flow defined by

$$\dot{z} = i\mu z + i\sum_{1 \le j \le \frac{q}{2}} \Psi_j |z|^{2j} z + iA(z^*)^{q-1} + iBz^{q+1} + iCz(z^*)^q, \quad (1)$$

where $z = \rho e^{i\varphi}$, $z^* = \rho e^{-i\varphi}$. Note that the reversibility requires the coefficients $A$, $B$, $C$, and $\Psi_j$ in (1) to be real. Importantly, for conservative maps the coefficients $B$ and $C$ must be zero. However, they can take arbitrary real values in the case of reversible systems if we do not a priori require the conservativity. We rewrite (1) as

$$\dot{\rho} = \rho^{q-1}(A + (C-B)\rho^2)\sin\phi,$$
$$\dot{\phi} = q\mu + \sum_{1 \le j \le \frac{q}{2}} q\Psi_j \rho^{2j} + q\rho^{q-2}(A + (C+B)\rho^2)\cos\phi, \quad (2)$$

where $\phi = q\varphi$. Assume $\Psi_1 \ne 0$. If $A \ne 0$, then the behavior of system (2) at small $\rho$ is essentially conservative: there are only two equilibrium states at $\mu\Psi_1 < 0$ and they both are symmetric (i.e., $\sin\phi = 0$), so one is a saddle and one is a center. However, if $A = 0$, then by changing $\mu$ and $A$ non-symmetric equilibria can be created in system (2). Namely, if $C \ne B$, then system (2) has a pair of equilibria $\rho^2 = \frac{A}{B-C}$ with $\sin\phi \ne 0$ for $\mu$ running a small interval that lies near the point $\mu = \frac{A\Psi_1}{C-B}$. One checks that the non-symmetric equilibria are a sink and a source if $B(B - C) > 0$. They correspond to $q$-periodic sink and source for the Poincare map $T$.

Now we make the following

**Proposition.** *Given an elliptic point with the multipliers $e^{\pm i\psi}$, an arbitrarily small $C^r$-perturbation can be added such that it does not lead the map out of the class of reversible maps and makes the elliptic point resonant $\left(\text{i.e., } \psi = \frac{2\pi p}{q}\right)$ with any given real values of the coefficients A, B, and C in the normal form* (1).

We do not prove this claim when $r = \omega$, i.e. it has to be considered as a conjecture in the analytic case. For small $C^\infty$-perturbations the proof of the proposition is achieved as follows. When $\frac{\psi}{\pi}$ is irrational, the normal form for the Poincare map $T$ times the rotation $R_{-\psi}$ coincides, up to flat terms, with the time-1 shift by the orbits of the equation $\dot{z} = i\Psi(zz^*)z$ for some real function $\Psi$ where $\Psi(0) = 0$. Therefore, by a $C^\infty$-small perturbation of the original map one can make the map $R_{-\psi} \circ T$ coincide exactly with the time 1 shift by this flow in a sufficiently small neighborhood of $z = 0$ (in the analytic case, $r = \omega$, this is, in general, impossible to do). Now, take a sufficiently large $q$ such that $\frac{2\pi p}{q}$ is close to $\psi$, change the equation $\dot{z} = i\Psi(zz^*)z$ to $\dot{z} = i\Psi(zz^*)z + iA(z^*)^{q-1} + iBz^{q+1} + iCz(z^*)^q$ (in a sufficiently small neighbourhood of zero this perturbation is small in $C^{q-2}$), and change the value of $\psi$ to $\frac{2\pi p}{q}$.

Thus, there exist arbitrarily small perturbations of elliptic points which produce sinks and sources, and, at least in the $C^\infty$-case, these perturbations do not need to be related to “unpredictable subtleties” of homoclinic tangles [7].

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