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Bifurcations of Systems with Structurally Unstable Homoclinic Orbits and Moduli of Ω-Equivalence

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Abstract—Bifurcations of both two-dimensional diffeomorphisms with a homoclinic tangency and three-dimensional flows with a homoclinic loop of an equilibrium state of saddle-focus type are studied in one- and two-parameter families. Due to the well-known impossibility of a complete study of such bifurcations, the problem is restricted to the study of the bifurcations of the so-called low-round periodic orbits. In this connection, the idea of taking Ω-moduli (continuous invariants of the topological conjugacy on the nonwandering set) as the main control parameters (together with the standard splitting parameter) is proposed. On this way, new bifurcational effects are found which do not occur at a one-parameter analysis. In particular, the density of cusp-bifurcations is revealed.

Keywords—Bifurcation, Homoclinic tangency, Modulus.

1. INTRODUCTION

As it is well known, the development of the theory of global bifurcations of multidimensional systems was started in 1960's. In particular, there was discovered a remarkable phenomenon [1,2] that a multidimensional system with a homoclinic loop of a saddle equilibrium state can possess an infinite number of periodic orbits, in distinction with the two-dimensional case. The first example of such complicated behavior is given by a homoclinic loop of an equilibrium state of saddle-focus type (Figure 1) in a three-dimensional space. Such equilibrium state has the characteristic roots $-\lambda \pm i\omega$ and γ, where γ, λ, and ω are positive; besides, the so-called saddle index $\rho = \lambda/\gamma$ is less than unity.

It was found in [1,2] that the structure of the set N composed by the orbits lying entirely in a small neighborhood of the homoclinic loop is not just nontrivial but it also depends essentially on the value of ρ. This dependence is such that, when ρ varies continuously, the structure of the set N permanently varies in any one-parameter family X_ρ of systems holding a saddle-focus homoclinic loop.[1]

In modern terms, the results of [1,2] imply that the value ρ is a *modulus* of the Ω-equivalence of systems with a homoclinic loop of a saddle-focus. Recall the following definition.

DEFINITION. *We say that a system X has a modulus if, in the space of dynamical systems, a Banach subspace $\mathcal{M}$ passes through X, and on $\mathcal{M}$ a locally nonconstant continuous functional h is defined such that, in order for two systems X_1 and X_2 from $\mathcal{M}$ to be equivalent, it is necessary*

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[1]In particular, it was established in [3,4] that the values of ρ for which X_ρ has a structurally unstable periodic orbit compose a dense set.

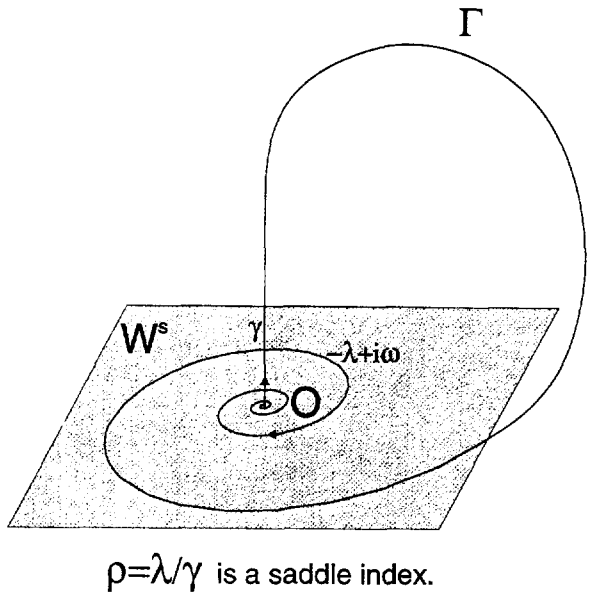


Figure 1. A three-dimensional flow X with the homoclinic loop Γ of the equilibrium state O of saddle-focus type (i.e., the equilibrium state O possesses characteristic roots $-\lambda \pm i\omega$ and γ, where λ, γ, and ω are positive). Shil'nikov has shown that if the value $\rho = \lambda/\gamma$ is less than 1, then the set N of orbits lying entirely in a small neighborhood of Γ has a nontrivial structure which depends essentially on the value of ρ.

that $h(X_1) = h(X_2)$. We shall say that X has at least m moduli if a Banach subspace passes through X on which m independent moduli are defined, and that X has a countable number of moduli if X has an arbitrary finite number of moduli.

Among different types of equivalences in the space of dynamical systems, the most known are the topological and the Ω-equivalence (the topological equivalence on nonwandering sets). The *topological* moduli in systems with simple dynamics were discovered by Palis [5] for diffeomorphisms of a plane which have an orbit of heteroclinic tangency. Figure 2a represents such a diffeomorphism. It has two saddle fixed points O_1 and O_2 with multipliers λ_i and γ_i, where $|\lambda_i| < 1$, $|\gamma_i| > 1$ $(i = 1, 2)$. It also has a heteroclinic orbit Γ_0 at the points of which the manifolds $W^u(O_2)$ and $W^s(O_1)$ have a tangency. Palis established that two such diffeomorphisms f and f' can be topologically conjugated in some neighborhoods $U(\bar{\Gamma}_0)$ and $U'(\bar{\Gamma}'_0)$ only in the case where the values of the invariant $\alpha = -\ln|\lambda_2|/\ln|\gamma_1|$ are the same for f and f'.

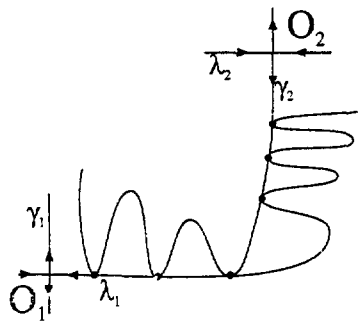
This means that α is a modulus of the topological equivalence for diffeomorphisms with a heteroclinic tangency. At the same time, any two diffeomorphisms of the Palis example are Ω-conjugate; i.e., the value α is not a modulus with respect to the Ω-equivalence.

If we identify the saddles O_1 и O_2, we get a diffeomorphism with a homoclinic tangency (Figure 2b). The invariant α is equal in this case to the value which is traditionally denoted as θ:

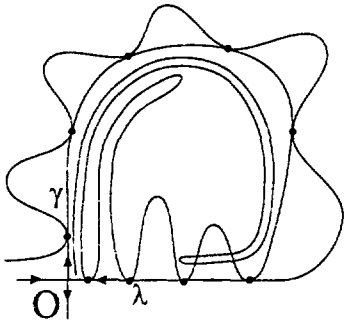
$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|},$$

where λ and γ are the multipliers of O.

Note that in distinction with the heteroclinic situation, the invariant θ may be a modulus not only for the topological but also for the Ω-equivalence. It should be mentioned here, that topological moduli appear, mainly, as obstacles to the existence of a conjugating homeomorphism, whereas Ω-moduli have an essentially different sense. To our opinion, Ω-moduli should be considered as parameters determining the structure of the nonwandering set. Historically, it is exactly the context in which Ω-moduli were found (the mentioned value ρ for a saddle-focus



(a) A two-dimensional diffeomorphism with a heteroclinic tangency is represented. It possesses two saddle fixed points O_1 and O_2 with multipliers λ_i and γ_i, where $|\lambda_i| < 1$, $|\gamma_i| > 1$, $i = 1, 2$. There exists also a structurally unstable heteroclinic orbit Γ_0 at the points of which the manifolds $W^u(O_2)$ and $W^s(O_1)$ are tangent. Palis established that two such diffeomorphisms f and f' may be topologically conjugate only in the case if the value $\alpha = -\ln|\lambda_2|/\ln|\gamma_1|$ is the same for f and f'.



(b) A two-dimensional diffeomorphism with a homoclinic tangency is represented. It has a saddle fixed point O with multipliers λ and γ, where $|\lambda_i| < 1$, $|\gamma_i| > 1$, $i = 1, 2$. It possesses also a structurally unstable periodic orbit Γ at the points of which the manifolds $W^u(O)$ and $W^s(O)$ are tangent. The value $\theta = -\ln|\lambda|/\ln|\gamma|$ introduced by Gavrilov and Shil'nikov is an analogue of the invariant α. Note that here, in distinction with a heteroclinic situation, the value θ may be a modulus of the Ω-equivalence.

Figure 2.

homoclinic loop and the value θ for homoclinic tangencies [6] essentially earlier than the notion of a topological modulus were introduced in the theory of dynamical systems.

For the bifurcation theory, importance of the study of specifically Ω-moduli is obvious. Indeed, it is clear that if a system is perturbed so that the value of an Ω-modulus is changed, then bifurcations of nonwandering orbits (periodic, homoclinic, etc.) must occur. First, this phenomenon was revealed in [6] at the study of bifurcations of periodic orbits on the bifurcational surface $\mathcal{H}$ composed by systems with a quadratic homoclinic tangency. Namely, there was shown that for any one-parameter family X_θ of systems on $\mathcal{H}$ the values of θ are dense for which X_θ has a structurally unstable periodic orbit.

Note also, that θ is not a unique Ω-modulus for the systems with a homoclinic tangency. It was established in [7] that systems may be dense in $\mathcal{H}$ which have a countable number of independent Ω-moduli. Since an independent variation of the values of each of the Ω-moduli leads to bifurcations in the nonwandering set, a joint variation of the infinite series of the Ω-moduli may lead to *infinitely degenerate* bifurcations. Specifically, it was shown in [7] that systems with arbitrarily degenerate periodic orbits and with homoclinic orbits of any order of tangency may be dense in $\mathcal{H}$.

Immediately, there arise a number of problems. On one hand, systems with homoclinic tangencies compose bifurcational surfaces of codimension one in the space of dynamical systems. Therefore, such systems occur in general one-parameter families. On the other hand, the proven presence of systems with arbitrarily degenerate periodic and homoclinic orbits in an arbitrarily small neighborhood of any system with a simple homoclinic tangency shows that no finite number of control parameters is sufficient for a complete study of bifurcations of such systems. The analogous result can be also shown to hold for systems with a homoclinic loop of a saddle-focus.

In principle, we have to give up the ideology of "complete description" and to restrict ourself to the study of some most typical features and properties of such systems. Particularly, the problem of the study of *main* bifurcations in *low-parameter* families takes a sense.

In the latter sentence, we must, of course, clarify the term "main bifurcations." We must also solve the question on the choice of the control parameters.

We will study the structure of the set N of the orbits lying entirely in a small neighborhood U of a homoclinic orbit. In the case of a two-dimensional diffeomorphism with a homoclinic tangency, this neighborhood is the union of a small disc U_0, containing the fixed point O, and a finite number of small neighborhoods of the homoclinic points which lie outside U_0 (Figure 3). In the case of a three-dimensional system with a homoclinic loop of a saddle-focus, the neighborhood U is a solid torus composed by a ball U_0 with the point O in the center and by a handle U_1 which contains the piece of the homoclinic orbit that lies outside U_0 (Figure 4). As we mentioned, the complete study of all bifurcations in U is impossible and we restrict ourself to the study of *low-round* periodic and homoclinic orbits (single-, double-, triple-, ...). A periodic orbit lying in U will be called *k-round* if it leaves U_0 (and re-enters it) k times for the period. Analogously, the roundness of a homoclinic orbit is defined. The low-round orbits are, naturally, most interesting from the applied point of view. Moreover, the high-order degenerations occur only for quite high roundnesses.

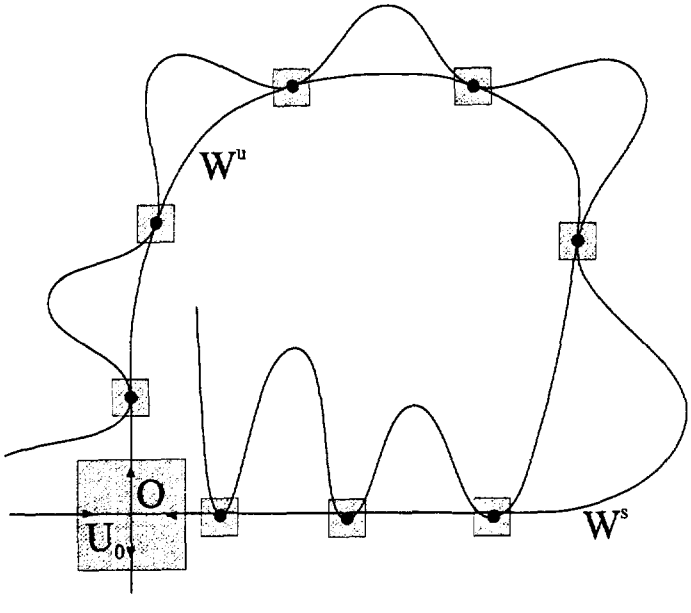


Figure 3. The neighborhood of a structurally unstable homoclinic orbit. The neighborhood U is a union of a small neighborhood U_0 of the saddle fixed point O and of a finite number of small neighborhoods of homoclinic points lying outside U_0.

What concerns the right choice of the control parameters, this question has a principal meaning for the systems with complex dynamics. There is no problem with finding appropriate control parameters in the classical bifurcation theory going back to the studying of flows on a plane: here, each parameter is responsible for unfolding some definite degeneration of the system (for instance, the control parameters govern independently the splitting of separatrices, variation of values of

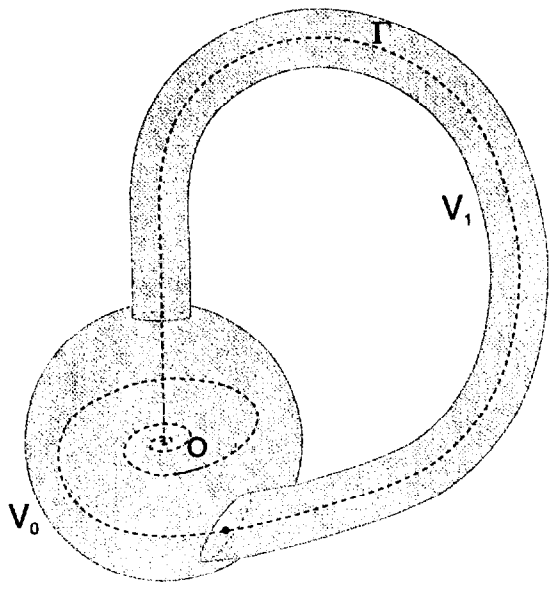


Figure 4. The neighborhood of a homoclinic loop of saddle-focus. The neighborhood U is a solid torus composed by a small neighbourhood U_0 of the point O and by a handle U_1 glued to U_0.

critical characteristic exponents of equilibrium states and multipliers of periodic orbits, variation of Lyapunov values, etc.). For the multidimensional systems with homoclinic tangencies, the so-called *splitting parameters* must clearly be taken as one of the main control parameters for the study of bifurcations.

However, according to what was said above about Ω-moduli, it becomes clear that to obtain a more detailed bifurcational picture one must take Ω-moduli as additional bifurcation parameters (or such values whose variation leads to variation of values of the Ω-moduli).

In the present paper, we demonstrate the effectiveness of this approach to the study of main bifurcations in systems with complex dynamics for two cases:

(1) two-dimensional diffeomorphisms with a homoclinic tangency (Section 2), and
(2) three-dimensional systems with a homoclinic loop of a saddle-focus (Section 3).

2. TWO-DIMENSIONAL DIFFEOMORPHISMS WITH A HOMOCLINIC TANGENCY

2.1. Geometric Constructions

2.1.1. The neighborhood of a structurally unstable homoclinic orbit

We consider a C^{r+2}-smooth ($r \geq 3$) two-dimensional diffeomorphism f which has a saddle fixed point O with multipliers λ and γ, where $0 < |\lambda| < 1$, $|\gamma| > 1$. We consider the case where $|\lambda\gamma| < 1$. The case $|\lambda\gamma| > 1$ is reduced to that under consideration by transition to the inverse map f^{-1} instead of the initial map f; the special case $|\lambda\gamma| = 1$ requires a separate investigation (see, for instance, [8]).

Suppose the stable and unstable manifolds of O have a quadratic tangency at the points of a homoclinic orbit Γ.

Let U be a small neighborhood of the set $O \cup \Gamma$. The neighborhood U is the union of a small disc U_0 containing O and of a finite number of small discs surrounding the points of Γ which are located outside U_0 (Figure 3). The subject of our study is the set N of orbits of the map f that lie entirely in U.

2.1.2. The local and global maps T_0 and T_1

Let T_0 be the restriction of f onto U_0 (it is called *the local map*). Note that the map T_0 in some C^{r+1}-coordinates (x, y) can be written in the form [9,10]

$$\bar{x} = \lambda x + f(x,y)x^2 y, \qquad \bar{y} = \gamma y + g(x,y)xy^2. \tag{2.1}$$

By (2.1), the equations of the local stable manifold $W^s_{\rm loc}$ and local unstable manifold $W^u_{\rm loc}$ are $y = 0$ and $x = 0$, respectively. Representation (2.1) for the local map is convenient, because in these coordinates the map T_0^k for any sufficiently large k is linear in the lowest order. Specifically, we have the following representation [9,10] for the map $T_0^k : (x_0, y_0) \mapsto (x_k, y_k)$:

$$\begin{aligned} x_k &= \lambda^k x_0 + |\lambda|^k |\gamma|^{-k} \phi_{k1}(x_0, y_k), \\ y_0 &= \gamma^{-k} y_k + |\gamma|^{-2k} \phi_{k2}(x_0, y_k), \end{aligned} \tag{2.2}$$

where ϕ_{k1} and ϕ_{k2} are functions uniformly bounded at all k along with their derivatives up to the order r.

Let $M^+(x^+, 0)$ and $M^-(0, y^-)$ be a pair of points of Γ which lie in U_0 and belong to $W^s_{\rm loc}$ and $W^u_{\rm loc}$, respectively. Without loss of generality, we can assume $x^+ > 0$ and $y^- > 0$. Let Π_0 and Π_1 be sufficiently small neighborhoods of the homoclinic points M^+ and M^- such that $T_0(\Pi_0) \cap \Pi_0 = \emptyset$ and $T_0(\Pi_1) \cap \Pi_1 = \emptyset$. Evidently, there exists an integer m such that $f^m(M^-) = M^+$. We denote the map $f^m : \Pi_1 \to \Pi_0$ as T_1 (it is called *the global map*). The map T_1 can obviously be written in the form

$$\begin{aligned} \bar{x} - x^+ &= ax + b\left(y - y^-\right) + \cdots, \\ \bar{y} &= cx + d\left(y - y^-\right)^2 + \cdots, \end{aligned} \tag{2.3}$$

where $bc \neq 0$ since T_1 is a diffeomorphism, and $d \neq 0$ since the tangency is quadratic.

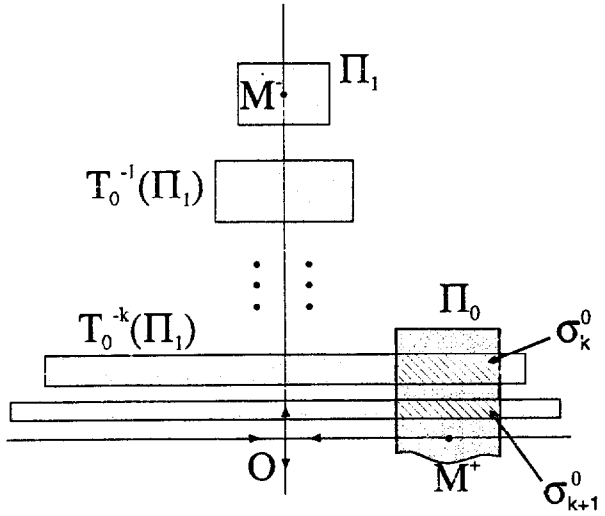


Figure 5. This figure illustrates the method of a construction of strips σ_k^0, $k = \overline{k}, \overline{k}+1, \ldots$, which lie in Π_0 and which are the domains of the maps $T_0^k : \Pi_0 \to \Pi_1$. The points of Π_0 targeting to Π_1 after k iterations of the map T_0 belong to the set $T_0^{-k} \cap \Pi_0$. The neighborhood Π_1 is contracted in γ^{-1} times along the vertical direction and it is expanded in λ^{-1} times along the horizontal direction under the action of the map T_0^{-1}. Moreover, $T_0^{-1}(\Pi_1) \cap \Pi_1 = \emptyset$. Respectively, the set $T_0^{-k}(\Pi_1)$ is a narrow rectangle which is stretched along the x-axis and lies at a distance of order γ^{-k} from it. Besides, the rectangles $T_0^{-k}(\Pi_1)$ and $T_0^{-(k+1)}(\Pi_1)$ are not intersected. For sufficiently large k, the intersection $T_0^{-k}(\Pi_1) \cap \Pi_1$ is the strip σ_k^0 as is shown. The strips σ_k^0 accumulate on the segment $W^s \cap \Pi_0$ as $k \to \infty$.

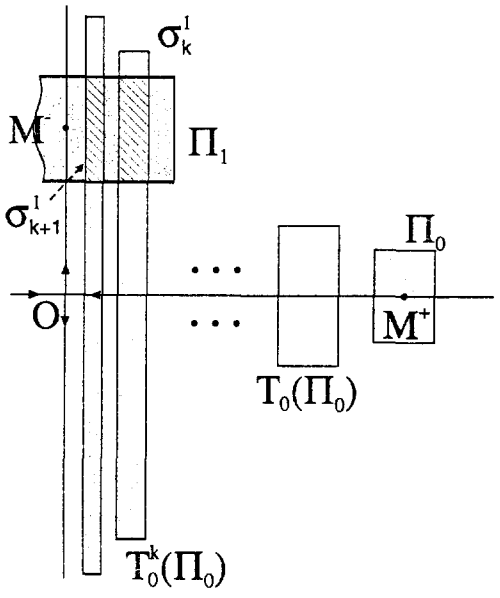


Figure 6. This figure illustrates a method of construction of the strips σ_k^1, $k = \overline{k}, \overline{k}+1, \ldots$, which lie in Π_1 and which are the ranges of the maps $T_0^k : \Pi_0 \to \Pi_1$.

2.1.3. Strips and horseshoes

Note that orbits of N must intersect the neighborhoods Π_0 and Π_1 (otherwise, these orbits would be far from Γ). However, not all orbits that start in Π_0 arrive in Π_1. The set of the points whose orbits get into Π_1 fills a countable number of strips $\sigma_k^0 = \Pi_0 \cap T_0^{-k}\Pi_1$ which accumulate on W^s. The way of constructing these strips is obvious from Figure 5. In turn, the images of the strips σ_k^0 under the maps T_0^k give on Π_1 a sequence of vertical strips σ_k^1 which accumulate on W^u_{loc} (Figure 6).

Neighborhoods Π_0 and Π_1 may be taken so that to contain all the strips σ_k^0 and σ_k^1 with numbers $k \geq \overline{k}$ and not to intersect with σ_k^0 and σ_k^1 for $k < \overline{k}$. Obviously, if $\text{diam}\Pi_0 \cdot \text{diam}\Pi_1 \to 0$, then $\overline{k} \to \infty$.

The images $T_1\sigma_k^1$ of the strips σ_k^1 have a shape of horseshoes accumulated on $T_1 W^u_{\text{loc}}$ as $k \to \infty$ (Figure 7). It is clear that orbits of N must intersect Π_0 in points lying in intersections of horseshoes $T_1\sigma_i^1$ and strips σ_j^0 for $i, j \geq \overline{k}$. Hence, the structure of N depends essentially on geometrical properties of such intersections.

2.1.4. The types of intersections of the strips and horseshoes

Different types of intersections of a horseshoe $T_1\sigma_i^1$ with the strips are shown in Figure 8. The horseshoe has a regular intersection with the strip σ_j^0, an irregular intersection with the strip σ_k^0 and empty intersection with the strip σ_l^0.

The intersection is called *regular* if the set $T_1\sigma_i^1 \cap \sigma_j^0$ is nonempty and consists of two connected components σ_{ji}^{01} and σ_{ji}^{02} (Figure 9), and the maps $T_{i\alpha} \equiv T_1 T_0^i : \sigma_i^{0\alpha} \to \sigma_j^0$, $\alpha = 1, 2$, are saddle (i.e., they are contracting along the coordinate x and expanding along the coordinate y). Here σ_i^{01} and σ_i^{02} are upper and lower parts of the strip σ_i^0. They are separated by the central part of σ_i^0 (denoted as σ_i^{0c} in Figure 9) which is mapped by $T_1 T_0^i$ onto the top of the horseshoe $T_1\sigma_i^1$.

2.1.5. The conditions of regular and irregular intersections of the strips and horseshoes

It is established in [8] that if the inequality

$$d\left[\gamma^{-j} y^- - c\lambda^i x^+\right] > S_{\overline{k}}(i, j) \tag{2.4}$$

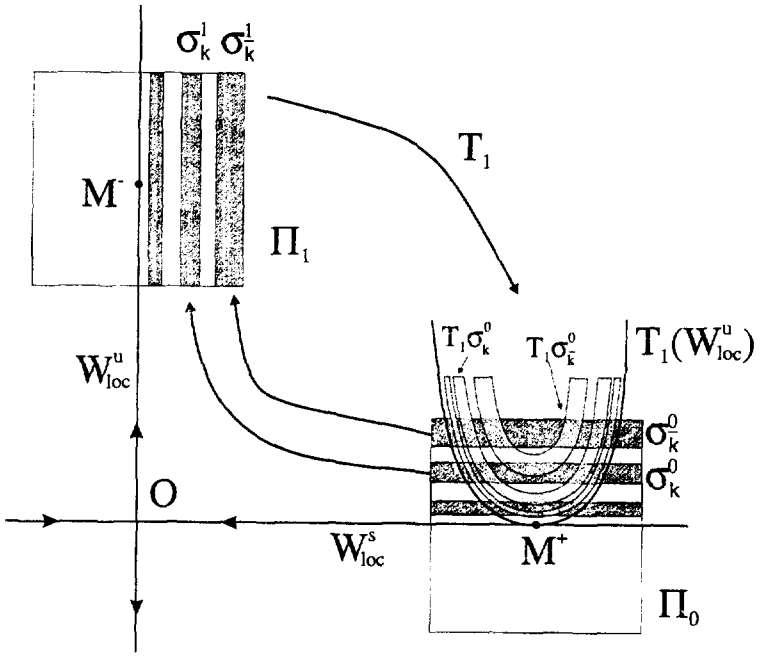


Figure 7. The images $T_1\sigma_k^1$ of the strips σ_k^1 have a shape of horseshoes which are accumulated on $T_1(W^u_{loc})$ as $k \to \infty$. Orbits of the set N must intersect Π_0 in the points of intersections of the "horseshoes" $T_1\sigma_i^1$ and the strips σ_j^0 for $i, j \geq \bar{k}$.

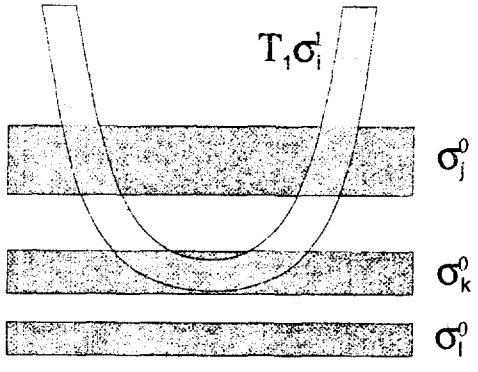


Figure 8. Various types of intersections of the "horseshoe" $T_1\sigma_i^1$ with the strips. The horseshoe has a regular intersection with the strip σ_j^0; it has an irregular intersection with the strip σ_k^0 and an empty intersection with the strip σ_l^0.

is satisfied where $S_{\bar{k}}(i,j) = S_1(|\lambda|^i + |\gamma|^{-j}) \cdot |\gamma|^{-\bar{k}/2}$, and S_1 is some positive constant independent from i, j, and $\bar{k}$, then the intersection of $T_1\sigma_i^1$ with σ_j^0 is regular.

The inequality

$$d\left[\gamma^{-j}y^- - c\lambda^i x^+\right] < -S_{\bar{k}}(i,j) \tag{2.5}$$

is a sufficient condition for an intersection of $T_1\sigma_i^1$ and σ_j^0 to be empty.

It is clear from (2.4) and (2.5) that the inequality

$$\left|d\left[\gamma^{-j}y^- - c\lambda^i x^+\right]\right| \leq S_{\bar{k}}(i,j) \tag{2.6}$$

is necessary in order for the horseshoe $T_1\sigma_i^1$ to have an irregular intersection with the strip σ_j^0.

Inequalities (2.4)–(2.6) have a quite simple geometrical sense (Figure 10). The strip σ_j^0 is a thin rectangle with the central line $y = \gamma^{-j}y^-$. The strip σ_i^1 is a thin rectangle with the central line $x = \lambda^i x^+$. The strip σ_i^1 is mapped by the map T_1 onto a horseshoe with the parabola $y = c\lambda^i x^+ + d((x - x^+)/b)^2$ as a central line. The condition $d[\gamma^{-j}y^- - c\lambda^i x^+] > 0$ means

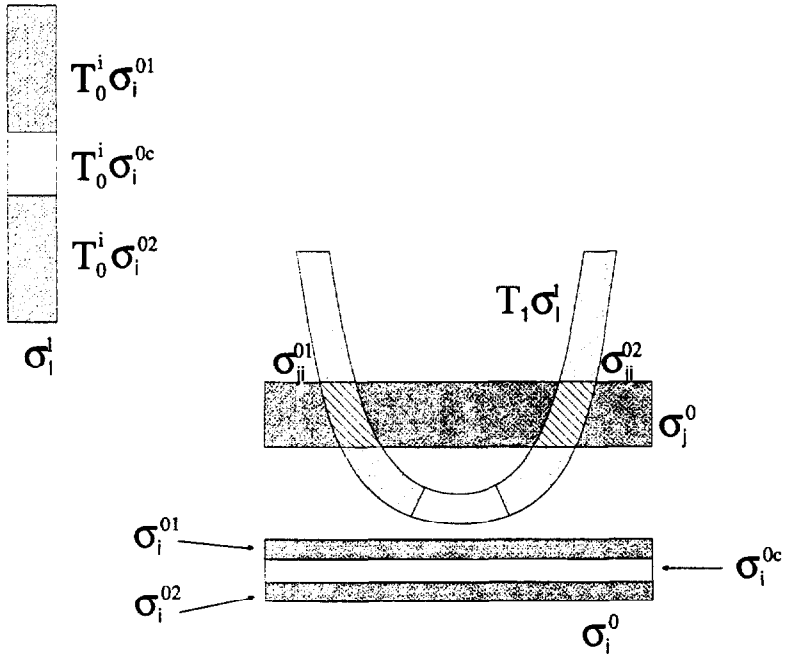


Figure 9. The case of regular intersection of the horseshoe $T_1\sigma_i^1$ with the strip σ_j^0. The intersection is called regular if (a) the set $T_1\sigma_i^1 \cap \sigma_j^0$ is nonempty and consists of two connected components σ_{ji}^{01} and σ_{ji}^{02}, and (b) the maps $T_{i\alpha} \equiv T_1T_0^i : \sigma_{ji}^{0\alpha} \to \sigma_j^0$, $\alpha = 1, 2$, are of saddle-type (i.e., they are contracting along the x-coordinate and expanding along the y-coordinate). Here, σ_i^{01} and σ_i^{02} are upper and lower parts of the strip σ_i^0. They are separated by the central part σ_i^{0c} of σ_i^0 which is mapped by $T_1T_0^i$ onto the top of the horseshoe $T_1\sigma_i^1$.

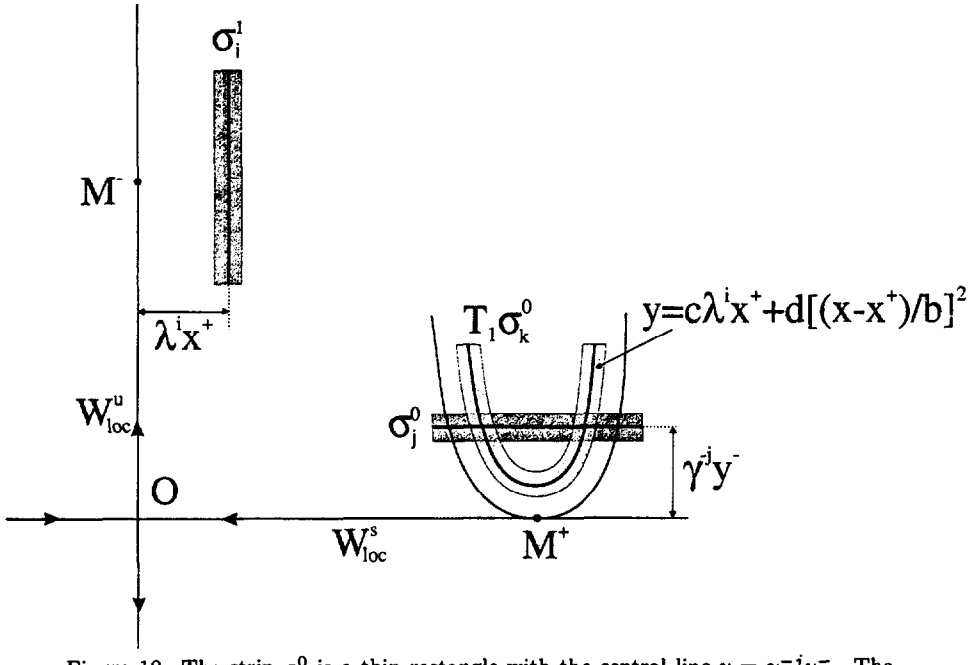


Figure 10. The strip σ_j^0 is a thin rectangle with the central line $y = \gamma^{-j}y^-$. The strip σ_i^1 is a thin rectangle with the central line $x = \lambda^i x^+$. The strip σ_i^1 is mapped by the map T_1 onto a horseshoe with the parabola $y = c\lambda^i x^+ + d((x - x^+)/b)^2$ as a central line.

that the straight line $y = \gamma^{-j}y^-$ and the parabola intersect in two points, and the condition $d[\gamma^{-j}y^- - c\lambda^i x^+] < 0$ means that they have no intersection. The coefficient $S_{\bar{k}}(i, j)$ in (2.4)–(2.6) is due to the nonzero thicknesses of the strip and horseshoe.

2.1.6. Codes

Let Q be an orbit lying in U entirely and nonasymptotic to O. This orbit intersects Π_0 in an infinite sequence of points M_s. Each point M_s belongs to some strip σ_{k_s}; hereat, successive points M_s and M_{s+1} are connected by the relation

$$M_{s+1} = T_1 T_0^{k_s}(M_s).$$

The infinite sequence of integers $\{k_s\}$ is called *a natural code* of the orbit Q.

DEFINITION. *A pair of integers (i, j) is called inadmissible if $i < \bar{k}$, or $j < \bar{k}$, or inequality (2.5) is fulfilled. Otherwise, the pair (i, j) is called admissible. An admissible pair is called regular if it satisfies inequality (2.4). A sequence of integers $\{k_s\}$ is called inadmissible if at least one of the pairs (k_s, k_{s+1}) is inadmissible, and it is called admissible otherwise. An admissible sequence $\{k_s\}$ is called regular if each pair (k_s, k_{s+1}) is regular.*

PROPOSITION 2.1.

(1) *For each orbit Q lying in U entirely, the code is an admissible sequence.*
(2) *If a sequence $\{k_s\}$ is regular, then there exists a continuum of saddle orbits in N which have the given sequence as the code.*

The first part of this assertion is evident because $M_{s+1} \in T_1\sigma^1_{k_s} \cap \sigma^0_{k_{s+1}}$. The second part of the assertion was proved in [6,11].

The last is connected with the fact that inequalities (2.4) guarantee that the intersection $T_1\sigma^1_{k_s} \cap \sigma^0_{k_{s+1}}$ is regular. It consists of two connected components $\sigma^{01}_{k_{s+1}k_s}$ and $\sigma^{02}_{k_{s+1}k_s}$ (Figure 9), and points belonging to different components may be distinguished. Therefore, for the orbits in U with the regular natural codes, a more precise code can be constructed. Namely, it is a sequence $\{(k_s, \alpha_s)\}$ ($\alpha_s \in \{1, 2\}$) such that the point M_s belongs to $\sigma^{0\alpha_s}_{k_{s+1}k_s} \subset \sigma^0_{k_s}$ (we will also use an equivalent notation for the code $\{(k_s, \alpha_s)\}$ as a sequence of the symbols "0", "1," and "2":

$$\ldots, \alpha_{s-1}, \overbrace{0, \ldots, 0}^{k_s}, \alpha_s, \overbrace{0, \ldots, 0}^{k_{s+1}}, \alpha_{s+1}, \ldots).$$

By the definition,

$$M_{s+1} = \tilde{T}_{k_s\alpha_s} M_s, \tag{2.7}$$

where the map $\tilde{T}_{k_s\alpha_s} \equiv T_1 T_0^{k_s}|_{\sigma^{0\alpha_s}_{k_s}}$ is saddle. By the "lemma on a saddle fixed point in a countable product of spaces" from [12], there exists a unique sequence of points satisfying equation (2.7). Thus, to each code $\{(k_s, \alpha_s)\}$ where $\{k_s\}$ is regular and $\{\alpha_s\}$ is an arbitrary fixed sequence of the symbols "1" and "2," there corresponds a unique orbit $Q \in N$ (the set of the orbits which correspond to different sequences $\{\alpha_s\}$ has the cardinality of continuum).

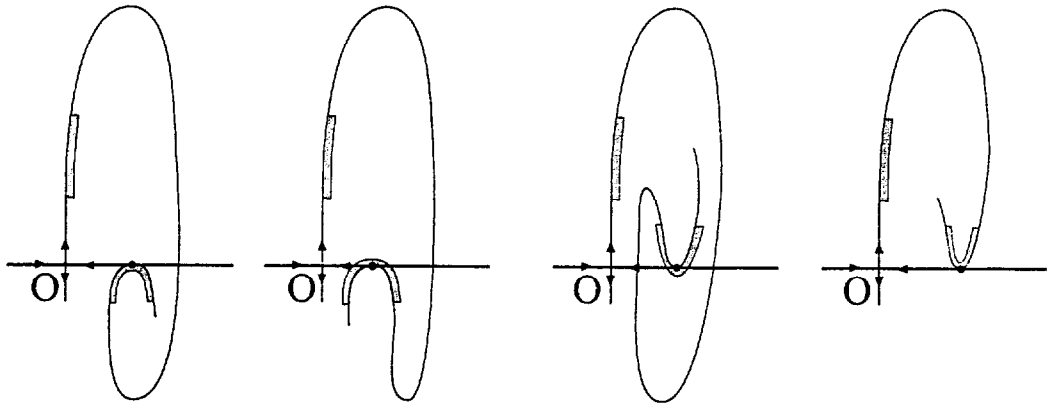
Note also that if a nonsaddle orbit exists in N, then its code $\{k_s\}$ must be such that inequality (2.6) is satisfied for at least one of the pairs $(i = k_s, j = k_{s+1})$.

2.2. The Types of Two-Dimensional Diffeomorphisms with a Homoclinic Tangency

Thus, an analysis of the structure of integer solutions of inequalities (2.4)–(2.6) is an essential part of the study of orbits of the set N. The sets of such solutions obviously depend on the signs of parameters λ, γ, c, and d. Geometrically, it is connected with the fact that the signs of these values determine the character of the reciprocal position of the manifolds W^s_{loc} and $T_1 W^u_{\text{loc}}$ in a neighborhood of the homoclinic point M^+. We restrict ourself to the case of positive λ and γ (see footnote[2]. Different cases possible here, in dependence on signs of c and d are shown in Figure 11.

According to [6], the diffeomorphisms under consideration are divided into the three classes for which the structure of the set N is essentially different.

[2]The cases of different signs of λ and γ are considered, for instance, in [9,10].



(a) $c < d$, $d < 0$. (b) $c > 0$, $d < 0$. (c) $c < 0$, $d > 0$. (d) $c > 0$, $d > 0$.

Figure 11. Various types of diffeomorphisms with a homoclinic tangency in the case where both λ and γ are positive. These types correspond to different cases of a reciprocal position of the stable and unstable manifolds which depends on the signs of the values c and d. When d is negative, W^u is tangent to W^s_{loc} from below (a) and (b); when d is positive, W^u is tangent to W^s_{loc} from above (c) and (d). The sign of the value c determines how the shaded semineighborhood of the homoclinic point M^- is mapped onto the neighborhood of the point M^+.

2.2.1. Systems of the first class

The systems of the first class are those for which $\lambda > 0$, $\gamma > 0$, $d < 0$. The following theorem takes place.

THEOREM 1. *[6] Let f be a diffeomorphism of the first class. Then the set N is trivial: $N = \{O, \Gamma\}$.*

This result can be obtained from the analysis of the set of integer solutions of inequalities (2.4)–(2.6): one can prove that if $\lambda > 0$, $\gamma > 0$, $d < 0$, any sequence $\{k_s\}$ is inadmissible. Geometrically, this can be verified in the following way. If $c < 0$, $d < 0$, then the horseshoes $T_1\sigma_i^1$ and the strips σ_j^0 do not intersect since they lie at the opposite sides from W^s_{loc} (Figure 12a). Thus, in this case, the set N has a trivial structure indeed: $N = \{O, \Gamma\}$.

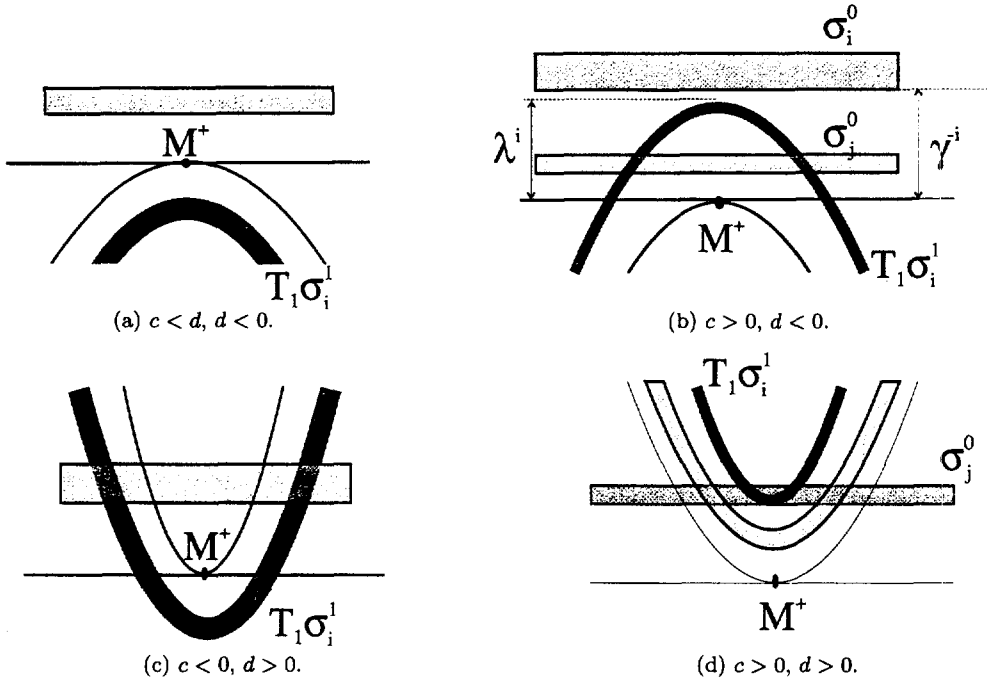
In the case $c > 0$, $d < 0$ (i.e., when "parabola" $T^1W^u_{\text{loc}}$ is tangent to W^s_{loc} from below; see Figure 12b), the set N has a trivial structure also. It is connected with the fact that here the intersection $T_1\sigma_i^1 \cap \sigma_j^0$ may be nonempty only for $j > i$. Indeed, the strip σ_j^0 lies at a distance of an order γ^{-j} from W^s_{loc} and the top of the horseshoe $T_1\sigma_i^1$ lies at a distance of an order λ^i from W^s_{loc} (Figure 12b). Since $\lambda\gamma < 1$, it follows that $\lambda^i \ll \gamma^{-i}$, so any horseshoe $T_1\sigma_i^1$ lies below the corresponding strip σ_i^0, and hence, below any strip σ_j^0 with $j < i$. As a consequence, we have that the negative semiorbit of any initial point on Π_0 (except M^+) leaves the neighbourhood U.

2.2.2. Systems of the second class

The systems of the second class are those for which $\lambda > 0$, $\gamma > 0$, $c < 0$, $d > 0$. In this case, evidently, inequality (2.4) is fulfilled for any sufficiently large i and j; i.e., the intersection of $T_1\sigma_i^1$ with σ_j^0 is regular for any $i, j \geq \bar{k}$ (Figure 12c). Correspondingly, any sequence of integers $k_s \geq \bar{k}$ is regular in this case. Therefore, the following statement [6] takes place.

THEOREM 2. *In the case $c < 0$, $d > 0$, all orbits from $N \backslash \Gamma$ have a saddle-type and N is in one-to-one correspondence with the a quotient-system Ω_0 which is obtained from the Bernoulli scheme on three symbols $\{0, 1, 2\}$ by identification of two homoclinic orbits: $(\ldots, 0, \ldots, 0, 1, 0 \ldots)$ and $(\ldots, 0, \ldots, 0, 2, 0, \ldots)$.*[3]

[3]Both these two codes correspond to the orbit Γ.



(a) $c < d$, $d < 0$. (b) $c > 0$, $d < 0$.

(c) $c < 0$, $d > 0$. (d) $c > 0$, $d > 0$.

Figure 12. The reciprocal position of the strips and the horseshoes. In the case $c < 0$, $d < 0$ (a), the horseshoes $T_1\sigma_i^1$ and the strips σ_j^0 lie from the opposite sides from $W^s_{\rm loc}$. In the case $c > 0$, $d < 0$ (b), any horseshoe $T_1\sigma_i^1$ lies below its "own" strip σ_i^0, and hence, below any strip σ_k^0 with $k \le i$. Really, the strip σ_j^0 lies here at a distance of order γ^{-j} from $W^s_{\rm loc}$; the top of the horseshoe $T_1\sigma_i^1$ lies at a distance of order λ^i from $W^u_{\rm loc}$. Since $|\lambda|\,|\gamma|$ is less than unity, then $\lambda^i \ll \gamma^{-i}$. In the cases of (a),(b), the set N has the trivial structure: $N = \{O, \Gamma\}$. If $c < 0$, $d > 0$ (c), then the intersection of $T_1\sigma_i^1$ with σ_j^0 is regular for any $i, j \ge \bar{k}$. In this case, the set N has a nontrivial structure which may be described completely. In the case $c > 0$, $d > 0$ (d), any horseshoe $T_1\sigma_i^1$ is intersected regularly with its "own" strip σ_i^0, with all strips lying above it and with some number (depending on i) of strips lying below. The set N has a nontrivial structure for this case. But, the description of N depends essentially on parameters of a homoclinic structure (such as the Ω-moduli θ and τ_0).

2.2.3. Systems of the third class

The systems of the third class are those for which $\lambda > 0$, $\gamma > 0$, $c > 0$, $d > 0$. In this case, $T_1W^u_{\rm loc}$ is tangent to $W^s_{\rm loc}$ from above ($c > 0$, $d > 0$) (Figure 12d). The study of systems of the third class (and nearby systems) is the main scope of the present paper.

2.3. Nontrivial Hyperbolic Subsets of Systems of the Third Class

Taking logarithm of both parts of (2.4), we rewrite the condition of regular intersection as

$$j < i\theta - \tau - S\gamma^{-\bar{k}/2}, \tag{2.8}$$

and the condition of empty intersection as

$$j > i\theta - \tau + S\gamma^{-\bar{k}/2}, \tag{2.9}$$

where

$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|},$$
$$\tau = \frac{1}{\ln|\gamma|}\ln\left|\frac{cx^+}{y^-}\right|,$$

and S is some positive constant. It is convenient to rewrite inequalities (2.8) and (2.9) in an "invariant" form

$$(j+m) < (i+m)\theta - \tau_0 - \hat{S}\gamma^{-(\overline{k}+m)/2} \tag{2.10}$$

and

$$(j+m) > (i+m)\theta - \tau_0 + \hat{S}\gamma^{-(\overline{k}+m)/2}, \tag{2.11}$$

respectively, where m is the constant defined by the condition $M^+ = f^m(M^-)$ and τ_0 is defined as

$$\tau_0 = \tau - m(\theta - 1).$$

Note that the value θ is independent of smooth transformations of the coordinates. The value τ_0 can also be proved [9] to be independent of smooth coordinate transformations preserving form (2.1) for the map T_0 as well as of the choice of homoclinic points in U_0. The number $(\overline{k}+m)$ is invariant also in that sense that it equals to the minimal period of periodical orbits of N.

Let us consider the subsystem Ω_1 (see [6,11]) belonging to Ω_0 (see Theorem 2) and composed by the orbits of the form

$$(\ldots, \alpha_{s-1}, 0, \ldots, 0, \alpha_s, 0, \ldots, 0, \alpha_{s+1}, \ldots),$$

where

(1) $\alpha_s \in \{1, 2\}$;
(2) the length of any complete string of zeros is not less than $(\overline{k}+m)$;
(3) the lengths (k_s+m) and $(k_{s+1}+m)$ of successive complete strings of zeros separated by a nonzero symbol satisfy inequality (2.10) with $i = k_s$ and $j = k_{s+1}$ (i.e., the sequence $\{k_s\}$ is regular).

THEOREM 3. *If $\overline{k}$ is large enough, then in the case $c > 0, d > 0$ there exists a subsystem $\tilde{N}$ in N which is conjugate to the symbolic system Ω_1 and such that all orbits of $\tilde{N}$ have a saddle-type.*

Note that the set $\tilde{N}$ may not coincide with N but it anyway forms a substantial part of N. Indeed, for a nonsaddle orbit, at least two subsequent points M_s and M_{s+1} of intersection with Π_0 must lie in the strips $\sigma^0_{k_s}$ and $\sigma^0_{k_{s+1}}$ whose numbers satisfy the inequality

$$|(k_{s+1}+m) - (k_s+m)\,\theta + \tau_0| \le \hat{S}\gamma^{-(\overline{k}+m)/2}, \tag{2.12}$$

which is equivalent to (2.6).

The set of integer solutions (i, j) of the last inequality will lie in the narrow strip on the plane (the greater $\overline{k}$, the more narrow is the strip). This set depends essentially on the values θ and τ_0. For instance, if θ is rational: $\theta = p/q$ and $\tau_0 q \notin Z$, then this set is empty for $\overline{k}$ large enough. This implies the following statement.

THEOREM 4. *If $\theta = p/q$ and $\tau_0 q \notin Z$, then there exists such $\overline{k} = \overline{k}(\theta, \tau_0)$ that all orbits from $N \backslash \Gamma$ are saddle and $N \backslash \{\Gamma, O\}$ is conjugate with Ω_1.*

Geometrically, the fact that the set of integer solutions of inequality (2.12) is empty for rational θ and suitable τ_0 means that for such θ and τ_0, tops of all horseshoes get to the "holes" between the strips.

2.4. Moduli of the Ω-Equivalence for Systems of the Third Class

As we mentioned, the structure of the set N of all orbits lying in the neighborhood U entirely is in a close connection with the structure of the sets of integer solutions of inequalities (2.10) and (2.12). These sets are different for different values of the invariants θ and τ_0. Therefore, the structure of the set N depends essentially on the values of θ and τ_0. Moreover, the following result shows that the invariants θ and τ_0 are moduli of the Ω-equivalence.

THEOREM 5. *[9,10] Let f and f' be diffeomorphisms of the third class and let f and f' be locally Ω-conjugate.*[4] *Then, $\theta = \theta'$. If, moreover, the value θ is irrational, then $\tau_0 = \tau_0'$. If θ is rational ($\theta = p/q$), then there exists such integer s that τ_0 and τ_0' satisfy simultaneously the inequalities $s \le \tau_0 q \le s+1$ and $s \le \tau_0' q \le s+1$.*

We give a sketch of the proof of the theorem. Let f and f' be locally Ω-conjugate and let M^+, M^- and M'^+, M'^- be conjugate pairs of homoclinic points. Evidently, $f^m(M^-) = M^+$ and $f'^m(M'^-) = M'^+$ for some natural m. Suppose that $\theta > \theta'$. Consider the set of pairs of integers (i,j) satisfying the inequality

$$(i+m)\theta - \tau_0 - \hat{S}\gamma^{-(\overline{k}+m)/2} > j+m > (i+m)\theta' - \tau_0' + \hat{S}\gamma'^{-(\overline{k}+m)/2}. \tag{2.13}$$

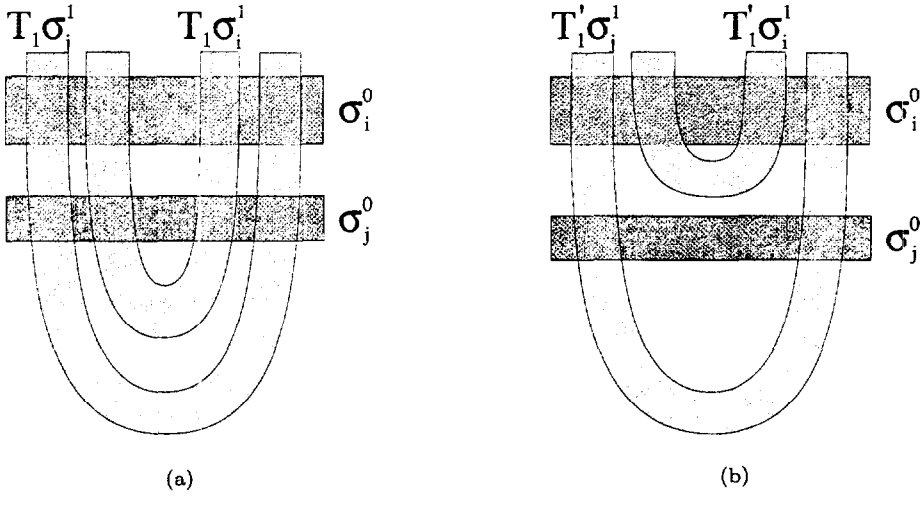
According to conditions (2.10) and (2.11), this inequality means that the pair (i,j) is regular for the diffeomorphism f, but it is inadmissible for the diffeomorphism f'. Since $\theta > \theta'$, the set of pairs (i,j) satisfying condition (2.13) is infinite.

Note that if a pair (i,j) satisfies condition (2.13), then the pair (j,i) is also regular for the diffeomorphism f because here $j > i$ and the inequality

$$(i+m) < (j+m)\theta - \tau_0 - \hat{S}\gamma^{-(\overline{k}+m)/2},$$

obtained from inequality (2.10) by substitution j instead of i, and i instead of j is automatically fulfilled (we take into account that $\theta > 1$ and $\theta' > 1$, since we consider the case $\lambda\gamma < 1$).

For such i and j, the intersection of the horseshoe $T_1(\sigma_j^1)$ with the strip σ_i^0 and the intersection of the horseshoe $T_1(\sigma_i^1)$ with the strip σ_j^0 are regular (Figure 13a), and the code $\{\dots ijijij\dots\}$ is regular. By Theorem 3, the диффеоморфизм f has a double-round saddle periodic orbit which intersects successively the strips σ_i^0 and σ_j^0.



(a) (b)

Figure 13. Since $\theta > \theta'$, it follows that there exists a countable set of pairs (i,j) which are regular for the diffeomorphism f, but they are inadmissible for the diffeomorphism f'. If (i,j) is such a pair, then the corresponding strips and horseshoes are positioned as follows. For the diffeomorphism f: the intersection of the horseshoe $T_1(\sigma_j^1)$ with the strip σ_i^0 is regular as well as the intersection of the horseshoe $T_1(\sigma_i^1)$ with the strip σ_j^0 (a). For the diffeomorphism f': the intersection of the horseshoe $T_1'(\sigma_j^1)$ with the strip σ_i^0 is regular, but the horseshoe $T_1'(\sigma_i^1)$ does not intersect the strip σ_j^0 (b).

[4]For example, there exist such neighborhoods U and U' for which the sets N and N' have the same structure.

On the other hand, for the diffeomorphism f', the horseshoe $T_1'(\sigma_i^1)$ does not intersect the strip σ_j^0 (Figure 13b), though i and j are the same as above. This follows from the fact that the pair (i, j) is inadmissible by virtue of the right of inequalities (2.13). Therefore, f' does not have periodic orbits intersecting successively the strips σ_i^0 and σ_j^0. It is clear that the diffeomorphisms f and f' are not Ω-conjugate in this case. Thus, for the Ω-conjugacy it is necessary that $\theta = \theta'$.

Now let $\theta = \theta'$. Suppose $\tau_0' > \tau_0$. If $\theta = \theta'$ is irrational, then inequality (2.13) again possesses infinitely many natural solutions for sufficiently large $\overline{k}$ and the diffeomorphisms f and f' are not Ω-conjugate. Hence, for the Ω-conjugacy of the diffeomorphisms, the equality $\tau_0' = \tau_0$ must hold in this case.

Let θ be rational, $\theta = p/q$. If, for some integer s_0, inequality $\tau_0 q > s_0 > \tau_0' q$ holds, then the integer points on the straight line

$$j = i\frac{p}{q} - \frac{s_0}{q}$$

satisfy inequality (2.13) and the diffeomorphisms f and f' are not Ω-conjugate again. Hence, for the Ω-conjugacy of f and f' in this case, it is necessary that $\tau_0 q, \tau_0' q \in [s, s+1]$ for some integer s, what completes the proof of the theorem.

2.5. Infinite Degenerations in Systems of the Third Class

We see that the cases of rational and irrational θ are principally different. In the rational case, almost all systems admit a complete description (Theorem 4) and all orbits of $N \backslash \Gamma$ are saddle. In the irrational case, condition (necessary) (2.12) of an irregular intersection has a countable set of integer solutions for any $\overline{k}$. Correspondingly, here a countable number of strips and horseshoes may have irregular intersections which leads to a very nontrivial dynamics. Namely, the following result [7] takes place.

THEOREM 6. *If H_3 is a bifurcational surface composed by diffeomorphisms of the third class, then systems with a countable number of saddle periodic orbits each of which has a homoclinic tangency are dense on H_3.*

The values θ calculated for these periodic orbits are Ω-moduli, according to Theorem 6. These values are independent of each other. Therefore, we arrive at the following theorem [7].

THEOREM 7. *Systems with a countable number of Ω-moduli are dense on H_3.*

As we mentioned in the Introduction, when the value of an Ω-modulus is changed, bifurcations of periodic, homoclinic, etc., orbits occur inevitably. The presence of an infinite number of independent Ω-moduli may lead to infinitely degenerate bifurcations. Indeed, the following result [7] takes place.

THEOREM 8. *Systems with homoclinic tangencies of any order and with structurally unstable periodic orbits of any degree of degeneracy are dense in H_3.*

It should be noted that the degenerations indicated in this theorem may exist only for periodic and homoclinic orbits of extremely high roundnesses. In the present paper, we will not consider the questions connected with the infinite degeneracies. Further, we will study bifurcations of low-round periodic orbits in the framework of low-parameter families. The control parameters be the splitting parameter as well as the Ω-moduli.

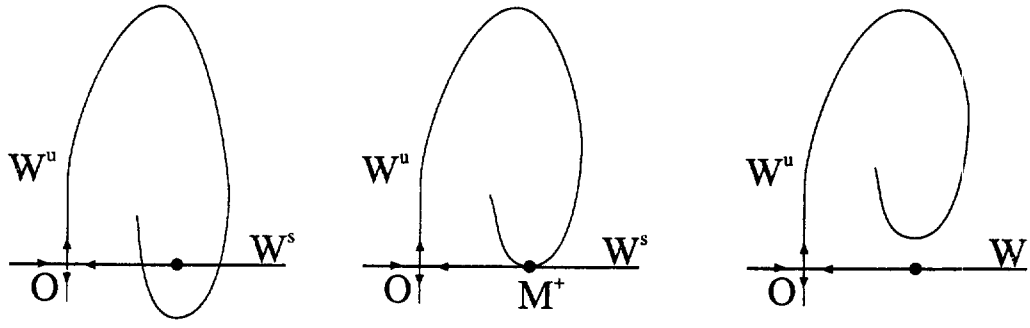
First, let us consider the bifurcations of single-round periodic orbits.

2.6. Bifurcations of Single-Round Periodic Orbits

Note that for any system belonging to the bifurcational surface H_3, single-round periodic orbits are structurally stable. Here, for any i sufficiently large, the horseshoe $T_1(\sigma_i^1)$ intersects its "own" strip σ_i^0 regularly (inequality (2.8) is evidently fulfilled for $j = i$ due to the condition $\theta > 1$).

Hereat, the structure of the nonwandering set for the map $T_1T_0^i : \sigma_i^0 \to \sigma_i^0$ is the same as for the famous Smale's horseshoe example.

However, when the system is perturbed such that the homoclinic tangency is destroyed, the single-round periodic orbits may undergo bifurcations. To study the bifurcations, we imbed the diffeomorphism f into a one-parameter family f_μ, where μ is the splitting parameter for the tangency. We assume that when $\mu < 0$, the parabola $T_1W^u_{\rm loc}$ intersects $W^s_{\rm loc}$ at two points; when $\mu = 0$, the parabola $T_1W^u_{\rm loc}$ is tangent to $W^s_{\rm loc}$ at one point, and when $\mu > 0$, there is no intersection (Figure 14). The family f_μ is supposed to depend smoothly on μ. The requirement of general position is that the family f_μ is transverse to the bifurcational surface H_3 in the space of dynamical systems.



(a) $\mu < 0$. (b) $\mu = 0$. (c) $\mu > 0$.

Figure 14. The behavior of $W^u(O)$ for one-parametric family f_μ, where μ is the splitting parameter. When $\mu < 0$, the parabola $T_1W^u_{\rm loc}$ intersects $W^s_{\rm loc}$ at two points; when $\mu = 0$, the parabola $T_1W^u_{\rm loc}$ is tangent to $W^s_{\rm loc}$ at one point, and when $\mu > 0$, there is no intersection.

Clearly, the local and global maps T_0 and T_1 depend now on μ. The map $T_0(\mu)$ can be represented in the form

$$\bar{x} = \lambda x + f(x, y, \mu)x^2y, \qquad \bar{y} = \gamma y + g(x, y, \mu)xy^2, \tag{2.14}$$

and the global map $T_1(\mu)$ is represented in the form

$$\begin{aligned} \bar{x} - x^+ &= ax + b\left(y - y^-\right) + \cdots, \\ \bar{y} &= cx + d\left(y - y^-\right)^2 + \mu + \cdots. \end{aligned} \tag{2.15}$$

Below, we will denote the coordinates on Π_0 as (x_0, y_0) and the coordinates on Π_1 as (x_1, y_1). If $(x_0, y_0) \in \sigma_k^0$ and $(x_1, y_1) = T_0^k(x_0, y_0) \in \sigma_k^1$, the following formula takes place (we change slightly notations in comparison with (2.2)):

$$\begin{aligned} x_1 &= \lambda^k x_0 + |\lambda|^k|\gamma|^{-k}\phi_{k1}\left(x_0, y_k, \mu\right), \\ y_0 &= \gamma^{-k}y_1 + |\gamma|^{-2k}\phi_{k2}\left(x_0, y_k, \mu\right). \end{aligned} \tag{2.16}$$

It is clear that if the bottom of the parabola $T_1W^u_{\rm loc}$ descends sufficiently low (large and negative μ), then each horseshoe intersects each strip. In this case, the set N_μ is a hyperbolic set similar to the invariant set in the Smale horseshoe. However, if μ is sufficiently large and positive, then the horseshoes and the strips do not intersect at all, and all of the orbits except O will escape from U.

The main question is what happens when the parameter μ varies from the negative to the positive values. First of all, it is necessary to study the structure of the bifurcation set corresponding to one strip, that is, to study the bifurcations in the family of the first return maps

$T_k(\mu) \equiv T_1 T_0^k : \sigma_k^0 \to \sigma_k^0$. The following result (see [13]) makes the analysis of the map T_k very simple.

LEMMA 2.1. *By means of a transformation of the coordinates and the parameter, the map $T_k(\mu)$ can be brought to the form*

$$\begin{aligned} \bar{x} &= y + \varepsilon_{1k}(x, y, \mu), \\ \bar{y} &= M - y^2 + \varepsilon_{2k}(x, y, \mu), \end{aligned} \tag{2.17}$$

where

$$\varepsilon_{ik}(x, y, \mu) = O\left(\lambda^k \gamma^k + \gamma^{-k}\right). \tag{2.18}$$

Here the rescaled splitting parameter $M = -d\gamma^{2k}(\mu - \gamma^{-k}y^- + c\lambda^k x^+ \cdots)$ may take arbitrary finite values for sufficiently large k.

PROOF. It is convenient to use the pair (x_0, y_1) as the coordinates for points on σ_k^0. We can use the coordinate y_1 instead of y_0 because the value y_0 is determined uniquely by formula (2.16) as a function of (x_0, y_1) for a fixed k. By virtue of equations (2.16),(2.15), the map $T_k(\mu)$ is written in the form

$$\begin{aligned} \bar{x_0} - x^+ &= a\lambda^k x_0(1 + \cdots) + b\left(y_1 - y^-\right) + \cdots, \\ \gamma^{-k}\bar{y_1}\left(1 + \gamma^{-k}\eta_k\left(\bar{x_0}, \bar{y_1}\right)\right) &= \mu + c\lambda^k x_0(1 + \cdots) + d\left(y_1 - y^-\right)^2 + \cdots. \end{aligned} \tag{2.19}$$

With the shift of the origin: $y_1 \to y + y^-$, $x_0 \to x + x^+$, we write the map $T_k(\mu)$ in the form

$$\begin{aligned} \bar{x} &= by + O\left(\lambda^k\right) + O\left(y^2\right), \\ \gamma^{-k}\bar{y} + \gamma^{-2k}O\left(\bar{y}\right) &= M_1 + dy^2 + \lambda^k O(|x| + |y|) + O\left(y^3\right), \end{aligned} \tag{2.20}$$

where

$$M_1 = \mu + c\lambda^k x^+ - \gamma^{-k}y^- + \cdots. \tag{2.21}$$

Now, rescaling the variables

$$x \to -\frac{b}{d}\gamma^{-k}x, \qquad y \to -\frac{1}{d}\gamma^{-k}y$$

brings equations (2.20) to form (2.17), where $M = -d\gamma^{2k}M_1$. This completes the proof of the lemma.

Map (2.17) is close to the one-dimensional parabola map

$$\bar{y} = M - y^2, \tag{2.22}$$

whose bifurcations have been well studied, so that it is possible to recover the bifurcation picture for the initial map T_k. For the parabola map, the bifurcation set is contained in the interval $[-(1/4), 2]$ of values of M: at $M = -(1/4)$, there appears a fixed point with the multiplier equal to $+1$, this fixed point is attractive at $M \in (-(1/4), 3/4)$ and it undergoes a period-doubling bifurcation at $M = 3/4$; the cascade of period-doubling bifurcations lead to chaotic dynamics which alternates with stability windows and the bifurcations stop at $M = 2$, when the restriction of the map onto the nonwandering set becomes conjugate to the Bernoulli shift of two symbols and the map no longer bifurcates as M increases.

By Lemma 2.1, similar bifurcations take place for the map T_k. The map has an attractive fixed point O_k at $\mu \in (\mu_k^{+1}, \mu_k^{-1})$ which arises at the saddle-node bifurcation at $\mu = \mu_k^{+1}$ and loses stability at $\mu = \mu_k^{-1}$ at the period-doubling bifurcation. Here,

$$\begin{aligned} \mu_k^{+1} &= \gamma^{-k}y^- - c\lambda^k x^+ + \frac{1}{4d}\gamma^{-2k} + \cdots, \\ \mu_k^{-1} &= \gamma^{-k}y^- - c\lambda^k x^+ - \frac{3}{4d}\gamma^{-2k} + \cdots. \end{aligned}$$

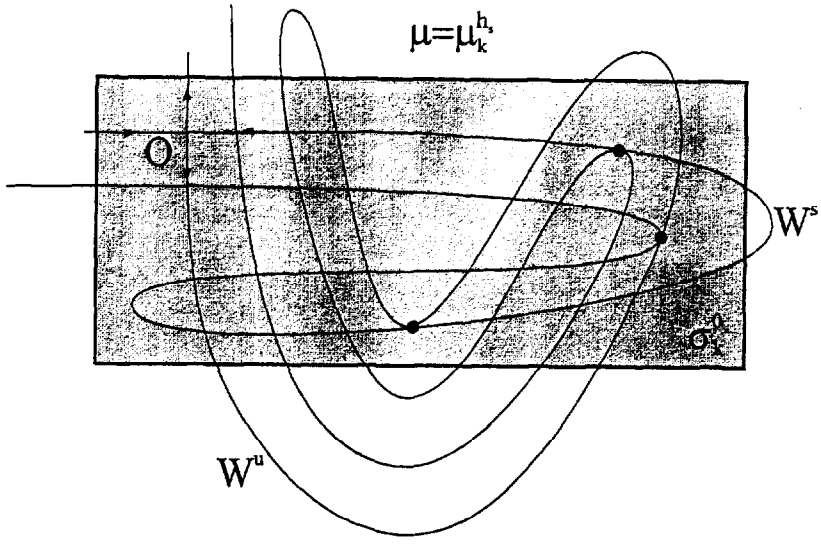


Figure 15. The last homoclinic tangency of the manifolds of the fixed point of T_k at $\mu = \mu_k^{hs}$. An invariant set similar to those of the Smale's horseshoe example arises after this bifurcation.

The bifurcation set of the map T_k is contained in the interval $[\mu_k^{+1}, \mu_k^{hs}]$, where

$$\mu_k^{hs} = \gamma^{-k} y^- - c\lambda^k x^+ - \frac{2}{d}\gamma^{-2k} + \cdots .$$

At $\mu = \mu_k^{hs}$, the fixed point of T_k has the last homoclinic tangency (Figure 15) and an invariant set similar to those of the Smale's horseshoe example arises after this bifurcation. Note that these bifurcational intervals do not intersect each other for different k.

2.7. Bifurcations of Double-Round Periodic Orbits

The study of double-round periodic orbits is reduced to the study of the fixed points of the second return maps $T_{ij} = T_1 T_0^i T_1 T_0^j$: $\Pi_0 \longrightarrow \Pi_0$, which by virtue of equations (2.16),(2.15), are represented in the form

$$\begin{aligned}
\bar{x}_0 - x^+ &= a\lambda^j x_0 + b\left(y_1 - y^-\right) + \cdots , \\
\gamma^{-i}\bar{y}_1(1 + \cdots) &= c\lambda^j x_0 + d\left(y_1 - y^-\right)^2 + \mu + \cdots , \\
\bar{\bar{x}}_0 - x^+ &= a\lambda^i \bar{x}_0 + b\left(\bar{y}_1 - y^-\right) + \cdots , \\
\gamma^{-j}\bar{\bar{y}}_1(1 + \cdots) &= c\lambda^i \bar{x}_0 + d\left(\bar{y}_1 - y^-\right)^2 + \mu + \cdots ,
\end{aligned} \tag{2.23}$$

where (x_0, y_1) and $(\bar{\bar{x}}_0, \bar{\bar{y}}_1)$ are the coordinates on the strip σ_j^0 for an initial point and its image by the map T_{ij}, respectively, and $(\bar{x}_0, \bar{y}_1)$ are the coordinates for the intermediate point $T_1 T_0^j(x_0, y_1)$ on the strip σ_i^0.

The map T_{ij} is a composition of the successively acting maps $T_j \equiv T_1 T_0^j$ and $T_i \equiv T_1 T_0^i$, which are defined, respectively, on the strips σ_j^0 and σ_i^0. The map T_i transforms the strip σ_i^0 into the horseshoe $T_1\sigma_i^1$, and the map T_j transforms the strip σ_j^0 into the horseshoe $T_1\sigma_j^1$.

2.7.1. Bifurcations on H_3

Let us consider here the case $\mu = 0$. Different cases of the reciprocal position of the strips and horseshoes σ_i^0, σ_j^0, $T_1\sigma_i^1$, $T_1\sigma_j^1$ are shown in Figures 16a–16c. We assume here $j > i$ (we do not consider the case $i = j$). The horseshoe $T_1\sigma_j^1$ intersects both strips σ_i^0 and σ_j^0 regularly, and the horseshoe $T_1\sigma_i^1$ intersects regularly the strip σ_i^0. For the intersection of $T_1\sigma_i^1 \cap \sigma_j^0$, different

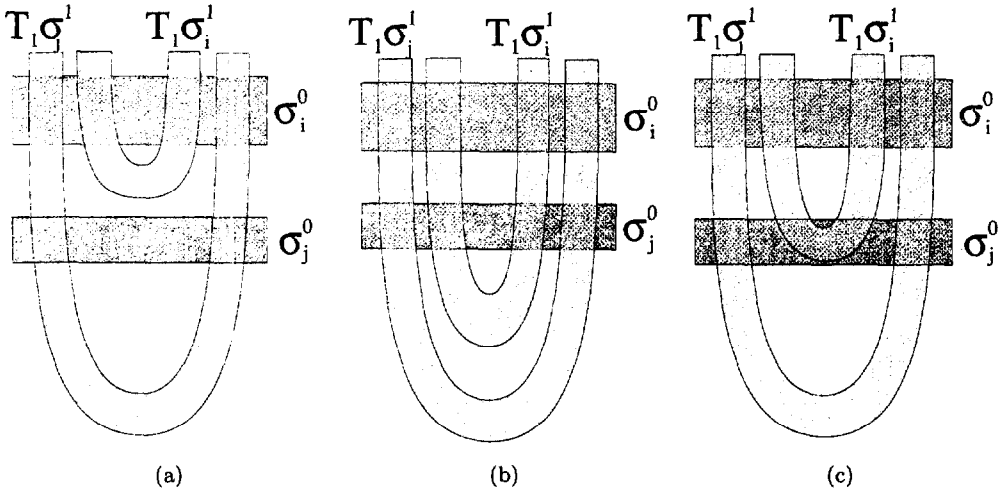


Figure 16. The various cases of the reciprocal position of the strips and horseshoes σ_i^0, σ_j^0, $T_1\sigma_i^1$, $T_1\sigma_j^1$ are shown for the case $\mu = 0$.

possibilities may take place: $T_1\sigma_i^1 \cap \sigma_j^0 = \emptyset$ in the case of Figure 16a; the intersection of $T_1\sigma_i^1$ with σ_j^0 is regular in the case of Figure 16b, and irregular in the case of Figure 16c.

The conditions of the regular, irregular, and empty intersection of the corresponding strips and horseshoes are written by the use of inequalities (2.8),(2.9). Note that since we are interested now by the bifurcations of the double-round periodic orbits which do not intersect Π^0 above the strip σ_i^0, we may assume $i = \bar{k}$ in these inequalities.

If $T_1\sigma_i^1 \cap \sigma_j^0 = \emptyset$, then the map T_{ij} has no fixed points. In this case, i and j satisfy the inequality

$$j - i\theta + \tau > S\gamma^{-i/2}. \tag{2.24}$$

On the other hand, if i and j satisfy the inequality

$$j - i\theta + \tau < -S\gamma^{-i/2}, \tag{2.25}$$

the intersection of $T_1\sigma_i^1$ with σ_j^0 is regular, and the map T_{ij} has saddle fixed points: there are exactly four such points, two of them have positive multipliers, and two have negative multipliers.[5]

It is clear that if one changes the system on H_3 so that to come from the situation of Figure 16a to the situation of Figure 16b, then bifurcations connected with the appearance of fixed points of T_{ij} (double-round periodic orbits f) will occur on the way.

To follow these bifurcations, it is convenient to consider one-parameter families of systems on H_3, where the invariant θ is the control parameter (note that when proving Theorem 5, which establishes that θ is an Ω-modulus for the systems on H_3, we just used the fact that the variation of θ is connected with the changes in the structure of intersections of the strip and horseshoes).

Let f_θ be such a family. Let i and j be sufficiently large fixed integers. By virtue of (2.24), if

$$\theta < \theta_1 \equiv \frac{j}{i} + \frac{1}{i}\tau - \frac{1}{i}S\gamma^{-i/2}, \tag{2.26}$$

then $T_1\sigma_i^1 \cap \sigma_j^0 = \emptyset$ and the map T_{ij} does not have fixed points. When θ increases, the bottom of the horseshoe $T_1\sigma_i^1$ moves down, and for the values of θ such that

$$\theta > \theta_2 \equiv \frac{j}{i} + \frac{1}{i}\tau + \frac{1}{i}S\gamma^{-i/2}, \tag{2.27}$$

[5]Moreover, Theorem 3 implies that the nonwandering set of the map T_{ij} is nontrivial in this case and has a hyperbolic structure.

the intersection of $T_1\sigma_i^1$ with σ_j^0 will be regular, and the map T_{ij} will have four saddle fixed points.

We, therefore, get that all bifurcations of the double-round periodic orbits which intersect the strips σ_i^0 and σ_j^0 occur for the values of θ belonging to the interval

$$\frac{j}{i} + \frac{1}{i}\tau - \frac{1}{i}S\gamma^{-i/2} \equiv \theta_1 \le \theta \le \theta_2 \equiv \frac{j}{i} + \frac{1}{i}\tau + \frac{1}{i}S\gamma^{-i/2}. \tag{2.28}$$

To clarify how the bifurcations go, we give a more detailed geometric construction (see Figure 17). The horseshoe $T_1\sigma_j^1$ intersects the strip σ_i^0 in two connected components which are denoted as $\tilde{\Delta}_{ij}^1$ and $\tilde{\Delta}_{ij}^2$. The preimages of these components with respect to the map T_j are the two "substrips" Δ_{ij}^1 and Δ_{ij}^2 lying on σ_j^0 (so, $T_j(\tilde{\Delta}_{ij}^\alpha) = \Delta_{ij}^\alpha$, $\alpha = 1, 2$). The image of the strip Δ_{ij}^α with respect to the map T_{ij} is a thin horseshoe $T_{ij}(\Delta_{ij}^\alpha)$ lying in $T_1\sigma_i^1$.

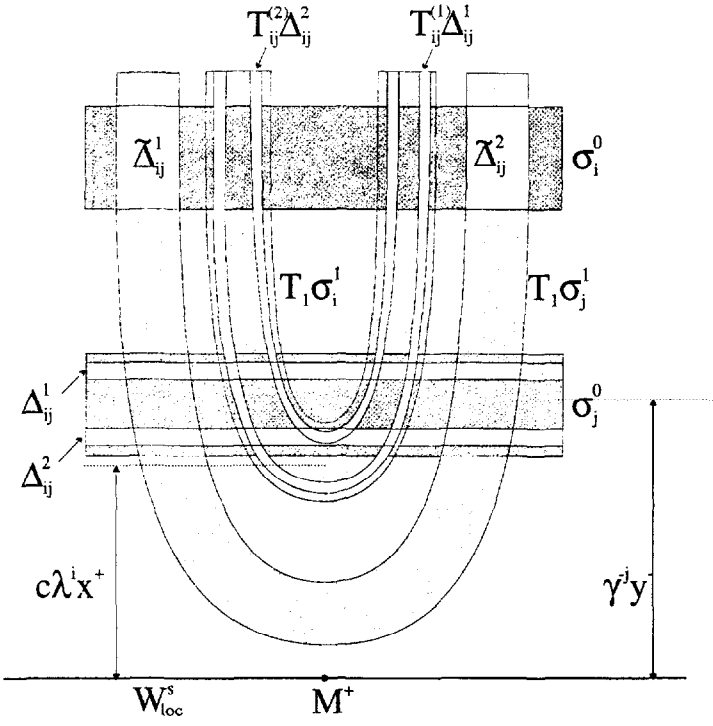


Figure 17. Details of the geometric structure for the bihorseshoe composed by the strips and horseshoes σ_i^0, σ_j^0, $T_1\sigma_i^1$, $T_1\sigma_j^1$. The horseshoe $T_1\sigma_j^1$ intersects the strip σ_i^0 on two connected components $\tilde{\Delta}_{ij}^1$ and $\tilde{\Delta}_{ij}^2$. Two "substrips" Δ_{ij}^1 and Δ_{ij}^2 in σ_j^0 are the preimages of these components; i.e., $T_j(\tilde{\Delta}_{ij}^\alpha) = \Delta_{ij}^\alpha$, $\alpha = 1, 2$. The image of the strip Δ_{ij}^α under the map T_{ij} is the narrow horseshoe $T_{ij}(\Delta_{ij}^\alpha)$ belonging to $T_1\sigma_i^1$.

The dynamics of the map $T_{ij} : \sigma_j^0 \to \sigma_j^0$ is determined by that how it acts in restriction onto the substrips Δ_{ij}^1 and Δ_{ij}^2. Particularly, the fixed points of T_{ij} are divided into two groups: the first are the fixed points of the map $T_{ij}^{(1)} = T_{ij}|_{\Delta_{ij}^1}$, and the second are the fixed points of the map $T_{ij}^{(2)} = T_{ij}|_{\Delta_{ij}^2}$. Since the regions Δ_{ij}^1 and Δ_{ij}^2 do not intersect for all θ, the fixed points of each of the maps bifurcate independently.

It can be shown (see [8]) that exactly two bifurcations take place in each group when θ varies; namely, a pair of saddle and stable fixed points of $T_{ij}^{(\alpha)}$ appears at $\theta_{ij}^{\alpha+}$ through the saddle-node bifurcation corresponding to the presence of a multiplier equal to "+1," and the stable fixed point loses its stability at $\theta_{ij}^{\alpha-}$ through the period-doubling bifurcation corresponding to the presence of a multiplier equal to "−1." The following asymptotic takes place:

$$\theta_{ij}^{\alpha\pm} = \frac{j}{i} + \frac{\tau}{i} + (-1)^\alpha \frac{1}{i}\frac{1 - by^-/x^+}{y^- \ln\gamma}\sqrt{\frac{y^-}{d}}\gamma^{-i/2} + \cdots, \tag{2.29}$$

where $\alpha = 1$ is assumed to correspond to the upper of the substrips Δ_{ij} and $\alpha = 2$ to the lower one (Figure 17). Note that the asymptotics for the moments corresponding to the "+1" and "−1" bifurcations coincide in the main order. Nevertheless, the intervals

$$\delta_{ij\alpha} = (\theta_{ij}^{\alpha+}, \theta_{ij}^{\alpha-})$$

are, evidently, nonempty and they correspond to the presence of a stable double-round periodic orbit.

2.7.2. Systems on H_3 with infinitely many stable periodic orbits

Since $\sigma = |\lambda\gamma| < 1$, it follows that the Jacobian of the map T_{ij} equal to $(bc)^2(\lambda\gamma)^{i+j}(1+\cdots)$ is less than unity if i and j are sufficiently large. Thus, the saddle-node bifurcations of double-round periodic orbits lead to the appearance of the stable periodic orbits indeed.

The following assertion was established in [8,14].

PROPOSITION 2.2. *Let f_θ be a one-parameter family of systems on H_3. Then, in the interval $\theta > 1$, the values θ^* are dense such that the diffeomorphism f_{θ^*} possesses infinitely many stable double-round periodic orbits.*

This result follows from the fact that the stability regions $\delta_{ij\alpha}$ may intersect for different (i, j). Indeed, as it follows from (2.29), the map T_{ij}^{α} has a stable fixed point if

$$\nu_{ij}^1 < j - \theta i + \tau - \nu_0\gamma^{-i/2} < \nu_{ij}^2, \tag{2.30}$$

where $\nu_{ij}^1 < \nu_{ij}^2$, $\nu_{ij}^{1,2} = o(\gamma^{-i/2})$ and ν_0 does not depend on i and j.

In order for an infinite number of stable double-round periodic orbits to exist for the diffeomorphism f_θ, it is necessary and sufficient that inequality (2.30) would have infinitely many integer solutions (i, j). The standard fact from the number theory is that for any functions $\nu_{ij}^{1,2}$ tending to zero as $i, j \to +\infty$ such inequality do have infinitely many integer solutions for a dense set of values of θ.

Note that inequality (2.30) is satisfied only if the invariants θ and τ admit "exponentially well" nonhomogeneous approximations by rational fractions.

2.7.3. Bifurcations in the case $\mu \neq 0$

Let us now consider bifurcations of double-round periodic orbits for the diffeomorphisms which are close to f and which may now not lie on H_3.

First, consider a one-parameter family f_μ. Recall that the absolute value of the splitting parameter μ is exactly the distance between the bottom of the parabola $T_1(W_{\text{loc}}^u)$ and the manifold W_{loc}^s. The sign of μ corresponds to where the bottom of the parabola lies: above or below W_{loc}^s. If $\mu > 0$, the diffeomorphism f_μ does not have single-round homoclinic orbits close to Γ, and when $\mu < 0$, the diffeomorphism has two such orbits.

When μ increases, the bottom of the parabola $T_1(W_{\text{loc}}^u)$ will move up and when μ decreases, it will move down. Accordingly, the bottoms of all horseshoes will move up and down. It follows from equations (2.14),(2.15) that the bottom of the horseshoe $T_1\sigma_i^1$ lies on a distance of the order

$$\mu + c\lambda^i x^+ \tag{2.31}$$

from the manifold W_{loc}^s. Recall also that the strip σ_j^0 lies on a distance of the order

$$\gamma^{-j} y^- \tag{2.32}$$

from the manifold W_{loc}^s.

Take some i and j such that, for $\mu = 0$, the horseshoe $T_1\sigma_i^1$ does not intersect the strip σ_j^0 (Figure 16a). Evidently, there is infinitely many such pairs (i,j). Since, for $\mu = 0$, the horseshoe $T_1\sigma_i^1$ lies above the strip σ_j^0 (i.e., $c\lambda^i x^+ > \gamma^{-j}y^-$), we have by virtue of (2.31),(2.32), that it lies above this strip for all positive μ. Therefore, for the given i and j, the map T_{ij} does not undergo bifurcations for positive μ. However, when μ is negative, the horseshoe $T_1\sigma_i^1$ may have a nonempty intersection with the strip σ_j^0 (this intersection will be nonempty and regular for sufficiently large negative μ). Thus, it is clear that there exists $\mu = \mu_{ij}^* < 0$ for which the map T_{ij} has a structurally unstable fixed point. Evidently, $\mu_{ij}^* \to 0$ as $i,j \to \infty$.

Take another pair of i and j such that, for $\mu = 0$, the horseshoe $T_1\sigma_i^1$ has a regular intersection with the strip σ_j^0 (Figure 16b); the set of such pairs is also infinite. Note that, for the given i and j, the horseshoe $T_1\sigma_i^1$ has regular intersection with the strip σ_j^0 for all negative μ. Therefore, in this case, the map T_{ij} does not undergo bifurcations for negative μ. On the other hand, if μ is positive, the horseshoe $T_1\sigma_i^1$ may have empty intersection with the strip σ_j^0 (if $\mu + c\lambda^i x^+ > \gamma^{-j}y^-$; see (2.31),(2.32)). It is clear, therefore, that there exists $\mu = \tilde{\mu}_{ij}^* > 0$ for which the map T_{ij} has a structurally unstable fixed point. Note also that $\tilde{\mu}_{ij}^* \to 0$ as $i,j \to \infty$.

We arrive at the following statement [6].

PROPOSITION 2.3. *There exists an infinite number of values of μ accumulating at $\mu = 0$ from both sides which correspond to the presence of the structurally unstable double-round periodic orbits.*

If, similar to the case $\mu = 0$, consider the substrips $\Delta_{ij}^1(\mu)$, $\Delta_{ij}^2(\mu)$ and the corresponding horseshoes $T_{ij}(\mu)\Delta_{ij}^1$ and $T_{ij}(\mu)\Delta_{ij}^2$, then repeating the arguments of [8], one can show that the following asymptotics take place for the bifurcational values of μ:

$$\mu_{ij}^{\alpha\pm} = \gamma^{-j}y^- - c\lambda^i x^+ + (-1)^\alpha \gamma^{-i/2}\lambda^i \frac{cx^+}{y^-}\sqrt{\frac{y^-}{d}}\left(1 - \frac{by^-}{x^+}\right)(1 + \cdots), \qquad \alpha = 1,2. \tag{2.33}$$

Here $\alpha = 1$ corresponds to the bifurcations of the fixed points of the map $T_{ij}(\mu)|_{\Delta_{ij}^1}$, and $\alpha = 2$ corresponds to the bifurcations of the fixed points of the map $T_{ij}(\mu)|_{\Delta_{ij}^1}$. The signs $\pm$ in the left-hand side of formula (2.33) denote the bifurcation moments corresponding to the multiplier equal to "+1" or to "−1", respectively.

Note that these bifurcation moments differ on a small value of order $o(\gamma^{-i/2})$. In spite of the intervals $\tilde{\delta_{ij}} = (\mu_{ij}^{\alpha-}, \mu_{ij}^{\alpha+})$ of existence of a stable double-round periodic orbit are extremely small, they, nevertheless, may intersect each other (which is not the case for the analogous intervals corresponding to single-round orbits; see above), and even an infinite number of these intervals may intersect. We have already seen this in the previous section, when proved that the value $\mu = 0$ belongs to the intersection of infinitely many regions of existence and stability of double-round periodic orbits if θ and τ admits exponentially well nonhomogeneous approximations by rational fractions.

The structure of these intersections cannot be studied in a one-parameter family f_μ because it depends essentially on, for instance, the values of θ and τ. Indeed, as we have shown, the structure of the set of the values of μ corresponding to the bifurcations of double-round periodic orbits of f_μ depends essentially on the reciprocal position of the strips and horseshoes for the diffeomorphism f_0. The latter is mainly determined by the values of θ and τ. If, for instance, $\theta > \theta'$, then there would exist infinitely many pairs (i,j) such that, for the diffeomorphism f_0, the horseshoe $T_1\sigma_i^1$ have regular intersection with the strip σ_j^0, and the horseshoe $T_1'\sigma_i^1$ has no intersection with the strip σ_j^0 for the diffeomorphism f_0' (see Theorem 5). Therefore, for the family f_μ, bifurcations of the double-round periodic orbits corresponding to the given values of i and j would happen at positive μ, and for the family f_μ', they would happen at negative μ. In other words, an arbitrary variation of θ changes the order of "double-round" bifurcations in the family f_μ.

In fact, using the machinery of "infinite degenerations" from [7], one can show that, by an arbitrary small perturbation of the family f_μ in the space of one-parameter families of dynamical systems, a family can be obtained for which values of μ accumulates at $\mu = 0$ corresponding to infinitely many coexisting structurally unstable double-round periodic orbits.

This implies that no finite number of control parameters is sufficient to obtain a stable picture of the bifurcation set corresponding to all double-round periodic orbits. At the same time, we have seen that if we restrict ourself to the study of the bifurcations of one double-round periodic orbit corresponding to an arbitrary code $\{i, j\}$, the one-parameter bifurcation analysis is quite satisfactory: there is a value of μ corresponding to the saddle-node bifurcation and a value of μ corresponding to the period-doubling bifurcation and no other bifurcation values.

2.8. Bifurcations of Triple-Round Periodic Orbits. Cusp-Bifurcations

2.8.1. Bifurcations on H_3

In this section, we consider the bifurcations of triple-round periodic orbits. In particular, we show that, in distinction with the single- and double-round periodic orbits, structurally unstable triple-round periodic orbits can have additional degenerations; namely, the first Lyapunov value may vanish. This means that *cusp-bifurcations* take place here.

This fact was established in [15] at the study of two-parameter families of systems on H_3 for which the Ω-moduli θ and τ are taken as the control parameters.

Let $f_{\theta,\tau}$ be a two-parameter family in H_3. Then, the following result holds.

THEOREM 9. *The values of (θ, τ) for which the system has a structurally unstable triple-round periodic orbit with one multiplier equal to unity and with first Lyapunov value equal to zero are dense in the region $\tilde{L} = \{(\theta, \tau) : \theta > 1\}$ on the parameter plane.*[6]

PROOF. The study of triple-round periodic orbits is reduced to the study of the fixed points of the third-return maps $T_{ijk} \equiv T_1 T_0^k T_1 T_0^j T_1 T_0^i$: $\sigma_i^0 \longrightarrow \sigma_i^0$. We will suppose $i < j < k$ (this condition can be shown to be necessary for the existence of the cusp-bifurcation).

The analysis carried out in [1] shows that the additional degeneration may take place only for the following structure of the intersections of the corresponding horseshoes and strips (Figure 18): the horseshoe $T_1\sigma_i^1$ intersects the strip σ_i^0 regularly and intersects the strip σ_j^0 irregularly, the horseshoe $T_1\sigma_j^1$ intersects the strips σ_i^0, σ_j^0 regularly and intersects the strip σ_k^0 irregularly, the horseshoe $T_1\sigma_k^1$ intersects all the strips regularly.

The study of triple-round periodic orbits is obviously reduced to the study of a system of equations connected the coordinates (x_0, y_0) and (x_1, y_1) of the points of intersection of the orbit with the neighborhoods Π_0 and Π_1, respectively. We do not write down the system here. Note that the system is easily resolved with respect to the coordinates x_0, y_0, and x_1. If $\{ijk\}$ is the code of the periodic orbit under consideration, then the system takes the form 16]

$$\begin{aligned} \gamma^{-j}\eta &= d\xi^2 + \left(cx^+\lambda^i - \gamma^{-j}y^-\right) + bc\lambda^i\zeta + \cdots, \\ \gamma^{-k}\zeta &= d\eta^2 + \left(cx^+\lambda^j - \gamma^{-k}y^-\right) + bc\lambda^j\xi + \cdots, \\ \gamma^{-i}\eta &= d\zeta^2 + \left(cx^+\lambda^k - \gamma^{-i}y^-\right) + bc\lambda^k\eta + \cdots, \end{aligned} \tag{2.34}$$

where we denote the value $y_1 - y^-$ as ξ for the point of intersection of the orbit with the strip σ_i^1, as η for the point of intersection with the strip σ_j^1, and as ζ for the point of intersection with the strip σ_k^1. The degenerate periodic orbits (i.e., having one multiplier equal to unity) correspond to the degenerate solutions of system (2.34).

[6] Note that the second Lyapunov value does not equal to zero here [15], so these points are the cusp-points from which a pair of curves corresponding to saddle-node bifurcations go.

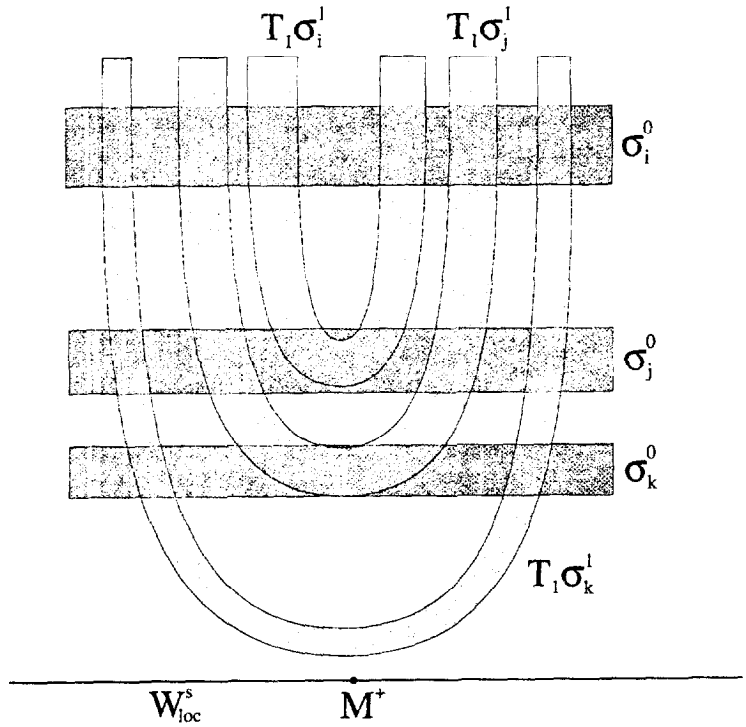


Figure 18. The geometric construction leading to the appearance of doubly-degenerate triple-round periodic orbits (the cusp-bifurcation). The horseshoe $T_1\sigma_i^1$ intersects the strip σ_i^0 regularly and it intersects the strip σ_j^0 irregularly; the horseshoe $T_1\sigma_j^1$ intersects the strips σ_i^0, σ_j^0 regularly and it intersects the strip σ_k^0 irregularly; the horseshoe $T_1\sigma_k^1$ intersects regularly all the strips.

Since $i < k$ and $\lambda\gamma < 1$, the last equation of system (2.34) is resolved with respect to ζ:

$$\zeta = \pm\sqrt{\frac{y^-}{d}}\gamma^{-i/2}(1 + \cdots). \tag{2.35}$$

The substitution of expression (2.35) in the first and second equations of system (2.34) and a shift of coordinates ξ and η on some small constants bring the system to the form

$$\begin{aligned} \gamma^{-j}\eta &= d\xi^2 + \left(cx^+\lambda^i - \gamma^{-j}y^-\right) + \cdots, \\ -bc\lambda^j\xi &= d\eta^2 + \left(cx^+\lambda^j - \gamma^{-k}y^-\right) + \cdots. \end{aligned} \tag{2.36}$$

Thus, the question about the degenerate triple-round periodic orbits is reduced to the question about the degenerate solutions of the system (2.36) corresponding to large i, j, k, and to small ξ and η.

Let us show that the system has a triple solution. Make the following rescaling of the variables: $\xi = \epsilon_1 \cdot u$; $\eta = \epsilon_2 \cdot v$, where

$$\epsilon_1 = -\frac{(bc)^{1/3}}{d} \cdot \lambda^{j/3} \cdot \gamma^{-2j/3}, \qquad \epsilon_2 = \frac{(bc)^{2/3}}{d} \cdot \lambda^{2j/3} \cdot \gamma^{-j/3}.$$

Dividing the first and second equations of (2.36) on $d \cdot \epsilon_1^2$ and $d \cdot \epsilon_2^2$, respectively, we arrive at the following system:

$$\begin{aligned} u^2 &= v + A + \delta_1(u, v), \\ v^2 &= u + B + \delta_2(u, v), \end{aligned} \tag{2.37}$$

where $\delta_{1,2} \to 0$ as $i, j, k \to +\infty$ and the quantities A and B are as follows:

$$\begin{aligned} A &= \frac{d}{(bc)^{2/3}}\lambda^{-2j/3}\gamma^{4j/3}\left[y^-\gamma^{-j} - cx^+\lambda^i + \cdots\right], \\ B &= \frac{d}{(bc)^{4/3}}\lambda^{-4j/3}\gamma^{2j/3}\left[y^-\gamma^{-k} - cx^+\lambda^j + \cdots\right]. \end{aligned} \tag{2.38}$$

Evidently, A and B may take arbitrary finite values if i and j are sufficiently large.

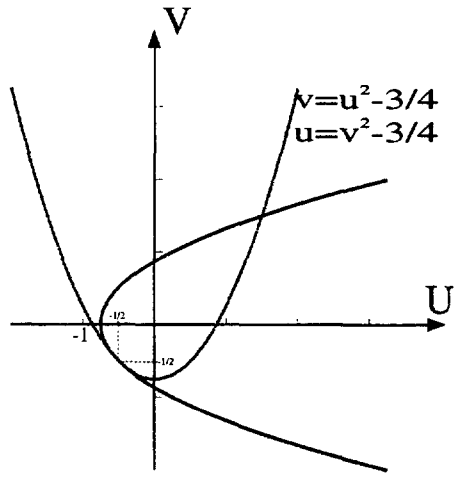


Figure 19. The case of the cubic tangency of two parabolas.

It is easy to see that the triple solution of system (2.37) exists when $A \approx 3/4$, $B \approx 3/4$. The geometric illustration of this fact is represented in Figure 19.

We obtained a necessary and sufficient condition for existence of triple solution of system (2.34). This condition can be rewritten as

$$\begin{aligned} y^{-}\gamma^{-j} - cx^{+}\lambda^{i} + \cdots = 0, \\ y^{-}\gamma^{-k} - cx^{+}\lambda^{j} + \cdots = 0. \end{aligned} \tag{2.39}$$

Taking the logarithm of the both parts of each of the equations of the system obtained we arrive at the equivalent system

$$\begin{aligned} j = \theta i - \tau + \cdots, \\ k = \theta j - \tau + \cdots. \end{aligned} \tag{2.40}$$

This system can be shown to have arbitrarily large integer solutions for a dense set of values of the parameters (θ, τ). So we can conclude that there exists a dense set L^* on the parameter plane such that for any pair $(\theta^*, \tau^*) \in L^*$, there exists a triple solution of system (2.34) for some i, j, k. This means that the dynamical system has an associated structurally-unstable triple-round periodic orbit arising as the result of the coalescence of three periodic orbits. Such orbit has a multiplier equal to unity and the first Lyapunov value is equal to zero. The theorem is proved.

Let us now construct the bifurcational curves, starting at the cusp points, which correspond to saddle-node triple-round periodic orbits. Let $\alpha = A - 3/4$ and $\beta = B - 3/4$. System (2.37) takes the form

$$\begin{aligned} u^2 = v + \frac{3}{4} + \alpha + \cdots, \\ v^2 = u + \frac{3}{4} + \beta + \cdots. \end{aligned} \tag{2.41}$$

On the plane (α, β), the bifurcational curves corresponding to the degenerate solutions of system (2.41) have the following form (see Figure 20):

$$\begin{aligned} \alpha = -\frac{3}{4} + \frac{1}{16t^2} - t + \cdots, \\ \beta = -\frac{3}{4} + t^2 - \frac{1}{4t} + \cdots, \end{aligned} \tag{2.42}$$

where t is some parameter; a triple solution exists when $t = -1/2$.

Since

$$\begin{aligned} \frac{3}{4} + \alpha = \frac{d}{(bc)^{2/3}}\lambda^{-2j/3}\gamma^{4j/3}\ \left[y^{-}\gamma^{-j} - cx^{+}\lambda^{i} + \cdots\right], \\ \frac{3}{4} + \beta = \frac{d}{(bc)^{4/3}}\lambda^{-4j/3}\gamma^{2j/3}\ \left[y^{-}\gamma^{-k} - cx^{+}\lambda^{j} + \cdots\right], \end{aligned} \tag{2.43}$$

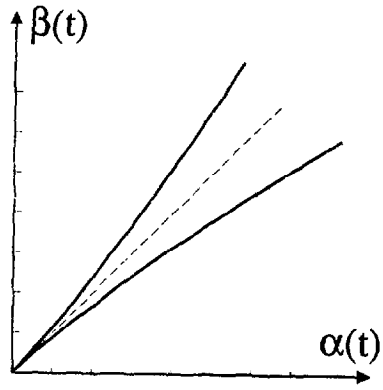


Figure 20. The cusp-point on the plane (α, β).

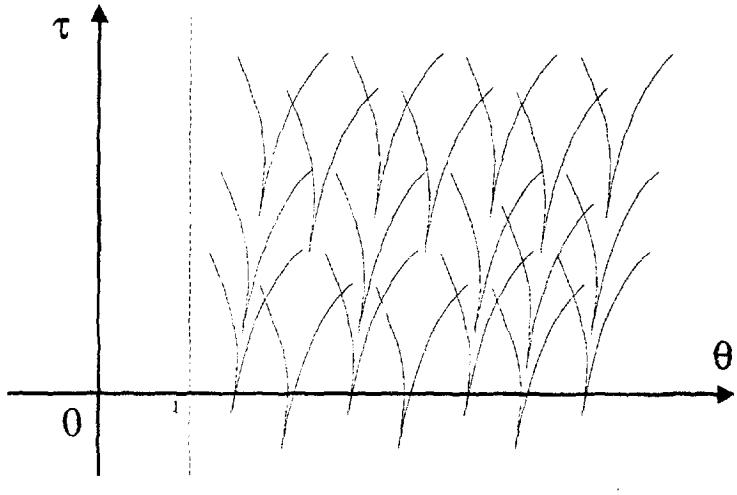


Figure 21. A fragment of the bifurcation diagram on the plane (θ, τ).

(see (2.38)) and since $\lambda = \gamma^{-\theta}$, $\gamma^{\tau} = cx^{+}/y^{-}$, we can write the following formula connecting the values of (α, β) with the values of (θ, τ):

$$\begin{aligned} \frac{3}{4} + \alpha &= \frac{d}{(bc)^{2/3}} \gamma^{2/3(\theta+2)j} \left[y^{-}\gamma^{-j} - y^{-}\gamma^{\tau-\theta i} + \cdots \right], \\ \frac{3}{4} + \beta &= \frac{d}{(bc)^{4/3}} \gamma^{4/3(\theta+1/2)j} \left[y^{-}\gamma^{-k} - y^{-}\gamma^{\tau-\theta j} + \cdots \right]. \end{aligned} \tag{2.44}$$

This formula allows one to map the curves (2.42) onto the (θ, τ)-plane (see Figure 21).

2.9. Cusp-Bifurcations in Two-Parameter Families $f_{\mu,\theta}$

For a two-parameter family $f_{\mu,\theta}$, the condition of existence of a triply-degenerate triple-round periodic orbit is written in the form

$$\begin{aligned} j &= \theta i - \tau + \cdots, \\ y^{-}\gamma^{-k} - cx^{+}\lambda^{j} + \mu + \cdots &= 0, \end{aligned} \tag{2.45}$$

which is analogous to condition (2.40) obtained for $\mu = 0$. One can see that in an arbitrarily small neighborhood of any point $(\theta, \mu = 0)$, there exists a point (θ^*, μ^*) for which system (2.45) has an integer solution. This implies that the following theorem holds.

THEOREM 10. *In an arbitrarily small neighborhood of any point $(\theta, \mu = 0)$, there exists a point (θ^*, μ^*) for which the map f_{θ^*,μ^*} has a doubly-degenerate triple-round periodic orbit.*

Note that μ^* can be of arbitrary sign: $\mu^* < 0$ when $k > \theta j - \tau$, and $\mu^* > 0$ when $k < \theta j - \tau$. The corresponding bifurcation diagram is represented in Figure 22.

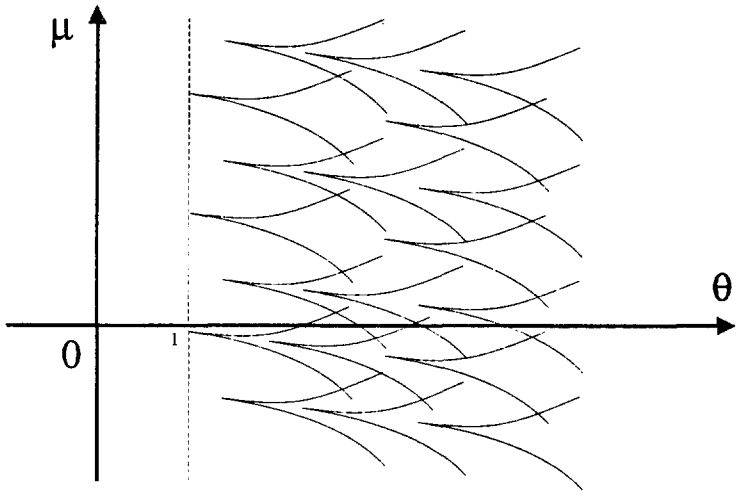


Figure 22. A fragment of the bifurcation diagram on the plane (θ, μ).

3. BIFURCATIONS OF HOMOCLINIC LOOPS TO A SADDLE-FOCUS

In this section, we will examine the dependence of the structure of the bifurcation set of homoclinic loops to an equilibrium point of saddle-focus type on the value of the Ω-modulus ρ for the case of three-dimensional flows. The main results of this section were obtained in [16].

Consider a smooth three-dimensional dynamical system X satisfying the following conditions:

A. X possesses an equilibrium state O of the saddle-focus type; i.e., the characteristic exponents ν_1, ν_2, ν_3 of O are such that $\nu_3 = \gamma > 0$, $\nu_{1,2} = -\lambda \pm i\omega$ $(\lambda > 0, \omega > 0)$; and

B. the saddle index $\rho = \lambda/\gamma$ is less than 1.

The unstable manifold W^u of O is one-dimensional. The point O divides it into two branches called *separatrices*. All orbits of the two-dimensional stable manifold W^s have a shape of spirals tending to O as $t \to +\infty$. We suppose that the following condition is also satisfied:

C. one of the separatrices (we denote it as Γ) comes back to O as $t \to +\infty$, forming a *homoclinic loop* (Figure 1).

Let us consider a sufficiently small neighbourhood U of the loop. U is a solid torus composed by a small neighbourhood U_0 of the point O and by a handle U_1 glued to U_0 as in Figure 4. We are interested in the bifurcations of orbits lying in U. Since systems with homoclinic loops of a saddle-focus form surfaces of codimension one in the space of dynamical systems, the standard way to study bifurcations of such a system is to include it into a one-parameter family X_μ, where μ controls the splitting of the loop. The parameter μ can be defined as the distance between the point of intersection of Γ with some surface of section and the line of intersection of W^s with the same surface of section. In this respect, the system forms the loop Γ when $\mu = 0$.

When μ changes, multiround homoclinic loops can appear; i.e., such loops that come back to O after a number of passages along the handle U_1. In a one-parameter family, bifurcations of such loops were studied in [17,18]. In the present section, we describe bifurcations of homoclinic loops in two-parameter families, and we choose the saddle index ρ as a second control parameter.

This choice is justified by the fact that the structure of the nonwandering set of systems with homoclinic loops of a saddle-focus depends essentially upon the saddle index ρ (see [1,2]). Systems with different values of ρ are not topologically equivalent, so that ρ is a genuine bifurcational parameter. Moreover, we shall show that the bifurcations of multiround loops in a one-parameter family X_μ depend on the value of the saddle index ρ.

We start with the bifurcations of double-round loops. As shown in [17], the region $\mu > 0$—which corresponds to the inward splitting of the loop—possesses a countable set of smooth curves L_j^α

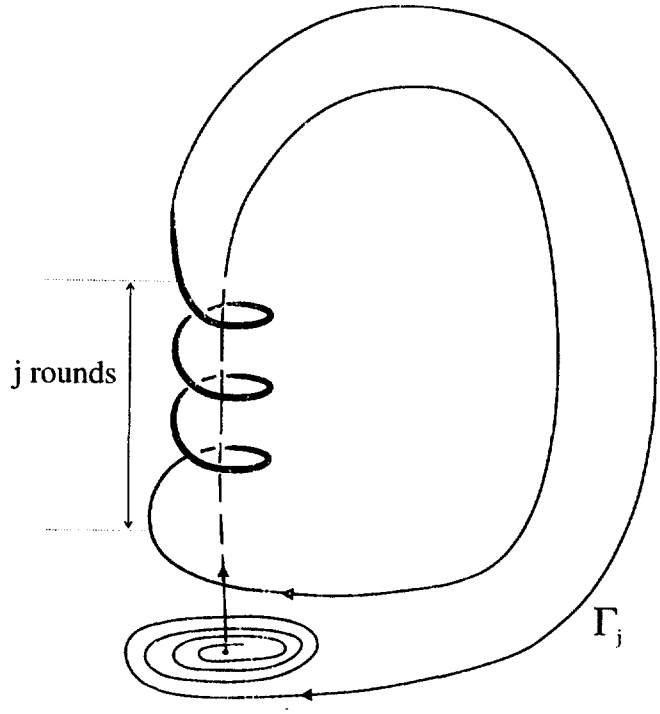


Figure 23. A moment of existence of a double-round homoclinic loop Γ_j to the saddle-focus O (where the index j means that the loop circles j times around O).

($\alpha = 0, 1$) of the form $\mu = f_j^\alpha(\rho) \sim \exp[-\frac{2\pi\rho}{\omega}(j - (1-\alpha)/2)]$ which correspond to the existence of double-round loops Γ_j (where the index j means that the loop circles j times around O, see Figure 23). In the cases where ρ is close to one and to zero (the last case corresponding to a pair of pure imaginary characteristic exponents of O), the behavior of the curves L_j^α was studied in [19,20]. It turns out that L_j^1 and L_j^0 merge at some $\rho = \rho_j^*$ (the greater j, the closer ρ_j^* to 1). On the other hand, L_j^1 and L_j^0 have different terminating points at $\rho = 0$ (Figure 24).

We see that if ρ lies between 0 and 1, the sequence of bifurcations of double-round loops is the same for all values of ρ in $(0, 1)$. However, this property does not extend to the triple-round loops. Thus, it is established in [16] that for sufficiently large j, in the region bounded by L_{j+1}^0 and L_j^1, there exist smooth curves L_{jk}^α, $\alpha = 0, 1$, corresponding to the existence of triple-round loops Γ_{jk}; i.e., loops which start with O, pass along the handle U_1, circle j times around O, pass along U_1 again, circle k times around O, pass along U_1 once more and enters O finally (Figure 25). Each of these bifurcation curves have a vertical tangent at some $\rho = \rho_{jk\alpha}^*$, at the left side of which the curve lies entirely (Figure 26). The following asymptotic behaviors hold:

$$\rho_{jk\alpha}^* = \frac{k}{j} + \left(\tau\left(\rho_{jk\alpha}^*\right) - \frac{\alpha}{2}\right)\frac{1}{j} + \cdots, \qquad \text{for } j > k, \tag{3.1}$$

and, when $j < k$,

$$\rho_{jk\alpha}^* = \frac{j}{k - 1/2} + \left(\tau\left(\rho_{jk\alpha}^*\right) + \frac{\alpha}{2}\right)\frac{1}{k} + \cdots, \qquad \text{for } j < k, \tag{3.2}$$

where $\tau(\rho)$ is a smooth function (see [16]) determined by the system at $\mu = 0$. These implicit equations admit solutions when j and k are large enough while k/j, or, respectively, j/k, is separated from 0 and 1.

Therefore, the following picture takes place for any small segment $0 < \rho_1 < \rho < \rho_2 < 1$: in any strip between L_{j+1}^0 and L_j^1, there is a finite number of curves L_{jk}^α consisting of two components which are either "parallel" to the ρ-axis or are connected together and have a parabola-like shape.

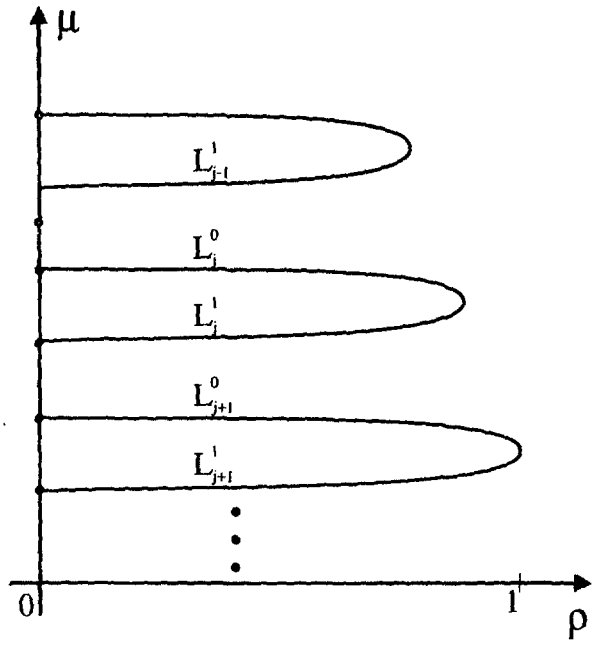


Figure 24. On the parameter plane (ρ, μ), the bifurcational curves L_j^α $(\alpha = 0, 1)$ corresponding to existence of double-round loops Γ_j are represented. The curves L_j^1 and L_j^0 merge at some $\rho = \rho_j^*$ (the greater j, the closer ρ_j^* to 1). On the other hand, L_j^1 and L_j^0 have different terminating points at $\rho = 0$.

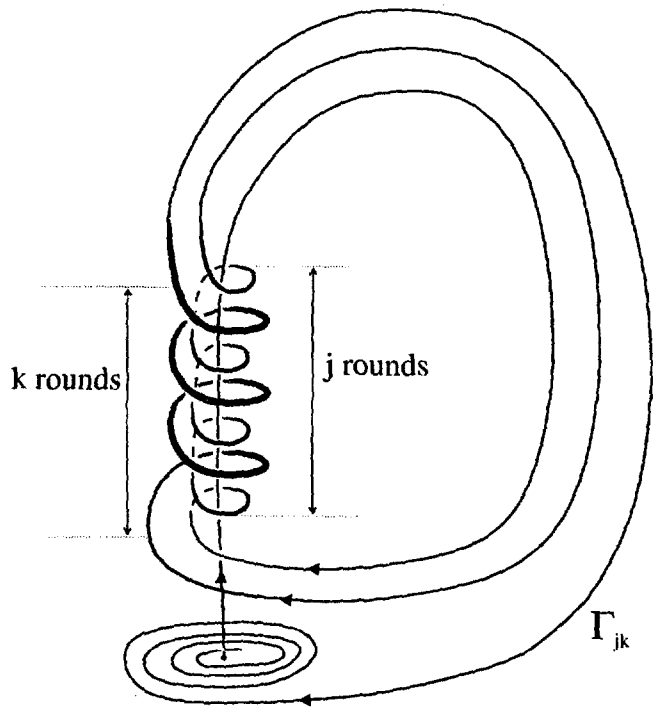


Figure 25. A triple-round loop Γ_{jk}; i.e., the loop which starts with O, passes along the handle U_1, circles j times around O, passes along U_1 again, circles k times around O, passes along U_1 once more and finally enters O.

The number of curves of both types grows linearly with the integer j. The closure of the set $\{\rho_{jk\alpha}^*\}$ taken for j, k large enough coincides with the segment $[0, 1]$. Therefore, we have the following theorem.

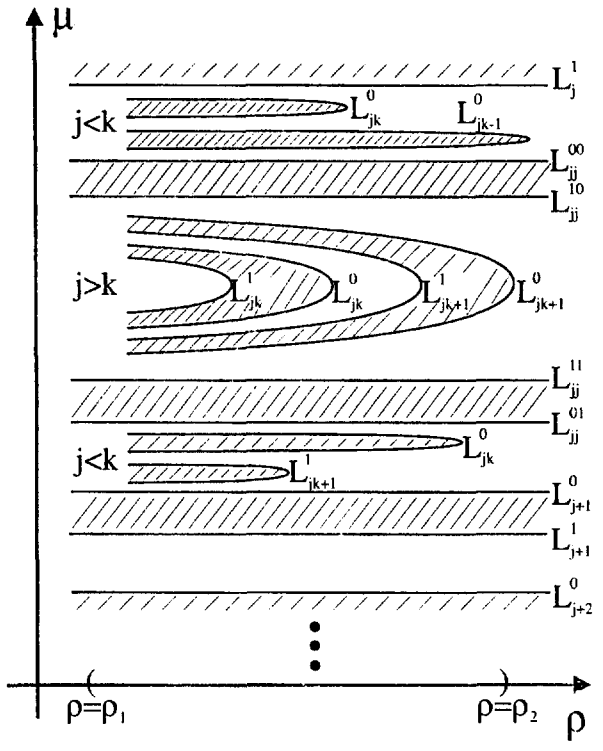


Figure 26. Bifurcational curves L^{α}_{jk}, $\alpha = 0, 1$, corresponding to existence of triple-round loops Γ_{jk} are shown in the region of parameters which is bounded by the curves L^0_{j+1} and L^1_j.

THEOREM 11. *[16] Let X_μ be a one-parameter subfamily of $X_{\mu,\rho}$ with the curve $\{(\mu,\rho) \mid \rho = \varphi(\mu)\}$ being transverse to the line $\mu = 0$. There exists a small variation $\rho = \varphi(\mu) + \delta$ which makes X_μ be tangent to some line of existence of a triple-round homoclinic loop.*

When multiround homoclinic loops are considered, the structure becomes more complicated for the corresponding set of bifurcation curves in the plane (μ, ρ). Indeed, folded lines of nine-round loops accumulate on the lines of triple-round loops in a way similar to the accumulation of the folded (parabola-like) lines of triple-round at the line of single-round loops ($\mu = 0$). It is geometrically evident (see Figure 27) that any curve transverse to $\mu = 0$ can be varied (in a more general way that in Theorem 1) such that to achieve a cubic tangency with some of these lines of nine-round loops.

Actually, the following general statement holds.

THEOREM 12. *[16] Consider a one-parameter subfamily of vector fields $X_{\mu,\rho}$ with $\rho = \varphi(\mu)$, which is transverse to the line $\mu = 0$ in the plane (ρ, μ). Then, a small smooth perturbation of the curve $\rho = \varphi(\mu)$ may have a tangency of arbitrarily high order with some of the lines of existence of homoclinic loops.*

This theorem shows the arbitrarily high structural instability of one-parameter families of vector fields near homoclinic loops of a saddle-focus. We emphasize consequences of this result for nonlinear partial differential equations modelling travelling waves in spatially extended systems, what will be discussed below.

Theorems 11 and 12 can also be applied to the theory of nonlinear partial differential equations modelling travelling waves in spatially extended systems. Let us imagine that $X_{\mu,\rho}$ is a family of ordinary differential equations describing the plane travelling waves of some distributed system; μ is the wave velocity while ρ is an internal parameter of the system. Let the saddle-focus O be at the origin. It is known that homoclinic loops correspond to self-localised waves in such

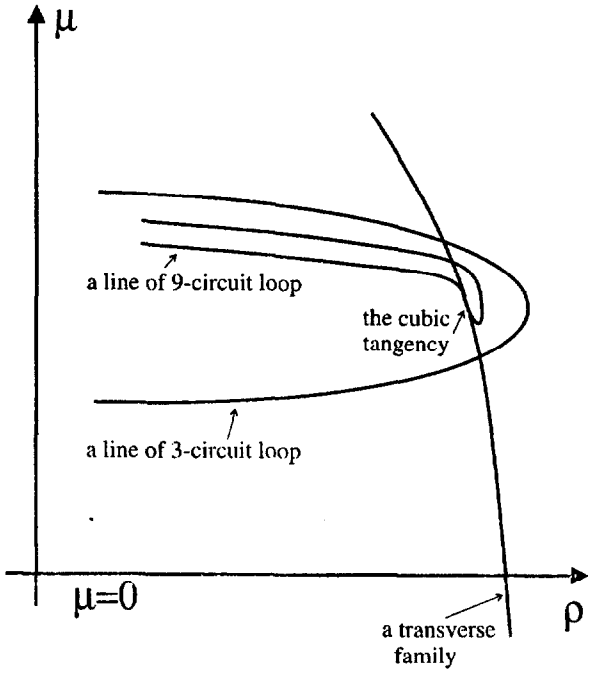


Figure 27. A curve transverse to $\mu = 0$ has a cubic tangency with a bifurcation curve of existence of a nine-round loop.

systems. Suppose that the system has such a wave and that Conditions A–C of this section are fulfilled for some parameter value $\mu = \mu_0$. It follows from Theorem 11 that bifurcations generating "three-pulsed" self-localized travelling waves occur for arbitrary small variations of ρ in this system. In turn, Theorem 12 implies that the complete description of bifurcations of plane self-localised waves is impossible in systems of such kind.

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