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Arnold Diffusion in A Priori Chaotic Symplectic Maps

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Abstract: We assume that a symplectic real-analytic map has an invariant normally hyperbolic cylinder and an associated transverse homoclinic cylinder. We prove that generically in the real-analytic category the boundaries of the invariant cylinder are connected by trajectories of the map.

1. Introduction

A Hamiltonian dynamical system is defined with the help of a Hamilton function $H : M \to \mathbb{R}$ on a symplectic manifold M of dimension $2n$. Let M_c be a connected component of a level set $\{H = c\}$. Since H remains constant along the trajectories of the Hamiltonian system, the set M_c is invariant. Depending on the Hamilton function H and the energy c, the restriction of the dynamics onto M_c may vary from uniformly hyperbolic (e.g., in the case of a geodesic flow on a surface of negative curvature) to completely integrable.

Since Poincarés works, it has been accepted that a typical Hamiltonian system does not have any additional integral of motion independent of H (unless the system possesses some symmetries and the Noether theorem applies). On the other hand a generic Hamiltonian system is nearly integrable in a neighbourhood of a totally elliptic equilibrium (a generic minimum or maximum of H) or totally elliptic periodic orbit. Then the Kolmogorov–Arnold–Moser (KAM) theory implies that the Hamiltonian system is not ergodic (with respect to the Liouville measure) on some energy levels [72]. Indeed, the KAM theory establishes that a nearly integrable system possesses a set of invariant tori of positive measure.

Each of the KAM tori has dimension n. For $n > 2$ a KAM torus does not divide M_c which has dimension $(2n - 1)$, moreover, the complement to the union of all KAM tori is connected and dense in M_c. Thus the KAM theory does not contradict to the existence of a dense orbit in M_c. It is unknown whether such orbits really exist in nearly integrable systems. The question goes back to Fermi [39] who suggested the following notion: a Hamiltonian system is called *quasi-ergodic* if in every M_c any two open sets are connected by a trajectory. This property is equivalent to topological transitivity of the Hamiltonian flow on M_c. This property can also be restated in slightly different terms: (a) in every M_c there is a dense orbit or (b) in every M_c dense orbits form a residual subset.

Fermi conjectured [39] that quasi-ergodicity is a generic property of Hamiltonian systems, but proved a weaker statement only: if a Hamiltonian system with $n > 2$ degrees of freedom has the form

$$H = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon), \tag{1}$$

where H_0 is integrable and (I, φ) are action-angle variables, then generically M_c does not contain an invariant $(2n-2)$-dimensional hyper-surface that is analytic in ε. Obviously, such surface would prevent the quasi-ergodicity. However, non-analytic invariant hyper-surfaces cannot be excluded from consideration as it is not known whether they can exist generically or not. So Fermi's quasi-ergodic hypothesis remains unproved. The recent papers [21,64,65,71,74] make an important step in the understanding of the underlying dynamics by showing that for the generic (in a certain smooth category) near-integrable case with $2\frac{1}{2}$ or more degrees of freedom, there are trajectories that visit an a priori prescribed sequence of balls. The paper [52] provides examples of systems having orbits whose closure contains a Lebesgue positive measure set of KAM-tori.

This problem is closely related to the problem of stability of a totally elliptic fixed point of a symplectic diffeomorphism, or stability of a totally elliptic periodic orbit for a Hamiltonian flow. It was proved in [35,36] that stability can be broken by an arbitrarily small smooth perturbation. It is believed that a totally elliptic periodic orbit is generically unstable but the time scales for this instability to manifest itself are extremely long, see e.g. [15,59].

For $\varepsilon = 0$, the unperturbed system (1) is described by the Hamiltonian $H = H_0(I)$. Then the actions I are constant along trajectories, so the equation $I = I_0$ defines an invariant torus, and the angles φ are quasi-periodic functions of time with the frequency vector $\omega_0(I) = H_0'(I)$. KAM theory implies that the majority of invariant tori survive under perturbation. Tori with rationally dependent frequencies are called *resonant* and are destroyed by a typical perturbation [2]. The frequency of a resonant torus satisfies a condition of the form $\omega_0(I) \cdot \mathbf{k} = 0$ for some $\mathbf{k} \in \mathbb{Z}^n \setminus \{0\}$. The resonant tori form a "resonant web", typically (e.g. if ω_0 is a local diffeomorphism) a dense set of measure zero.

Arnold's example [1] shows that a trajectory of the perturbed system (1) can slowly drift along a resonance. Arnold's paper inspired a large number of studies in the long-time stability of actions, the problem which is known as "Arnold diffusion". It has been attracting significant attention recently and we refer the reader to papers [11,31,33,40–43,54,57,60–62,68–70,76,82,84] for a more detailed discussion.

It should be noted that the motion along the resonant web is very slow: Nekhoroshev theory [14,78] provides a lower bound on the instability times in the analytic case. Let $\{\cdot, \cdot\}$ denote the Poisson brackets. Then $\dot{I} = \{H, I\} = \varepsilon\{H_1, I\}$ is of the order of ε. On the other hand, if the system satisfies assumptions of the KAM theory, $|I(t) - I(0)|$ remains small for all times and the majority of initial conditions, i.e., for the set of initial conditions of asymptotically full measure. If H satisfies assumptions of the Nekhoroshev theory, there are some exponents $a, b > 0$ such that $|I(t) - I(0)| < \varepsilon^a$ for all $|t| < \exp \varepsilon^{-b}$ and for all initial conditions. This estimate establishes an exponentially large lower bound for the times of Arnold Diffusion in analytic systems.

It is important to stress that the upper bound on the speed of Arnold diffusion strongly depends on the smoothness of the system. Indeed, the stability times are exponentially large in ε^{-1} for analytic systems, but only polynomial bounds can be obtained in the C^k category. In particular, papers [21,64,74] study the Arnold diffusion for non-analytic Hamiltonians and therefore the bounds established by the analytical Nekhoroshev theory are likely to be violated, see e.g. [13]. The problem of genericity of Arnold diffusion in analytic category remains fully open. We believe the methods proposed in our paper will help to advance the theory in the analytic case.

The normal form theory suggests that for small positive ε the system (1) has a normally hyperbolic cylinder with a pendulum-like separatrix located in a neighbourhood of a simple resonance. Indeed, Bernard proved the existence of normally-hyperbolic cylinders in a priori stable Hamiltonian systems [6], the size of such cylinder being bounded away from zero for arbitrarily small size of the perturbation.

A model for this situation is often obtained by assuming that the integrable part of the Hamiltonian already possesses a normally-hyperbolic cylinder and an associated homoclinic loop (e.g. by considering $H_0 = P(p, q) + h_0(I)$ where P is a Hamiltonian of a pendulum). A system of this type is called a priori *unstable*. The drift of orbits along the cylinder has been actively studied in the last decade[3–5,7–9,12,18–20,22,27,28,31–34,70,87,88], including the problem of genericity of this phenomenon and instability times. It should be noted that the Arnold diffusion can be much faster in this case.

In these studies, a drifting trajectory typically stays most of the time near the normally-hyperbolic cylinder, occasionally making a trip near a homoclinic loop. The process can be described using the notion of a scattering map introduced by Delshams et al. [32]. Earlier Moeckel [75] suggested that Arnold diffusion can be modelled by random application of two area-preserving maps on a cylinder (this approach was recently continued in [17, 46–49,53,66]). In this way the deterministic Hamiltonian dynamics is modelled by an iterated function system, and the obstacles to a drift along the cylinder appear in the form of essential curves which are invariant with respect to both maps simultaneously [67,75,77].

This problem is closely related to the Mather problem on the existence of trajectories with unbounded energy in a periodically forced geodesic flow [10,29]. The criteria for the existence of trajectories of the energy that grows up to infinity are known for sufficiently large initial energies [10,24,29,30,44,45,80,81]. The results of the present paper can be used to establish the generic existence of orbits of unbounded energy for all possible values of initial energy.

In our paper we depart from the near-integrable setting and study the dynamics of an exact symplectic map in a homoclinic channel, a neighbourhood of a normally-hyperbolic two-dimensional cylinder A along with a sequence of homoclinic cylinders B at a transverse intersection of the stable and unstable manifolds of A. We conduct a rigorous reduction of the problem to the study of an iterated function system and show that the existence of a drifting trajectory (i.e. the instability of the Arnold diffusion type) is guaranteed when the exact symplectic maps of the cylinder A that constitute the iterated function system do not have a common invariant curve. The reduction scheme is in the same spirit as in [50,77] while the setting and proofs are different. The completely novel result is that the existence of drifting orbits is a generic phenomenon, i.e. it holds for an open and dense subset of a neighbourhood, in the space of analytic symplectic maps, of the given map with a homoclinic channel, provided the restriction of the map on the cylinder A has a twist property. All the known similar genericity results for the Arnold diffusion have been proven so far in the smooth category and use the non-analiticity of the perturbations in an essential way.

In one respect, the situation we consider is more general than in the near-integrable setting, as we do not assume the existence of a large set of KAM curves on the invariant cylinder A. On the other side, as one can extract from the example of [25], our assumption of the strong transversality of the homoclinic intersections, which we need in order to define the scattering maps that form the iteration function system, seems to fail for a generic analytic near-integrable system in a neighbourhood of a resonance in the a priori stable case. Therefore, our results do not admit an immediate translation to the a priori stable case. Rather, the problem we consider here is related to the a priori *chaotic* case, e.g. we assume certain transversality of invariant manifolds associated with the normally hyperbolic cylinder.

The technical assumptions of our main theorem can be found in Sect. 2. As an example, we can consider a 4-dimensional symplectic map that is a direct product of a twist map and a standard map. Namely, $\Phi_0 : (\varphi, I, x, y) \mapsto (\bar{\varphi}, \bar{I}, \bar{x}, \bar{y})$ where

$$\begin{aligned} \bar{\varphi} &= \varphi + \omega(I), & \bar{x} &= x + \bar{y}, \\ \bar{I} &= I, & \bar{y} &= y + k \sin x, \end{aligned} \tag{2}$$

where $k > 0$ is a positive parameter and ω is an analytic function. We assume φ and x to be angular variables, so the map is a symplectic diffeomorphism of $(\mathbb{T} \times \mathbb{R})^2$. The map Φ_0 has a normally hyperbolic invariant cylinder A given by $x = y = 0$. The cylinder A is filled with invariant curves as the map Φ_0 preserves the value of the I variable. The (x, y) component of Φ_0 coincides with the standard map, which has transversal homoclinic points for all $k > 0$. Thus Φ_0 verifies the assumptions of the main theorem. Then a generic analytic perturbation of Φ_0 produces orbits which connects neighbourhoods of any two essential curves in A.

A more interesting example is obtained when the integrable twist map is replaced by another standard map, so the new unperturbed map is given by $\Phi_0 : (\varphi, I, x, y) \mapsto (\bar{\varphi}, \bar{I}, \bar{x}, \bar{y})$ where

$$\begin{aligned} \bar{\varphi} &= \varphi + \bar{I}, & \bar{x} &= x + \bar{y}, \\ \bar{I} &= I + k_1 \sin \varphi, & \bar{y} &= y + k_2 \sin x. \end{aligned} \tag{3}$$

The cylinder $A = \{x = y = 0\}$ is still invariant but it is no longer filled with invariant curves. Instead the cylinder contains a Cantor set of invariant curves provided k_1 is not too large. These tori prevent trajectories of Φ_0 from traveling in the direction of the I axis.

The theory presented in this paper allows us to treat both cases equally and implies that an arbitrarily small generic analytic perturbation creates trajectories which travel between regions $I < I_a$ and $I > I_b$ for any $I_a < I_b$ [provided $\omega'(I)$ is separated from 0 for (2), and $k_2 > C(4|k_1| + k_1^2)$ for (3)]. Indeed, in order to apply Theorem 1 to these examples, we note first, that the invariant cylinder A is normally hyperbolic. This cylinder has a stable and unstable separatrices $W^u(A)$ and $W^s(A)$ which coincide with the product of A and the stable (reps., unstable) separatrix of the standard map $W^{u,s}_{sm}$, so we can write (slightly abusing notation) $W^s(A) = A \times W^s_{sm}$ and $W^u(A) = A \times W^u_{sm}$. This product also describes the structure of the foliation of $W^{u,s}(A)$ into strong stable and strong unstable manifolds of points in A. For a point $v \in A$, we let $E^{uu}(v) = \{v\} \times W^u_{sm}$ and $E^{ss}(v) = \{v\} \times W^s_{sm}$. The assumption $k_2 > C(4|k_1| + k_1^2)$ for (3) ensures that these strong stable and strong unstable foliations remain C^1-smooth after the perturbation.

It can be proved that the standard map has infinitely many transversal homoclinic orbits for any $k > 0$. Let $\boldsymbol{p}_h = (x_h, y_h)$ be one of these orbits. The cylinder $B = A \times \{\boldsymbol{p}_h\} \subset W^u(A) \cap W^s(A)$ is homoclinic to A. Since the strong stable and strong unstable foliations of a point $v \in A$ coincide with the product of the base point and the separatrices of the standard map, we see that $(v, p_h) \in E^{ss}(v) \cap E^{uu}(v)$, and the cylinder B satisfies the strong tansversality assumption described in the next section giving rise to a *simple* homoclinic intersection (defined in the next section). Then Theorem 2 implies that generic perturbation of Φ_0 has orbits traveling in the direction of the cylinder A.

Similar maps were considered in Easton et al. [37] (motivated by the "stochastic pump model" of Tennyson et al. [89]). In [37] the existence of drift orbits was shown for all non-integrable Lagrangian perturbations provided k_2 is large enough (i.e. in the "anti-integrable" limit). Our methods allow us to obtain the drifting orbits without the large k_2 assumption, i.e., without a detailed knowledge of the dynamics of the system.

2. Set-up, Assumptions, and Results

Consider a real-analytic diffeomorphism $\Phi : \Sigma \to \mathbb{R}^{2d}$, $d \geq 2$, defined on an open set $\Sigma \subseteq \mathbb{R}^{2d}$. We assume that Φ preserves the standard symplectic form Ω, and that Φ is exact (e.g. the latter is always true if Σ is simply-connected). Let Φ have an invariant smooth two-dimensional cylinder A diffeomorphic to $\mathbb{S}^1 \times [0, 1]$ and $\psi : \mathbb{S}^1 \times [0, 1] \to \Sigma$ be the corresponding embedding. Then the boundary of A consists of two invariant circles: $\partial A = \psi\left(\mathbb{S}^1 \times \{0\}\right) \cup \psi\left(\mathbb{S}^1 \times \{1\}\right)$. Let $\mathrm{int}(A) = A \backslash \partial A$ and $F_0 = \Phi|_A$.

We assume that the cylinder A is normally-hyperbolic. More precisely, we assume that at each point $v \in A$ the tangent space is decomposed into a direct sum of three non-zero subspaces: $T_v\mathbb{R}^{2d} = \mathbb{R}^{2d} = N_v^c \oplus N_v^u \oplus N_v^s$, where N_v^c is the two-dimensional plane tangent to A at the point v. The subspaces $N^{s,u}$ depend continuously on v and are invariant with respect to the derivative Φ' of the map, i.e. $\Phi' N_v^s = N_{F_0(v)}^s$ and $\Phi' N_v^u = N_{F_0(v)}^u$. We note that $\Phi' N_v^c = N_{F_0(v)}^c$ as A is invariant with respect to Φ. We assume that for some choice of norms in $N^{s,u,c}$ there exist $\alpha > 1$ and $\lambda \in (0, 1)$ such that at every point $v \in A$

$$\|F_0'(v)\| < \alpha, \quad \|(F_0'(v))^{-1}\| < \alpha, \tag{4}$$

$$\|\Phi'(v)|_{N_v^s}\| < \lambda, \quad \|(\Phi'(v)|_{N_v^u})^{-1}\| < \lambda, \tag{5}$$

where

$$\alpha^2\lambda < 1. \tag{6}$$

Note that these assumptions are more restrictive in comparison with the standard definition of a normally hyperbolic manifold. In particular, the large spectral gap condition (6) implies the C^1-regularity of the strong stable and strong unstable foliations while in the general case these foliations are Hölder continuous only (see e.g. [83]).

We also note that in (4) and (5) the same pair of exponents α and λ bound both Φ' and $(\Phi')^{-1}$, so we say that A is *symmetrically* normally-hyperbolic. The symmetric form of the spectral gap assumption implies that the restriction of the symplectic form on A is non-degenerate (see Proposition 4). Thus A is a symplectic submanifold of $\mathbb{R}^{2d}$ and the map $F_0 = \Phi|_A$ inherits the (exact) symplecticity of Φ.

We have no doubts that our results can be extended to cover the case when λ and α of inequalities (4) and (5) depend on the point $v \in A$. However, for the sake of simplicity, we conduct the proofs for the case of constant λ and α only.

The points in a small neighbourhood of the normally-hyperbolic cylinder A, whose forward iterations do not leave the neighbourhood and tend to A exponentially with the rate at least λ, form a smooth (at least C^2 in our case) invariant manifold, the local stable manifold $W_{loc}^s \supset A$, which is tangent to $N^s \oplus N^c$ at the points of A (see e.g. [56]). The points whose backward iterations tend to A exponentially with the rate at least λ (and without leaving the neighbourhood) form a C^2-smooth invariant manifold $W_{loc}^u \supset A$ (the local unstable manifold), which is tangent to $N^u \oplus N^c$ at the points of A. The invariant cylinder A is the intersection of W^u_{loc} and W^s_{loc}. The global stable and unstable manifolds of A are defined by iterating the local invariant manifolds: $W^u(A) := \bigcup_{m\geq 0} \Phi^m W^u_{loc}$ and $W^s(A) := \bigcup_{m\geq 0} \Phi^{-m} W^s_{loc}$.

In each of the manifolds there exists a uniquely defined C^1-smooth invariant foliation transverse to A, the strong-stable invariant foliation E^{ss} in $W^s(A)$ and the strong-unstable invariant foliation E^{uu} in $W^u(A)$, such that for every point $v \in A$ there is a unique leaf of E^{ss}_v and a unique leaf of E^{uu}_v which pass through this point and are tangent to N^s_v and, respectively, N^u_v (see [86]). The C^1-regularity of a foliation means that the leaves of the foliation are smooth and, importantly, the field of tangents to the leaves is also smooth, which implies that for any two smooth cross-sections transverse to the foliation the correspondence defined by the leaves of the foliation between the points in the cross-sections is a local diffeomorphism.

Let us discuss the question of the persistence of A at small perturbations. It is a standard fact from the theory of normal hyperbolicity [56] that any strictly-invariant normally-hyperbolic compact smooth manifold with a boundary can be extended to a locally-invariant normally-hyperbolic manifold without a boundary. In our case this means that the smooth embedding ψ that defines the invariant cylinder $A = \psi(\mathbb{S}^1 \times [0, 1])$ can be extended onto $\mathbb{S}^1 \times I$ where I is an open interval containing $[0, 1]$, and the image $\tilde{A} = \psi(\mathbb{S}^1 \times I) \supset A$ is normally-hyperbolic and locally-invariant with respect to the map Φ. Here, by the local invariance we mean that there exists a neighbourhood Z of $\tilde{A}$ such that the iterations of each point of $\tilde{A}$ stay in $\tilde{A}$ until they leave Z. An important property of the locally-invariant normally-hyperbolic manifold without a boundary is that it persists at C^2-small perturbations, i.e. for all maps C^2-close to Φ there exists a locally-invariant normally-hyperbolic cylinder $\tilde{A} \subset Z$. It is not defined uniquely, but it can be chosen in such a way that it will depend on the map continuously as a C^2-manifold.[1] The continuous dependence on the map implies that the cylinder $\tilde{A}$ remains symplectic and symmetrically normally-hyperbolic for all maps C^2-close to Φ.

Note that the normal hyperbolicity implies that $\tilde{A}$ contains all the orbits that never leave Z. In particular, any invariant curve that lies in Z must lie in $\tilde{A}$. We call a smooth invariant essential[2] simple curve $\gamma \subset \tilde{A}$ a KAM-curve if the map Φ restricted to γ is smoothly conjugate to the rigid rotation to a Diophantine angle and the map $F_0 = \Phi|_{\tilde{A}}$ near γ satisfies the twist condition. As the Lyapunov exponent at every point of γ is zero, the gap with the contraction/expansion in the directions transverse to $\tilde{A}$ is infinitely large. Therefore, the cylinder $\tilde{A}$ is of class C^r (for any given finite r) in a sufficiently small neighborhood of γ (see [38,56]). This holds true for every map C^r-close to Φ, i.e. the map F_0 stays C^r-smooth and the twist condition also holds. Now, by applying KAM-theory to the map F_0, we conclude that the invariant curve γ persists for every symplectic map which is at least C^4-close to Φ. Namely, every such map has a uniquely defined, continuously depending on the map, invariant KAM-curve with the same rotation number.

We further assume that *the boundary of A is a pair of KAM-curves*. These curves persist for all C^4-small symplectic perturbations hence they lie in $\tilde{A}$ and bound a compact invariant sub-cylinder $A \subset \tilde{A}$. Every orbit in A stays in Z, so the same cylinder A is a sub-cylinder of $\tilde{A}$ for every choice of the cylinder $\tilde{A}$. This means that even though

[1] Throughout this paper we assume the large spectral gap assumption (6) in the notion of normal hyperbolicity. This guarantees the C^2-smoothness of the manifold, and the C^1-smoothness of the corresponding strong-stable and strong-unstable invariant foliations for every map C^2-close to Φ.

[2] I.e. non-contractible to a point.

the cylinder $\tilde{A}$ is not uniquely defined, the cylinder A is defined uniquely for all symplectic maps C^4-close to Φ, and it depends continuously on the map. The stable and unstable manifolds and the strong-stable and strong-unstable foliations of A also depend continuously, in the C^1-topology, on the map.

We now assume that *the symmetrically normally-hyperbolic cylinder A has a homoclinic*, i.e., the intersection of $W^u(A)$ and $W^s(A)$ has a point x outside A. If $W^u(A)$ and $W^s(A)$ are transverse at x, the implicit function theorem implies that x has an open neighbourhood U_x in $W^u(A) \cap W^s(A)$, which is diffeomorphic to a two-dimensional disk.

For any $x \in W^u(A) \cap W^s(A)$ there is a unique leaf of E^{uu}_x and a unique leaf of E^{ss}_x which pass through this point. We call the homoclinic intersection at x *strongly transverse* if

$$\mathcal{T}_x E^{ss}_x \oplus \mathcal{T}_x E^{uu}_x \oplus \mathcal{T}_x (W^u(A) \cap W^s(A)) = \mathbb{R}^{2d}. \tag{7}$$

This property is equivalent to the condition that the leaf E^{uu}_x is transverse to $W^s(A)$ and the leaf E^{ss}_x is transverse to $W^u(A)$ at the point x.

The holonomy maps $\pi^s : U_x \to A$ and $\pi^u : U_x \to A$ are projections along the leaves of the foliations E^{ss} and E^{uu}, respectively. Since the foliations are smooth, the strong transversality implies that the foliation E^{uu} is transverse to the disc U_x in $W^u(A)$ and the foliation E^{ss} is transverse to U_x in $W^s(A)$ provided U_x is sufficiently small. Then $\pi^u : U_x \to A$ and $\pi^s : U_x \to A$ are local diffeomorphisms.

In this case, following [32], one can define the *scattering map* on $\pi^u(U_x)$:

$$F_x = \pi^s \circ (\pi^u)^{-1}.$$

It is a local diffeomorphism which does not always extends to the whole cylinder A. However, in this paper we consider the case where the scattering map can be globally defined on a large portion of A.

Let $\bar{A} \subset \mathrm{int}(A)$ be a compact invariant sub-cylinder in A, i.e. it is a closed region in $\mathrm{int}(A)$ bounded by two non-intersecting invariant essential simple curves γ^+ and γ^-. Let the set of points homoclinic to A contain a smooth two-dimensional manifold $B \subset W^u(\mathrm{int}(A)) \cap W^s(\mathrm{int}(A)) \setminus A$. We call B a *homoclinic cylinder, simple relative to the cylinders $\bar{A}$ and A*, if the following assumptions hold:

[S1] The *strong transversality* condition (7) holds for all $x \in B$.

[S2] For every point $x \in \bar{A}$, the corresponding leaf of the foliation E^{uu} intersects the homoclinic cylinder B at exactly one point each, and no two points in B belong to the same leaf of the foliation E^{ss}. In other words, the scattering map $F_B = \pi^s_B \circ (\pi^u_B)^{-1} : \bar{A} \to \mathrm{int}(A)$ is well-defined.

[S3] The image of $\bar{A}$ by the scattering map F_B contains an essential curve.

Under these conditions the scattering map is a diffeomorphism $\bar{A} \to F_B(\bar{A}) \subset \mathrm{int}(A)$. Indeed, assumption [S1] implies that the projections $\pi^{s,u}_B : B \to \mathrm{int}(A)$ are local diffeomorphisms and assumption [S2] implies that the maps $\pi^u_B : (\pi^u_B)^{-1}(\bar{A}) \to \bar{A}$ and $\pi^s_B : B \to \pi^s_B(B)$ are bijective. Condition [S3] means that the scattering map is homotopic to identity on $\bar{A}$.

We conclude that $F_B(\bar{A}) \subset A$ is a sub-cylinder bounded by two essential simple curves $F_B(\gamma^+)$ and $F_B(\gamma^-)$. Obviously, $F_B(\gamma^+) \cap F_B(\gamma^-) = \emptyset$. Proposition 7 implies that F_B is an exact symplectic map. In particular, the cylinder $F_B(\bar{A})$ has the same area as $\bar{A}$, and $F_B(\bar{A}) \cap \bar{A} \neq \emptyset$. Note also that the fulfillment of condition [S2] depends on both invariant cylinders, $\bar{A}$ and A, as the cylinder A must be large enough to incorporate $F_B(\bar{A})$.

If γ_+ and γ_- are KAM-curves, then the cylinder $\bar{A}$ bounded by these curves persists for all C^4-small symplectic perturbations. The transversality condition [S1] implies that the C^1-smooth homoclinic cylinder B also persists and remains simple relative to $\bar{A}$ and A. Let $\mathcal{V}_N$ be a set of real-analytic exact symplectic diffeomorphisms $\Phi : \Sigma \to \mathbb{R}^{2d}$ such that:

- each map $\Phi \in \mathcal{V}_N$ has two invariant, bounded by KAM-curves, symmetrically normally-hyperbolic, two-dimensional closed cylinders A and $\bar{A}$ such that $\bar{A} \subset int(A)$,
- each map $\Phi \in \mathcal{V}_N$ has N different[3] homoclinic cylinders $B_1, \ldots, B_N$ simple relative to $\bar{A}$ and A,
- the cylinders A, $\bar{A}$, $B_1, \ldots, B_N$ depend continuously (as C^2-smooth manifolds) on the map Φ,
- for each $\Phi \in \mathcal{V}_N$ the map $F_0 = \Phi|_A$ has a twist property in some symplectic coordinates (y, φ).[4]

We define the topology in the space of real-analytic exact symplectic diffeomorphisms as follows. Take any compact $K \subset \mathbb{R}^{2d}$ and let an analyticity domain Q be a compact complex neighbourhood of K. We consider exact symplectomorphisms $K \to \mathbb{R}^{2d}$ which admit a holomorphic extension onto some open neighbourhood of Q. Two such maps are considered to be close if they are uniformly close on Q. For any given r, two holomorphic maps which are sufficiently close on Q are C^r-close on K. As we explained, the C^4-closeness is enough for the persistence of the cylinders A, $\bar{A}$, $B_1, \ldots, B_N$ [if all of their orbits by Φ lie in $int(K)$], so the set $\mathcal{V}_N$ is open.

Theorem 1. (Main theorem) *Let $N \geq 8$. Then, there is an open and dense subset $\tilde{\mathcal{V}}$ of $\mathcal{V}_N$, such that for each map $\Phi \in \tilde{\mathcal{V}}$ for every two open neighbourhoods U^- of γ^- and U^+ of γ^+ the image of U^- by some forward iteration of the map Φ intersects U^+.*

Remark 1. It is obvious that given any two open sets U^+ and U^- the set of maps whose orbits connect U^- and U^+ is open. The theorem makes a stronger claim that the intersection of all these sets (over all possible choices of the neighbourhoods U^- and U^+ of the given curves γ^- and γ^+) is open and dense in $\mathcal{V}_N$. The theorem implies that for any map $\Phi \in \mathcal{V}_N$ there is an open set of arbitrarily small perturbations of Φ within $\mathcal{V}_N$ such that each of these perturbations creates, for each pair of neighbourhoods U^- and U^+ of the curves $\gamma^\pm$, an orbit that connects U^- and U^+.

Note that the existence of at least 8 different homoclinic cylinders required by Theorem 1 is not a restrictive condition. Namely, under an additional mild assumption the existence of one homoclinic cylinder implies the existence of infinitely many different homoclinic cylinders (see Sect. 3.3). Using this, we can infer the following result from our main theorem.

Consider the set $\mathcal{V}$ of real-analytic exact symplectic diffeomorphisms $\Phi : \Sigma \to \mathbb{R}^{2d}$ such that:

- each map $\Phi \in \mathcal{V}$ has an invariant, bounded by KAM-curves, symmetrically normally-hyperbolic, two-dimensional closed cylinder A,
- in A there exist two invariant sub-cylinders $\bar{A}$ and $\hat{A}$ such that $\bar{A} \subset int(\hat{A}) \subset int(A)$, each of them is bounded by KAM-curves,

[3] I.e. none intersects any image of another by the iterations of the map Φ.

[4] In these coordinates, Birkhoff theorem [55] implies that the boundary curves $\gamma^\pm$ of the invariant sub-cylinder $\bar{A}$ are graphs of Lipschitz functions, $y = y^\pm(\varphi)$.

– Φ has a homoclinic cylinder B simple relative to $\hat{A}$ and A,
– the cylinder B is simple relative $\bar{A}$ and $\hat{A}$; i.e. $F_B(\bar{A}) \subset \mathrm{int}\hat{A}$,
– the map $F_0 = \Phi|_A$ has a twist property.

As all the invariant cylinders involved are bounded by KAM-curves, they persist at C^4-small symplectic perturbations. Thus the set $\mathcal{V}$ is an open subset of the space of real-analytic symplectomorphisms. Let γ^- and γ^+ be the boundary curves of $\bar{A}$.

Theorem 2. *In $\mathcal{V}$ there is an open and dense subset $\tilde{\mathcal{V}}$ such that for each map $\tilde{\Phi} \in \tilde{\mathcal{V}}$ and for every two open neighbourhoods U^- of γ^- and U^+ of γ^+ the image of U^- by some forward iteration of $\tilde{\Phi}$ intersects U^+.*

Remark 2. Statements similar to Theorems 1 and 2 are known for non-analytic (smooth) case, see e.g. [19,20,77]. The main difference between the analytic and smooth case is that the class of perturbations small in the real-analytic sense is narrower than the class of perturbations that are small in the C^∞-sense. In particular, for a typical real-analytic map the normally-hyperbolic invariant cylinder A is not analytic (it has only finite smoothness), so no real-analytic perturbations can vanish on A. Consequently, methods of [19,20,77] are not applicable in the analytic category (in the crucial part that concerns removing the barriers to diffusion by a small perturbation). On the other hand, the proofs of the present paper hold true for the case of C^k maps as well.

Remark 3. The symplectic diffeomorphism Φ can be a Poincare map of a certain cross-section Σ for a Hamiltonian flow inside a level of constant energy. We do not need to assume that the Poincare map Φ is defined outside a small neighbourhood of the invariant cylinder A in this case: the global stable and unstable manifolds of A, as well as the global strong-stable and strong-unstable foliations on these manifolds are defined by continuation of the corresponding local objects by the orbits of the Hamiltonian system. As above, one defines scattering maps by the orbits homoclinic to A. One can easily adjust the proof of the two main theorems in order to show that if the Poincare map Φ and the scattering map (maps) for some Hamiltonian system satisfy the assumptions of theorem 1 or 2, then a generic small perturbation of the Hamiltonian function H in the space of real-analytic Hamiltonians leads to creation of orbits that connect U^- to U^+.

The strategy of the proof of our two main theorems is as follows. We show in Proposition 2 that the existence of one homoclinic cylinder B which is simple relative to the invariant cylinders $\hat{A}$ and A where $\hat{A}$ is such that $\bar{A} \subseteq \hat{A}$ and $F_B(\bar{A}) \subseteq \hat{A}$ implies the existence of infinitely many different secondary homoclinic cylinders which are simple relative to $\bar{A}$ and A. Thus, Theorem 2 is immediately reduced to Theorem 1, and we will further consider $N \geq 8$ homoclinic cylinders $B_1, \dots, B_N$, all of which are simple relative to the same pair of compact invariant cylinders $\bar{A}$ and A, and all are different in the sense that $\Phi^m(B_i) \cap B_j = \emptyset$ for all m and all $i, j = 1, \dots, N$ such that $i \neq j$. Let $F_n : \bar{A} \to \mathrm{int}(A)$ denote the scattering map defined by the homoclinic cylinder B_n. By condition [S1], F_n is a local diffeomorphism. By condition [S2] F_n is a bijection, hence F_n is a diffeomorphism of $\bar{A}$ onto the set $F_n(\bar{A})$. Obviously, condition [S1] implies that the scattering maps are defined in an open neighbourhood A' of $\bar{A}$ in A.

Take any map $\Phi \in \mathcal{V}$. Let $(v_s)_{s=0}^m \subset A$ be a part of an orbit of the iterated function system $\{F_0, \dots, F_N\}$, i.e. for each $s = 0, \dots, m-1$ there exists $n_s = 0, \dots, N$ such that $v_{s+1} = F_{n_s}(v_s)$. In order to ensure that $F_{n_s}(v_s)$ is well-defined we assume that $v_s \in A'$ for $n_s \neq 0$. In Sect. 4 we show that for any such orbit and any $\varepsilon > 0$, there is a point x_0 and a positive integer ℓ such that

$$\operatorname{dist}(x_0, v_0) < \varepsilon, \quad \text{and} \quad \operatorname{dist}(\tilde{\Phi}^\ell(x_0), v_m) < \varepsilon$$

(see Lemma 4). Note that we do not use hyperbolicity or index arguments in this lemma. We also do not use the symplecticity of the maps $F_1, \ldots, F_N$, nor the twist property of the map F_0. However, the fact that the large cylinder A is an invariant domain for the area-preserving map F_0 is crucial, as we use the Poincare Recurrence Theorem in an essential way (we first prove a certain weak shadowing result, Lemma 2, that holds without this assumption on the map F_0, then Lemma 4 is deduced from it in the case of area-preserving F_0).

According to this shadowing lemma (Lemma 4), in order to show that two open sets are connected by a forward orbit of the map Φ, it is sufficient to show that the intersections of these sets with A are connected by orbits of the iterated function system $\{F_0, \ldots, F_N\}$. A generalisation (Theorem 3) of a classical Birkhoff theorem states that if F_n for $n = 0, \ldots, N$ are exact symplectomorphisms homotopic to identity, and F_0 is a twist map, then for any two essential curves $\gamma^\pm \subset A'$ there is a trajectory of the iterated function system with $v_0 \in \gamma^-$ and $v_m \in \gamma^+$ *unless the functions F_n have a common invariant essential curve.*

Thus, if the maps $F_0, \ldots, F_N$ have no common invariant essential curves between γ^- and γ^+, every pair of neighbourhoods, U^- of γ^- and U^+ of γ^+, is connected by orbits of the map Φ. Theorem 3 also implies that the absence of a common invariant essential curve is an open property.

Theorem 4, the most difficult part of the argument, establishes that this property is also dense in $\mathcal{V}$ (provided $N \geq 8$). Thus, for every map Φ from an open and dense subset of $\mathcal{V}$, the corresponding scattering maps $F_1, \ldots, F_N$ ($N \geq 8$) and F_0 do not have any common essential invariant curve. As we just explained, this implies that every two neighbourhoods $U^\pm$ of $\gamma^\pm$ are connected by forward orbits of each such map Φ, and Theorem 1 follows.

Theorem 4 is the crucial step in the proof of Theorem 1. An analogue of Theorem 4 for generic *non-analytic* maps can be derived from [19,20,77]. However, the methods of destroying common invariant curves that are used in those papers cannot be used in the analytic case (as the real-analytic perturbations cannot, in general, vanish on the finitely smooth normally-hyperbolic cylinder A; the same concerns C^∞ perturbations, for that matter). Therefore, we develop a completely different perturbation technique in order to prove Theorem 4 for the analytic case.

3. Estimates in a Neighbourhood of a Symmetrically Normally-Hyperbolic Invariant Cylinder

In this section we study dynamics in a small neighbourhood of a normally-hyperbolic cylinder. This analysis does not require the map to be either symplectic or analytic.

3.1. Fenichel coordinates, cross form of the map, and estimates for the local dynamics. Let A be a compact, symmetrically normally-hyperbolic, smooth, invariant cylinder of a C^r-smooth map Φ ($r \geq 2$). As we already mentioned, A can be extended to a larger, smooth normally-hyperbolic locally-invariant cylinder $\tilde{A}$. Let us introduce coordinates in a small neighbourhood of A such that this larger invariant cylinder is straightened. Moreover, the local stable and unstable manifolds $W^{s,u}_{loc}(A)$ are straightened as well, along with the strong-stable and strong-unstable foliations E^{ss} and E^{uu} on them. Note that the foliations are at least C^1. The straightening of the manifolds and foliations means that one can introduce C^1-coordinates (u, v, z) in a neighbourhood of A such that the manifold W^s_{loc} will have equation $z = 0$, the manifold W^u_{loc} will be given by $u = 0$, and the leaves of the foliations E^{ss} and E^{uu} will all have the form $\{z = 0, v = \text{const}\}$ and $\{u = 0, v = \text{const}\}$ respectively (cf. [58]). The cylinder A thus lies in $\{u = 0, z = 0\}$. Here $v = (\varphi, y)$ with $\varphi \in \mathbb{S}^1$ being the angular variable and y taking values from an interval I of the real line.

Note that the manifolds $W^u(A)$ and $W^s(A)$ can be non-orientable. In this case we use the same coordinates (u, v, z) with $v = (\varphi, y)$ assuming that the hyperplanes $\varphi = 0$ and $\varphi = 2\pi$ are glued together by means of a linear involution in the space of (v, z). This modification does not affect our estimates.

In these coordinates the map Φ near A takes the form $\Phi : (u, v, z) \mapsto (\bar{u}, \bar{v}, \bar{z})$,

$$\bar{u} = h_1(u, v, z), \quad \bar{z} = h_2(u, v, z), \quad \bar{v} = F_0(v) + h_3(u, v, z), \tag{8}$$

where $h_{1,2,3}$ and F_0 are C^1-functions such that

$$\begin{aligned} h_1(0, v, z) &\equiv 0, \quad h_2(u, v, 0) \equiv 0, \\ h_3(0, v, z) &\equiv 0, \quad h_3(u, v, 0) \equiv 0. \end{aligned} \tag{9}$$

The identities $\Phi(0, v, z) = (0, \bar{v}, \bar{z})$ and $\Phi(0, v, z) = (0, \bar{v}, \bar{z})$ imply the first line of (9). The second line follows from the observation that the v-component of $\Phi(0, v, z)$ and $\Phi(u, v, 0)$ is independent of z and u respectively.

Differentiating Eq. (8) and taking into account that the local stable and unstable manifolds are given by the equations $z = 0$ and, respectively, $u = 0$, we find that

$$\left.\frac{\partial h_1}{\partial u}\right|_{u=z=0} = \Phi'(v)|_{N^s_v}, \quad \left.\frac{\partial h_2}{\partial z}\right|_{u=z=0} = \Phi'(v)|_{N^u_v}.$$

Then the assumption (5) implies that in an appropriately chosen norm

$$\left\| \frac{\partial h_1}{\partial u} \right\| < \lambda, \quad \left\| \left(\frac{\partial h_2}{\partial z} \right)^{-1} \right\| < \lambda.$$

The implicit function theorem implies that for small u and z the second equation of (8) can be resolved with respect to z. Therefore there is a neighbourhood of the closed invariant cylinder A, where the map $\Phi : (u, v, z) \mapsto (\bar{u}, \bar{v}, \bar{z})$ can be written in the following "cross" form:

$$\bar{u} = p(u, v, \bar{z}), \quad z = q(u, v, \bar{z}), \tag{10}$$

$$\bar{v} = F_0(v) + f(u, v, \bar{z}), \tag{11}$$

where

$$p(0, v, \bar{z}) \equiv 0, \quad q(u, v, 0) \equiv 0, \tag{12}$$

$$f(0, v, \bar{z}) \equiv 0, \quad f(u, v, 0) \equiv 0, \tag{13}$$

$$\|F_0'(v)\| < \alpha, \quad \|(F_0'(v))^{-1}\| < \alpha, \tag{14}$$

$$\left\| \frac{\partial p}{\partial u} \right\| < \lambda, \quad \left\| \frac{\partial q}{\partial \bar{z}} \right\| < \lambda, \tag{15}$$

$$\alpha^2 \lambda < 1, \quad 0 < \lambda < 1 < \alpha. \tag{16}$$

These estimates follow from the upper bounds (4)–(6) and the equalities (8), (9) provided the neighbourhood is sufficiently small.

Let Z_δ denote a δ-neighbourhood of A.

Lemma 1. *There is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and any $k \geq 0$ the following statements hold.*

1. *Any trajectory of length k such that $(u_i, v_i, z_i) := \Phi^i(u_0, v_0, z_0) \in Z_\delta$ for $i = 0, \dots, k$ satisfies the following estimates for $i = 0, \dots, k$:*

$$\|u_i\| \leq \delta\lambda^i, \quad \|z_i\| \leq \delta\lambda^{k-i}, \tag{17}$$

$$\|v_i - F_0^i(v_0)\| \leq \delta(\alpha\lambda)^{k/2}, \quad \|v_i - F_0^{i-k}(v_k)\| \leq \delta(\alpha\lambda)^{k/2}. \tag{18}$$

2. *The orbit (u_i, v_i, z_i) is determined in a unique way for any given u_0, v_0, z_k such that $\|u_0\|, \|z_k\| \leq \delta$ and $v_0 \in A$, as well as for any given u_0, v_k, z_k such that $\|u_0\|, \|z_k\| \leq \delta$ and $v_k \in A$.*
3. *Moreover, as $k \to +\infty$,*

$$\left\| \frac{\partial z_0}{\partial(u_0, v_0)} \right\| + \left\| \frac{\partial(u_k, v_k)}{\partial z_k} \right\| \to 0, \quad \left\| \frac{\partial u_k}{\partial(v_k, z_k)} \right\| + \left\| \frac{\partial(v_0, z_0)}{\partial u_0} \right\| \to 0, \tag{19}$$

uniformly for all $\|u_0\|, \|z_k\| \leq \delta$ and all $v_k \in A$ or $v_0 \in A$.

4. *We also have for all k large enough*

$$\left\| \frac{\partial z_0}{\partial z_k} \right\| \leq \lambda^k, \quad \left\| \frac{\partial(u_k, v_k)}{\partial(u_0, v_0)} \right\| \leq \alpha^k \tag{20}$$

[*at any given (u_0, v_0) in the first inequality, and at any given z_k in the second one*], *and*

$$\left\| \frac{\partial u_k}{\partial u_0} \right\| \leq \lambda^k, \quad \left\| \frac{\partial(v_0, z_0)}{\partial(v_k, z_k)} \right\| \leq \alpha^k, \tag{21}$$

[*at any given (v_k, z_k) in the first inequality, and at any given u_0 in the second one*).

Proof. Using (10) and (11), we get

$$u_{i+1} = p(u_i, v_i, z_{i+1}), \quad z_i = q(u_i, v_i, z_{i+1}), \quad v_{i+1} = F_0(v_i) + f(u_i, v_i, z_{i+1}), \tag{22}$$

for all $i = 0, \dots, k-1$. For a trajectory inside Z_δ Eqs. (12) and (15) imply

$$\|u_{i+1}\| = \|p(u_i, v_i, z_{i+1})\| \leq \lambda\|u_i\|, \quad \|z_i\| = \|q(u_i, v_i, z_{i+1})\| \leq \lambda\|z_{i+1}\|. \tag{23}$$

Since $\|u_0\|, \|z_k\| \leq \delta$, it follows that the orbit $\{(u_i, z_i, v_i)\}_{i=0}^k$ satisfies (17).

For the future convenience let us define

$$C_0(\delta) = \sup_{Z_\delta} \left\{ \|p'_v\|, \ \|p'_z\|, \ \|q'_u\|, \ \|q'_z\|, \|f'_u\|, \|f'_v\|, \|f'_z\| \right\}. \tag{24}$$

Note that $C_0(\delta)$ can be made as small as we need by decreasing δ because (12) and (13) imply that $p'_v = 0$, $p'_z = 0$, $q'_u = 0$, $q'_v = 0$, $f'_u = 0$, $f'_v = 0$, $f'_z = 0$ at $(u = 0, z = 0)$, for all $v \in A$.

In order to prove inequalities (18), let $V_i := v_i - F_0^i(v_0)$. In particular $V_0 = 0$. Equation (22) implies

$$\|V_{i+1}\| \leq \sup_{v \in A} \|F_0'(v)\| \cdot \|V_i\| + \|f(u_i, v_i, z_{i+1})\|. \tag{25}$$

Then Eq. (13) implies

$$\|f(u_i, v_i, z_{i+1})\| \leq \sup_{(u,v,z)\in Z_\delta} \|f_u'\| \cdot \|u_i\| \leq C_0(\delta)\|u_i\|,$$
$$\|f(u_i, v_i, z_{i+1})\| \leq \sup_{(u,v,z)\in Z_\delta} \|f_z'\| \cdot \|z_{i+1}\| \leq C_0(\delta)\|z_{i+1}\|.$$

Using Eq. (17) we get

$$\|f(u_i, v_i, z_{i+1})\| \leq \delta C_0(\delta)\ \min\{\lambda^i, \lambda^{k-i-1}\}. \tag{26}$$

Now using Eqs. (14), (26) and (25) we conclude that

$$\|V_{i+1}\| \leq \alpha\|V_i\| + \delta C_0(\delta)\ \min\{\lambda^i, \lambda^{k-i-1}\}.$$

Using $V_0 = 0$ and inequalities (16) we find that for all $1 \leq j \leq k$

$$\|V_j\| \leq \delta C_0(\delta) \sum_{0\leq i\leq j-1} \alpha^{j-i-1} \min\{\lambda^i, \lambda^{k-i-1}\} \leq \delta C_0(\delta) \sum_{0\leq i\leq k-1} \alpha^{k-i-1} \min\{\lambda^i, \lambda^{k-i-1}\}$$
$$= \delta C_0(\delta) \left\{ \sum_{0\leq i\leq (k-1)/2} (\alpha\lambda)^{k-i-1} + \alpha^{k-1} \sum_{(k-1)/2<i\leq k-1} (\lambda/\alpha)^i \right\} \leq \delta(\alpha\lambda)^{k/2},$$

when δ_0 is chosen small enough to ensure $\dfrac{C_0(\delta)}{\sqrt{\alpha\lambda}}\left[\dfrac{1}{1-\alpha\lambda} + \dfrac{\lambda}{\alpha-\lambda}\right] \leq 1$. The first of inequalities (18) is proved. The second inequality follows immediately by the symmetry of the problem (if we replace the map Φ by its inverse, then F_0 changes to F_0^{-1}, i to $(k-i)$, (u_0, z_k) to (z_k, u_0) and v_0 to v_k).

Given u_0, v_0, z_k, the orbit $\{(u_i, z_i, v_i)\}_{i=0}^k$ is a fixed point of the operator

$$Q : \{(u_i, v_i, z_i)\}_{i=0}^k \mapsto \{(\hat{u}_i, \hat{v}_i, \hat{z}_i)\}_{i=0}^k,$$

which acts on a sequence $\{(u_i, v_i, z_i)\}_{i=0}^k$ by

$$\begin{cases} \hat{u}_{i+1} = p(u_i, v_i, z_{i+1}), \qquad \hat{z}_i = q(u_i, v_i, z_{i+1}), \\ \hat{v}_{i+1} = F_0(v_i) + f(u_i, v_i, z_{i+1}) \quad \text{for}\, i = 0, \ldots, k-1, \\ \hat{u}_0 = u_0, \quad \hat{v}_0 = v_0, \quad \hat{z}_k = z_k. \end{cases} \tag{27}$$

Recall that $v = (y, \varphi)$, where $\varphi \in \mathbb{S}^1$, and y runs an interval I such that for all sufficiently small δ the points in the δ-neighbourhood Z_δ of the cylinder A have the y-coordinates strictly inside I. It is convenient to extend the functions p, q, F_0, f in (10) and (11) to all $y \in \mathbb{R}^1$ in such a way that they remain smooth, have uniformly continuous derivatives, moreover the identities (12) and (13) hold, and the estimates (14) and (15) remain true with a margin of safety. We assume that the functions p, q, F_0, f are not changed for all points with $y \in I$. If a sequence $\{(u_i, v_i, z_i)\}_{i=0}^k$ is a fixed point of the extended operator Q and lies entirely in Z_δ, then this sequence is also an orbit for the original map Φ.

It is convenient to consider the lift of the original map so that φ runs the whole real axis and the functions p, q and $F_0 + f - v$ are periodic in φ. So, in the analysis of the operator Q given by (27), we assume $v \in \mathbb{R}^2$.

Denote by $X = X_{k,u_0,v_0,z_k}$ the set of all sequences $\{(u_i, v_i, z_i)\}_{i=0}^k$ with the given value of (u_0, v_0, z_k), which also satisfy $\|u_i, z_i\| \le \delta$ for all $i = 0, \dots, k$. By (23), if $\|u_i, z_i\| \le \delta$ for all $i = 0, \dots, k$, then $\|\hat{u}_i, \hat{z}_i\| \le \delta$ for all $i = 0, \dots, k$ as well, thus $QX \subseteq X$. Let us show that the operator Q is contracting on X in the norm

$$\|\{(u_i, v_i, z_i)\}_{i=0}^k\|_\alpha = \max_{i=0,\dots,k} \alpha^{-i} \|u_i, v_i, z_i\|.$$

Indeed, in this norm

$$\begin{aligned}\|Q'\|_\alpha \le \max\Bigg\{&\alpha^{-1}\left\|\frac{\partial p}{\partial u}\right\| + \alpha^{-1}\left\|\frac{\partial p}{\partial v}\right\| + \left\|\frac{\partial p}{\partial \bar z}\right\|, \quad \left\|\frac{\partial q}{\partial u}\right\| + \left\|\frac{\partial q}{\partial v}\right\| + \alpha\left\|\frac{\partial q}{\partial \bar z}\right\|,\\ &\alpha^{-1}\left\|F_0'\right\| + \alpha^{-1}\left\|\frac{\partial f}{\partial u}\right\| + \alpha^{-1}\left\|\frac{\partial f}{\partial v}\right\| + \left\|\frac{\partial f}{\partial \bar z}\right\|\Bigg\}\\ \le \max\Big\{&\alpha^{-1}\lambda + \alpha^{-1}C_0(\delta) + C_0(\delta),\ 2C_0(\delta) + \alpha\lambda,\ \alpha^{-1}\left\|F_0'\right\| + \alpha^{-1}C_0(\delta) + 2C_0(\delta)\Big\},\end{aligned}$$

where, for the derivatives in the right-hand side, we use the supremum norm taken over all $(u, v, \bar z)$ such that $\|u, \bar z\| \le \delta$, and $C_0(\delta)$ is defined by (24). By (12)–(16), if δ is sufficiently small, then $\|Q'\|_\alpha < 1$ uniformly for every element from X, independently of the value of $k \ge 0$. Since the set X is convex, it follows that the operator Q is indeed contracting.

Thus, by contraction mapping principle, given any (u_0, v_0, z_k) such that $\|u_0, z_k\| \le \delta$ there exists indeed a unique length-k orbit with the given values of u_0, v_0 and z_k. We already proved that this orbit must satisfy (17) and (18). Since $v_0 \in A$ implies $F_0^i v_0 \in A$ for all $i = 0, \dots, k$ by the invariance of A with respect to F_0, estimates (17) and (18) imply that the orbit lies in Z_δ as required.

By the symmetry of the problem, given any (u_0, v_k, z_k) such that $\|u_0, z_k\| \le \delta$ and $v_k \in A$, there exists a unique length-k orbit with the given values of u_0, v_k and z_k, and this orbit lies in Z_δ.

As a fixed point of a smooth contracting operator, the obtained orbit must depend smoothly on all data on which the operator depends smoothly. So (u_i, v_i, z_i) depend smoothly on (u_0, v_0, z_k) (and, by the symmetry of the problem, on (u_0, v_k, z_k) as well). To complete the proof of the lemma, it remains to prove estimates (19)–(21).

We prove only the first limit in (19), as the second one follows from the first one due to the symmetry of the problem with respect to change of Φ to Φ^{-1}. It is enough to prove (20) only, as (21) also follows by the symmetry. Denote $\beta_i = \|\partial(u_i, v_i)/\partial(u_0, v_0)\|$, $\gamma_i = \|\partial z_i/\partial(u_0, v_0)\|$, where the derivatives are taken at z_k fixed. By differentiating (22), we obtain

$$\beta_{i+1} \le \left\|\frac{\partial(p, F_0 + f)}{\partial(u, v)}\right\| \beta_i + \left\|\frac{\partial(p, f)}{\partial \bar z}\right\| \gamma_{i+1}, \quad \gamma_i \le \left\|\frac{\partial q}{\partial \bar z}\right\| \gamma_{i+1} + \left\|\frac{\partial q}{\partial(u, v)}\right\| \beta_i,$$

where the derivatives are taken at $(u, v, \bar z) = (u_i, v_i, z_{i+1})$. Since u_i and z_i satisfy (17), we obtain from (12) to (15) that for sufficiently small δ (independent of i and k)

$$\beta_{i+1} \le (\alpha - \rho)\beta_i + \mu_i \gamma_{i+1}, \quad \gamma_i \le (\lambda - \rho)\gamma_{i+1} + \mu_{k-i-1}\beta_i, \tag{28}$$

where ρ is a small positive constant, and

$$\mu_j = \sup_{\|u\| \le \delta\lambda^j,\ (u,v,z)\in Z_\delta} \left\| \frac{\partial(p,f)}{\partial \bar{z}} \right\| + \sup_{\|z\| \le \delta\lambda^j,\ (u,v,z)\in Z_\delta} \left\| \frac{\partial q}{\partial(u,v)} \right\| . \tag{29}$$

It follows from (12) and (13) that

$$\mu_j \to 0 \quad \text{as} \quad j \to +\infty. \tag{30}$$

Recall also that, by definition,

$$\beta_0 = 1, \quad \gamma_k = 0. \tag{31}$$

Define the sequence M_j by the rule

$$M_{j+1} = \alpha\lambda M_j + \mu_j, \tag{32}$$

for an arbitrarily chosen M_0. As $\alpha\lambda < 1$, it follows from (30) that

$$M_j \to 0 \quad \text{as} \quad j \to +\infty. \tag{33}$$

By (28)

$$\begin{aligned} \gamma_i - M_{k-i}\beta_i \le{}& \frac{\lambda-\rho}{1-\mu_i M_{k-i-1}}(\gamma_{i+1} - M_{k-i-1}\beta_{i+1}) \\ &+ \left[\mu_{k-i-1} - M_{k-i} + \alpha\frac{\lambda-\rho}{1-\mu_i M_{k-i-1}} M_{k-i-1}\right]\beta_i . \end{aligned}$$

As the sequences μ_j and M_j both tend to zero, it follows that

$$\lim_{k\to+\infty} \max_{i=0,\dots,k-1} \mu_i M_{k-i-1} = 0. \tag{34}$$

If k is large enough, then $\mu_i M_{k-i-1} < \rho/\lambda < 1$ for all $i = 0, \dots, k-1$. Thus,

$$\gamma_i - M_{k-i}\beta_i \le \lambda(\gamma_{i+1} - M_{k-i-1}\beta_{i+1}) + \left[\mu_{k-i-1} - M_{k-i} + \alpha\lambda M_{k-i-1}\right]\beta_i,$$

which, by (32), implies

$$\gamma_i - M_{k-i}\beta_i \le \lambda(\gamma_{i+1} - M_{k-i-1}\beta_{i+1}),$$

hence, for all k large enough, for every $i = 0, \dots, k-1$

$$\gamma_i - M_{k-i}\beta_i \le \lambda^{k-i}(\gamma_k - M_0\beta_k), \tag{35}$$

in particular

$$\gamma_0 - M_k\beta_0 \le \lambda^k(\gamma_k - M_0\beta_k). \tag{36}$$

Now, by (31), we have $\gamma_0 \le M_k$, so (33) implies $\partial z_0/\partial(u_0, v_0) \to 0$ as $k \to +\infty$, which agrees with (19). Note also that by (35) we have $\gamma_{i+1} \le M_{k-i-1}\beta_{i+1}$. By (28), (34), this gives us that for all k large enough, for every $i = 0, \dots, k-1$

$$\beta_{i+1} \le \alpha\beta_i,$$

which [see (31)] implies the second inequality in (20).

It remains to estimate $\partial(u_k, v_k, z_0)/\partial z_k$ as $k \to +\infty$. To this aim, let $\beta_i = \|\partial(u_i, v_i)/\partial z_k\|$ and $\gamma_i = \|\partial z_i/\partial z_k\|$, where the derivatives are taken at (u_0, v_0) fixed.

Then by differentiating (10), (11), we will obtain the inequalities (29), hence the estimate (36) holds at all sufficiently large k for the newly defined γ_i, β_i. However, instead of (31) we have now

$$\beta_0 = 0, \quad \gamma_k = 1.$$

Thus, we find from (36) that

$$\gamma_0 \le \lambda^k, \quad \beta_k \le 1/M_0$$

for all k sufficiently large. This immediately gives us the first inequality in (20), and since M_0 can be taken arbitrary, we also obtain that $\partial(u_k, v_k)/\partial z_k \to 0$ as $k \to +\infty$, which completes the proof of (19). □

Estimates provided by this lemma are close to those obtained by Shilnikov in [85].

3.2. "Lambda-lemma". The following analogue of the "lambda-lemma" [23,79] follows from Lemma 1.

Proposition 1. *If $L \subset Z_\delta$ is a surface of the form $u = w(v, z)$, where w is a smooth function defined for all $v \in A$ and all small z, then the images $\Phi^m(L) \cap Z_\delta$ converge to $W^u_{loc}(A) \cap Z_\delta$ as $m \to +\infty$ in the C^1-topology. If $L \subset Z_\delta$ is a surface of the form $z = w(v, u)$, where w is a smooth function defined for all $v \in A$ and all small u, then the images $\Phi^{-m}(L) \cap Z_\delta$ converge to $W^s_{loc}(A) \cap Z_\delta$ as $m \to +\infty$ in the C^1-topology.*

Proof. By the symmetry of the problem, it is enough to consider only the case where L is a surface of the form $u = w(v, z)$. By Lemma 1, given any (u_0, v_k, z_k) the corresponding orbit (u_i, v_i, z_i) is defined uniquely. Denote as η_k the operator that sends (u_0, v_k, z_k) to (v_0, z_0), and as ξ_k the operator that sends (u_0, v_k, z_k) to u_k. The point (u, v, z) belongs to $\Phi^k L$ if and only if $u_0 = w(v_0, z_0)$, i.e. the equation of $\Phi^k L$ is

$$u_k = \xi_k(u_0, v_k, z_k) \tag{37}$$

where u_0 is defined from

$$u_0 = w(\eta_k(u_0, v_k, z_k)). \tag{38}$$

By (17) and (19),

$$\|\eta_k\| + \|\partial\eta_k/\partial u_0\| \to 0 \quad \text{as} \quad k \to +\infty,$$

therefore at each k large enough equation (38) defines u_0 uniquely as a smooth function of (v_k, z_k). It follows from (21) that

$$\|\partial u_0/\partial(v_k, z_k)\| = O(\alpha^k).$$

Thus, Eq. (37) defines u_k as a smooth function $w_k(v_k, z_k)$, for all $\|z_k\| \le \delta$ and $v_k \in A$. By (17), $\|u_k\| \to 0$ as $k \to +\infty$. Moreover, since by (21) and (19) we have $\|\partial\xi_k/\partial u_0\| = O(\lambda^k)$ and $\|\partial\xi_k/\partial(v_k, z_k)\| \to 0$ as $k \to +\infty$, it follows that

$$\left\|\frac{dw_k}{d(v_k, z_k)}\right\| \le \left\|\frac{\partial\xi_k}{\partial u_0}\right\| \cdot \left\|\frac{\partial u_0}{\partial(v_k, z_k)}\right\| + \left\|\frac{\partial\xi_k}{\partial(v_k, z_k)}\right\| = O((\alpha\lambda)^k) + \left\|\frac{\partial\xi_k}{\partial(v_k, z_k)}\right\| \to 0$$

as $k \to +\infty$ (recall that $\alpha\lambda < 1$). We see that for all k large enough the surface $\Phi^k L$ is given by the equation $u = w_k(v, z)$ where w_k tends to zero along with the first derivative as $k \to +\infty$. Since equation of W^u_{loc} is $u = 0$, this proves the proposition. □

3.3. Secondary homoclinic cylinders. Using Proposition 1 we can establish a sufficient condition for the existence of infinitely many independent homoclinic cylinders. Let $\bar{A}$ and $\hat{A}$ be compact invariant cylinders such that $\bar{A} \subset \hat{A} \subset A$. Let the intersection of $W^u(A)$ and $W^s(A)$ contain a homoclinic cylinder B which is simple relative to $\hat{A}$ and A, and $F_B(\bar{A}) \subseteq \hat{A}$, i.e.

$$W^u(\bar{A}) \cap B \subseteq W^s(\hat{A}) \cap B. \tag{39}$$

Proposition 2. *There are infinitely many homoclinic cylinders B_i, each corresponds to a simple (relative to $\bar{A}$ and A) intersection of $W^u(A)$ with $W^s(A)$, and none of the cylinders belongs to the orbit of another cylinder: $B_i \cap \Phi^m(B_j) = \emptyset$ for every m and every $i \neq j$.*

Proof. Since the manifolds $W^u(A)$ and $W^s(A)$ are invariant with respect to Φ, the cylinder $\Phi^m(B)$ lies in $W^u(A) \cap W^s(A)$ for all $m \in \mathbb{Z}$. This homoclinic sequence of cylinders $\Phi^m(B)$ tends to A as $m \to \pm\infty$. Therefore, there are positive numbers m_+ and m_- such that the cylinders $B^- = \Phi^{-m_-}(B)$ and $B^+ = \Phi^{m_+}(B)$ belong to a small neighbourhood of A.

It is easy to show that if B is a simple homoclinic cylinder relative to $\hat{A}$ and A then the cylinder $\Phi^m(B)$ with any m also has this property. Indeed, conditions [S1] and [S2] follow directly from the invariance of the foliations E^{ss} and E^{uu} and the invariance of the cylinders $\hat{A}$ and A. Moreover, the invariance of the foliations implies $\Phi(\pi^s(x)) = \pi^s(\Phi(x))$ and $\Phi(\pi^u(x)) = \pi^u(\Phi(x))$ for every point x in $W^s(A)$ and $W^u(A)$ respectively. Then

$$\pi^{s,u}_{\Phi(B)} = \Phi \circ \pi^{s,u}_B \circ \Phi^{-1} \tag{40}$$

and the scattering map takes the form

$$F_{\Phi(B)} = F_0 \circ F_B \circ F_0^{-1} \tag{41}$$

where $F_0 = \Phi|_A$. Consequently, the scattering maps, which correspond to any two cylinders such that one is the image of the other by an mth iteration of Φ, are conjugate to each other by means of the mth iteration of F_0. Condition [S3] follows immediately as F_0 maps an essential curve to an essential curve. Thus the fulfilment of the simplicity conditions for the cylinder B implies the fulfilment of the simplicity conditions for all its iterations by Φ.

Thus, the cylinders B^- and B^+ satisfy $B^- \subset W^u_{loc}(A)$ and $B^+ \subset W^s_{loc}(A)$, and they are simple relative to $\hat{A}$ and A. In the Fenichel coordinates, $W^u_{loc}(A)$ has the equation $u = 0$ and the leaves of the foliation E^{uu} in $W^u_{loc}(A)$ are given by $\{u = 0, v = \text{const}\}$. By the simplicity conditions [S1] and [S2], each leave of E^{uu} in $W^u_{loc}(\hat{A})$ intersects the cylinder B^- at a single point and is transverse to $W^s(A)$ at this point. It follows that there is a piece W of the manifold $W^s(A)$ which contains the homoclinic cylinder B^- and has the form $z = w(v, u)$ where w is a smooth function defined for all v from some neighbourhood of $\hat{A}$ and all small u.

Proposition 1 (where the invariant cylinder A is replaced by the invariant cylinder $\hat{A}$) implies that the images $W_i = \Phi^{-i}(W)$ by the backward iterations of Φ accumulate on $W^s_{loc}(\hat{A})$ in C^1. Equation (39) implies that each of W_i with i sufficiently large has a non-empty and transverse intersection with $W^u(\bar{A})$ near B^+. Since W_i are, by construction, pieces of $W^u(A)$, this gives us the sought infinite set of homoclinic cylinders B_i converging to the cylinder B^+; obviously none of them belongs to the orbit of another one. Since W_i are C^1-close to $W^s_{loc}(\hat{A})$ near B^+, it follows from the relative to $\hat{A}$ simplicity of B^+ that W_i intersect transversely each leaf of the foliation E^{uu} in $W^u(\bar{A})$, the uniqueness of the intersections is also inherited.

Thus, the scattering maps $F_i : \bar{A} \to \mathrm{int}(A)$ are defined for each of the cylinders B_i. In order to check the simplicity of the homoclinic intersection at B_i, we need to show that the projections $\pi^s_{B_i} : B_i \to \mathrm{int}(A)$ by the leaves of the strong-stable foliation are injective for all i (condition [S2]), and that the scattering maps are homotopic to identity (condition [S3]). To check the injectivity, notice that

$$\pi^s_{\Phi^i(B_i)} = \Phi^i \circ \pi^s_{B_i} \circ \Phi^{-i} \tag{42}$$

by (40). So, it is enough to show the injectivity of $\pi^s_{\Phi^i(B_i)}$. To do this, note that the cylinders $\Phi^i(B_i)$ are close to B^- at large i, so the maps $\pi^s_{\Phi^i(B_i)}$ are close to $\pi^s_{B^-}$, and the latter map is injective by the simplicity of the homoclinic intersection at B^-.

It remains to show that the scattering maps F_i are homotopic to identity. As we just mentioned, the maps $\hat{\pi}^s_i = \pi^s_{\Phi^i(B_i)} \circ (\pi^s_{B^-})^{-1}$ are close to identity at large i. The same is true for the maps $\hat{\pi}^u_i = \pi^u_{B^+} \circ (\pi^u_{B_i})^{-1}$. Using (42), we find

$$F_{B_i} = \pi^s_{B_i} \circ (\pi^u_{B_i})^{-1} = \Phi^i \circ \hat{\pi}^s_i \circ F_{B^-} \circ \pi^u_{B^-} \circ \Phi^{-i} \circ (\pi^s_{B^+})^{-1} \circ F_{B^+} \circ \hat{\pi}^u_i, \tag{43}$$

where $F_{B^+} = \pi^s_{B^+} \circ (\pi^u_{B^+})^{-1}$ and $F_{B^-} = \pi^s_{B^-} \circ (\pi^u_{B^-})^{-1}$ are the scattering maps corresponding to the cylinders B^+ and B^-. By the simplicity of the homoclinic intersection at B, these maps are homotopic to identity diffeomorphisms. The map Φ^i in formula (43) acts in a small neighbourhood Z of A and is homotopic to identity in Z. The maps $\pi^s_{B^+}$ and $\pi^u_{B^-}$ are projections along the foliations in the local stable and unstable manifolds, so they are homotopic to identity in Z. Thus, all the maps in the right-hand side of formula (43) are homotopic to identity, which implies that the scattering maps F_{B_i} are homotopic to identity for all i large enough. The proposition is proved. □

This proposition shows that the assumptions of Theorem 2 imply the existence of an infinite series of different homoclinic cylinders which are simple relative to $\bar{A}$ and A, i.e. Theorem 2 reduces to Theorem 1. For our purposes, the existence of $N \geq 8$ such cylinders is enough, so it will be our standing assumption for the rest of the paper. We do not need the auxiliary invariant cylinder $\hat{A}$ anymore.

4. Shadowing in the Homoclinic Channel

4.1. Homoclinic channel. Let $B_1, \ldots, B_N$ be homoclinic cylinders, each corresponds to a simple homoclinic intersection relative to the compact invariant subcylinder $\bar{A}$ of A, and none of the cylinders B_n belongs to the orbit of another cylinder. Let us repeat the definition of the scattering maps F_n. Since the homoclinic intersections are simple, it follows that two maps, π^u_n and π^s_n, from B_n into $\mathrm{int}(A)$ are defined for every n by the leaves of the foliations E^{uu} and E^{ss}, respectively. Namely, $v = \pi^u_n(x)$ if the points $x \in B_n$ and $v \in A$ belong to the same leaf of the foliation E^{uu}, and $v = \pi^s_n(x)$ if $x \in B_n$ and $v \in A$ belong to the same leaf of the foliation E^{ss}. The smoothness of the maps π^s_n and π^u_n and their inverse maps follows from the transversality of the intersections of the leaves with B_n. By assumption, $\bar{A} \subset \pi^u_n(B_n)$. Thus, for each homoclinic cylinder B_n we have a diffeomorphism $F_n = \pi^s_n \circ (\pi^u_n)^{-1}$ which acts from $\bar{A}$ into $\mathrm{int}(A)$. In fact, as the strong transversality condition [S1] is open, there is a neighbourhood A' of $\bar{A}$ such that all the scattering maps $F_1, \ldots, F_N$ are diffeomorphisms of A' into $\mathrm{int}(A)$.

Take sufficiently large positive m_+ and m_- such that all the cylinders $B_n^+ = \Phi^{m_+}(B_n)$ and $B_n^- = \Phi^{-m_-}(B_n)$ $(n = 1, \ldots, N)$ lie in the δ-neighbourhood of A, where δ is small enough. As $B_n^+ \in W_{loc}^s$ and $B_n^- \in W_{loc}^u$, it follows that in the Fenichel coordinates $z = 0$ on B_n^+, and $u = 0$ on B_n^-. Since the homoclinic cylinders are simple, the cylinder B_n^+ intersects the leaves $\{v = \text{const}\}$ of the foliation E^{ss} in W_{loc}^s transversely, no more than at one point each, hence B_n^+ is a graph of a function, $B_n^+ = \{u = u_n^+(v), z = 0\}$, where u^+ is a smooth function whose domain of definition contains $\bar{A}$. Analogously, $B_n^- := \{z = z_n^-(v), u = 0\}$ for a smooth function z^-. Thus, points on B_n^+ and B_n^- are uniquely determined by their v-coordinates. Since in the Fenichel coordinates the projections π^u and π^s do not change the v-components of a point, we may formally treat the maps F_n, $n = 0, \ldots, N$, as acting from B_n^- to B_n^+ in the same way these maps act on A.

Since the foliations E^{ss} and E^{uu} are invariant with respect to the map Φ, it follows that $\Phi(E_v^{uu}) = E_{F_0(v)}^{uu}$ and $\Phi(E_v^{ss}) = E_{F_0(v)}^{ss}$. Consequently, for any $x \in B_n$ the points $F_0^{-m_-} \circ \pi_n^u(x)$ and $\Phi^{-m_-}(x)$ have the same v-coordinate. The same is true for the points $F_0^{m_+} \circ \pi_n^s(x)$ and $\Phi^{m_+}(x)$. Thus, in the v-coordinates, we have

$$\Phi^{m_+ + m_-}|_{B_n^-} = F_0^{m_+} \circ F_n \circ F_0^{m_-} \tag{44}$$

Denote by $T_n : (u, v, z) \mapsto (\bar{u}, \bar{v}, \bar{z})$ the map $\Phi^{m_+ + m_-}$ from a sufficiently small neighbourhood of B_n^- to a small neighbourhood of B_n^+. The transversality condition implies that the image by the map T_n of any leaf of the foliation E^{uu} in W_{loc}^u, given by $\{u = 0, v = \text{const}\}$, is transverse to $W_{loc}^s = \{\bar{z} = 0\}$. Consequently the derivative $\partial\bar{z}/\partial z$ is invertible. Therefore, given any small $(u, \bar{z})$ and $v \in \bar{A}$ we have a uniquely defined $(\bar{u}, z, \bar{v})$ such that $(\bar{u}, \bar{z}, \bar{v}) = T_n(u, z, v)$. So, we may write the map T_n in the following form:

$$\bar{u} = p_n(u, v, \bar{z}), \quad \bar{v} = G_n(u, v, \bar{z}) = \bar{F}_n(v) + f_n(u, v, \bar{z}), \quad z = q_n(u, v, \bar{z}), \tag{45}$$

where p_n, q_n, f_n are smooth functions defined for small $(u, \bar{z})$ and for v from a small neighbourhood A'' of $\bar{A}$ in A. We define $\bar{F}_n$ of (45) in such a way that

$$f_n(0, v, 0) \equiv 0. \tag{46}$$

As $u = 0$ corresponds to an initial point in W_{loc}^u, and $\bar{z} = 0$ corresponds to the image of this point (by T_n) that lies in W_{loc}^s, the equalities $u = 0$ and $\bar{z} = 0$ correspond to an initial point in B_n^- which has its image in B_n^+. Thus, by (44), we have

$$\bar{F}_n = \Phi^{m_+ + m_-}|_{B_n^-} = F_0^{m_+} \circ F_n \circ F_0^{m_-}, \tag{47}$$

where F_n is the scattering map. Since the cylinder $\bar{A}$ is invariant with respect to F_0, the maps $\bar{F}_n$ are defined in a neighbourhood of $\bar{A}$, as the scattering maps F_n are. Thus, we will further assume that the open neighbourhood A'' of $\bar{A}$ in A is chosen such that the *modified scattering maps* $\bar{F}_n$ are all defined there, and theay are homotopic to identity diffeomorphisms of A'' into A, moreover

$$F_0^{-m_-}(A') \subseteq A'' \tag{48}$$

where A' is a small neighbourhood of $\bar{A}$ in A where the scattering maps F_n are defined.[5]

Let us denote by T_0 the map Φ restricted to the δ-neighbourhood Z_δ of A. Let us call the union of the δ-neighbourhood of A with certain, sufficiently small neighbourhoods of the cylinders $\tilde{\Phi}(B^-), \dots, \tilde{\Phi}^{m_+ + m_- - 1}(B^-)$ *a homoclinic channel*. For every finite orbit in the homoclinic channel with the initial point $P_0 \in Z_\delta$ there is a uniquely defined sequence of points $(P_s)_{s=0}^{2J+1}$ of this orbit which lie in Z_δ and satisfy

$$\begin{aligned} P_{2j+1} &= T_0^{k_j} P_{2j} \qquad \text{for } j = 0, \dots, J, \\ P_{2j} &= T_{n_j} P_{2j-1} \quad \text{for } j = 1, \dots, J, \end{aligned}$$

where n_j may take values from $1, \dots, N$ and $k_j \geq 0$. We call the sequence P_j a *channel orbit*, and the sequence $\omega = (k_0, n_1, k_1, \dots, n_J, k_J)$ is called *the code* of the orbit. Given a code ω, we say that a sequence $(v_s^*)_{s=0}^{2J}$ of points in A is a *shadow orbit*, if $v_{2j+1}^* = F_0^{k_j}(v_{2j}^*)$ and $v_{2j}^* = \bar{F}_{n_j}(v_{2j-1}^*)$. In the last definition, we assume that

$$v_{2j-1}^* \in A'' \quad \text{for } j = 1, \dots, J, \tag{49}$$

so these points belong to the domain of $\bar{F}_{n_j}$ and the sequence is well defined. We note that it is possible that some codes do not correspond to any shadow orbit. On the other hand, any channel orbit $(P_s)_{s=0}^{2J}$ has a code ω and defines a shadow orbit with the code ω and v_0^* equal to the v-coordinate of P_0.

4.2. Shadowing orbits of proper codes. Our next goal is to estimate the deviation of the channel orbit P_s from its shadow. In this section we restrict our attention to orbits which correspond to a special class of codes. Namely, a finite code is called *proper* if for all s

$$k_s \geq \bar{k} \quad \text{and} \quad k_s \geq \gamma k_{s+1} + D, \tag{50}$$

for some $\bar{k} \geq 0$, $D \geq 0$ and $\gamma > 1$. In other words, k_s is a sufficiently fast decreasing sequence of sufficiently large numbers.

Lemma 2. *Given any sufficiently large $\bar{k}$, γ and D, for any shadow orbit $v_0^*, \dots, v_{2J+1}^*$ with a proper code $k_0, \{n_s, k_s\}_{1 \leq s \leq J}$, given any u^{in} and z^{out} such that $\|u^{in}\| \leq \delta$, $\|z^{out}\| \leq \delta$, in the δ-neighbourhood of A' there exists a uniquely defined channel orbit $(P_s)_{s=0}^{2J+1}$ with $P_s = (u_s, v_s, z_s)$ such that $u_0 = u^{in}$, $v_0 = v_0^*$, $z_{2J+1} = z^{out}$, and $P_{2j+1} = T_0^{k_j} P_{2j}$, $P_{2j} = T_{n_j} P_{2j-1}$. Moreover,*

$$\|v_s - v_s^*\| \leq 2\delta(\alpha\lambda)^{k_J/2} \leq 2\delta(\alpha\lambda)^{\bar{k}/2}, \tag{51}$$

and

$$\|u_{2J+1}\| \leq \delta\lambda^{\bar{k}}, \quad \|z_0\| \leq \delta\lambda^{\bar{k}}. \tag{52}$$

Remark 4. Usual shadowing results would require hyperbolicity (or its topological analogues) from the maps F_0 and $F_1, \dots, F_N$, see e.g. [26]. We, however, do not make any assumption on the dynamics of these maps in this lemma (e.g. we have not assumed the symplecticity so far). Therefore we need to restrict here the class of shadow orbits to those with proper codes only; we believe any significantly stronger shadowing statement can not hold in this situation without further assumptions.

[5] As $F_0(\bar{A}) = \bar{A} \subset A''$, the inclusion (48) can always be achieved by choosing A' to be close enough to $\bar{A}$.

Proof of the lemma. For $J = 0$ the statement of the lemma is contained in Lemma 1, so we will proceed by induction in J. Suppose that for any $\tilde{z}$ with $\|\tilde{z}\| \le \delta$ there is a unique sequence (u_s, v_s, z_s), $s = 0, \dots, 2J-1$ with the code $k_0, n_1, \dots, k_{J-1}$, which satisfies the condition $u_0 = u^{in}$, $v_0 = v_0^*$ and $z_{2J-1} = \tilde{z}$ and such that the inequalities

$$\|v_s - v_s^*\| \le 2\delta(\alpha\lambda)^{k_{J-1}/2} \tag{53}$$

hold for all $s \le 2J-1$. In order to shorten our notation we suppress dependence on u^{in} and v_0^* which are assumed to be fixed. Then $u_{2J-1} = \tau(\tilde{z})$ and $v_{2J-1} = \phi(\tilde{z})$ for some functions τ and ϕ respectively. Equations (49) and (53) imply that

$$\phi(\tilde{z}) \in A''_\rho \quad \text{for any} \quad \rho > 2\delta(\alpha\lambda)^{\bar{k}/2}, \tag{54}$$

where A''_ρ is the closed ρ-neighbourhood of A''.

Since $(u_{2J-1}, v_{2J-1}, \tilde{z}) = T_0^{k_{J-1}}(u_{2J-2}, v_{2J-2}, z_{2J-2})$, equation (17) of Lemma 1 implies that

$$\|\tau\| \le \delta\lambda^{k_{J-1}} \le \delta\lambda^{\bar{k}}. \tag{55}$$

We will also include in our induction assumption a bound for the derivatives:

$$\|\tau', \phi'\| \le \nu \tag{56}$$

for some sufficiently small constant ν. Thus, in order to carry out the induction, when we prove that the sought sequence (u_j, v_j, z_j) is uniquely defined for all $j = 0, \dots, 2J+1$ we must also show that

$$\|\partial(u_{2J+1}, v_{2J+1})/\partial z^{out}\| \le \nu \tag{57}$$

with the same ν.

Since $(u_{2J+1}, v_{2J+1}, z^{out}) = T_0^{k_J}(u_{2J}, v_{2J}, z_{2J})$, Lemma 1 implies that z_{2J} is a uniquely defined smooth function of (u_{2J}, v_{2J}) and z^{out}. We denote it by $\sigma : (u_{2J}, v_{2J}, z^{out}) \mapsto z_{2J}$. Equations (17) and (19) imply

$$\|\sigma\| \le \delta\lambda^{k_J} \le \delta\lambda^{\bar{k}}, \tag{58}$$

and

$$\|\sigma'\| \le \nu, \tag{59}$$

for any $\nu > 0$ chosen in advance (if $\bar{k}$ is large enough), and

$$\|\partial\sigma/\partial z^{out}\| \le \lambda^{k_J}. \tag{60}$$

Taking into account that $(u_{2J}, v_{2J}, z_{2J}) = T_{n_J}(u_{2J-1}, v_{2J-1}, z_{2J-1})$ where the map T_{n_j} has the form (45) with $n = n_J$, we obtain the following system of equations

$$\begin{aligned}
u_{2J} &= p_{n_J}(\tau(z_{2J-1}), \phi(z_{2J-1}), \sigma(u_{2J}, v_{2J}, z^{out})),\\
v_{2J} &= G_{n_J}(\tau(z_{2J-1}), \phi(z_{2J-1}), \sigma(u_{2J}, v_{2J}, z^{out})),\\
z_{2J-1} &= q_{n_J}(\tau(z_{2J-1}), \phi(z_{2J-1}), \sigma(u_{2J}, v_{2J}, z^{out})).
\end{aligned} \tag{61}$$

In order to show that this system has a unique solution $(u_{2J}, v_{2J}, z_{2J-1})$ for every z^{out} we use the contraction mapping theorem. Indeed, take any z^{out} with $\|z^{out}\| \le \delta$ and consider the map

$$\begin{aligned} \bar{u} &= p_n(\tau(z), \phi(z), \sigma(u, v, z^{out})), \\ \bar{v} &= G_n(\tau(z), \phi(z), \sigma(u, v, z^{out})), \\ \bar{z} &= q_n(\tau(z), \phi(z), \sigma(u, v, z^{out})), \end{aligned} \tag{62}$$

where $\|z\| \le \delta$, $\|u\| \le \delta$ and $v \in A''_\rho$ (for some ρ small enough).

The functions p_n, q_n, G_n are defined by Eq. (45). By (54), (55), and (58), if $\bar{k}$ is sufficiently large, then the values of τ and σ can be made arbitrarily small, and the range of values of ϕ can be confined to an arbitrarily small neighbourhood of A'', i.e. (τ, ϕ, σ) belong to the domain of definition of $(p_n \cdot q_n, G_n)$, and the map (62) is well-defined.

As the functions p_n, q_n, G_n are smooth, their derivatives are bounded:

$$\|p'_n, q'_n, G'_n\| \le C.$$

We chose ν in (56) and (59) such that $C\nu < 1$. Then $\|\bar{z}\| \le \delta$, $\|\bar{u}\| \le \delta$ and $\bar{v} \in A''_\rho$. The first two inequalities hold as p_n and q_n are components of the map T_n which acts from a small neighbourhood of the cylinder B_n^- to a small neighbourhood of the cylinder B_n^+, and both cylinders belong to the δ-neighbourhood of A. In order to show that $\bar{v} \in A''_\rho$ we note that the induction assumption (53) implies $\|\phi(z) - v^*_{2J-1}\| \le 2\delta(\alpha\lambda)^{k_{J-1}/2}$. Since $G_n := \bar{F}_n + f_n$, and $v^*_{2J} = \bar{F}_n(v^*_{2J-1})$, it follows that

$$\|G_n(\tau, \phi, \sigma) - v^*_{2J}\| \le C\|\phi - v^*_{2J-1}\| + \|f_n(\tau, \phi, \sigma)\|.$$

Taking into account that f_n vanishes at $\tau = 0, \sigma = 0$ [see (46)], we obtain

$$\|f_n\| \le C\|\tau, \sigma\| \le C\lambda^{k_J}$$

due to (55) and (58). Combining these inequalities, we find that

$$\|\bar{v} - v^*_{2J}\| \le C\delta\left(2(\alpha\lambda)^{k_{J-1}/2} + \lambda^{k_J}\right). \tag{63}$$

Since $k_J > \bar{k}$ and $\bar{k}$ is large, we obtain that

$$\|\bar{v} - v^*_{2J}\| \le \rho.$$

Since $v^*_{2J} \in A''$, we have $\bar{v} = G_n(\tau, \phi, \sigma) \in A''_\rho$.

Thus the map (62) maps the set $\|\bar{z}\| \le \delta$, $\|\bar{u}\| \le \delta$ and $\bar{v} \in A''_\rho$ into itself. The chain rule together with the bounds (56) and (59) imply that this map is a contraction. Consequently, system (61) has a unique solution $(u_{2J}, v_{2J}, z_{2J-1})$ as required.

Moreover, after differentiating Eq. (61) and using (60) we obtain

$$\|\partial(u_{2J}, v_{2J})/\partial z^{out}\| \le \frac{C}{1 - C\nu}\lambda^{k_J} = o(\alpha^{-k_J}) \tag{64}$$

where the last bound follows from $\alpha\lambda < 1$ [see (16)]. Recalling that $(u_{2J+1}, v_{2J+1}, z^{out}) = T_0^{k_J}(u_{2J}, v_{2J}, z_{2J})$ and using (19), (20) and (64), we find that $\|\partial(u_{2J+1}, v_{2J+1})/\partial z^{out}\|$ can be made as small as we need by taking k_J large enough. Thus (57) holds true indeed for $\bar{k}$ large enough.

So we have proved the existence and uniqueness of the sequence (u_s, v_s, z_s) with $s = 0, \dots, 2J + 1$. It remains to demonstrate inequalities (51) and (52).

For $s \le 2J - 1$, inequality (51) follows from the induction assumption (53) as $k_{J-1} > k_J$. For $j = 2J$ inequality (51) follows from (63) applied to the fixed point of the map. In order to check (51) for $s = 2J + 1$, we recall that $(u_{2J+1}, v_{2J+1}, z_{2J+1}) = T_0^{k_J}(u_{2J}, v_{2J}, z_{2J})$ and $v^*_{2J+1} = F_0^{k_J}(v^*_{2J})$. Then Eqs. (14), (18) and (63) with $\bar v = v_{2J}$ imply

$$\begin{aligned}\left\| v_{2J+1} - v^*_{2J+1}\right\| &\le \| F_0^{k_J}(v_{2J}) - F_0^{k_J}(v^*_{2J})\| + \delta(\alpha\lambda)^{k_J/2}\\ &\le \| v_{2J} - v^*_{2J}\|\alpha^{k_J} + \delta(\alpha\lambda)^{k_J/2}\\ &\le C\delta\left(2(\alpha\lambda)^{k_{J-1}/2} + \lambda^{k_J}\right)\alpha^{k_J} + \delta(\alpha\lambda)^{k_J/2}.\end{aligned}$$

This inequality implies (51) for $s = 2J + 1$ provided the first term in the last line is not larger than the second one, i.e.,

$$2C(\alpha\lambda)^{k_J/2}\left((\alpha\lambda)^{(k_{J-1}-k_J)/2}(\alpha/\lambda)^{k_J/2} + \frac{1}{2}\right) \le 1.$$

Taking into account (50) we see that this inequality can be achieved if $2C(\alpha\lambda)^{\bar k/2} \le 1$ and $(\alpha\lambda)^{(\gamma-1)k_J+D}(\alpha/\lambda)^{k_J} \le \frac{1}{4}$. The first inequality holds if $\bar k$ is sufficiently large and the second one follows from

$$\gamma > 2\ln\frac{1}{\lambda}\Big/\ln\frac{1}{\alpha\lambda}\,, \quad (\alpha\lambda)^D \le \frac{1}{4}.$$

Finally, inequality (52) is an immediate corollary of (17). □

4.3. Replacing a code with a proper code. Since the diffeomorphism F_0 is area-preserving, the Poincare Recurrence Theorem implies that recurrent (Poisson stable) orbits of F_0 are dense in the invariant cylinder A. This fact, as the following lemma shows, allows an arbitrary orbit of the iterated function system $\{F_0, F_1, \dots, F_N\}$ to be approximated by a shadow with a proper code. We recall that $F_0 : A \to A$ is the restriction of the map Φ onto A, and $F_n : A' \to \mathrm{int}(A)$ with $n \ge 1$ are scattering maps.

Lemma 3. *Let $v_0, \dots, v_{2J+1}$ be a sequence of points, $i_j \ge 0$ and $n_j \in \{1, \dots, N\}$, such that*

$$\begin{aligned} v_{2j+1} &= F_0^{i_j}(v_{2j}) \qquad j = 0, \dots, J,\\ v_{2j} &= F_{n_j}(v_{2j-1}) \quad j = 1, \dots, J,\end{aligned} \tag{65}$$

$v_{2j-1} \in A'$ and $v_{2j} \in \mathrm{int}(A)$. Let U_0, U_{2J+1} be open subsets of A such that $v_0 \in U_0$ and $v_{2J+1} \in U_{2J+1}$. Then for any positive $\bar k$, γ and D, there exists a sequence of points $v^*_s \in \mathrm{int}(A)$ such that $v^*_0 \in U_0$, $v^*_{2J+1} \in U_{2J+1}$ and

$$\begin{aligned} v^*_{2j+1} &= F_0^{k_j}(v^*_{2j}) \qquad j = 0, \dots, J,\\ v^*_{2j} &= \bar F_{n_j}(v^*_{2j-1}) \quad j = 1, \dots, J,\end{aligned}$$

with the same n_j as in (65), $v^*_{2j-1} \in A''$ (the domain of the maps $\bar F_n$) for $j = 1, \dots, J$, and the numbers k_j form a proper sequence in the sense of (50).

Proof. The definition of the modified scattering maps $\bar{F}_n$ [see (47)] implies that it is enough to show that there exists a sequence of points $\hat{v}_s$ such that $\hat{v}_0 \in U_0$, $\hat{v}_{2J+1} \in U_{2J+1}$, and

$$\begin{aligned} \hat{v}_{2j+1} &= F_0^{\hat{k}_j}(\hat{v}_{2j}) \qquad (j = 0, \ldots, J), \\ \hat{v}_{2j} &= F_{n_j}(\hat{v}_{2j-1}) \quad (j = 1, \ldots, J) \end{aligned} \tag{66}$$

where n_j are taken from (65), $\hat{v}_{2j-1} \in A'$ for $j = 1, \ldots, J$, and the numbers $\hat{k}_j$ are such that numbers $k_j = \hat{k}_j - (m_+ + m_-)$ for $0 \le j \le J-1$ and $k_J = \hat{k}_J - m_+$ form a proper sequence. Then the sequence v_s^* is defined by the following equations

$$v_0^* = \hat{v}_0, \quad v_{2J+1}^* = \hat{v}_{2J+1}, \quad v_{2j-1}^* = F_0^{-m_-}\hat{v}_{2j-1}, \quad v_{2j}^* = F_0^{m_+}\hat{v}_{2j} \quad (j = 1, \ldots, J).$$

Note that (48) implies $v_{2j-1}^* \in A''$.

We construct the sequence $\hat{v}_j$ by induction in J. Let $J = 0$. Since $v_0 \in U_0$ and $v_1 = F_0^{i_0}(v_0) \in U_1$, there is $\hat{U} \subset U_1$, a small open neighbourhood of v_1 in A' such that $F_0^{-i_0}\hat{U} \subset U_0$. The Poincaré recurrence theorem implies that for any K there is $k > K$ such that $F_0^{-k}\hat{U} \cap \hat{U} \neq \emptyset$. Let $K = \bar{k} - i_0 + m_+ + m_-$ and $\hat{k}_0 = k + i_0$. Then

$$k_0 = \hat{k}_0 - m_- - m_+ \ge \bar{k}. \tag{67}$$

Moreover, for any $\hat{v}_0 \in F_0^{-k-i_0}\hat{U} \cap F^{-i_0}\hat{U} \neq \emptyset$, we have $\hat{v}_0 \in U_0$ and $\hat{v}_1 := F_0^{\hat{k}_0}(\hat{v}_0) \in U_1$.

Now let $J \ge 1$. The induction assumption implies that for any open subset $U_2 \subset A'$ such that $v_2 \in U_2$ there is a point $v' \in U_2$ such that $\mathcal{F}(v') \in U_{2J+1}$, where $\mathcal{F} = \left(\prod_{2\le j\le J} F_0^{\hat{k}_j} \circ F_{n_j}\right) \circ F_0^{\hat{k}_1}$ and the numbers $\hat{k}_j$ are such that the sequence k_j defined by

$$k_J = \hat{k}_J - m_+ \quad \text{and} \quad k_j = \hat{k}_j - m_- - m_+, \quad 2 \le j \le J-1, \tag{68}$$

is proper.

There is a small open neighbourhood U_1 of v_1 in A' such that $\mathcal{F} \circ F_{n_1}(U_1) \subseteq U_{2J+1}$. Since $v_0 \in U_0$ and $v_1 = F_0^{i_0}(v_0) \in U_1$, there is $\hat{U} \subset U_1$, a small open neighbourhood of v_1 in A' such that $F_0^{-i_0}\hat{U} \subset U_0$. The Poincaré recurrence theorem implies that for any K there is $k > K$ such that $F_0^{-k}\hat{U} \cap \hat{U} \neq \emptyset$. Let $K = \gamma k_1 + D - i_0 + m_+ + m_-$ and $\hat{k}_0 = k + i_0$. Then

$$\hat{k}_0 - m_- - m_+ \ge \gamma k_1 + D, \tag{69}$$

where k_1 given by (68). Let $k_0 = \hat{k}_0 - m_+ - m_-$. Then the sequence $k_0, \ldots, k_J$ is proper [see (50)], and Eq. (66) define a sequence $\hat{v}_s$ such that $\hat{v}_{2J+1} \in U_{2J+1}$, as $\hat{v}_{2J+1} = \mathcal{F} \circ F_{n_1}(\hat{v}_1) \in \mathcal{F} \circ F_{n_1}(U') \subseteq U_{2J+1}$. □

We say that two points v_0 and v_m are connected by an orbit of the iterated function system $\{F_0, \ldots, F_N\}$ if v_{2J+1} is an image of v_0 by a certain sequence of maps F_n. Obviously, this means that v_0 and v_m are the first and the last points in a sequence of points v_s constructed by the rule (65) with $m = 2J+1$. Since the corresponding sequence v_s^* constructed in Lemma 3 is a shadow of proper code, we may use Lemma 2. Thus, combining Lemmas 3 and 2, we obtain the following statement.

Lemma 4. *Let the map F_0 be area-preserving. Let two points $v_0 \in A$ and $v_m \in A$ be connected by an orbit of the iterated function system $\{F_0, \ldots, F_N\}$. Then, for any $\varepsilon > 0$ the ε-neighbourhoods of v_0 and v_m in $\mathbb{R}^{2d}$ are connected by an orbit of the map Φ.*

5. Symplectic Properties of Scattering Maps

Let $N \subset M$ be an open subset of a smooth symplectic manifold M endowed with a closed non-degenerate symplectic form Ω. We consider a diffeomorphism $\Phi : N \to \tilde{\Phi}(N) \subset M$ which preserves Ω. We assume that $A \subset N$ is a symmetrically normally hyperbolic invariant manifold. An important example is $M = \mathbb{R}^{2d}$ and A is a two-dimensional compact cylinder bounded by two invariant curves of Φ. We review some properties of the manifold A, its stable and unstable manifolds, and homoclinics to A. Similar results can be found e.g. in [32].

We start with establishing some useful geometric properties of the stable and unstable manifolds and the scattering maps. These properties are based on a symplectic orthogonality property of the next proposition.

Proposition 3. *If A is a symmetrically normally-hyperbolic invariant manifold and $x \in A$, then $T_y W^s(A) \perp_\Omega T_y E^{ss}(x)$ for any $y \in E^{ss}(x)$ and $T_y E^{uu}(x) \perp_\Omega T_y W^u(A)$ for any $y \in E^{uu}(x)$.*

Proof. Let $y \in W^s(A)$. Take any $w \in T_y E^{ss}(x)$ and $u \in T_y W^s(A)$. Since the map Φ preserves the form Ω, we have for any $m \in \mathbb{N}$:

$$\Omega(w,u) = \Omega((\Phi')^m w, (\Phi')^m u) = (\alpha\lambda)^m \Omega(\alpha^{-m}(\Phi')^m w, \lambda^{-m}(\Phi')^m u) = O((\alpha\lambda)^m).$$

Taking the limit $m \to +\infty$, we find that $\Omega(w,u) = 0$, i.e. $u \perp_\Omega v$. Thus, we have proved $T_y E^{ss}(x) \perp_\Omega T_y W^s(A)$. In a similar way we conclude that $T_y E^{uu}(x) \perp_\Omega T_y W^u(A)$ for any $y \in W^u(A)$. $\square$

Proposition 4. *The restriction of the symplectic form Ω to the symmetrically normally-hyperbolic invariant manifold A is non-degenerate.*

Proof. If the proposition is not true and the restriction of the symplectic form is degenerate, then there are $x \in A$ and a non-zero vector $w \in T_x(A)$ such that $w \perp_\Omega T_x(A)$. On the over hand $w \in T_x A = T_x W^s(A) \cap T_x W^u(A)$ implies that $w \perp_\Omega T_x E^{ss}_x$ and $w \perp_\Omega T_x E^{uu}_x$. The normal hyperbolicity assumptions imply that $TM = T_x E^{ss}_x \oplus T_x E^{uu}_x \oplus T_x A$ for any $x \in A$. Consequently, $w \perp_\Omega T_x M$, which contradicts to the non-degeneracy of Ω, and the proposition follows immediately. $\square$

We remind that a homoclinic intersection of $W^u(A)$ and $W^s(A)$ at a point y is *strongly transverse* if E^{uu}_y is transverse to $W^s(A)$ and E^{ss}_y is transverse to $W^u(A)$ at the point y.

Proposition 5. *If $y \in E^{uu}(x_1) \cap E^{ss}(x_2)$ for some $x_1, x_2 \in A$ and $T_y M = T_y E^{uu}(x_1) \oplus T_y W^s(A)$ then $T_y M = T_y E^{ss}(x_2) \oplus T_y W^u(A)$ and, consequently, the homoclinic intersection at y is strongly transverse.*

The proof of this proposition is completely straightforward: it is sufficient to note that under the assumptions of the proposition any vector from $T_y E^{ss}(x_2) \cap T_y W^u(A)$ is Ω-orthogonal to all vectors due to Proposition 3. The proposition implies that the strong transversality is equivalent to the transversality of the strong stable leaves to the unstable manifold (or the transversality of the strong unstable leaves to the stable manifold). This property reduces the number of conditions which are necessary to verify the strong transversality of a homoclinic intersection.

For every $y \in W^s(A)$ there is a unique $x \in A$ such that $y \in E^{ss}(x)$. We define the projection $\pi^s : W^s(A) \to A$ by setting $\pi^s(y) = x$. Let $v = (v_1, v_2, \ldots)$ be some coordinates on A defined in a small neighbourhood U of the point x. Define coordinates

(u, v) in $(\pi^s)^{-1}(U)$ such that $u = 0$ corresponds to a point in A and $v = \text{const}$ corresponds to a strong-stable leave of E^{ss}. In these coordinates $\pi^s : (u, v) \mapsto (u, 0)$.

Since $T_y E^{ss}(x) \perp_\Omega T_y W^s(A)$, we see that in these coordinates $\Omega|_{W^s(A)} = \sum_{i,j} a_{ij}(u, v) dv_i \wedge dv_j$. On the other hand, the symplectic form is closed, i.e., $d\Omega = 0$. So we have $d\Omega|_{W^s(A)} = \sum_{i,j,k} \frac{\partial a_{ij}}{\partial u_k} du_k \wedge dv_i \wedge dv_j = 0$. Consequently the coefficients a_{ij} do not depend on u and $\Omega|_{W^s(A)} = \sum_{i,j} a_{ij}(v) dv_i \wedge dv_j$.

Let B be any section of $W^s(A)$ transverse to the strongly stable leaves. Then the restriction $\pi^s|_B : B \to A$ is a local diffeomorphism. Moreover, since the projection is the identity in the coordinates u, we find that $\pi^s|_B$ is a symplectomorphism, i.e. it transforms $\Omega|_B$ into $\Omega|_A$. In particular, $\Omega|_B$ is non-degenerate, i.e. B is a symplectic manifold.

Obviously, a similar statement is true for the stable manifolds replaced by the unstable ones: for any section B of $W^u(A)$ transverse to the strongly unstable leaves, the projection $\pi^u : B \to A$ by the strongly unstable leaves is locally a symplectomorphism. Thus, we obtain the following

Proposition 6. *If $y \in W^s(A) \cap W^u(A)$ is a strongly transverse homoclinic point and B is a sufficiently small neighbourhood of y inside $W^s(A) \cap W^u(A)$, then the scattering map $F_B = \pi^s|_B \circ (\pi^u|_B)^{-1} : B^u \to B^s$ is a symplectomorphism, where $B^{u,s} = \pi^{u,s}(B) \subset A$.*

We can define the scattering map F_B relative to any connected subset B of $W^s(A) \cap W^u(A)$ that consists of strongly transverse homoclinic points. When B is not a small neighbourhood of a single point, the scattering map F_B does not need to be single-valued nor injective (eventhough every branch of it is a local diffeomorphism). In this paper we assume B to be a simple homoclinic cylinder. Then the scattering map is single-valued and injective, so it is a symplectic diffeomorphism defined on a large open subset A' of A.

Assume the symplectic form is exact, i.e., $\Omega = d\vartheta$, where ϑ is a differential 1-form. For example, in the case of our interest, $M = \mathbb{R}^{2d}$, $\Omega = dp \wedge dq$, and $\vartheta = pdq$. The symplectic map Φ is exact if

$$\int_\gamma \vartheta = \int_{\Phi(\gamma)} \vartheta$$

for every smooth closed curve γ. Obviously, the exactness of Φ implies the exactness of the map $F_0 = \Phi|_A$.

Proposition 7. *Let $A' \subseteq A$ be a region such that the scattering map F_B is a diffeomorphism $A' \to F_B(A') \subseteq A$. If for each point $x \in A'$ the corresponding leaves $E^{uu}(x)$ and $E^{ss}(F_B(x))$ intersect B exactly at one point, then the restriction of F_B on A' is exact.*

Proof. Let us prove that the map $(\pi^u|_B)^{-1}$ is exact on A'. The proof of the exactness of the map $(\pi^s|_B)^{-1}$ on $F_B(A')$ is exactly the same, so the exactness of F_B will follow immediately. Take any smooth closed curve $\gamma \subset A'$. By assumption, for any $x \in \gamma$ there is a unique point $y(x) \in B$ such that $y \in E^{uu}(x)$, the union of the points $y(x)$ over all $x \in \gamma$ gives the curve $(\pi^u|_B)^{-1}\gamma = \tilde{\gamma} \subset B$. As the strongly unstable leaves are simply-connected [each is a diffeomorphic copy of $\mathbb{R}^k$ where $2k = \dim(M) - \dim(A)$] and depend smoothly on the base point x, one can connect each point $x \in \gamma$ with the corresponding point $y(x) \in \tilde{\gamma}$ by a smooth arc $\ell(x)$ that lies in $E^{uu}(x)$ so that the union of these arcs forms a smooth two-dimensional surface $S \subset W^u(A)$, an annulus bounded by γ and $\tilde{\gamma}$. By Stokes theorem,

$$\int_\gamma \vartheta - \int_{\tilde{\gamma}} \vartheta = \int_S \Omega.$$

At every point $y \in S$ the tangent plane contains a vector tangent to one of the curves $\ell(x)$ which lies in the $E^{uu}(x)$, so Ω vanishes on $T_y S$ by Proposition 3. Thus, $\int_S \Omega = 0$, which gives us the required identity $\int_\gamma \vartheta = \int_{\tilde{\gamma}} \vartheta$ for every smooth closed curve γ in A'. □

Note that, surprisingly, the exactness of the scattering map in the statement above does not require the exactness of the map Φ itself.

6. Transport in an Iterated Functions System and Obstruction Curves

The symplecticity of the map $F_0 = \Phi|_A$ established in Proposition 4 means that this map is area-preserving (with the area of a domain obtained by integrating $\Omega|_A$ over this domain). Therefore, as shown in Sect. 4.3, for two open sets to be connected by an orbit from the homoclinic channel it is enough for these sets to be connected by the orbits of the iterated function system $\{F_0, F_1, \ldots, F_N\}$. As we showed in Sect. 5 all these maps are exact symplectomorphisms. The diffeomorphism F_0 is defined everywhere on the cylinder A which is invariant with respect to F_0, i.e. $F_0(A) = A$. The scattering maps F_n, $n = 1, \ldots, N$, are defined on a subset A' of the cylinder A and, as follows from the simplicity assumptions [S1]–[S3], they are homotopic to identity diffeomorphisms $A' \to A$. The exact symplecticity of the maps F_n implies that the area between any curve γ and its image $F_n(\gamma)$ is zero. Hence, $F_n(\gamma) \cap \gamma \neq \emptyset$ for any simple essential curve $\gamma \subset A'$.

We assume that there exist coordinates $v = (y, \varphi)$ in A such that the map $F_0 : (y, \varphi) \mapsto (\bar{\varphi}, \bar{y})$ in these coordinates satisfies the *twist condition*, i.e.

$$\frac{\partial \bar{\varphi}}{\partial y} \neq 0$$

everywhere in this cylinder (we assume that $\varphi \in \mathbb{S}^1$ is the angular variable).

Let $\bar{A}$ be a compact cylinder in A' bounded by two simple essential curves γ^+ and γ^- such that $\gamma^- \cap \gamma^+ = \emptyset$ (we no longer need to assume that $\bar{A}$ is invariant). Let γ^+ corresponds to larger values of y than γ^- does. The set $A \backslash \mathrm{int}(\bar{A})$ consists of two connected components, the upper component A^+ contains γ^+ and the lower component A^- contains γ^-. If $\bar{A}$ contains an essential curve γ^* which is *invariant for all of the maps* F_n, $n = 0, \ldots, N$, then the curve γ^* divides the cylinder $\bar{A}$ into two invariant parts, so no trajectory of the iterated function system $\{F_0, F_1, \ldots, F_N\}$ which starts within $\bar{A^-}$ can get to A^+. In other words, the absence of essential common invariant curves in $\bar{A}$ is a necessary condition for the orbits of iterated function system to connect $\bar{A^-}$ with A^+. The following theorem shows that this condition is also sufficient. This theorem generalises a result by Moeckel [75].

Theorem 3. *Let $F_1, \ldots, F_N$ be exact symplectomorphisms $A' \to A$, homotopic to identity. Let A be invariant with respect to a symplectic diffeomorphism F_0 which satisfies the twist condition on A. Suppose no essential curve in $\bar{A}$ is a common invariant curve* *for the maps F_n with $n = 0, 1, \ldots, N$. Then there is a finite trajectory $(v_i)_{i=0}^m \subset \bar{A}$ of the iterated function system $\{F_0, F_1, \ldots, F_N\}$ that starts on γ^- and ends on γ^+ (i.e. $v_0 \in \gamma^-$, $v_m \in \gamma^+$, and $v_{i+1} = F_{k_i}(v_i)$ for some sequence of $k_i \in \{0, \ldots, N\}$).*

Remark 5. As the common invariant curve is, in particular, an invariant curve of the twist map F_0, the Birkhoff theory implies that it is necessarily a graph of a Lipschitz function $y = y^*(\varphi)$, so it is sufficient to verify the absence of common invariant Lipschitz curves.

Remark 6. Our statement makes an important change in the setup of the problem compared to e.g. [67,75] as we do not ask the boundaries γ^- and γ^+ to be invariant with respect to any of the maps $F_n, n = 0, \ldots, N$. Indeed, it is not natural to assume that the scattering maps preserve the boundaries as this would require certain non-transversality of stable and unstable manifolds associated with the Φ-invariant curves on the boundary.

Proof of Theorem 3. We say that a map $F : A' \to A$ has a *strong intersection property* if $F(\gamma) \cap \gamma \neq \emptyset$ for any simple essential curve $\gamma \subset A'$ and, moreover, if $F(\gamma) \neq \gamma$, then $F(\gamma)$ has points in both components of $A \backslash \gamma$. The symplectomorphisms $F_n : A' \to A$, which are exact and homotopic to identity, have the intersection property.

The boundary of the F_0-invariant cylinder A consists of two non-intersecting essential curves. We refer to the boundary curve with larger values of the coordinate y as the upper boundary of A. Let $\gamma \subset A'$ be a simple essential curve and let γ^n be the boundary of the connected component of $A \backslash (\gamma \cup F_n(\gamma))$ adjacent to the upper boundary of A. This is also a simple essential curve. Denote by $\mathcal{F}_n$ the operator that replaces the curve γ by γ^n. By construction, $\mathcal{F}_n(\gamma)$ has no points below γ and the intersection property of F_n implies that $\gamma \cap \mathcal{F}_n(\gamma) \neq \emptyset$. If $\mathcal{F}_n(\gamma^-) \cap \gamma^+ \neq \emptyset$ for some n, we have found a connecting orbit. Indeed, take $v_1 \in \mathcal{F}_n(\gamma^-) \cap \gamma^+$ and let $v_0 = F_n^{-1}(v_1)$.

We continue by induction. Let $m = 0$, $\gamma_0 = \gamma^-$. Let us construct, inductively, a sequence of simple essential curves $\gamma_m \subset \bar{A}$, such that each point of γ_m can be reached by a trajectory which starts on γ^- and has the length not larger than m. Suppose we have constructed such γ_m for some $m \geq 0$. If $\mathcal{F}_n(\gamma_m) \cap \gamma^+ \neq \emptyset$ for some n, the inductive process is terminated as the intersection point belongs to a trajectory which starts on γ^- and finishes on γ^+ as required. Otherwise define γ_{m+1} as the boundary of that connected component of $A \backslash (\cup_n \mathcal{F}_n(\gamma_m))$ which is adjacent to the upper boundary of A. Obviously, γ_{m+1} is a simple essential curve. The intersection property implies $\mathcal{F}_n(\gamma_m) \cap \gamma_m \neq \emptyset$. Then taking into account that for every n the curve $\mathcal{F}_n(\gamma_m)$ has no points below γ_m and does not intersects γ^+, we conclude that the curve γ_{m+1} belongs to a cylinder bounded by γ_m and γ^+. So $\gamma_{m+1} \subset \bar{A}$.

We claim that this process terminates after a finite number of steps because otherwise the maps F_n would have a common invariant essential curve in $\bar{A}$.

Indeed, suppose that the process does not terminate. Then the curves $\gamma_m \subset \bar{A}$ form a "bounded and monotone" sequence. Namely, if we denote as γ_0^+ the upper boundary of A, then the closed cylinders $[\gamma_m, \gamma_0^+]$ bounded by the curves γ_m and γ_0^+ form a monotone sequence of closed sets (as γ_{m+1} has no points below γ_m). Then $\tilde{U}^* = \cap_{m \geq 0} [\gamma_m, \gamma_0^+]$ is closed and has non-empty interior since $[\gamma^+, \gamma_0^+] \subset \tilde{U}^*$. Let U^* be the connected component of $\operatorname{int}(\tilde{U}^*)$ adjacent to the upper boundary γ_0^+. Let $\gamma^* = \partial U^* c \gamma_0^+$ (i.e. ∂U^* is the disjoint union of γ^* and γ_0^+).

Let us show that γ^* is an essential curve, invariant with respect to F_0. First, we note that for any point $p^* \in \gamma^*$ there is a sequence of points $p_m \in \gamma_m$ such that $\lim_{m \to \infty} p_m = p^*$. Indeed, otherwise there is an open neighbourhood Q of p^* and an unbounded subsequence m_k such that $\gamma_{m_k} \cap Q = \emptyset$. Then $Q \subset \operatorname{int}[\gamma_{m_k}, \gamma_0^+]$ (recall that Q intersects γ^* and $\gamma^* \subset [\gamma_{m_k}, \gamma_0^+]$ for each m_k). Since the sequence of cylinders is monotone, it follows that $Q \subset \mathrm{int}[\gamma_m, \gamma_0^+]$ for all m. Thus $Q \subset \mathrm{int}(\tilde{U}^*)$, which contradicts to $p^* \in \gamma^*$.

We can approximate the sequence p_m by a sequence of points $p'_m \to p^*$ such that p'_m lies outside $[\gamma_m, \gamma_0^+]$ (below γ_m) for each m, i.e. $p'_m \notin \tilde{U}^*$. Thus, each point of γ^* is a limit of a sequence of points which do not lie in U^*, i.e. γ^* forms the boundary of the closure of U^* (a priori, some points of the boundary of an opens set may not lie in the boundary of the closure of the set).

It also follows that $F_n(\gamma^*) \cap U^* = \emptyset$ for all n. Indeed, suppose $F_n(p^*) \in U^*$ for some $p^* \in \gamma^*$. Then, since p^* is a limit of points lying in the curves γ_m and U^* is open, there is $p_m \in \gamma_m$ such that $F_n(p_m) \in U^*$, which is impossible as, by the construction, $F_n(\gamma_m)$ lies below γ_{m+1} and, hence, has no points inside U^*.

In particular, we have $F_0(\gamma^*) \cap U^* = \emptyset$, which means that $U^* \subseteq F_0(U^*)$ and

$$cl(U^*) \subseteq F_0(cl(U^*)). \tag{70}$$

Would the image of any point $q \in cl(U^*)$ by the map F_0 lies outside $cl(U^*)$, then the images of all points from U^* which are close enough to q would also lie outside $cl(U^*)$, i.e. the set $F_0(cl(U^*))$ would have an open subset outside of $cl(U^*)$ (recall that U^* is open). Thus, the Lebesgue measure of $F_0(cl(U^*))$ would be strictly greater than the measure of $cl(U^*)$, which is a contradiction with the area-preservation property of F_0. Therefore, it follows from (70) that, in fact, $F_0(cl(U^*)) = cl(U^*)$, i.e. U^* is an invariant domain for the twist map F_0. Now, Birkhoff theorem implies that the boundary γ^* of U^* is a simple essential curve, invariant with respect to F_0.

The set U^* is one of the two connected components of $A \backslash \gamma^*$. Since $F_n(\gamma^*) \cap U^* = \emptyset$ for all n, the strong intersection property implies that $F_n(\gamma^*) = \gamma^*$ for all n. We have proved that the non-existence of a connecting trajectory is equivalent to the existence of a common invariant curve. □

Theorem 3 is valid for any two non-intersecting essential curves in A': either they are connected by an orbit of the iterated function system, or there is an essential curve γ^* between them which is invariant with respect to all maps F_n. It follows that the absence of a common invariant essential curve in $\bar{A}$ is equivalent to the existence of an orbit of the iterated function systems which connects $\mathrm{int}(A^+)$ with $\mathrm{int}(A^-)$ (move the curves γ^+ and γ^- inside $\mathrm{int}(A^+)$ and, respectively, $\mathrm{int}(A^-)$, and apply Theorem 3 to these curves). Since the existence of such orbit is an open property, Theorem 3 implies that the cylinder $\bar{A}$ contains no essential curve invariant with respect to all maps $F_0, \dots, F_N$ for an *open* set of maps from $\mathcal{V}_N$. In the next Section we show that this set of maps is also *dense* in $\mathcal{V}_N$. This will finish the proof of the Main Theorem: it follows immediately from Theorem 3 and Lemma 4 that for any map Φ from this open and dense set any two neighbourhoods of γ^- and γ^+ are connected by Φ.

7. Simultaneous Destruction of All Obstruction Curves

We finish the proof of the Main Theorem by showing that for a map Φ from a dense subset of the set $\mathcal{V}_N$ the corresponding maps $F_0, F_1, \dots, F_N$ do not have a common essential invariant curve, provided $N \geq 8$. As F_0 is a twist map, we can restrict the problem to Lipshitz invariant curves only. Recall that for any map Φ from $\mathcal{V}_N$ there exists a compact normally-hyperbolic invariant cylinder A. We introduce coordinates

(y, φ) on A such that the restriction F_0 of Φ on A has a twist property. In these coordinates $F_0 : (y, \varphi) \mapsto (\bar{y}, \bar{\varphi})$ and

$$\frac{\partial \bar{y}}{\partial \varphi} \neq 0$$

for all $(y, \varphi) \in A$. By the Birkhoff theorem, every essential invariant curve of F_0 is Lipschitz:

$$y = y(\varphi), \quad |y(\varphi_1) - y(\varphi_2)| \leq L|\varphi_1 - \varphi_2|,$$

where the Lipschitz constant L satisfies

$$L \leq \sup_{v \in \bar{A}} \max \left\{ \left| \frac{\partial \bar{\varphi}}{\partial \varphi} \right| / \left| \frac{\partial \bar{\varphi}}{\partial y} \right|, \quad \left| \frac{\partial \bar{y}}{\partial y} \right| / \left| \frac{\partial \bar{\varphi}}{\partial y} \right| \right\}.$$

Given map $\Phi \in \mathcal{V}_N$, we can choose the constant L the same for all maps from a neighbourhood of Φ in $\mathcal{V}_N$ (since the maps which are close in $\mathcal{V}_N$ are also C^1-close, and the corresponding cylinders A are C^1-close as well).

By the assumptions of the Main Theorem, we have a compact subcylinder $\bar{A}$ in A such that $N \geq 8$ scattering maps are defined on a neighbourhood A' of $\bar{A}$. The cylinder $\bar{A}$ depends continuously on the map Φ, so we can choose A' to be the same (in appropriately chosen coordinates (y, φ)) for all maps close to Φ. We can also assume that the maps $F_1, \ldots, F_N$ are defined in some neighbourhood of the closure of A'. Note that the scattering maps depend continuously on the map Φ in the following sense: if two maps Φ are C^2-close, then the corresponding scattering maps are C^1-close.

Theorem 4. *Arbitrarily close to any map Φ, in $\mathcal{V}_N$ there exists a map for which the corresponding scattering maps $F_1, \ldots, F_8$ have no common L-Lipschitz invariant curves in A'.*

Proof. Consider the space of all L-Lipshitz (periodic) functions $y = y(\varphi)$ endowed with the C^0-metric. Let $\mathcal{L}$ be the subset of this space which consists of all functions whose graphs lie in the closure of A' and are invariant, simultaneously, for all the scattering maps $F_1, \ldots, F_8$ generated by the map Φ. If $\mathcal{L} = \emptyset$, there is nothing to prove. If $\mathcal{L} \neq \emptyset$, we note that $\mathcal{L}$ is compact, so given any $\delta > 0$ there is a finite set of L-Lipshitz curves $C_1, \ldots, C_q$ such that each of them is invariant with respect to all the maps $F_1, \ldots, F_8$ and every other common invariant L-Lipshitz curve lies in the δ-neighbourhood of one of the curves C_s, i.e. it belongs to the cylinder $A_s := \{|y - y_s(\varphi)| \leq \delta\}$ where $y = y_s(\varphi)$ is the equation of the curve C_s. Moreover, the set of the L-Lipshitz common invariant curves of the scattering maps depends upper-semicontinuously on the map Φ [if we have a sequence of maps $\Phi^{(k)}$ that converges to Φ in C^2, then the corresponding scattering maps $F_j^{(k)}$ converge to the scattering maps F_j in C^1; and if the maps $F_j^{(k)}$ each have an L-Lipshitz invariant curve, then the set of the limit points of these curves as $k \to +\infty$ is the union of a set of L-Lipshitz curves each of which is invariant with respect to the scattering maps F_j]. Thus, for all maps from $\mathcal{V}_N$ which are sufficiently close to Φ, every common invariant L-Lipshitz curve of the scattering maps that lies in A' lies entirely in one of the cylinders $A_1, \ldots, A_q$.

Below [see (73)] we will fix, once and for all, a certain value of $\delta > 0$ which will give us a finite set of these cylinders A_s. We will show for each such cylinder A_s that arbitrarily close to Φ in $\mathcal{V}_N$ there exists a map for which the corresponding scattering maps $F_1, \ldots, F_8$ have no common L-Lipschitz invariant curves in A_s. This will prove the theorem. Indeed, the absence of the common invariant L-Lipshitz curves in any given (open) cylinder is an open property. So, we first perturb the map Φ to get rid of all common invariant L-Lipshitz curves in the cylinder A_1, then we add another small perturbation to kill all common invariant L-Lipshitz curves in A_2—by choosing the perturbation small enough we guarantee that no new common invariant L-Lipshitz curves emerge in A_1, etc. Then, after finitely many steps of the procedure, we will have all the cylinders $A_1, \ldots, A_q$ cleaned of common invariant L-Lipshitz curves.

Let $R > 1$ be a constant that bounds the derivatives of the scattering maps:

$$\left\| \frac{\partial F_j}{\partial (y, \varphi)} \right\| < R \tag{71}$$

for all $(y, \varphi) \in A'$, $j = 1, \ldots, 8$, and all maps that are close enough to Φ in $\mathcal{V}_N$. Recall that φ is an angular variable that runs a circle $\mathbb{S}^1$; we assume that the length of the circle is 2π. Choose 4 arcs $J_i \subsetneq \mathbb{S}^1$, $i \in \{1, 2, 3, 4\}$, such that $J_1 \cup J_2 = J_3 \cup J_4 = \mathbb{S}^1$. Moreover, denote $J_{ik} = J_i \backslash J_k$ and let us assume that J_{12}, J_{34}, J_{21} and J_{43} are disjoint and located in the circle in the same order as they are listed here (following the orientation of the circle). Neither of the arcs J_i constitutes the whole circle, so their lengths are smaller than 2π. Choose any L-Lipshitz curve $C : y = y_C(\varphi)$ which is invariant with respect to all maps $F_1, \ldots, F_8$. Each arc J_i corresponds to an arc $\hat{J}_i : \{y = y_C(\varphi), \varphi \in J_i\}$ of the curve C. Since C is invariant with respect to each of the maps F_j, the image $F_j(\hat{J}_i)$ also lies in C. Hence it is given by $F_j(\hat{J}_i) := \{y = y_C(\varphi), \varphi \in \bar{J}_i^j\}$ where $\bar{J}_i^j$ is an arc in $\mathbb{S}^1$ which does not cover the whole of $\mathbb{S}^1$, so its length is strictly less than 2π. Since the set $\mathcal{L}$ of all common invariant L-Lipshitz curves is compact, we have

$$K = \max_{C \in \mathcal{L}} \max_{i,j} \text{ length}(\bar{J}_i^j) < 2\pi. \tag{72}$$

Now, we choose

$$\delta = \frac{2\pi - K}{R} > 0. \tag{73}$$

As it was explained above, the compactness of $\mathcal{L}$ implies that every possible common invariant L-Lipshitz curve lies in one of a finitely many cylinders A_s; each of these cylinders is the δ-neighbourhood of some invariant L-Lipshitz curve $C_s : \{y = y_s(\varphi)\}$. Take any of these cylinders. Note that, by virtue of (71), the image $F_j(A_s \cap \{\varphi \in J_i\})$ lies inside the $(R\delta)$-neighbourhood of the curve $F_j(C_s \cap \{\varphi \in J_i\})$. This curve is a subset of the invariant curve C_s, and it corresponds to an interval of φ values such that the length of this interval does not exceed the constant K defined by (72). Thus, by (73),

$$F_j(A_s \cap \{\varphi \in J_i\}) \subset \{|y - y_s(\varphi)| < R\delta, \quad \varphi \in \hat{J}_{sij}\} \tag{74}$$

where $\hat{J}_{sij}$ is a certain arc whose length is strictly less than 2π, i.e. it does not cover the entire $\mathbb{S}^1$. As F_j depends continuously on the map Φ, inclusion (74) holds for all maps from $\mathcal{V}_N$ which are close enough to Φ.

Now, let us imbed the map Φ into a two-parameter analytic family of maps Φ_{μ_1,μ_2} from $\mathcal{V}_N$ such that $\Phi_0 = \Phi$. We will show (Lemmas 5, 6) that this family can be chosen such that there exist arbitrarily small values of $\mu = (\mu_1, \mu_2)$ for which the scattering maps $F_1, \ldots, F_8$ defined by the map Φ_μ have no common L-Lipschitz invariant curves in the cylinder A_s. The map Φ_μ that corresponds to a small value of μ is a small perturbation of Φ, so this gives us the required arbitrarily small perturbations that clear the cylinder A_s of the common L-Lipshitz invariant curves of the scattering maps. By performing this perturbations consecutively for each of the cylinders $A_1, \dots, A_q$ we will obtain the result of the theorem.

Note that the invariant cylinder A, its stable and unstable manifolds, as well as the strong stable and strong unstable foliations depend smoothly on μ, therefore the scattering maps also depend smoothly on μ. This means that for all small μ we can introduce coordinates (y, φ) on the μ-dependent cylinder A such that the maps F_j, $j = 0, \dots, N$, will be given each by a pair of smooth functions Y_j, Ψ_j of (y, φ, μ):

$$F_j : (y, \varphi) \mapsto (Y_j(y, \varphi, \mu), \Psi_j(y, \varphi, \mu)).$$

Let our family Φ_μ be chosen such that for all $(\varphi, y) \in A_s$

$$\left\| \frac{\partial \Psi_j}{\partial(\mu_1, \mu_2)} \right\| < 1 \qquad \text{for all } j = 1, \dots, 8, \tag{75}$$

$$\left| \frac{\partial Y_{1,2,3,4}}{\partial \mu_2} \right| < 1, \qquad \left| \frac{\partial Y_{5,6,7,8}}{\partial \mu_1} \right| < 1, \tag{76}$$

$$j = 1, 2\colon \quad \frac{\partial Y_j}{\partial \mu_1} > 2(L+1) \quad \text{and} \quad \frac{\partial Y_{j+4}}{\partial \mu_2} > 2(L+1) \quad \text{when } \Phi_j(\varphi, y, \mu) \in J_j, \tag{77}$$

$$j = 3, 4\colon \quad \frac{\partial Y_j}{\partial \mu_1} < -2(L+1) \quad \text{and} \quad \frac{\partial Y_{j+4}}{\partial \mu_2} < -2(L+1) \quad \text{when } \Phi_j(\varphi, y, \mu) \in J_j, \tag{78}$$

where L is the Lipschitz constant in the condition of the theorem, and J_j are the four arcs defined above. Lemma 6 establishes the existence of a family Φ_μ which satisfies these properties. Then the main theorem follows from the following statement.

Lemma 5. *For every family of maps Φ_μ, $\mu = (\mu_1, \mu_2)$, such that the derivatives of the scattering maps $F_1, \dots, F_8$ satisfy estimates (75)–(78) for all $(\varphi, y) \in A_s$, the set of parameter values for which the scattering maps $F_1, \dots, F_8$ have an L-Lipshitz common invariant essential curve in A_s has measure zero. In particular, there exist arbitrarily small values of μ for which the maps $F_1, \dots, F_8$ have no L-Lipshitz common invariant essential curves in the cylinder A_s.*

Proof. Take any two, may be equal, values of μ: $\mu = \mu^*$ and $\mu = \mu^{**}$, such that at $\mu = \mu^*$ the maps $F_1, \dots, F_8$ have a common L-Lipschitz invariant curve $\mathcal{L}^* : \{y = y^*(\varphi), \varphi \in \mathbb{S}^1\} \subset A_s$ and at $\mu = \mu^{**}$ they have a common L-Lipschitz invariant curve $\mathcal{L}^{**} : \{y = y^{**}(\varphi), \varphi \in \mathbb{S}^1\} \subset A_s$. Let us show that the following condition holds:

$$\|\mu^* - \mu^{**}\| \le R|y^*(0) - y^{**}(0)|, \tag{79}$$

where R is defined in (71) and $\|\mu\| = \max\{|\mu_1|, |\mu_2|\}$.

We note that without losing in generality we may assume that

$$y^*(0) \ge y^{**}(0), \tag{80}$$

$$|\mu_2^* - \mu_2^{**}| \le |\mu_1^* - \mu_1^{**}| \qquad \text{and} \qquad \mu_1^* \ge \mu_1^{**}. \tag{81}$$

If necessary, these inequalities can be achieved by swapping y and $(-y)$, μ and $(-\mu)$, $F_1 \leftrightarrow F_3$, $F_2 \leftrightarrow F_4$, $F_5 \leftrightarrow F_7$, $F_6 \leftrightarrow F_8$, as well as $\mu_1 \leftrightarrow \mu_2$ and $F_{1,2,3,4} \leftrightarrow F_{5,6,7,8}$. Conditions (75)–(78) are symmetric with respect to these changes.

Now suppose (79) is not true, i.e.

$$0 \le y^*(0) - y^{**}(0) < \frac{\Delta\mu}{R}, \tag{82}$$

where

$$\Delta\mu = \mu_1^* - \mu_1^{**} > 0.$$

Since $J_3 \cup J_4 = \mathbb{S}^1$, we have that $\varphi = 0$ lies at least in one of the arcs J_3 or J_4. For definiteness, we assume $0 \in J_3$. Let $(\varphi^*, \bar{y}^*) = F_3(0, y^*(0), \mu^*)$ and $(\varphi^{**}, \bar{y}^{**}) = F_3(0, y^{**}(0), \mu^{**})$, i.e.

$$\begin{aligned} \varphi^* &= \Psi_3(0, y^*(0), \mu^*), & \bar{y}^* &= Y_3(0, y^*(0), \mu^*), \\ \varphi^{**} &= \Psi_3(0, y^{**}(0), \mu^{**}), & \bar{y}^{**} &= Y_3(0, y^{**}(0), \mu^{**}). \end{aligned}$$

Standard estimates based on the mean value theorem and formulas (71), (75), (76), (78), (81), (82) imply that

$$|\varphi^{**} - \varphi^*| < 2\Delta\mu, \quad \bar{y}^* - \bar{y}^{**} < -2L\Delta\mu.$$

Since the curves $y = y^*(\varphi)$ and $y = y^{**}(\varphi)$ are invariant with respect to F_3 (at $\mu = \mu^*$ and $\mu = \mu^{**}$ respectively), it follows that $\bar{y}^* = y^*(\varphi^*)$, $\bar{y}^{**} = y^{**}(\varphi^{**})$. Because of the L-Lipschitz property, we find that

$$y^*(\varphi^*) - y^{**}(\varphi^*) = \bar{y}^* - \bar{y}^{**} + y^{**}(\varphi^{**}) - y^{**}(\varphi^*) < -2L\Delta\mu + 2L\Delta\mu < 0.$$

Then taking into account (80) we conclude that

$$\mathcal{L}^* \cap \mathcal{L}^{**} \neq \emptyset.$$

Recall that the cylinder A depends on μ, so the two curves $\mathcal{L}^*$ and $\mathcal{L}^{**}$ lie, strictly speaking on different cylinders. Therefore, in order to stay completely rigorous, when we say that these two curves intersect, we mean that there is a value of φ such that $y^*(\varphi) = y^{**}(\varphi)$.

Now, let us call an arc $I \subset \mathbb{S}^1$ *positive* if $y^*(\varphi) > y^{**}(\varphi)$ for all $\varphi \in \mathrm{int}(I)$ and $y^*(\varphi) = y^{**}(\varphi)$ at the end points of I. We call an arc *negative*, if $y^*(\varphi) = y^{**}(\varphi)$ at its end points and $y^*(\varphi) < y^{**}(\varphi)$ on its interior. It is convenient to allow arcs to have empty interiors, i.e. any point from $\mathcal{L}^* \cap \mathcal{L}^{**}$ is considered to be both a positive and a negative arc at the same time.

We have just proved that there is at least one negative and at least one positive arc. For a positive arc I, let $\mathcal{L}_I^* = \{y = y^*(\varphi), \varphi \in I\}$ and $\mathcal{L}_I^{**} = \{y = y^{**}(\varphi), \varphi \in I\}$ be the corresponding pieces of the curves $\mathcal{L}^*$ and $\mathcal{L}^{**}$, and let $\mathcal{D}_I = \{y^*(\varphi) \ge y \ge y^{**}(\varphi), \varphi \in I\}$ be the region bounded by $\mathcal{L}_I^*$ and $\mathcal{L}_I^{**}$. Let us show that if $I \subseteq J_j$ for $j = 1$ or $j = 2$, then, with this j, the image of $\mathcal{L}_I^*$ by the map F_j at $\mu = \mu^*$ lies strictly inside $\mathcal{L}_{I'}^*$ which corresponds to a positive arc I' and

$$\mathrm{length}(I') > \Delta\mu > 0, \tag{83}$$

$$\mathrm{area}(\mathcal{D}_{I'}) > \mathrm{area}(\mathcal{D}_I). \tag{84}$$

Indeed, denote as F_j^* the map F_j at $\mu = \mu^*$ and F_j^{**} the map F_j at $\mu = \mu^{**}$. Take any point $M = (\varphi, y^*(\varphi)) \in \mathcal{L}_I^*$, so $\varphi \in I$. Let $M^* = (\varphi^*, y^*(\varphi^*)) \in \mathcal{L}^*$ be the image of M by the map F_j^*, and $M' = (\varphi', y') \in F_j^{**}(\mathcal{L}_I^*)$ be the image of M by the map F_j^{**}.

Since I is a positive arc, we have that for any $\varphi \in I$ the point M is either on the curve $\mathcal{L}^{**}$ or above it. Since $\mathcal{L}^{**}$ is invariant with respect to F_j^{**}, the point M' also does not lie below $\mathcal{L}^{**}$, i.e.

$$y' \geq y^{**}(\varphi'). \tag{85}$$

We have

$$\varphi' = \Psi_j(\varphi, y^*(\varphi), \mu^{**}), \quad y' = Y_j(\varphi, y^*(\varphi), \mu^{**}),$$
$$\varphi^* = \Psi_j(\varphi, y^*(\varphi), \mu^*), \quad y^*(\varphi^*) = Y_j(\varphi, y^*(\varphi), \mu^*).$$

Then inequalities (75)–(77) imply that

$$|\varphi^* - \varphi'| < \Delta\mu, \quad y^*(\varphi^*) - y' > (2L+1)\Delta\mu$$

(recall that we assume $I \subseteq J_j$, hence $\varphi \in J_j$). By (85) and the L-Lipschitz property of $\mathcal{L}^{**}$ we obtain

$$y^*(\varphi^*) - y^{**}(\varphi^*) > (L+1)\Delta\mu > 0, \tag{86}$$

and

$$y^*(\varphi') - y' > (L+1)\Delta\mu > 0. \tag{87}$$

Denote $\tilde{F}_j(\varphi) = \Psi_j(\varphi, y^*(\varphi), \mu^*)$, i.e., $\tilde{F}_j$ is the restriction of the map F_j^* on the invariant curve $\mathcal{L}^*$. We have just showed that if $\varphi \in I$, where $I \subseteq J_j$ is a positive arc, then $\varphi^* = \tilde{F}_j(\varphi)$ satisfies (86), i.e. it is inside some positive arc I'. Moreover, at the end points of I' we must have $y^* - y^{**} = 0$ while at the points of $\tilde{F}_j(I) \Subset I'$ we have $y^* - y^{**} > L\Delta\mu$ by (86), hence the length of I' is bounded from below as in (83), by virtue of the $2L$-Lipschitz property of the function $y^*(\varphi) - y^{**}(\varphi)$.

We have shown that $F_j^*(\mathcal{L}_I^*) \subset \mathcal{L}_{I'}^*$ and $F_j^{**}(\mathcal{L}_I^{**}) \subset \mathcal{L}_{I'}^{**}$ where I' is a positive arc. As the point M runs $\mathcal{L}_I^*$, the point M' runs the curve $\mathcal{L}' = F^{**}(\mathcal{L}_I^*)$, and it follows from (85), (87) that the curve $\mathcal{L}'$ lies between $\mathcal{L}^*$ and $\mathcal{L}^{**}$, strictly below $\mathcal{L}^*$. Since the end points of $\mathcal{L}'$ coincide with the end points of $F_j^{**}(\mathcal{L}_I^{**})$ and the latter lie inside $\mathcal{L}_{I'}^{**}$, it follows that $\mathcal{L}'$ lies between $\mathcal{L}_{I'}^*$ and $\mathcal{L}_{I'}^{**}$, strictly below $\mathcal{L}_{I'}^*$. Therefore the area of the region $F_j^{**}(\mathcal{D}_I)$ bounded by the curves $\mathcal{L}'$ and $F_j^{**}(\mathcal{L}_I^{**})$ is strictly smaller than the area of the region $\mathcal{D}_{I'}$ bounded by the curves $\mathcal{L}_{I'}^*$ and $\mathcal{L}_{I'}^{**}$. As the map F_j is area-preserving, $\text{area}(F_j^{**}\mathcal{D}_I) = \text{area}(\mathcal{D}_I)$, and (84) follows.

Thus, we start with any positive arc I which is contained entirely inside J_1 or J_2 and obtain a sequence I_s of positive arcs such that $I_0 = I$ and $\tilde{F}_{j_s}(I_s) \subset I_{s+1}$, where we chose $j_s = 1$ if $I_s \subseteq J_1$, and $j_s = 2$ if $I_s \subseteq J_2$ and $I_s \nsubseteq J_1$. If for some s the arc I_s is not entirely contained neither in J_1 nor in J_2, the sequence is terminated. By (84), the area of the region $\mathcal{D}_{I_s}$ is a strictly increasing function of s, so the arcs with different s can never coincide. The definition of a positive arc implies that the intersection of interiors for two different positive arcs is always empty. Thus, the arcs $int(I_s)$ are mutually disjoint. By (83), no more than $\frac{2\pi}{\Delta\mu}$ of such arcs can coexist in $\mathbb{S}^1$. We conclude that the sequence I_s must terminate. This means the last arc in the sequence is not contained entirely neither in J_1 nor in J_2, i.e. we have proved that there is a positive arc I^+ such that both $I^+ \cap J_{12} \neq \emptyset$ and $I^+ \cap J_{21} \neq \emptyset$.

Similarly, one proves that there exists a negative arc I^- such that $I^- \cap J_{34} \neq \emptyset$ and $I^- \cap J_{43} \neq \emptyset$. Since J_{12}, J_{34}, J_{21} and J_{43} are placed on $\mathbb{S}^1$ in this order, we find that the interiors of I^+ and I^- intersect, which is impossible by the definition of positive and negative arcs. Thus, by contradiction, we have established estimate (79).

Let $\mathcal{M} \subset \mathbb{R}^2$ be the set of all μ such that the maps $F_1, \ldots, F_8$ have at least one common L-Lipschitz invariant curve in the cylinder A_s. Let $\mathcal{Y}$ be the set which consists of all intersection points of these curves with the axis $\varphi = 0$. By (79), for each $y_0 \in \mathcal{Y}$ there is exactly one $\mu \in \mathcal{M}$ such that the corresponding system of scattering maps has a common L-Lipshitz invariant curve that lies in A_s and intersects the line $\varphi = 0$ at $y = y_0$. Estimate (79) also implies that $y_0 \mapsto \mu$ is an R-Lipschitz function $\mathcal{Y} \to \mathcal{M}$. For any Lipschitz function from a subset of $\mathbb{R}$ to $\mathbb{R}^2$, the Lebesgue measure of the image vanishes. Thus, as $\mathcal{Y}$ is a subset of an interval, it follows that $\text{mes}(\mathcal{M}) = 0$. The lemma is proved. □

We stress that Lemma 5 holds for any family of symplectic maps Φ_μ which satisfy the conditions (75)–(78). In order to finish the prove of the main theorem, it remains to show that such family can be constructed inside the space $\mathcal{V}_N$ of analytic exact-symplectic maps. This is given by the lemma below.

Lemma 6. *Any map $\Phi \in \mathcal{V}_N$ can be imbedded into an analytic family of analytic exact-symplectic maps Φ_μ that satisfies conditions* (75)–(78).

Proof. We define $\Phi_\mu = X_\mu \circ \Phi$, where X_μ is an analytic family of exact-symplectic maps such that $X_0 = \text{id}$. We set $X_\mu = X^{(1)}_{\mu_1} \circ X^{(2)}_{\mu_2}$ where $X^{(i)}_{\mu_i}$ is the time-μ_i shift along the orbits of the vector field defined by an analytic Hamiltonian function H_i $(i = 1, 2)$. Since we are interested in small μ, it is enough to check the conditions (75)–(78) at $\mu = 0$ only. Therefore the family $\Phi_\mu = X_\mu \circ \Phi$ satisfies (75)–(78) for all small μ, provided the conditions

$$\text{for all } \varphi \in \mathbb{S}^1: \quad \begin{aligned} &\left|\frac{\partial \Psi_j}{\partial \mu_i}\right|_{\mu_i=0} < 1 \qquad (j = 1, \ldots, 8), \\ &\left|\frac{\partial Y_j}{\partial \mu_i}\right|_{\mu_i=0} < 1 \qquad (j = 9 - 4i, \ldots, 12 - 4i), \end{aligned} \tag{88}$$

$$\begin{aligned} &\text{for all } \varphi \in J_j \text{ with } j = 1, 2: \qquad \left.\frac{\partial Y_{j+4(i-1)}}{\partial \mu_i}\right|_{\mu_i=0} > \ 2(L+1), \\ &\text{for all } \varphi \in J_j \text{ with } j = 3, 4: \qquad \left.\frac{\partial Y_{j+4(i-1)}}{\partial \mu_i}\right|_{\mu_i=0} < -2(L+1), \end{aligned} \tag{89}$$

are satisfied by the scattering maps for the families $\Phi^{(i)}_{\mu_i} = X^{(i)}_{\mu_i} \circ \Phi$, $i = 1, 2$, for all $(\varphi, y) \in A_s$.

Let us construct a family of maps $X^{(1)}_{\mu_1}$ for which these conditions are satisfied (the construction for $i = 2$ is essentially the same). Inequalities (88) and (89) are strict and involve only the first derivatives of the scattering maps. A C^2-small change of the family $\Phi^{(1)}_{\mu_1}$ leads to a C^1-small change of the strong-stable and strong-unstable foliations and, therefore, a C^1-small change of the scattering maps. Thus, it is enough to build a C^2-smooth family of maps $X^{(1)}_{\mu_1}$ [generated by a C^3-smooth Hamiltonian $H^{(1)}$] such that the corresponding scattering maps satisfy (88) and (89). Then for any sufficiently C^3-close approximation of $H^{(1)}$ by an analytic Hamiltonian [the analiticity of $H^{(1)}$ and $H^{(2)}$ is needed for the family Φ_μ to be analytic, i.e. lie in $\mathcal{V}_N$] conditions (75) and (78) will still be satisfied (the idea of constructing analytic perturbations by approximating smooth parametric families of perturbations can be traced back to [16], it was also used in [51]).

We construct the C^3-smooth Hamiltonian $H^{(1)}$ localised in a small neighbourhood of the cylinders $\Phi(B_1)$, $\Phi(B_2)$, $\Phi(B_3)$, $\Phi(B_4)$. Thus, the maps $X^{(1)}_{\mu_1}$ differ from identity only in a small neighbourhood of these cylinders, so the maps $\Phi^{(1)}_{\mu_1}$ differ from Φ in a small neighbourhood of the cylinders $B_1, \ldots, B_4$ only. The perturbation we build near one of these cylinders does not affect the scattering maps near the other cylinders, so we restrict our attention to the cylinder B_1 only. We further omit the subscript "1" whenever possible and let $\tau = \mu_1$. Thus we consider a homoclinic cylinder B and continue with building a C^3-smooth Hamiltonian H localised in a small neighbourhood of the cylinder $\Phi(B)$ such that for the corresponding flow map X_τ the derivative with respect to τ of the scattering map F defined by the map $\Phi_\tau = X_\tau \circ \Phi$ satisfies, for all $(\varphi, y) \in A_s$, the following inequalities:

$$\begin{aligned} \left|\frac{\partial \Psi}{\partial \tau}\right|_{\tau=0} &< 1 \quad \text{for all } \varphi \in \mathbb{S}^1, \\ \left.\frac{\partial Y}{\partial \tau}\right|_{\tau=0} &> 2(L+1) \quad \text{for all } \varphi \in J, \end{aligned} \tag{90}$$

where J is a certain arc that does not contain the whole $\mathbb{S}^1$, and

$$F(A_s \cap \{\varphi \in J\}) \subset \{\varphi \in \hat{J}\} \tag{91}$$

where $\hat{J}$ is an arc that does not contain the whole of $\mathbb{S}^1$ [see (74)].

Let w^u denote a piece of the unstable manifold $W^u(A)$ that contains the cylinder B [i.e. w^u is a small neighbourhood of the cylinder B in $W^u(A)$] and w^s be a small neighbourhood of $\Phi(B)$ in $W^s(A)$, so $B = \Phi(w^u) \cap w^s$. Since the map Φ_τ differs from Φ in a small neighbourhood of the cylinder B only, the pieces w^u and w^s do not depend on τ, nor the strong unstable foliation of the piece of $W^u(A)$ between A and w^u depends on τ, neither the strong stable foliation of the piece of $W^s(A)$ between w^s and A does. Thus, given any C^1-family of cylinders B_τ close to B the projection map $\pi^u_{B_\tau} : B_\tau \to A$ by the leaves of the strong unstable foliation is of class C^1; moreover, if two such families of cylinders are C^1-close, then the corresponding projection maps $\pi^u_{B_\tau}$ are also C^1-close. The same holds true for the projection map $\pi^s_{B'_\tau} : B'_\tau \to A$ by the leaves of the strong stable foliation, where we denote as B'_τ any C^1-family of cylinders close to $\Phi(B)$. As the perturbation X_τ is localised in a small neighbourhood of the cylinder $\Phi(B)$, we find that the scattering map F satisfies

$$F = F_0^{-1} \circ \pi^s_{B'_\tau} \circ X_\tau \circ \Phi \circ (\pi^u_{B_\tau})^{-1}, \tag{92}$$

where $B_\tau = w^u \cap \Phi_\tau^{-1}(w^s)$ is a homocinic cylinder close to B, and $B'_\tau = \Phi_\tau(B_\tau)$. If we add to the family X_τ any C^1-small perturbation localised in a small neighbourhood of $\Phi(B)$, this will result in C^1-small perturbations of the family of cylinders B'_τ and B_τ. Thus, the perturbation to the corresponding family of scattering maps defined by (92) will be also C^1-small. It follows that it is enough to build a C^1-family of maps X_τ (generated by a C^2-smooth Hamiltonian H) localised in a small neighbourhood of the cylinder $\Phi(B)$ such that the corresponding family of scattering maps satisfies (90). Then any C^3-Hamiltonian which is C^2-close to H and is localised in a small neighbourhood of $\Phi(B)$ produces a family of scattering maps that still satisfies (90).

This reduction of smoothness requirement (from $H \in C^3$ to $H \in C^2$) is important since it allows to construct the Hamiltonian H such that the vector field it generates is tangent to the given homoclinic cylinder B (for which only C^2-smoothness can be guaranteed by our spectral gap assumptions). Once this is done, the cylinder $\Phi(B)$ will be invariant with respect to the map X_τ, i.e. $\Phi_\tau(B) = \Phi(B)$ for all τ. This means the trajectory of B remains the same for all τ, i.e. it remains a homoclinic cylinder. Thus, formula (92) for the scattering map will recast as

$$F = F_0^{-1} \circ \pi^s_{\Phi(B)} \circ X_\tau \circ \Phi \circ (\pi^u_B)^{-1}, \tag{93}$$

and the only τ-dependent term in the right-hand side is X_τ.

In order to build the required Hamiltonian, we introduce C^2-coordinates (x, v) near $\Phi(B)$ such that the cylinder $\Phi(B)$ is given by $x = 0$ (so v gives the coordinates on the cylinder and x runs a neighbourhood of zero in $\mathbb{R}^{2d-2}$). The cylinder is transverse to the strong-stable and strong-unstable foliations, so if we denote as $N(v)$ the direct sum of the tangents to the leaves of the strong-stable and unstable foliations that pass through the point $(x = 0, v) \in \Phi(B)$, then the field $N(v)$ will have a form $dv = P(v)dx$. Note that N depends smoothly on v [as the fields of tangents to the strong stable and strong unstable leaves are smooth when the large spectral gap assumption (6) is fulfilled], i.e. the function $P(v)$ is at least C^1. As the homoclinic cylinder $\Phi(B)$ belongs both to the stable and unstable manifolds of A, it follows from Proposition 3 that a vector is tangent to $\Phi(B)$ if an only if it is Ω-orthogonal to N. Thus, the vector field $\tilde{X} = \Omega^{-1}\nabla H$ generated by the Hamiltonian H will be tangent to $\Phi(B)$ if the gradient of H is orthogonal to N at the points of $\Phi(B)$, i.e.

$$\frac{\partial H}{\partial x}(0, v) + \frac{\partial H}{\partial v}(0, v)P(v) = 0. \tag{94}$$

This condition is satisfied e.g. by any function of the form

$$H(x, v) = h(v) - \sum_{i=1}^{2d-2} x_i \int p_i(v_1 + s_1 x_i, v_2 + s_2 x_i)\xi(s_1, s_2)d^2 s$$

where h is any C^2-function on $\Phi(B)$, the vector-function $p(v) = (p_1(v), \ldots, p_{2d-2}(v))$ is given by $p(v) = h'(v)P(v)$, the x_i's are the coordinates of the vector x, and $(v_1, v_2) = v$, and ξ is a C^2-smooth function on a plane, localised in a small neighbourhood of zero, such that $\int \xi(s)d^2 s = 1$. Integrating by parts, we find

$$\frac{\partial H}{\partial x_i} = \int p_i(v + s x_i)[s\xi'(s) + \xi(s)]d^2 s, \quad \frac{\partial H}{\partial v_j} = \frac{\partial h}{\partial v_j}(v) + \sum_{i=1}^{2d-2} \int p_i(v + s x_i)\frac{\partial x_i}{\partial s_j} d^2 s.$$

After substituting $x = 0$ into these formulas, we see that (94) is satisfied indeed. Since $q \in C^1$ and $\xi \in C^2$, it follows that $H \in C^2$, so given any C^2-function h on the cylinder $\Phi(B)$ we can extend it to a C^2-function H defined in a neighbourhood of this cylinder, such that the vector field generated by the Hamiltonian H is tangent to the cylinder.

As we explained above, under this condition the scattering map is given by (93), so the vector field

$$\tilde{F} = \left(\tilde{\Psi} = \left.\frac{\partial \Psi}{\partial \tau}\right|_{\tau=0}, \tilde{Y} = \left.\frac{\partial Y}{\partial \tau}\right|_{\tau=0}\right)$$

of the τ-derivatives of the scattering map F on the cylinder A is given by

$$\tilde{F} = \frac{\partial}{\partial v}\left(F_0^{-1} \circ \pi^s_{\Phi(B)}\right) \circ \tilde{X} \circ \Phi \circ (\pi^u_B)^{-1}, \tag{95}$$

where $\tilde{X} = \Omega^{-1}(v)h'(v)$ is the vector field of the flow on the cylinder $\Phi(B)$, which is generated by the Hamiltonian h. Let $\Omega(v)$ denote the antisymmetric (2×2)-matrix that defines the restriction of the symplectic form on the cylinder at the point v. In order to satisfy (90), we need to have

$$\begin{aligned} |\tilde{\Psi}| < 1 \quad & \text{for all } \varphi \in \mathbb{S}^1, \\ \tilde{Y} > 2(L+1) \quad & \text{for all } \varphi \in J. \end{aligned} \tag{96}$$

It is seen from (95) that if conditions (96) are satisfied by $\tilde{F}$ for some choice of the vector field $\tilde{X}$, they are satisfied by $\tilde{F}$ for any C^0-small perturbation of $\tilde{X}$. Thus, it is enough to find any C^1-smooth Hamiltonian function $h(v)$ such that the field $\tilde{F}$ defined by (95) satisfies (96), then for any C^2-smooth function which is C^1-close to h the derivative of the scattering map F with respect to τ will satisfy (90), and the lemma will be proven.

In order to build the sought C^1-function $h(v)$, we introduce C^1-coordinates $v = (\varphi, y)$ on the cylinder $\Phi(B)$ such that the diffeomorphism $F_0^{-1} \circ \pi^s_{\Phi(B)} : \Phi(B) \to A$ is identity. Then (95) recasts as

$$\tilde{F} = \tilde{X} \circ F|_{\tau=0}$$

[see (93)]. As $\tilde{X}$ is a Hamiltonian vector field, its φ-component is given by $-\omega^{-1}\dfrac{\partial h}{\partial y}$ and the y-component is $\omega^{-1}\frac{\partial h}{\partial \varphi}$, where the C^0-function $\omega(\varphi, y) > 0$ is such that $\omega(\varphi, y)\, dy \wedge d\varphi$ is the symplectic form on the cylinder $\Phi(B)$. Thus, conditions (96) take the form

$$\begin{aligned} \left|\frac{\partial h}{\partial y}\right| &< \omega(\varphi, y) \quad \text{for all } (\varphi, y) \in F(A_s), \\ \frac{\partial h}{\partial \varphi} &> 2(L+1)\omega(\varphi, y) \quad \text{for all } (\varphi, y) \in F(A_s \cap \{\varphi \in J\}). \end{aligned}$$

We finish the proof of the lemma by noticing that these conditions are satisfied by a y-independent function h such that

$$h(\varphi) = M\varphi \quad \text{at} \quad \varphi \in \hat{J}$$

where the constant M is given by $M = 1 + 2(L+1)\sup_{F(A_s)} \omega$, and the arc $\hat{J}$ is defined by (91). Since h must be periodic in φ, it is important that $\hat{J}$ does not cover the whole of $\mathbb{S}^1$. $\square$

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References

1. Arnold, V.I.: Instability of dynamical systems with many degrees of freedom. Dokl. Akad. Nauk SSSR **156**, 9–12 (1964) **(Russian)**
2. Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Mathematical Aspects of Classical and Celestial Mechanics. Dynamical Systems III, 3rd edn. Encyclopaedia of Mathematical Sciences, vol. 3. Springer, Berlin (2006)
3. Bernard, P.: Perturbation of a partially hyperbolic Hamiltonian system. C. R. Acad. Sci. Paris Sr. I Math. **323**(2), 189–194 (1996) **(French)**
4. Bernard, P.: The dynamics of pseudographs in convex Hamiltonian systems. J. Am. Math. Soc. **21**(3), 615–669 (2008)
5. Bernard, P.: Arnold' diffusion: from the a priori unstable to the a priori stable case. In: Proceedings of the International Congress of Mathematicians, Hyderabad, India, vol. III, pp. 1680–1700. Hindustan Book Agency, New Delhi (2010)
6. Bernard, P.: Large normally hyperbolic cylinders in a priori stable Hamiltonian systems. Ann. Henri Poincarè **11**(5), 929–942 (2010)
7. Berti, M., Biasco, L., Bolle, P.: Drift in phase space: a new variational mechanism with optimal diffusion time. J. Math. Pures Appl. (9) **82**(6), 613–664 (2003)
8. Berti, M., Bolle, P.: Fast Arnold diffusion in systems with three time scales. Discrete Contin. Dyn. Syst. **8**(3), 795–811 (2002)
9. Bessi, U.: Arnold's diffusion with two resonances. J. Differ. Equ. **137**(2), 211–239 (1997)
10. Bolotin, S., Treschev, D.: Unbounded growth of energy in nonautonomous Hamiltonian systems. Nonlinearity **12**(2), 365–388 (1999)
11. Bourgain, J., Kaloshin, V.: On diffusion in high-dimensional Hamiltonian systems. J. Funct. Anal. **229**(1), 1–61 (2005)
12. Bounemoura, A., Pennamen, E.: Instability for a priori unstable Hamiltonian systems: a dynamical approach. Discrete Contin. Dyn. Syst. **32**(3), 753–793 (2012)
13. Bounemoura, A.: Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians. J. Differ. Equ. **249**(11), 2905–2920 (2010)
14. Bounemoura, A., Marco, J.-P.: Improved exponential stability for near-integrable quasi-convex Hamiltonians. Nonlinearity **24**(1), 97–112 (2011)
15. Bounemoura, A., Fayad, B., Niederman, L.: Double Exponential Stability for Generic Real-Analytic Elliptic Equilibrium Points (2015). arXiv:1509.00285
16. Broer, H.W., Tangerman, F.M.: From a differentiable to a real analytic perturbation theory, applications to the Kupka–Smale theorems. Ergod. Theory Dyn. Syst. **6**, 345–362 (1986)
17. Castejon, O., Kaloshin, V.: Random Iteration of Maps on a Cylinder and Diffusive Behavior (2015). arXiv:1501.03319
18. Chierchia, L., Gallavotti, G.: Drift and diffusion in phase space. Ann. Inst. H. Poincarè Phys. Théor. **60**(1),144 (1994) **[erratum, Ann. Inst. H. Poincarè Phys. Théor. 68(1),135 (1998)]**
19. Cheng, C.-Q., Yan, J.: Existence of diffusion orbits in a priori unstable Hamiltonian systems. J. Differ. Geom. **67**(3), 457–517 (2004)
20. Cheng, C.-Q., Yan, J.: Arnold diffusion in Hamiltonian systems: a priori unstable case. J. Differ. Geom. **82**(2), 229–277 (2009)
21. Cheng, C.-Q.: Arnold Diffusion in Nearly Integrable Hamiltonian Systems, p. 127 (2013) **(preprint)**. arXiv:1207.4016v2
22. Cresson, J.: Symbolic dynamics and Arnold diffusion. J. Differ. Equ. **187**(2), 269–292 (2003)
23. Cresson, J., Wiggins, S.: A λ-Lemma for Normally-Hyperbolic Invariant Manifolds (2005). arXiv:math/0510645 **(preprint)**
24. de la Llave, R.: Some recent progress in geometric methods in the instability problem in Hamiltonian mechanics. In: International Congress of Mathematicians, vol. II, pp. 1705–1729. European Mathematical Society, Zurich (2006)
25. Delshams, A., Gelfreich, V., Jorba, A., Seara, T.-M.: Exponentially small splitting of separatrices under fast quasiperiodic forcing. Commun. Math. Phys. **189**, 35–71 (1997)
26. Delshams, A., Gidea, M., Roldán, P.: Transition map and shadowing lemma for normally hyperbolic invariant manifolds. Discrete Contin. Dyn. Syst. **33**(3), 1089–1112 (2013)
27. Delshams, A., Huguet, G.: Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems. Nonlinearity **22**(8), 1997–2077 (2009)
28. Delshams, A., Huguet, G.: A geometric mechanism of diffusion: rigorous verification in a priori unstable Hamiltonian systems. J. Differ. Equ. **250**(5), 2601–2623 (2011)
29. Delshams, A., de la Llave, R., Seara, T.M.: A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbf{T}^2$. Commun. Math. Phys. **209**(2), 353–392 (2000)
30. Delshams, A., de la Llave, R., Seara, T.M.: Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows. Adv. Math. **202**(1), 64–188 (2006)

31. Delshams, A., de la Llave, R., Seara, T.M.: Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows. Adv. Math. **202**(1), 64–188 (2006)
32. Delshams, A., de la Llave, R., Seara, T.M.: A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. Memoirs of the American Mathematical Society, vol. 179, no. 844. American Mathematical Society, Providence, RI (2006)
33. Delshams, A., de la Llave, R., Seara T.M.: Geometric approaches to the problem of instability in Hamiltonian systems. An informal presentation. In: Chreg, W. (ed.) Hamiltonian Dynamical Systems and Applications, pp. 285–336. Springer, Berlin (2008)
34. Delshams, A., de la Llave, R., Seara, T.M. Instability of high Dimensional Hamiltonian Systems: Multiple Resonances Do Not Impede Diffusion. Adv. Math. **294**, 689–755 (2016)
35. Douady, R.: Stabilité ou instabilité des points fixes elliptiques [Stability or instability of elliptic fixed points]. Ann. Sci. école Norm. Sup. (4) **21**(1), 1–46 (1988) **(French)**
36. Douady, R., Le Calvez, P.: Example of a non-topologically stable elliptic fixed point in dimension 4. C. R. Acad. Sci. Paris Sér. I Math. **296**(21), 895–898 (1983) **(French)**
37. Easton, R.W., Meiss, J.D., Roberts, G.: Drift by coupling to an anti-integrable limit. Phys. D **156**(3–4), 201–218 (2001)
38. Fenichel, N.: Persistence and smoothness of invariant manifolds for flows. Indiana Univ. Math. J. **21**, 193–226 (1971-1972)
39. Fermi, E.: Beweis, dass ein mechanisches Normalsystem im allgemeinen quasi-ergodisch ist. Phys. Z. **24**, 261–265 (1923)
40. Fontich, E., Martin, P.: Arnold diffusion in perturbations of analytic exact symplectic maps. Nonlinear Anal. **42**(8), 1397–1412 (2000)
41. Fontich, E., Martin, P.: Arnold diffusion in perturbations of analytic integrable Hamiltonian systems. Discrete Contin. Dynam. Syst. **7**(1), 61–84 (2001)
42. Gallavotti, G.: Arnold's diffusion in isochronous systems. Math. Phys. Anal. Geom. **1**(4), 295–312 (1998/1999)
43. Gallavotti, G., Gentile, G., Mastropietro, V.: Hamilton–Jacobi equation, heteroclinic chains and Arnold diffusion in three time scale systems. Nonlinearity **13**(2), 323–334 (2000)
44. Gelfreich, V., Turaev, D.: Unbounded energy growth in hamiltonian systems with a slowly varying parameter. Commun. Math. Phys. **283**(3), 769–794 (2008)
45. Gelfreich, V., Turaev, D.: Fermi acceleration in non-autonomous billiards. J. Phys. A **41**, 212003 (2008)
46. Gidea, M., Robinson, C.: Shadowing orbits for transition chains of invariant tori alternating with Birkhoff zones of instability. Nonlinearity **20**(5), 1115–1143 (2007)
47. Gidea, M., Robinson, C.: Obstruction argument for transition chains of tori interspersed with gaps. Discrete Contin. Dyn. Syst. Ser. S **2**(2), 393–416 (2009)
48. Gidea, M., Zgliczynski, P.: Covering relations for multidimensional dynamical systems. II. J. Differ. Equ. **202**(1), 59–80 (2004)
49. Gidea, M., de la Llave, R.: Topological methods in the instability problem of Hamiltonian systems. Discrete Contin. Dyn. Syst. **14**(2), 295–328 (2006)
50. Gidea, M., de la Llave, R., Seara, T.: A General Mechanism of Diffusion in Hamiltonian Systems: Qualitative Results (2014). arXiv:1405.0866 **(preprint)**
51. Gonchenko, S.V., Turaev, D., Shilnikov, L.: Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps. Nonlinearity 20, 241–275 (2007)
52. Guardia, M., Kaloshin, V.: Orbits of Nearly Integrable Systems Accumulating to KAM Tori (2014). arXiv:1412.7088
53. Guardia, M., Kaloshin, V., Zhang, J.: A second order expansion of the separatrix map for trigonometric perturbations of a priori unstable systems. Commun. Math. Phys. **348**(1), 321–361 (2016)
54. Guzzo, M., Lega, E., Froeschlé, C.: A numerical study of Arnold diffusion in a priori unstable systems. Commun. Math. Phys. **290**(2), 557–576 (2009)
55. Herman, M.-R.: Sur les courbes invariantes par les diffeomorphismes de l'anneau, vol. 1 [On the Curves Invariant Under Diffeomorphisms of the Annulus, vol. 1] Astérisque, pp. 103–104. Soc. Math. de France, Paris (1983) **(French)**
56. Hirsch, M.W., Pugh, C.C., Shub, M.: Invariant Manifolds. Lecture Notes in Mathematics, vol. 583, p. 149. Springer, Berlin (1977)
57. Holmes, P.J., Marsden, J.E.: Melnikov's method and Arnold diffusion for perturbations of integrable Hamiltonian systems. J. Math. Phys. **23**(4), 669–675 (1982)
58. Jones, C.K.R.T., Tin, S.-K.: Generalized exchange lemmas and orbits heteroclinic to invariant manifolds. Discrete Contin. Dyn. Syst. Ser. S **2**(4), 967–1023 (2009)
59. Kaloshin, V., Mather, J., Valdinoci, E.: Instability of resonant totally elliptic points of symplectic maps in dimension 4. Analyse complexe, systemes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II. Astérisque **297**, 79–116 (2004)

60. Kaloshin, V.: Geometric proofs of Mather's connecting and accelerating theorems. In: Topics in Dynamics and Ergodic Theory. London Mathematical Society Lecture Note Series No. 310, pp. 81–106. Cambridge University Press, Cambridge (2003)
61. Kaloshin, V., Levi, M.: Geometry of Arnold diffusion. SIAM Rev. **50**(4), 702–720 (2008)
62. Kaloshin, V., Levi, M.: An example of Arnold diffusion for near-integrable Hamiltonians. Bull. Am. Math. Soc. **45**(3), 409–427 (2008)
63. Kaloshin, V., Saprykina, M.: An example of a nearly integrable Hamiltonian system with a trajectory dense in a set of maximal Hausdorff dimension. Commun. Math. Phys. **315**(3), 643–697 (2012)
64. Kaloshin, V., Zhang, K.: A Strong Form of Arnold Diffusion for Two and a Half Degrees of Freedom (2012). arXiv:1212.1150 **(preprint)**
65. Kaloshin, V., Zhang, K.: A Strong Form of Arnold Diffusion for Three and a Half Degrees of Freedom (2014) **(preprint)**
66. Kaloshin, V., Zhang, J., Zhang, K.: Normally Hyperbolic Invariant Laminations and Diffusive Behaviour for the Generalized Arnold Example Away from Resonances (2015). arXiv:1511.04835
67. Le Calvez, P.: Drift for families of twist maps on the annulus. Ergod. Theory Dyn. Syst. **27**, 869–879 (2007)
68. Lochak, P., Marco, J.-P.: Diffusion times and stability exponents for nearly integrable analytic systems. Cent. Eur. J. Math. **3**(3), 342–397 (2005)
69. Lochak, P., Marco, J.-P., Sauzin, D.: On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems. Memoirs of the American Mathematical Society, vol. 163, no. 775. American Mathematical Society, Providence, RI (2003)
70. Marco, J.-P.: Transition along chains of invariant tori for analytic Hamiltonian systems. Ann. Inst. H. Poincaré Phys. Thor. **64**(3), 205–252 (1996) **(French)**
71. Marco, J.-P.: Arnold diffusion for cusp-generic nearly integrable convex systems on $\mathbb{A}^3$ (2016). arXiv:1602.02403 **(preprint)**
72. Markus, L., Meyer, K.R.: Generic Hamiltonian dynamical systems are neither integrable nor ergodic. Mem. AMS **144**, 52 (1974)
73. Mather, J.N.: Arnold diffusion: announcement of results. J. Math. Sci. **124**(5), 5275–5289 (2004)
74. Mather, J.N.: Arnold diffusion by variational methods. In: Pardalos, P.M., Rassias, T. (eds.) Essays in Mathematics and Its Applications, pp. 271–285. Springer, Heidelberg (2012)
75. Moeckel, R.: Generic drift on Cantor sets of annuli. In: Chenciner, A., Cushman, R., Robinson, C., Xia, Z.J. (eds.) Celestial Mechanics (Evanston, IL, 1999). Contemporary Mathematics, vol. 292, pp. 163–171. American Mathematical Society, Providence (2002)
76. Moeckel, R.: Transition tori in the five-body problem. J. Differ. Equ. **129**, 290–314 (1996)
77. Nassiri, M., Pujals, E.R.: Robust transitivity in Hamiltonian dynamics. Ann. Sci. Norm. Sup. (4) **45**(2), 191–239 (2012)
78. Nekhoroshev, N.N.: An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. Russ. Math. Surv. **32**(6), 1–65 (1977)
79. Palis J., de Melo W. A Geometrical Introduction to Dynamical Systems. Springer, Berlin (1982)
80. Piftankin, G.N.: Diffusion speed in the Mather problem. Nonlinearity **19**, 2617–2644 (2006)
81. Piftankin, G.N., Treshchev, D.V.: Separatrix maps in Hamiltonian systems. Russ. Math. Surv. **62**(2), 219–322 (2007)
82. Procesi, M.: Exponentially small splitting and Arnold diffusion for multiple time scale systems. Rev. Math. Phys. **15**(4), 339–386 (2003)
83. Pugh, C., Shub, M., Wilkinson, A.: Hölder Foliations Revisited, (2011). arXiv:1112.2646 **(preprint)**
84. Robinson, C.: Symbolic dynamics for transition tori. In: Chenciner, A., Cushman, R., Robinson, C., Xia, Z.J. (eds.) Celestial Mechanics (Evanston, IL, 1999). Contemporary Mathematics, vol. 292, pp. 199–208. American Mathematical Society, Providence (2002)
85. Shilnikov, L.P.: On the question of the structure of the neighborhood of a homoclinic tube of an invariant torus. Soviet Math. Dokl. **9**, 624–628 (1968)
86. Shilnikov, L.P., Shilnikov, A., Turaev, D., Chua, L.: Methods of qualitative theory in nonlinear dynamics. Part I. World Scientific Publishing, Singapore (1998)
87. Treschev, D.V.: Evolution of slow variables in a priori unstable Hamiltonian systems. Nonlinearity **17**(5), 1803–1841 (2004)
88. Treschev, D.: Arnold diffusion far from strong resonances in multidimensional a priori unstable Hamiltonian systems. Nonlinearity **25**(9), 2717–2757 (2012)
89. Tennyson, J.L., Lieberman, M.A., Lichtenberg, A.J.: Diffusion in near-integrable Hamiltonian systems with three degrees of freedom. In: Month, M., Herrera, J.C. (eds.) Nonlinear Dynamics and the Beam–Beam Interaction, vol. 57, pp. 272–301. American Institute of Physics, New York (1979)

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